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Isolated singularities of super-polyharmonic functions

(Dedicated to Professor Hiroshi Yamaguchi on the occasion of his 60th birthday)

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Abstract. We consider a Riesz decomposition theorem for lower semicontinuous and locally integrable functions u on the punctured unit ball such that $(-\Delta)^m u$ is a nonnegative measure and u satisfies certain growth condition on surface integrals.

Key words: super-polyharmonic functions, isolated singularities, Bôcher's theorem, Laurent series expansion, Riesz decomposition.

1. Notation and statement of results

A function u on an open set $\Omega \subset \mathbf{R}^n$, where $n \geq 2$, is called polyharmonic of order m if $(-\Delta)^m u = 0$ on Ω, where m is a positive integer, $(-\Delta)^m = (-1)^m \Delta^m$ and Δ^m denotes the Laplace operator iterated m times. We say that a lower semicontinuous and locally integrable function u on Ω is super-polyharmonic of order m in Ω if every point of Ω is a Lebesgue point of u and $(-\Delta)^m u$ is a nonnegative measure on Ω, that is,

$$\int_\Omega u(x)(-\Delta)^m \varphi(x)\,dx \geq 0 \quad \text{for all nonnegative } \varphi \in C_0^\infty(\Omega).$$

We denote by $\mathcal{H}^m(\Omega)$ and $\mathcal{SH}^m(\Omega)$ the space of polyharmonic functions of order m on Ω and the space of super-polyharmonic functions of order m on Ω.

For a multi-index $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and a point $x = (x_1, x_2, \ldots, x_n)$, we set

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n,$$
$$\lambda! = \lambda_1!\lambda_2!\cdots\lambda_n!,$$
$$x^\lambda = {x_1}^{\lambda_1}{x_2}^{\lambda_2}\cdots{x_n}^{\lambda_n}$$

and

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$$D^\lambda = \left(\frac{\partial}{\partial x}\right)^\lambda = \left(\frac{\partial}{\partial x_1}\right)^{\lambda_1} \left(\frac{\partial}{\partial x_2}\right)^{\lambda_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\lambda_n}.$$

Consider the Riesz kernel of order $2m$ defined by

$$R_{2m}(x) = \begin{cases} |x|^{2m-n} & \text{if } n \text{ is odd or } n > 2m, \\ |x|^{2m-n}\log(1/|x|) & \text{if } n \text{ is even and } n \leq 2m, \end{cases}$$

and the remainder term in the Taylor expansion of R_{2m}, given by

$$R_{2m,L}(\zeta, x) = R_{2m}(\zeta - x) - \sum_{|\lambda| \leq L} \frac{\zeta^\lambda}{\lambda!}(D^\lambda R_{2m})(-x),$$

where L is a real number; in case $L < 0$, set $R_{2m,L}(\zeta, x) = R_{2m}(\zeta - x)$. Here note that $R_{2m} \in \mathcal{H}^m(\mathbf{R}^n \setminus \{0\})$ and $(-\Delta)^m R_{2m} = \alpha_m^{-1}\delta_0$ with the Dirac measure δ_x at x and a constant $\alpha_m \neq 0$.

The open ball and the sphere centered at x with radius r are denoted by $B(x, r)$ and $S(x, r)$. We write $B(r) = B(0, r)$ and $S(r) = S(0, r)$. We also denote by $\mathbf{B}$ and $\mathbf{B}_0$ the unit ball $B(1)$ and the punctured unit ball $\mathbf{B} \setminus \{0\}$ respectively.

For a Borel measurable function u on $\mathbf{R}^n$, we define the average integral over $S(r)$ by

$$⨍_{S(r)} u\, dS = \frac{1}{|S(r)|}\int_{S(r)} u\, dS,$$

where $|S(r)|$ denotes the surface measure of $S(r)$.

We know that $u \in \mathcal{H}^m(\mathbf{B}_0)$ can be expressed as a Laurent series expansion:

$$u(x) = h(x) + \sum_\lambda c(\lambda) D^\lambda R_{2m}(x), \tag{1.1}$$

where $h \in \mathcal{H}^m(\mathbf{B})$, $c(\lambda)$ are constants, and the series converges absolutely and uniformly on compact subsets of $\mathbf{B}_0$; see e.g. [6, Chapter 10].

First we give a condition for the sum in (1.1) to have only finitely many terms.

Theorem 1.1 *If $u \in \mathcal{H}^m(\mathbf{B}_0)$ satisfies*

$$\liminf_{r \to 0} r^s ⨍_{S(r)} |u|\, dS = 0 \tag{1.2}$$

for a real number s, then u is of the form

$$u(x) = h(x) + \sum_{|\lambda| < s+2m-n} c(\lambda) D^{\lambda} R_{2m}(x),$$

where $h \in \mathcal{H}^m(\mathbf{B})$ *and* $c(\lambda)$ *are constants.*

In particular, in case $s + 2m - n \leq 0$, *u can be extended to a polyharmonic function of order m on* $\mathbf{B}$.

Theorem 1.2 *Let* $s > n - 2$ *and suppose* $u \in \mathcal{H}^m(\mathbf{B}_0)$ *satisfies*

$$\liminf_{r \to 0} r^s \fint_{S(r)} u^+ \, dS = 0, \tag{1.3}$$

where $u^+ = \max\{u, 0\}$. *Then u is of the form*

$$u(x) = h(x) + \sum_{|\lambda| < s+2m-n} c(\lambda) D^{\lambda} R_{2m}(x),$$

where $h \in \mathcal{H}^m(\mathbf{B})$ *and* $c(\lambda)$ *are constants.*

Corollary 1.1 *If* $u \in \mathcal{H}^m(\mathbf{B}_0)$ *is nonnegative in* $\mathbf{B}_0$, *then u is of the form*

$$u(x) = h(x) + \sum_{|\lambda| \leq 2(m-1)} c(\lambda) D^{\lambda} R_{2m}(x),$$

where $h \in \mathcal{H}^m(\mathbf{B})$ *and* $c(\lambda)$ *are constants.*

Armitage kindly informed the second author that he had obtained the present theorems. But, for reader's convenience, we give proofs in the next section, along the same lines as Armitage [4].

Suppose $u \in \mathcal{SH}^m(2\mathbf{B}_0)$ and $\mu = (-\Delta)^m u$ is the Riesz measure on $2\mathbf{B}_0$, where $2\mathbf{B}_0 = B(2) \setminus \{0\}$. Then, as in the book of Hayman-Kennedy [11] and our paper [9], u can be represented as

$$u(x) = v(x) + \alpha_m \int_{\mathbf{B}_0} R_{2m,\, L(|\zeta|)}(\zeta, x) \, d\mu(\zeta) \tag{1.4}$$

where $v \in \mathcal{H}^m(\mathbf{B}_0)$ and $L(r)$ is a nonincreasing positive function on $(0, 1]$ such that $L(r) \geq 2m - n$.

The following theorem gives a condition which ensures that L is bounded.

Theorem 1.3 *Let $u \in \mathcal{SH}^m(2\mathbf{B}_0)$ and $\mu = (-\Delta)^m u$. If $s > n-2$ and*

$$\lim_{r\to 0} r^s \fint_{S(r)} u\, dS = 0, \tag{1.5}$$

then

$$u(x) = v(x) + \alpha_m \int_{\mathbf{B}_0} R_{2m,L}(\zeta, x) d\mu(\zeta)$$

holds for $x \in \mathbf{B}_0$, where $v \in \mathcal{H}^m(\mathbf{B}_0)$ and L is the integer such that $s + 2m - n - 1 < L \le s + 2m - n$.

Theorem 1.4 *Let u be as in* Theorem 1.3. *If in addition*

$$\liminf_{r\to 0} r^s \fint_{S(r)} u^+\, dS = 0, \tag{1.6}$$

then u is of the form

$$u(x) = h(x) + \sum_{|\lambda|\le L} c(\lambda) D^\lambda R_{2m}(x) + \alpha_m \int_{\mathbf{B}_0} R_{2m,L}(\zeta, x)\, d\mu(\zeta),$$

where $h \in \mathcal{H}^m(\mathbf{B})$, $c(\lambda)$ are constants and L is the integer such that $s + 2m - n - 1 < L \le s + 2m - n$.

Remark 1.1 In Theorem 1.4, we can replace condition (1.6) by

$$\liminf_{r\to 0} r^s \fint_{S(r)} u^-\, dS = 0, \tag{1.7}$$

where $u^- = \max\{-u, 0\}$.

2. Proofs of Theorems 1.1 and 1.2

We know the following Almansi expansion for polyharmonic functions (see [5, Proposition 1.1.3] and [15]).

Lemma 2.1 *If $h \in \mathcal{H}^m(\mathbf{B})$, then there exist harmonic functions $h_0, \dots, h_{m-1}$ such that*

$$h(x) = \sum_{k=0}^{m-1} |x|^{2k} h_k(x) \quad \textit{for } x \in \mathbf{B}.$$

We write $\mathcal{HP}_k^m$ for the space of all homogeneous polynomials of degree

k which are polyharmonic of order m.

By the orthogonality property [6, Theorem 5.3] and Almansi expansion, we have the following result.

Lemma 2.2 *If $P_k \in \mathcal{HP}_k^m$, $Q_\ell \in \mathcal{HP}_\ell^{m'}$ and $k - \ell \neq 2i$ with $-m' + 1 \leq i \leq m - 1$, then*

$$\int_{S(r)} P_k Q_\ell \, dS = 0. \tag{2.1}$$

Further we need the following easy fact.

Lemma 2.3 *For a multi-index λ, $D^\lambda R_{2m}$ is of the form*

$$D^\lambda R_{2m}(x) = \frac{P_\lambda(x) + Q_\lambda(x)\log|x|}{|x|^{n-2m+2|\lambda|}},$$

where $P_\lambda \in \mathcal{HP}_{|\lambda|}^m$ and $Q_\lambda \in \mathcal{HP}_{|\lambda|}^m$. (The logarithmic term does not appear unless n is even, $n \leq 2m$ and $|\lambda| \leq 2m - n$.)

Proof of Theorem 1.1. In view of (1.1), we see that

$$u(x) = h(x) + \sum_\lambda c(\lambda) D^\lambda R_{2m}(x),$$

where $h \in \mathcal{H}^m(\mathbf{B})$ and $c(\lambda)$ are constants. By Lemma 2.1, $h \in \mathcal{H}^m(\mathbf{B})$ has an Almansi expansion

$$h(x) = \sum_{j=0}^{m-1} |x|^{2j} h_j(x) \quad (h_j \in \mathcal{H}^1(\mathbf{B})).$$

Using homogeneous expansion for harmonic functions (see [6]), we write

$$h(x) = \sum_{j=0}^{m-1} |x|^{2j} \sum_{i=0}^{\infty} H_{j,i}(x) \quad (H_{j,i} \in \mathcal{HP}_i^1).$$

Here we set $H_k(x) = \sum_{j=0}^{m-1} |x|^{2j} H_{j,k-2j}(x)$ with $H_{j,i} \equiv 0$ for $i < 0$. Then $H_k \in \mathcal{HP}_k^m$ and

$$h(x) = \sum_{k=0}^{\infty} H_k(x).$$

With the aid of Lemma 2.3, we have

$$\sum_{|\lambda|=k} c(\lambda) D^{\lambda} R_{2m}(x) = \frac{P_k(x) + Q_k(x)\log|x|}{|x|^{n-2m+2k}},$$

where P_k, $Q_k \in \mathcal{HP}_k^m$ and $Q_k \equiv 0$ unless n is even, $2m \ge n$ and $k \le 2m-n$. Then we have

$$u(x) = \sum_{k=0}^{\infty} H_k(x) + \sum_{k=0}^{\infty} \frac{P_k(x) + Q_k(x)\log|x|}{|x|^{n-2m+2k}}.$$

In what follows it suffices to show that $P_\ell \equiv Q_\ell \equiv 0$ for $\ell \ge s+2m-n$. If n is even, $2m \ge n$ and $k \le (2m-n)/2$, then, since $P_k(x)/|x|^{n-2m+2k} \in \mathcal{H}^m(\mathbf{R}^n)$, we may assume that $P_k \equiv 0$; further, if n is even, $2m \ge n$ and $0 < k-(2m-n)/2 \le m-1$, then, noting that

$$\frac{P_k(x)}{|x|^{n-2m+2k}} = h_k'(x) + \frac{P_k'(x)}{|x|^{n-2m+2k}}$$

where $h_k' \in \mathcal{H}^{m-k+(2m-n)/2}(\mathbf{R}^n)$ and $P_k' \in \mathcal{HP}_k^{k-(2m-n)/2}$, we may assume from the beginning that $P_k \in \mathcal{HP}_k^{k-(2m-n)/2}$.

Suppose, on the contrary, that $P_\ell \not\equiv 0$ for some ℓ such that $\ell \ge s+2m-n$. In case $2m-n$ is a nonnegative even integer, we may assume that $P_\ell \in \mathcal{HP}_\ell^{m'}$ with $m' = \min\{\ell-(2m-n)/2, m\} \ge 1$; in case $2m-n$ is not a nonnegative even integer, set $m'=m$.

We define

$$F(r) = \fint_{S(r)} uP_\ell \, dS$$

and note with the aid of (1.2) that

$$\liminf_{r\to 0} r^{s-\ell} F(r) = 0. \tag{2.2}$$

On the other hand, we obtain by (2.1)

$$\begin{aligned} F(r) &= \fint_{S(r)} \left(\sum_{k=0}^{\infty} H_k(x) + \sum_{k=0}^{\infty} \frac{P_k(x) + Q_k(x)\log|x|}{|x|^{n-2m+2k}} \right) P_\ell(x)\, dS \\ &= \sum_{k=0}^{\infty} \fint_{S(r)} H_k P_\ell \, dS \\ &\quad + \sum_{k=0}^{\infty} r^{2m-n-2k} \left(\fint_{S(r)} P_k P_\ell \, dS + \log r \fint_{S(r)} Q_k P_\ell \, dS \right) \end{aligned}$$

$$= \sum_{i=-m'+1}^{m-1} A_i r^{2(\ell+i)} + \sum_{i=-m'+1}^{m-1} (B_i + C_i \log r) r^{2m-n-2i},$$

where $A_i = \fint_S H_{\ell+2i} P_\ell \, dS$, $B_i = \fint_S P_{\ell+2i} P_\ell \, dS$ and $C_i = \fint_S Q_{\ell+2i} P_\ell \, dS$. Note here that

$$A_i = B_i = C_i = 0 \quad \text{if } \ell + 2i < 0.$$

Our assumptions imply that $B_0 \neq 0$ and $2m - n \neq 2(\ell + i)$ when $-m' + 1 \leq i \leq m - 1$ and $\ell + 2i \geq 0$. Since $s - \ell \leq -2m + n$, we see from (2.2) that

$$\lim_{r \to 0} r^{-2m+n} F(r) = 0,$$

which gives a contradiction. Thus it follows that

$$u(x) = h(x) + \sum_{k=0}^{2m-n} \frac{Q_k(x) \log |x|}{|x|^{n-2m+2k}} + \sum_{k<s+2m-n} \frac{P_k(x)}{|x|^{n-2m+2k}}.$$

If $2m - n$ is a nonnegative even integer and $s \leq 0$, then we consider

$$G(r) = \fint_{S(r)} u Q_\ell \, dS$$

instead of $F(r)$ to obtain

$$Q_\ell \equiv 0 \quad \text{for } s + 2m - n \leq \ell \leq 2m - n.$$

Now the proof of Theorem 1.1 is completed. □

Proof of Theorem 1.2. In view of Lemma 2.3 and [8, Lemma 1],

$$\fint_{S(r)} u \, dS = O(R_2(r)) \quad \text{as } r \to 0, \tag{2.3}$$

where $R_2(r) = R_2(x)$ with $|x| = r$. Hence (1.3) implies (1.2) since $s > n-2$, so that the present theorem follows from Theorem 1.1. □

Remark 2.1 In Theorem 1.2, we need the assumption that $s > n - 2$, since $u(x) = -R_{2m}(x)$ satisfies (1.3) for all s.

3. Removable singularities for polyharmonic functions

In this section, we give some conditions for a function $u \in \mathcal{H}^m(\mathbf{B}_0)$ to be extendable to a polyharmonic function on $\mathbf{B}$.

Proposition 3.1 *If $u \in \mathcal{H}^m(\mathbf{B}_0)$ satisfies*

$$\liminf_{r\to 0} r^s \fint_{S(r)} |\Delta^p u|\, dS = 0 \tag{3.1}$$

for a real number s and an integer p, $0 \le p < m$, then u is of the form

$$u(x) = v(x) + |x|^{2p}h(x) + \sum_{|\lambda|<s+2(m-p)-n} c(\lambda)D^\lambda R_{2m}(x),$$

where $v \in \mathcal{H}^p(\mathbf{B}_0)$, $h \in \mathcal{H}^{m-p}(\mathbf{B})$ and $c(\lambda)$ are constants.

For a proof, we prepare several lemmas. The following two lemmas can easily be proved by direct calculation.

Lemma 3.1 *For a positive integer p such that $1 \le p < m$,*

$$R_{2(m-p)}(x) = \alpha(m, p, n)\Delta^p R_{2m}(x) + \beta(m, p, n)|x|^{2(m-p)-n}$$

with constants $\alpha(m, p, n)$ and $\beta(m, p, n)$ depending only on m, p and n; $\beta(m, p, n) = 0$ unless $2(m-p)-n$ is a nonnegative even integer.

Lemma 3.2 *If $P_i \in \mathcal{HP}_i^1$, then for positive integers j and k,*

$$|x|^{2j}P_i = c(i, j, k)\Delta^k(|x|^{2j+2k}P_i)$$

with $c(i, j, k)^{-1} = 2^k\Pi_{\ell=j+1}^{j+k}\ell(n+2\ell-2+2i)$

Lemma 3.3 *For $h \in \mathcal{H}^m(\mathbf{B})$ and a positive integer k, there is $h' \in \mathcal{H}^m(\mathbf{B})$ such that $h(x) = \Delta^k(|x|^{2k}h'(x))$ for $x \in \mathbf{B}$.*

Proof. By Lemma 2.1, $h \in \mathcal{H}^m(\mathbf{B})$ has an Almansi expansion

$$h(x) = \sum_{j=0}^{m-1} |x|^{2j}h_j(x) \quad (h_j \in \mathcal{H}^1(\mathbf{B})).$$

Considering homogeneous expansions for harmonic functions (see [6]), we write

$$h(x) = \sum_{j=0}^{m-1} |x|^{2j} \sum_{i=0}^{\infty} H_{j,i}(x)$$

with $H_{j,i} \in \mathcal{HP}_i^1$, for $x \in \mathbf{B}$. By Lemma 3.2, we have

$$h(x) = \Delta^k\Bigg(|x|^{2k} \sum_{j=0}^{m-1} |x|^{2j} \sum_{i=0}^{\infty} c(i,\, j,\, k) H_{j,i}(x)\Bigg).$$

Since $\{c(i,\, j,\, k)\}_{i=0}^{\infty}$ is bounded,

$$h' = \sum_{j=0}^{m-1} |x|^{2j} \Bigg(\sum_{i=0}^{\infty} c(i,\, j,\, k) H_{j,i}(x)\Bigg) \in \mathcal{H}^m(\mathbf{B}).$$

The proof of the lemma is completed. □

Proof of Proposition 3.1. Since $\Delta^p u \in \mathcal{H}^{m-p}(\mathbf{B}_0)$, it follows from Theorem 1.1 that $\Delta^p u$ is of the form

$$\Delta^p u = h_0 + \sum_{|\lambda| < s+2(m-p)-n} c(\lambda) D^\lambda R_{2(m-p)}$$

for some $h_0 \in \mathcal{H}^{m-p}(\mathbf{B})$ and constants $c(\lambda)$. By Lemma 3.1 there are constants $c'(\lambda)$ for which

$$\Delta^p \Bigg(\sum_{|\lambda| < s+2(m-p)-n} c'(\lambda) D^\lambda R_{2m}(x) \Bigg)$$
$$= h_1(x) + \sum_{|\lambda| < s+2(m-p)-n} c(\lambda) D^\lambda R_{2(m-p)}(x),$$

where $h_1 \in \mathcal{H}^{m-p}(\mathbf{B})$. With the aid of Lemma 3.3 we can find $h \in \mathcal{H}^{m-p}(\mathbf{B})$ such that

$$h_0(x) - h_1(x) = \Delta^p(|x|^{2p} h(x))$$

for $x \in \mathbf{B}$. We here see that

$$\Delta^p \Bigg(u(x) - |x|^{2p} h(x) - \sum_{|\lambda| < s+2(m-p)-n} c'(\lambda) D^\lambda R_{2m}(x)\Bigg) = 0 \quad (\text{on } \mathbf{B}_0),$$

so that there exists $v \in \mathcal{H}^p(\mathbf{B}_0)$ such that

$$u(x) = v(x) + |x|^{2p}h(x) + \sum_{|\lambda|<s+2(m-p)-n} c'(\lambda)D^{\lambda}R_{2m}(x)$$

for $x \in \mathbf{B}_0$. □

Theorem 3.1 *Suppose $u \in \mathcal{H}^m(\mathbf{B}_0)$. Then u can be extended to a function in $\mathcal{H}^m(\mathbf{B})$ if and only if there are positive integers p_0, p_1, ..., p_N ($N \geq 0$) satisfying*

(1) $m = p_0 > p_1 > \cdots > p_N \geq 1$ *and* $2p_N \leq n$,

(2) $2p_i - n \leq 2p_{i+1}$ *for* $i = 0, 1, \ldots, N-1$,

(3) $\liminf_{r\to 0} R_{2(p_i-p_{i+1})}(r)^{-1} \fint_{S(r)} |\Delta^{p_{i+1}}u|\, dS = 0$ *for* $i = 0, 1, \ldots, N-1$,

(4) $\liminf_{r\to 0} R_{2p_N}(r)^{-1} \fint_{S(r)} |u|\, dS = 0$.

Proof. By (3) for $i = 0$, it follows from Proposition 3.1 that u can be written as

$$u(x) = v_1(x) + |x|^{2p_1}h_1(x),$$

where $v_1 \in \mathcal{H}^{p_1}(\mathbf{B}_0)$ and $h_1 \in \mathcal{H}^{m-p_1}(\mathbf{B})$. Since $2(p_2 - p_1) - n \leq 0$, we see from (3) for $i = 1$ that

$$\liminf_{r\to 0} R_{2(p_1-p_2)}(r)^{-1} \fint_{S(r)} |\Delta^{p_2}v_1|\, dS = 0.$$

By Proposition 3.1 again, we see that

$$v_1(x) = v_2(x) + |x|^{2p_2}h_2(x),$$

where $v_2 \in \mathcal{H}^{p_2}(\mathbf{B}_0)$ and $h_2 \in \mathcal{H}^{p_1-p_2}(\mathbf{B})$. By repeating these arguments, we find that

$$u(x) = v_N(x) + \sum_{i=1}^{N} |x|^{2p_i}h_i(x),$$

where $v_N \in \mathcal{H}^{p_N}(\mathbf{B}_0)$ and $h_i \in \mathcal{H}^{p_{i-1}-p_i}(\mathbf{B})$ for $1 \leq i \leq N$. Here, in view of (4), we have

$$\liminf_{r\to 0} R_{2p_N}(r)^{-1} \fint_{S(r)} |v_N|\, dS = 0.$$

Theorem 1.1 shows that v_N can be extended to a function in $\mathcal{H}^{p_N}(\mathbf{B})$, and the proof is completed. □

In particular, in case $2m \le n$, we have the following result.

Theorem 3.2 *Suppose $2m \le n$ and $u \in \mathcal{H}^m(\mathbf{B}_0)$. Then the following are equivalent:*
(1) *u can be extended to a function in $\mathcal{H}^m(\mathbf{B})$;*
(2) *$\lim_{x\to 0} u(x)$ exists and is finite;*
(3) *u is bounded near the origin;*
(4) *$u(x) = o(R_{2m}(x))$ as $x \to 0$;*
(5) *$\liminf_{r\to 0} R_{2m}(r)^{-1} \fint_{S(r)} |u|\, dS = 0$.*

These theorems give extensions of Al-Fadhel-Anandam-Othman [1, Theorem 2.2, 2.5], Anandam-Damlakhi [2, Theorem 3.4] and Armitage [3].

4. Proofs of Theorems 1.3 and 1.4

We write $\Delta^k R_{2m}(x) = \Delta^k R_{2m}(r)$ when $|x| = r$. In view of Pizetti [16], we have the following results.

Lemma 4.1 *For $r > 0$,*

$$\fint_{S(r)} R_{2m}(\zeta - x)\, dS(x) = \begin{cases} \displaystyle\sum_{k=0}^{m-1} a_k^{-1} \Delta^k R_{2m}(\zeta) r^{2k} & \text{if } |\zeta| > r, \\ \displaystyle\sum_{k=0}^{m-1} a_k^{-1} \Delta^k R_{2m}(r) |\zeta|^{2k} & \text{if } |\zeta| < r, \end{cases}$$

where $a_0 = 1$ and $a_k = 2^k k! n(n+2) \cdots (n+2(k-1))$ for $k = 1, 2, \ldots, m-1$.

Lemma 4.2 *For $r > 0$,*

$$\sum_{|\lambda|=j} \fint_{S(r)} \frac{\zeta^\lambda}{\lambda!} (D^\lambda R_{2m})(-x)\, dS(x)$$
$$= \begin{cases} a_k^{-1} \Delta^k R_{2m}(r) |\zeta|^{2k} & \text{if } j = 2k,\ 0 \le k \le m-1, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemmas 4.1 and 4.2, we have the following result.

Lemma 4.3 *Let $L \ge 2(m-1)$. Then*

$$\fint_{S(r)} R_{2m,L}(\zeta, x)\, dS(x)$$

$$= \begin{cases} \sum_{k=0}^{m-1} a_k^{-1}(\Delta^k R_{2m}(\zeta)r^{2k} - \Delta^k R_{2m}(r)|\zeta|^{2k}) & \text{if } |\zeta| > r, \\ 0 & \text{if } |\zeta| < r. \end{cases}$$

Here we show that the above integral defines a monotone and continuous function of $|\zeta|$.

Lemma 4.4 *For fixed $r > 0$, set*

$$G_m(\zeta) = G_m(\zeta, r) = \sum_{k=0}^{m-1} \frac{\Delta^k R_{2m}(\zeta)r^{2k} - \Delta^k R_{2m}(r)|\zeta|^{2k}}{a_k}.$$

Then $G_m(\zeta)$ is strictly monotone as a function of $|\zeta|$.

Proof. For $m > 1$ we claim that

$$\Delta G_m(\zeta) = c_m G_{m-1}(\zeta), \tag{4.1}$$

where $c_m = (2m-n)(2m-2)$ if $2m \neq n$ and $c_m = 2-n$ if $2m = n$. First we show this in case $2m - n \neq 2\ell$, where $\ell \in \mathbf{N} = \{1, 2, \ldots\}$. Using $2k(n+2(k-1))a_k^{-1} = a_{k-1}^{-1}$ and $\Delta R_{2m}(x) = c_m R_{2(m-1)}(x)$, we see that

$$\begin{aligned} \Delta G_m(\zeta) =& \sum_{k=0}^{m-1} a_k^{-1}(\Delta^{k+1} R_{2m}(\zeta)r^{2k} - \Delta^k R_{2m}(r)(\Delta|\zeta|^{2k})) \\ =& c_m \sum_{k=0}^{m-2} a_k^{-1}\Delta^k R_{2(m-1)}(\zeta)r^{2k} \\ & - c_m \sum_{k=1}^{m-1} a_{k-1}^{-1}\Delta^{k-1} R_{2(m-1)}(r)|\zeta|^{2(k-1)} \\ =& c_m G_{m-1}(\zeta). \end{aligned}$$

Next, suppose $2m - n = 2\ell$ where $\ell \in \mathbf{N}$. Similarly, we have

$$\begin{aligned} \Delta G_m(\zeta) =& c_m G_{m-1}(\zeta) + (n-4m+2) \\ & \times \sum_{k=0}^{m-2} a_k^{-1}(\Delta^k I_{2(m-1)}(\zeta)r^{2k} - \Delta^k I_{2(m-1)}(r)|\zeta|^{2k}), \end{aligned}$$

where $I_\alpha(x) = |x|^{\alpha-n}$. Since $\Delta^k I_{2(m-1)}(x) = b_k|x|^{2(m-1-k)-n}$ with $b_0 = 1$

and $b_k = 2^k(m-2)\cdots(m-k-1)(2(m-1)-n)\cdots(2(m-k)-n)$, we have

$$\begin{aligned}&\sum_{k=0}^{m-2} a_k^{-1}\left(\Delta^k I_{2(m-1)}(\zeta)r^{2k} - \Delta^k I_{2(m-1)}(r)|\zeta|^{2k}\right)\\ =&\sum_{k=0}^{\ell-1} a_k^{-1}b_k\left(|\zeta|^{2(m-1-k)-n}r^{2k} - r^{2(m-1-k)-n}|\zeta|^{2k}\right)\\ =&\sum_{k=0}^{\ell-1}\left(\frac{b_k}{a_k} - \frac{b_{\ell-1-k}}{a_{\ell-1-k}}\right)|\zeta|^{2(m-1-k)-n}r^{2k} = 0.\end{aligned}$$

Thus (4.1) is obtained.

Since $\int_{S(r)} R_{2m}(\zeta - x)\,dS(x) \in C^{2(m-1)}$, we see from Lemma 4.3 that

$$|\nabla G_m(\zeta)| = 0 \quad \text{on } S(r).$$

Therefore, with the aid of (4.1), we can show inductively that $G_m(\zeta)$ is strictly monotone as a function of $|\zeta|$. □

Remark 4.1 In fact, $(-1)^m\alpha_m G_m(\zeta)$ is strictly increasing as a function of $|\zeta|$.

Lemma 4.5 *If u is as in* Theorem 1.3, *then*

$$\lim_{r\to 0} r^{s+2m-n}\mu(\{\zeta\colon r < |\zeta| \le 1\}) = 0. \tag{4.2}$$

Proof. We may assume that $L(|\zeta|) \ge 2m-2$ in (1.4). In view of (1.4), (2.3) and Lemma 4.3, we have

$$\begin{aligned}&⨍_{S(r)} u(x)\,dS(x)\\ &= ⨍_{S(r)}\left(\alpha_m\int_{\mathbf{B}_0} R_{2m,\,L(|\zeta|)}(\zeta,\,x)\,d\mu(\zeta) + v(x)\right)dS(x)\\ &= \alpha_m\int_{\mathbf{B}_0}\left(⨍_{S(r)} R_{2m,\,L(|\zeta|)}(\zeta,\,x)\,dS(x)\right)d\mu(\zeta) + ⨍_{S(r)} v(x)\,dS(x)\\ &= \alpha_m\int_{\{\zeta\colon r<|\zeta|<1\}} G_m(\zeta,\,r)\,d\mu(\zeta) + O(R_2(r)).\end{aligned} \tag{4.3}$$

Since $s > n-2$, we have by (1.5)

$$\lim_{r\to 0} r^s\int_{\{\zeta\colon r<|\zeta|<1\}} G_m(\zeta,\,r)\,d\mu(\zeta) = 0,$$

so that Lemma 4.4 gives

$$\lim_{r\to 0} r^s \int_{\{\zeta:\, 2r<|\zeta|<1\}} G_m(2r,\, r)\, d\mu(\zeta) = 0,$$

where $G_m(2r,\, r) = G_m(\zeta,\, r)$ for $|\zeta| = 2r$. Noting that

$$G_m(2r,\, r) = r^{2m-n} G_m(2,\, 1),$$

we have

$$\lim_{r\to 0} r^{s+2m-n} \mu(\{\zeta\colon 2r < |\zeta| < 1\}) = 0,$$

which proves (4.2). □

Corollary 4.1 *If u is as above, then*

$$\int_{\mathbf{B}_0} |\zeta|^{\ell}\, d\mu(\zeta) < \infty \quad \text{for } \ell > s + 2m - n. \tag{4.4}$$

From Remark 4.1 and (4.3), we obtain the following result:

Corollary 4.2 *If $u \in \mathcal{SH}^m(\mathbf{B}_0)$, then*

$$(-1)^m \fint_{S(r)} u\, dS \geq O(R_2(r)) \quad \text{as } r \to 0.$$

Remark 4.2 If u is a superharmonic function on $\mathbf{B}_0$, then one can show this by the convexity theorem ([11, Theorem 2.12]).

Lemma 4.6 ([9, Lemma 3]) *If μ is a nonnegative measure on $\mathbf{B}_0$ satisfying* (4.2) *and $s > n-2$, then*

$$\lim_{r\to 0} r^q \fint_{S(r)} \left(\int_{\mathbf{B}_0} |R_{2m,\, L}(\zeta,\, x)|\, d\mu(\zeta) \right) dS(x) = 0 \quad \text{for every } q > s,$$

where L is the integer such that $s + 2m - n - 1 < L \leq s + 2m - n$.

Proof. Note that $L - (2m - n) \geq n - 2$, so that $L \geq 2(m-1)$. Hence if $2m \geq n$, then we see from [14, Lemmas 6, 8, 9] that

$$\int_{\mathbf{B}_0} |R_{2m,L}(\zeta, x)|\, d\mu(\zeta) \leq M\big(I_1(x) + I_2(x) + I_3(x)\big),$$

$$I_1(x) = \int_{\{\zeta\in\mathbf{B}_0:\, |\zeta|<|x|/2\}} |\zeta|^{L+1} |x|^{2m-n-L-1}\, d\mu(\zeta),$$

$$I_2(x)=\int_{\{\zeta\in\mathbf{B}_0:\,|\zeta-x|<|\zeta|/2\}}\left(|\zeta|^{2m-n}+|\zeta-x|^{2m-n}\log\frac{|\zeta|}{|\zeta-x|}\right)d\mu(\zeta),$$

$$I_3(x)=\int_{\{\zeta\in\mathbf{B}_0:\,|\zeta-x|\geq|\zeta|/2,\,|\zeta|\geq|x|/2\}}|\zeta|^L|x|^{2m-n-L}\log\frac{4|\zeta|}{|x|}\,d\mu(\zeta);$$

if $2m<n$, then $I_2(x)$ might be replaced by

$$I_2(x)=\int_{\{\zeta\in\mathbf{B}_0:\,|\zeta-x|<|\zeta|/2\}}|\zeta-x|^{2m-n}\,d\mu(\zeta).$$

Since $L+1>s+2m-n$, we have by (4.2)

$$\begin{aligned}
r^q \fint_{S(r)} I_1(x)\,dS(x)&=r^{q+2m-n-L-1}\int_{\{0<|\zeta|<r/2\}}|\zeta|^{L+1}\,d\mu(\zeta)\\
&=r^{q+2m-n-L-1}\sum_{j=1}^{\infty}\int_{A(2^{-j-1}r)}|\zeta|^{L+1}\,d\mu(\zeta)\\
&\leq r^{q+2m-n-L-1}\sum_{j=1}^{\infty}(2^{-j}r)^{L+1}\mu(A(2^{-j-1}r))\\
&\leq Mr^{q+2m-n-L-1}\sum_{j=1}^{\infty}(2^{-j}r)^{L+1-s-2m+n}\\
&=Mr^{q-s}\sum_{j=1}^{\infty}(2^{-j})^{L+1-s-2m+n}\\
&=Mr^{q-s}\longrightarrow 0 \quad \text{as } r\to 0
\end{aligned}$$

for $q>s$, where $A(R_1)=A(R_1,\,2R_1)$ with $A(R_1,\,R_2)=\{\zeta\colon R_1<|\zeta|\leq R_2\}$.

Next, in order to estimate I_2 when $2m\geq n$ and $q>s$, we find from (4.2) that

$$\begin{aligned}
r^q\fint_{S(r)} I_2(x)\,dS(x)&\leq r^q\int_{A(2r/3,2r)}|\zeta|^{2m-n}\,d\mu(\zeta)+Mr^{q-n+1}\\
&\times\int_{A(2r/3,2r)}\left(\int_{\{x\in S(r):\,|\zeta-x|<|\zeta|/2\}}|\zeta-x|^{2m-n}\log\frac{|\zeta|}{|\zeta-x|}\,dS(x)\right)d\mu(\zeta)\\
&\leq Mr^{q+2m-n}\mu(A(2r/3,2r))+Mr^{q-n+1}\int_{A(2r/3,2r)}r|\zeta|^{2m-2}\,d\mu(\zeta)
\end{aligned}$$

$$\leq M r^{q+2m-n}\mu(A(2r/3, 2r)) \longrightarrow 0 \quad \text{as } r \to 0.$$

Since $q - s - (L - s - 2m + n) \geq q - s > 0$, we have by (4.2)

$$\begin{aligned}
& r^q \fint_{S(r)} I_3(x)\, dS(x) \\
\leq & r^{q+2m-n-L} \int_{\{\zeta:\, r/2<|\zeta|<1\}} |\zeta|^L \log \frac{4|\zeta|}{r}\, d\mu(\zeta) \\
= & r^{q+2m-n-L} \sum_{j=0}^{\infty} \int_{\mathbf{B}_0 \cap A(2^{j-1}r)} |\zeta|^L \log \frac{4|\zeta|}{r}\, d\mu(\zeta) \\
\leq & M r^{q+2m-n-L} \sum_{2^j \leq 2/r} (2^j r)^L \log(2^{j+2}) \mu(A(2^{j-1}r)) \\
\leq & M r^{q+2m-n-L} \sum_{2^j \leq 2/r} (2^j r)^L \log(2^{j+2}) (2^{j-1}r)^{-s-2m+n} \\
\leq & M r^{q-s} \sum_{2^j \leq 2/r} (2^j)^{L-s-2m+n} \log(2^{j+2}) \\
\leq & M r^{q-s-(L-s-2m+n)} \{\log(4/r)\}^2 \longrightarrow 0 \quad \text{as } r \to 0.
\end{aligned}$$

Thus we obtain

$$\lim_{r \to 0} r^q \fint_{S(r)} \left(\int_{\mathbf{B}_0} |R_{2m,L}(\zeta, x)| d\mu(\zeta) \right) dS(x) = 0,$$

as required. □

Proof of Theorem 1.3. From (4.4) and the proof of Lemma 4.6, we see that $\int_{\mathbf{B}_0} R_{2m,L}(\zeta, x)\, d\mu(\zeta)$ is defined and

$$u(x) - \alpha_m \int_{\mathbf{B}_0} R_{2m,L}(\zeta, x)\, d\mu(\zeta)$$

is polyharmonic in $\mathbf{B}_0$ in the sense of distributions, where $s + 2m - n - 1 < L \leq s + 2m - n$ with $s > n - 2$. By Weyl's lemma, there exists a polyharmonic function $v \in \mathcal{H}^m(\mathbf{B}_0)$ such that

$$v(x) = u(x) - \alpha_m \int_{\mathbf{B}_0} R_{2m,L}(\zeta, x)\, d\mu(\zeta)$$

for a.e. $x \in \mathbf{B}_0$, which proves Theorem 1.3. □

Proof of Theorem 1.4. According to Theorem 1.3, there exists a polyharmonic function $v \in \mathcal{H}^m(\mathbf{B}_0)$ such that

$$u(x) = v(x) + \alpha_m \int_{\mathbf{B}_0} R_{2m,L}(\zeta, x)\, d\mu(\zeta)$$

for $x \in \mathbf{B}_0$, where $\mu = (-\Delta)^m u$ and L is the integer such that $s + 2m - n - 1 < L \le s + 2m - n$. Further we see from Lemma 4.6 and (1.6) that v satisfies

$$\liminf_{r \to 0} r^{L-(2m-n-1)} \fint_{S(r)} v^+ \, dS(x) = 0.$$

Hence Theorem 1.2 shows that v is of the form

$$v(x) = h(x) + \sum_{|\lambda| < L+1} c(\lambda) D^\lambda R_{2m}(x) = h(x) + \sum_{|\lambda| \le L} c(\lambda) D^\lambda R_{2m}(x),$$

where $h \in \mathcal{H}^m(\mathbf{B})$. Thus Theorem 1.4 is proved. □

5. Remarks

In this section we give several remarks on our theorems.

Remark 5.1 We say that a sequence $\{r_j\}$ is regular at 0 if $r_j \to 0$ and $0 < r_j < c r_{j+1}$ for some constant $c > 0$. In Theorem 1.3, we can replace condition (1.5) by

$$\lim_{j \to \infty} r_j^s \fint_{S(r_j)} u \, dS = 0 \tag{5.1}$$

for some sequence $\{r_j\}$ regular at 0.

Corollary 5.1 (cf. [9, Theorem 1]) *Let $u \in \mathcal{SH}^m(2\mathbf{B}_0)$ and $\mu = (-\Delta)^m u$. If $s > -2$ and*

$$\int_{2\mathbf{B}_0} \big((-1)^m u(x)\big)^+ |x|^s \, dx < \infty, \tag{5.2}$$

then u is the form

$$u(x) = \alpha_m \int_{\mathbf{B}_0} R_{2m,L}(\zeta, x)\, d\mu(\zeta) + h(x) + \sum_{|\lambda| \le L} c(\lambda) D^\lambda R_{2m}(x)$$

for $x \in \mathbf{B}_0$, where L is the integer such that $s + 2m - 1 < L \le s + 2m$, $h \in \mathcal{H}^m(\mathbf{B})$ and $c(\lambda)$ denote constants.

Proof. In view of Corollary 4.2, (5.2) is equivalent to

$$\int_{2\mathbf{B}_0} |u(x)||x|^s \, dx < \infty$$

with $s > -2$, and so we can find a sequence $\{r_j\}$ (regular at 0) such that $2^{-j} \le r_j < 2^{-j+1}$ for all j and

$$\lim_{j\to\infty} r_j^{s+n} \fint_{S(r_j)} u \, dS = 0.$$

Hence our corollary follows from Remark 5.1 and Theorem 1.4. □

As a corollary to Theorem 1.2, we have the following result.

Corollary 5.2 *If u is a function in $\mathcal{H}^m(\mathbf{B}_0)$ such that*

$$\limsup_{x\to 0} u(x)|x|^s \le 0 \tag{5.3}$$

for some number $s > n-2$, then u is of the form

$$u = h + \sum_{|\lambda| < s+2m-n} c(\lambda) D^\lambda R_{2m},$$

where $h \in \mathcal{H}^m(\mathbf{B})$ and $c(\lambda)$ are constants.

In particular, in case $s = n-1$ and $m = 1$, u is of the form

$$u = h + cR_2,$$

where $h \in \mathcal{H}^1(\mathbf{B})$ and c is a constant. This was proved by Ishikawa-Nakai-Tada [12].

If u is superharmonic in $\mathbf{B}_0$, then $\min(u, 0)$ is also superharmonic in $\mathbf{B}_0$. Hence we can prove the following simple result.

Corollary 5.3 *If u is a superharmonic function on $2\mathbf{B}_0$ satisfying*

$$\liminf_{r\to 0} [R_2(r)]^{-1} \fint_{S(r)} u^- \, dS = 0, \tag{5.4}$$

then u can be extended to a superharmonic function on $\mathbf{B}$.

Proof. It suffices to show that $v = \min(u, 0)$ can be extended to a superharmonic function on $\mathbf{B}$. Since v is superharmonic on $2\mathbf{B}_0$, we find in the same way as Lemma 4.5 that

$$\liminf_{r\to 0} \alpha_1 \int_{\{\zeta:\, r<|\zeta|<1\}} \frac{R_2(r) - R_2(\zeta)}{R_2(r)}\, d\mu(\zeta) \quad \text{is finite,}$$

where $\mu = (-\Delta)v$. This implies that $\mu(\mathbf{B}_0) < \infty$. Hence it follows from Lemma 4.6 and Theorem 1.4 that v is of the form

$$v(x) = \alpha_1 \int_{\mathbf{B}_0} R_2(\zeta - x)\, d\mu(\zeta) + h(x) + cR_2(x)$$

for $x \in \mathbf{B}_0$, where $h \in \mathcal{H}^1(\mathbf{B})$ and c is a constant. Further Lemma 4.1 yields

$$\lim_{r\to 0} [R_2(r)]^{-1} \fint_{S(r)} \left(\int_{\mathbf{B}_0} R_2(\zeta - x)\, d\mu(\zeta) \right) dS(x) = 0.$$

In view of (5.4), we see that $c = 0$, so that v can be extended to a superharmonic function on $\mathbf{B}$. □

Remark 5.2 Let u be a C^{2m}-function on open set D. Then, in view of Haußmann-Kounchev [10], $u \in \mathcal{SH}^m(D)$ if and only if

$$u(x) \geq \frac{1}{A(r_1, r_2, \ldots, r_m)} \begin{vmatrix} \fint_{S(x,r_1)} u\, dS & r_1^2 & \ldots & r_1^{2(m-1)} \\ \fint_{S(x,r_2)} u\, dS & r_2^2 & \ldots & r_2^{2(m-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \fint_{S(x,r_m)} u\, dS & r_m^2 & \ldots & r_m^{2(m-1)} \end{vmatrix} \tag{5.5}$$

whenever $x \in D$ and $0 < r_1 < r_2 < \cdots < r_m < \operatorname{dist}(x, \partial D)$, where

$$A(r_1, r_2, \ldots, r_m) = \begin{vmatrix} 1 & r_1^2 & \ldots & r_1^{2(m-1)} \\ 1 & r_2^2 & \ldots & r_2^{2(m-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & r_m^2 & \ldots & r_m^{2(m-1)} \end{vmatrix}. \tag{5.6}$$

Moreover, Pizetti's formula [16] implies that

$$\lim_{h\to 0+} \frac{1}{h^{2m}} \left(u(x) - \frac{1}{A(h, 2h, \ldots, mh)} \right.$$

$$\times \begin{vmatrix} ⨍_{S(x,h)} u\,dS & h^2 & \dots & h^{2(m-1)} \\ ⨍_{S(x,2h)} u\,dS & (2h)^2 & \dots & (2h)^{2(m-1)} \\ \vdots & \vdots & \ddots & \vdots \\ ⨍_{S(x,mh)} u\,dS & (mh)^2 & \dots & (mh)^{2(m-1)} \end{vmatrix} \Bigg)$$
$$= a_m^{-1} 1^2 \cdot 2^2 \cdot \dots \cdot m^2 \{(-\Delta)^m u(x)\}$$

with a positive constant a_m in Lemma 4.1.

References

[1] Al-Fadhel T. Anandam V. and Othman S.I., *Biharmonic point singularities in* $\mathbf{R}^n$, Arab J. Math. Sc. **5** (1999), 29–41.

[2] Anandam V. and Damlakhi M., *Biharmonic Green domains in* $\mathbf{R}^n$, Hokkaido Math. J. **27** (1998), 669–680.

[3] Armitage D.H., *On polyharmonic functions in* $\mathbf{R}^n \setminus \{0\}$, J. London Math. Soc. **8** (2) (1974), 561–569.

[4] Armitage D.H., *A Liouville theorem for polyharmonic functions*, Hiroshima Math. J. **31** (2001), 367–370.

[5] Aronszajn N., Creese T.M. and Lipkin L.J., *Polyharmonic functions*, Clarendon Press, 1983.

[6] Axler S., Bourdon P. and Ramey W., *Harmonic function theory*, Springer-Verlag, 1992.

[7] Brambel J.H. and Payne L.E., *Mean value theorems for polyharmonic functions*, Amer. Math. Monthly **73** (II) (1966), 124–127.

[8] Futamura T., Kishi K. and Mizuta Y., *A generalization of Bôcher's theorem for polyharmonic functions*, Hiroshima Math. J. **31** (2001), 59–70.

[9] Futamura T., Kishi K. and Mizuta Y., *Removability of sets for sub-polyharmonic functions*, Hiroshima Math. J. **33** (2003), 31–42.

[10] Haußmann W. and Kounchev O., *Definiteness of the Peano kernel associated with the polyharmonic mean value property*, J. London Math. Soc. **62** (2) (2000), 149–160.

[11] Hayman W.K. and Kennedy P.B., *Subharmonic functions*, vol. 1, Academic Press, London, 1976.

[12] Ishikawa Y., Nakai M. and Tada T., *A form of classical Picard principle*, Proc. Japan Acad. Ser. A, Math. Sci. **72** (1996), 6–7.

[13] Mizuta Y., *Potential theory in Euclidean spaces*, Gakkōtosyo, Tokyo, 1996.

[14] Mizuta Y., *An integral representation and fine limits at infinity for functions whose Laplacians iterated m times are measures*, Hiroshima Math. J. **27** (1997), 415–427.

[15] Nicolesco M., *Recherches sur les fonctions polyharmoniques*, Ann. Sci. École Norm Sup. **52** (1935), 183–220.

[16] Pizetti P., *Sulla media deivalori che una funzione dei punti dello spazio assume alla superficie di una sfera*, Rend. Lincei **5** (1909), 309–316.

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