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Representing densities of the multi-sequences of moments

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Abstract. One considers a finite set $\gamma = (\gamma_\alpha)_{\alpha \in A}$ $(A \subset \mathbb{Z}_+^n)$ of moments $\gamma_\alpha \in \mathbb{R}$ of a measurable representing density $f = f(t) \geq 0$ $(t \in \mathbb{R}^n)$ with respect to the monomials t^α. We prove the existence in this case of some representing densities $f_* \geq 0$ of γ with a higher degree of smoothness, or having a particular form. The results are established also in a more general setting.

Key words: problem of moments, representing measure.

1. Introduction

A truncated problem of power moments on the n–dimensional Euclidian space can be stated as follows. Consider a measurable subset $T \subset \mathbb{R}^n$ and a finite subset $A \subset \mathbb{Z}_+^n$. Set $u_\alpha(t) := t^\alpha$ for $t \in \mathbb{R}^n$ and $\alpha \in A$. Given a set $\gamma = (\gamma_\alpha)_{\alpha \in A}$ of real numbers γ_α, one asks to establish if there exist nonnegative measures μ on $\mathbb{R}^n$ with $\operatorname{supp} \mu \subset T$, such that all $u_\alpha \in L^1(\mu)$ and

$$\int_T u_\alpha(t)\, d\mu(t) = \gamma_\alpha, \quad \alpha \in A. \tag{1}$$

In this case, we call γ_α the *moments* of μ. In particular, we are interested in measures $\mu = f\, dt$ absolutely continuous with respect to the Lebesgue measure dt, with $f \geq 0$ (almost everywhere). If (1) holds, then μ (resp. f) is called a *representing* measure (resp. density) for the finite multi–sequence γ. The problem is then to characterize those sets γ having nonnegative representing measures, to study the set of the solutions and find or approximate such measures μ.

For $n = 1$ these problems received good answers in a large class of cases, in terms of positive–definiteness. For example, a sequence $\gamma = (\gamma_\alpha)_{\alpha=0}^{2k}$ has nonnegative representing measures on $\mathbb{R}$ iff the quadratic form

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$(\gamma_{\alpha+\beta})_{\alpha,\beta=0}^{k}$ is nonnegative definite [1]. Similar characterizations hold for $A := \{0, 1, \ldots, k\}$ and $T :=$ an interval (or union of intervals) in $\mathbb{R}$, for $A := \{-k, \ldots, k\}$ and $T :=$ the unit circle (with respect to the moment functions $u_\alpha(e^{i\theta}) := e^{i\alpha\theta}$ and the measure $d\theta$), as well as in other 1-dimensional cases [1], [4], [11]. They are based on the possibility to represent any nonnegative polynomial as a sum of squares [9], which makes the condition $\int_T p^2\, d\mu \geq 0$ ($p \in \mathbb{R}[X]$) sufficiently strong.

The moment problem on $T := \mathbb{R}^n$ for the set $A := \{\alpha \in \mathbb{Z}_+^n; |\alpha| \leq 2\}$ has been solved in [6] by operator–theoretic methods. Generally one can not state similar results for larger A and $n \geq 2$, in which case the set of nonnegative polynomials is more complicated and difficult to handle [5].

There are also other approaches, providing the existence of certain maximum entropy–type solutions on the torus [4], or concerning singular measures on $\mathbb{R}^n$ (for instance, finitely atomic) for flat data in the sense of [6]. We also mention some results from [2] that we shall follow.

Note that the functional analytic techniques applied usually to the full moment problems $A := \mathbb{Z}_+^n$ (see for instance [7]) do not seem to apply to the truncated case.

For finite A and arbitrary n these problems have thus received only partial answers. In this context we show in the present paper that the existence of an arbitrary representing density $f \in L^1(T)$ for γ is equivalent to the existence of some representing densities of class C^∞ or having a concrete form (Theorems 6, 7). Certain generalizations are then straightforward (Theorem 8).

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2. Main results

Let T be a compact subset of $\mathbb{R}^n$, with nonempty interior $\operatorname{int} T = T \setminus \partial T$. Let m denote the Lebesgue measure on T. Assume that $m(\partial T) = 0$. Let u_α ($\alpha \in A$) be a finite set of functions of class C^1 on a neighbourhood of T, containing the constant function 1 ($= u_0$ for a distinguished element $0 \in A$). In particular, we can take $A \subset \mathbb{Z}_+^n$ and $u_\alpha(t) := t^\alpha = t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ for $t = (t_1, \ldots t_n) \in T$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in A$. Set $a := \operatorname{card} A$. Let $\gamma_\alpha \in \mathbb{R}$ for $\alpha \in A$, and set $\gamma := (\gamma_\alpha)_{\alpha \in A} \in \mathbb{R}^a$. In so far we are concerned with the equalities (1), we can suppose that $\gamma_0 > 0$ and, if necessary, that

u_α are linearly independent. In the particular case $u_\alpha(t) = t^\alpha$, this follows from the condition $\operatorname{int} T \neq \emptyset$. Let P be the a-dimensional linear space of functions on T generated by u_α, $\alpha \in A$. Consider on P the sup–norm $\|p\| := \max_{t\in T} |p(t)|$ $(p \in P)$. Let P^* denote the dual of P, endowed with the norm $\|\varphi\| := \sup_{\|p\|\le 1} |\varphi(p)|$. Any locally integrable function φ on $\mathbb{R}^n$ defines a functional, denoted by the same symbol $\varphi \in P^*$, by $p \mapsto \int_T \varphi p\, dt$. Then $\|\varphi\| \le \|\varphi\|_1$, where $\|\varphi\|_1 := \int_T |\varphi|\, dm$ is the norm on the space $L^1(T)$ of all (classes of) measurable functions that are Lebesgue integrable on T with respect to m. We will sometimes identify P and $\mathbb{R}^a$ by the algebraic isomorphism $\sum_{\alpha\in A} x_\alpha u_\alpha \equiv (x_\alpha)_{\alpha\in A}$. We consider only nonnegative Radon measures $\mu \in C(T)^*$, where $C(T)$ denotes the space of all continuous functions on T endowed with the sup norm. For $t \in T$, let $\delta_t \in P^*$ denote the evaluation functional $\delta_t p := p(t)$.

Lemma 1 *There exist $t(\alpha) \in \operatorname{int} T$ $(\alpha \in A)$ with $\delta_{t(\alpha)}$ linearly independent.*

Proof. Since $P \subset C(T)$, then $P^* \equiv C(T)^*/P^\perp$ where $P^\perp$ denotes the space of all functionals vanishing on P. Any $d \in P^*$ extends by the Hahn–Banach theorem to a functional $\tilde{d} \in C(T)^*$. Now $\tilde{d}$ can be weakly* approximated by a net of convex combinations of Dirac extremal measures d_t, via the Krein–Milman theorem. Applying this on the functions $u_\alpha \in P$ shows that d can be weakly* approximated by linear combinations of the functionals $\delta_t \equiv d_t + P^\perp$. Note that $(d_t + P^\perp)p = \delta_t p$ for $p \in P$. Since d was arbitrary, the linear span $P^{\cdot}$ of δ_t $(t \in T)$ is thus weakly* dense in P^*. But $\dim P^* = \dim P = a < \infty$, and so we have $P^* = P^{\cdot}$. Extract now from the set of generators $\{\delta_t\}_t$ of P^* a basis $\delta_{t(\alpha)}$ $(\alpha \in A)$. Since $(u_\beta)_\beta$ also is a basis (of P), we have $\det[\delta_{t(\alpha)} u_\beta]_{\alpha,\beta} \neq 0$. Now u_β are continuous, and hence a slight perturbation of the points $t(\alpha)$ let them belong to $\operatorname{int} T$ without affecting the linear independence of the corresponding $\delta_{t(\alpha)}$ (expressed by the condition $\det \neq 0$). Thus Lemma 1 is proved. □

For any set $b = (d_\alpha)_{\alpha\in A} \in (P^*)^a$ consisting of a functionals $d_\alpha \in P^*$, define the operator B_b on P by

$$B_b u_\alpha := \sum_{\beta\in A} d_\alpha(u_\beta) u_\beta \quad (\alpha \in A). \tag{2}$$

If $d_\alpha = \delta_{t(\alpha)}$ for some set $\tau := (t(\alpha))_{\alpha\in A} \in (\mathbb{R}^n)^a$ of points $t(\alpha) \in T$ $(\alpha \in A)$, then we set $\det(\tau) :=$ the determinant of B_b. Identify P with $\mathbb{R}^a$ by

$\sum_{\alpha\in A} x_\alpha u_\alpha \equiv (x_\alpha)_{\alpha\in A}$. Then B_b acts on $\mathbb{R}^a$ like the multiplication by the matrix $[c_{\beta\alpha}]_{\beta,\,\alpha}$ where $c_{\beta\alpha} := d_\alpha(u_\beta)$. Namely, $B_b x \equiv (\sum_{\alpha\in A} c_{\beta\alpha} x_\alpha)_{\beta\in A}$ for $x := (x_\alpha)_{\alpha\in A}$. In the context of (2) we have:

Lemma 2 (1) *b is a basis $\Leftrightarrow B_b$ is invertible*;
(2) *$\{\tau \in (\operatorname{int} T)^a \mid \det(\tau) \neq 0\}$ is dense in T^a.*

Proof. The equivalence $B_b x = 0 \Leftrightarrow \sum_{\alpha\in A} x_\alpha d_\alpha = 0$ gives (1). For $b := (\delta_{t(\alpha)})_{\alpha\in A}$ as in Lemma 1, $\det(\tau) \neq 0$. Thus $\det(\,\cdot\,)$ is a polynomial function $\not\equiv 0$ on $(\operatorname{int} T)^a$. Hence (2) follows. □

Let f be a measurable function on T, and fix $t_0 \in T \setminus \partial T$. We recall that if there exists a finite constant $c_f(t_0)$ such that

$$\lim_{k\to\infty} \frac{1}{m(B(t_0,\, r_k))} \int_{B(t_0,\, r_k)} |f(t) - c_f(t_0)|\, dt = 0$$

for any sequence of balls $B(t_0,\, r_k)$ of center t_0 and radius $r_k > 0$ $(k \geq 0)$ such that $r_k \to 0$, then t_0 is said to be a *continuity point* of f (see the more general Definition 8.2 from [10]). If f is continuous on T then $c_f(t_0) = f(t_0)$ for each t_0.

Lemma 3 *Let $f \in L^1(T)$. Assume $f \geq 0$ and $\int_T f\, dm \neq 0$. Then there exists a set $b = (\delta_{t(\alpha)})_{\alpha\in A} \in (P^*)^a$ such that:*
(1) *$t(\alpha) \in T \setminus \partial T$ $(\alpha \in A)$ and the operator B_b is invertible*;
(2) *all $t(\alpha)$ are continuity points of f with $0 < c_f(t(\alpha)) < \infty$.*

Proof. We can choose in the class of f some (measurable) representing function which takes only finite and nonnegative values. We extend this function f to $\mathbb{R}^n$ with the value 0 outside T. Set $E := \{t \in T \setminus \partial T\,;\, f(t) \neq 0\}$. Since $\int_T f\, dt \neq 0$ while ∂T has zero Lebesgue measure, then E has strictly positive measure. Set $F := \{t \in \mathbb{R}^n \mid t = \text{continuity point of } f\}$. Since f is Lebesgue integrable on $\mathbb{R}^n$, then almost all $t \in \mathbb{R}^n$ are continuity points of f such that $f(t) = c_f(t) = \text{finite}$, see Theorem 8.8 from [10]. Hence the measure of $E \cap F$ is positive. Then the measure of the product $(E \cap F)^a$ of a copies of $E \cap F$ in $(\mathbb{R}^n)^a$ is positive, too. By Lemma 2, $\det(\tau)$ is a nonnull polynomial function of the variable $\tau \in (\operatorname{int} T)^a$ $(\subset \mathbb{R}^{na})$. Then the set $Z := \{\tau \in (\mathbb{R}^n)^a;\, \det(\tau) = 0\}$ is a finite union of smooth manifolds of dimensions strictly less than na. Hence the Lebesgue measure of Z is zero. The measure of $G := (E \cap F)^a \setminus Z$ is then strictly positive. Hence $G \neq \emptyset$. Now take $\tau := (t(\alpha))_{\alpha\in A} \in G$. Thus Lemma 3 is proved. □

Lemma 4 (1) *Given $f \in L^1(T)$ with $f \geq 0$, there exists a sequence of nonnegative functions $f_k \in C_0^\infty(T \setminus \partial T)$, $k \geq 1$ such that $f_k \to f$ in $L^1(T)$.*
(2) If the restriction $f|_B$ of the function f above to a closed ball $B \subset \operatorname{int} T$ is continuous, we can take $(f_k)_k$ such that $f_{k+1} < f_k$ on B for all $k \geq 1$.
(3) Given a measure $\nu \geq 0$ on T, there exists a sequence of nonnegative functions $\varphi_k \in C_0^\infty(T \setminus \partial T)$, $k \geq 1$ such that $\varphi_k m \to \nu$ weakly in $C(T)^*$.*

Proof. (1) Whenever it is not specified, the convergence will be considered below in the space $L^1(T)$. We take a sequence of bounded measurable functions $f_k \geq 0$ such that $f_k \to f$. Moreover, since $m(\partial T) = 0$, we may assume that $\operatorname{supp} f_k \subset T \setminus \partial T$, by taking a sequence of compact sets T_k exhausting $T \setminus \partial T$ and truncating f_k with the characteristic functions of T_k respectively. We fix now a nonnegative function $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\int \varphi\, dt = 1$. For every $r > 0$ define $\varphi_r(t) := c_r \varphi(rt)$ where c_r is the positive constant such that $\int \varphi_r\, dt = 1$. For each k we can choose $r = r_k$ sufficiently big such that $\operatorname{supp}(f_k * \varphi_r) \subset T \setminus \partial T$ and $\|f_k * \varphi_r - f_k\|_1 < 1/k$. Replace then each f_k by the convolution $f_k * \varphi_{r_k}$ (≥ 0).
(2) Extend $f|_B$ to a nonnegative function g continuous on a closed ball B' such that $B \subset \operatorname{int} B'$. The Stone–Weierstrass theorem provides, for each $k \geq 1$, a real polynomial function such that $\sup_{B'} |g + k^{-1} - g_k| < (2(k+1)^2)^{-1}$. In particular, on B' the sequence $(g_k)_k$ is uniformly bounded and

$$\begin{aligned} g_k = g + k^{-1} - (g + k^{-1} - g_k) &\geq g + k^{-1} - |g + k^{-1} - g_k| \\ &\geq g + k^{-1} - (2(k+1)^2)^{-1} \end{aligned} \tag{3}$$

(whence $g_k \geq 0$). Set $c = \sup_k \max_{B'} |g_k|$. Let β denote the characteristic function of B. Take a sequence of functions $\psi_k \in C_0^\infty(\operatorname{int} B')$ such that $\psi_k = 1$ in the neighbourhood of B, $0 \leq \psi_k \leq 1$ and $\psi_k \to \beta$ in $L^1(T)$. By (1), let $h_k \in C_0^\infty(\operatorname{int} T)$ nonnegative such that $h_k \to (1-\beta)f$. Set $f_{lk} = \psi_k g_k + (1 - \psi_l) h_k$ (≥ 0 by (3)). Since $\psi_k \to \beta$, then $|(1-\beta)\psi_k g_k| \leq c|(1-\beta)\psi_k| \to 0$. Moreover, $\beta \psi_k g_k \to \beta f$ because $g_k \to f$ uniformly on B. Hence $\psi_k g_k \to \beta f$. Also, $(1-\psi_l)h_k \to (1-\psi_l)(1-\beta)f = (1-\psi_l)f$ as $k \to \infty$. Then $f_{lk} \to \beta f + f - \psi_l f$ as $k \to \infty$ and l is fixed. Now $\psi_l f \to \beta f$ almost everywhere as $l \to \infty$, with $|\psi_l f| \leq f$. By Lebesgue's theorem of dominated convergence, we obtain $\psi_l f \to \beta f$. Take then a sequence $(l_j)_j$ such that $\|\psi_{l_j} f - \beta f\|_1 < 1/j$. For every j there is $k = k_j$ such that $\|f + \beta f - f_{l_j k} - \psi_{l_j} f\|_1 < 1/j$ for all $k \geq k_j$. We can successively replace each k_j $(j = 1, 2, \ldots)$ by $\max_{i=1}^{j} k_i + 1$, so that $k_1 < k_2 < \cdots$. Set $f_j = f_{l_j k_j}$

$(j \geq 1)$. Then the inequalities

$$\|f - f_{l_j k_j}\|_1 \leq \|f - f_{l_j k_j} + \beta f - \psi_{l_j} f\|_1 + \|\psi_{l_j} f - \beta f\|_1 < \frac{2}{j}$$

show that $f_j \to f$. Moreover, on B we have $f_{lk} = g_k$ for all l, k. We obtain on B the inequality

$$\begin{aligned} f_{l\,k+1} = g_{k+1} &\leq |g_{k+1} - (g + (k+1)^{-1})| + g + (k+1)^{-1} \\ &\leq (2(k+2)^2)^{-1} + g + (k+1)^{-1} \end{aligned}$$

from which we substract $f_{l'\,k} = g_k$ as estimated from below by (3) to get $f_{l\,k+1} - f_{l'\,k} \leq (k+1)^{-2} - (k(k+1))^{-1} < 0$ for any l, l'. Hence $f_{l,k'} < f_{l'\,k}$ on B for any l, l' and $k' > k$. In particular, $f_{j+1} < f_j$ since $k_{j+1} > k_j$.
(3) By the Krein–Milman theorem, ν can be approximated by convex combinations of extremal (evaluation) functionals δ_t $(t \in T)$, with respect to the weak–* topology of $C(T)^*$. Any such t is the limit of a sequence of points $t' \in \operatorname{int} T$, so that $\delta_{t'} \to \delta_t$ weakly*. For any t' there is a sequence of nonnegative test functions $\varphi_k \to \delta_{t'}$ weakly*; take for example $\varphi_k(s) := c_k \varphi(k(s-t'))$, where $\varphi \in C_0^\infty(\mathbb{R}^n)$ is nonnegative with $\int \varphi\, dt = 1$ and $c_k > 0$ is such that $\int \varphi_k\, dt = 1$. We omit the details. Lemma 4 is thus proved. □

In what follows, we assume the existence of an arbitrary measurable representing density $f \geq 0$ for γ on T. Then we derive the existence of some representig densities f_* with a higher degree of regularity (of class C^∞ etc.). The idea is to approximate f in $L^1(T)$ by a sequence $f_k \to f$ with $f_k \geq 0$ having the desired regularity properties. In this case we will have only $\int_T u_\alpha f_k\, dt \approx \gamma_\alpha$, $\alpha \in A$. Then we try to correct f_k by some suitable linear combination $\sum_{\alpha \in A} x_\alpha f_\alpha$ in order to obtain an exact representing density for γ. Namely we take certain fixed bounded functions f_α with supports contained in some small disjoint balls $B(t(\alpha), r)$ respectively. Then for sufficiently large $k \geq 1$ we can determine a number of a small parameters $x_\alpha = x_{\alpha k}$, $\alpha \in A$ such that all the measures $g_k\, dt$, $k \geq 1$ defined by $g_k := f_k - \sum_{\alpha \in A} x_\alpha f_\alpha$ satisfy the system (1). Moreover, we shall have also $x_{\alpha k} \to 0$ as $k \to \infty$. Thus g_k are good candidates for f_*. Now to insure in addition the positivity of g_k, it suffices only to know that $\inf_{\operatorname{supp} f_\alpha} f > 0$ and $f_k(t) \to f(t)$ uniformly on each set $\operatorname{supp} f_\alpha$. To this aim we start by considering some continuity points $t(\alpha)$ of f with $c_f(t(\alpha)) > 0$ and letting each f_α be the characteristic function of a certain small set around $t(\alpha)$. In order to obtain in this way a smooth representing density g_k, we should have also $f_\alpha \in C^\infty$. This is the reason why two steps (Proposition 5 and Theorem 6) are necessary. Also, we will allow measures $\mu = f dt + \nu$ having a singular part ν, provided $f \neq 0$ in $L^1(T)$.

Proposition 5 *If $f \geq 0$ belongs to $L^1(T) \setminus \{0\}$, there exists $f_* \geq 0$ in $L^1(T)$ such that*

$$\int_T u_\alpha(t) f(t)\, dt = \int_T u_\alpha(t) f_*(t)\, dt \quad (\alpha \in A)$$

and f_* is continuous on some nonempty open subset T' of T with $f_*|_{T'} > 0$.

Proof. Set $\gamma_\alpha := \int_T u_\alpha f\, dt$ and $\gamma := (\gamma_\alpha)_{\alpha \in A}$. By Lemma 3, there exists a basis $b = (\delta_{t(\alpha)})_{\alpha \in A}$ of P^* such that all $t(\alpha) \in \operatorname{int} T$ are continuity points of f and the numbers $c_\alpha := c_f(t(\alpha))$ are finite and strictly positive. Moreover, the $a \times a$ matrix B_b defined by (2) is invertible. Fix such a basis b, then set $l := \min_{\alpha \in A} c_\alpha > 0$. Since $t(\alpha)$ are continuity points, there exist $r_\alpha > 0$, $\alpha \in A$ such that

$$\frac{1}{m_r} \int_{B(t(\alpha), r)} |f(t) - c_\alpha|\, dt < \frac{l}{2} \tag{4}$$

for any $r < r_\alpha$, where m_r denotes the Lebesgue measure of the open ball $B(t(\alpha), r)$ for $\alpha \in A$. For each $r \in (0, r_\alpha)$ the measure $e_{\alpha r} = m(E_{\alpha r})$ of the set

$$E_{\alpha r} := \left\{ t \in B(t(\alpha), r); f(t) > \frac{l}{4} \right\} \tag{5}$$

is positive. Indeed, if $f(t) \leq l/4$ for almost all $t \in B(t(\alpha), r)$, then by using the estimates $c_\alpha \geq l$ we obtain

$$|f - c_\alpha| \geq c_\alpha - f \geq l - f \geq \frac{3l}{4}$$

almost everywhere on the ball. By integrating on the ball and multiplying with $1/m_r$, we obtain a contradiction with the estimate (4). Thus $e_{\alpha r} > 0$, $\alpha \in A$, and so for every $r \in (0, r_\alpha)$ we may set $f_{\alpha r} := e_{\alpha r}^{-1} h_{\alpha r}$, where $h_{\alpha r}$ is the characteristic function of $E_{\alpha r}$. For any $\alpha \in A$ and every r in the interval $(0, r_\alpha)$, the function $f_{\alpha r}$ defines a functional on P by the equality

$$f_{\alpha r} p := \int_T f_{\alpha r}(t) p(t)\, dt \quad (p \in P). \tag{6}$$

Let u'_β be the Fréchet differential of u_β. Let C_β be the supremum of $\|u'_\beta\|$ on T, where $\|\cdot\|$ denotes the operator norm on $(\mathbb{R}^n)^*$. Then we have the estimates

$$|u_\beta(t) - u_\beta(t(\alpha))| \le C_\beta \|t - t(\alpha)\|_2 \quad (t \in T),$$

where $\|\cdot\|_2$ is the euclidian norm on $\mathbb{R}^n$. By these estimates, we obtain the inequalities

$$|(f_{\alpha r} - \delta_{t(\alpha)})u_\beta| = \left| e_{\alpha r}^{-1} \int_{E_{\alpha r}} u_\beta \, dt - u_\beta(t(\alpha)) \right| \tag{7}$$
$$\le e_{\alpha r}^{-1} \int_{E_{\alpha r}} |u_\beta(t) - u_\beta(t(\alpha))| \, dt \le C_\beta r \quad (0 < r \le r_\alpha,\ \alpha,\ \beta \in A).$$

We can take a positive $r_0 < \min\{r_\alpha,\ \alpha \in A\}$ enough small such that the balls $B(t(\alpha), r)$, $\alpha \in A$ be disjoint and the set

$$T_r := T \setminus \bigcup_{\alpha \in A} B(t(\alpha), r)$$

have nonempty interior for all $r \in (0, r_0]$. Also, if $r_0 > 0$ if sufficiently small, then by the inequalities (7) and Lemma 2, (1) it follows that any set $b_r := (f_{\alpha r})_{\alpha \in A}$, $0 < r \le r_0$ of functionals $f_{\alpha r} \in P^*$ as in (6) is still a basis of P^*, and the corresponding operators B_{b_r} remain invertible for $0 < r \le r_0$. Finally, by taking r_0 even smaller if necessary, we can assume also the strict inequality

$$\int_{\cup_{\alpha \in A} B(t(\alpha), r)} f \, dt < \gamma_0, \quad r \le r_0, \tag{8}$$

because $f \in L^1(T)$ and $\gamma_0 > 0$.

Now fix $r := r_0$ as before. Thus we have also fixed the basis $b_r = (f_{\alpha r})_\alpha$ of P^* etc. Let $T' = \operatorname{int} T_r$. By Lemma 2, there exists an $\epsilon = \epsilon(b_r) > 0$ such that if $b' = (f'_\alpha)_{\alpha \in A}$ is any other basis of P^* consisting of some functionals $f'_\alpha \in P^*$ such that

$$\|f'_\alpha - f_{\alpha r}\| < \epsilon e_{\alpha r}^{-1} \quad (\alpha \in A), \tag{9}$$

then $B_{b'}$ is invertible, too. We fix also such an ϵ.

By Lemma 4, (1) we can take a sequence of nonnegative functions $f_k \in C_0^\infty(T \setminus \partial T)$, $k \ge 1$ such that $f_k \to f$ in $L^1(T)$. Then use Egoroff's theorem of assymptotically uniform convergence. Hence by replacing $(f_k)_{k \ge 1}$ with a subsequence, we can assume the existence of a measurable set $S \subset T$ with $m(S) < \epsilon$ such that $f_k \to f$ uniformly on $T \setminus S$. Set $E'_\alpha := E_{\alpha r} \setminus S$ for $\alpha \in A$, and note that $E_{\alpha r} \setminus E'_\alpha = E_{\alpha r} \cap S$. Thus $f_k \to f$ uniformly on $\cup_{\alpha \in A} E'_\alpha$, and $m(E_{\alpha r} \setminus E'_\alpha) < \epsilon$. Fix S, let h'_α denote the characteristic function of E'_α and set $f'_\alpha := e_{\alpha r}^{-1} h'_\alpha$. Then for any $p \in P$ with $\|p\| \leq 1$ we have

$$|(f'_\alpha - f_{\alpha r})p| \leq e_{\alpha r}^{-1} \int_{E_{\alpha r} \setminus E'_\alpha} \|p\| \, dt \leq \epsilon e_{\alpha r}^{-1}.$$

According to the above choice of ϵ, the operator $B_{b'}$ defined by using the basis $b' := (f'_\alpha)_{\alpha \in A}$ is invertible, see (9). Note that $B_{b'}$ is fixed independently of $k \geq 1$. For arbitrary $k \geq 1$, define

$$\gamma_\alpha^k := \int_T u_\alpha f_k \, dt \quad (\alpha \in A,\ k \geq 1); \quad \gamma^k := (\gamma_\alpha^k)_{\alpha \in A}.$$

Let $x^k := (B_{b'})^{-1}(\gamma^k - \gamma)$ and set $x^k = (x_{\alpha k})_{\alpha \in A}$. Since $f_k \to f$ in $L^1(T)$ and

$$\gamma_\alpha^k - \gamma_\alpha = \int_T u_\alpha(t)(f_k(t) - f(t)) \, dt \quad (\alpha \in A),$$

then $\gamma^k \to \gamma$ as $k \to \infty$. Then $x^k \to 0$. Hence $x_{\alpha k} \to 0$ for any $\alpha \in A$. Define the functions

$$g_k := f_k - \sum_{\alpha \in A} x_{\alpha k} f'_\alpha \quad (k \geq 1).$$

By using the equality $B_{b'} x^k = \gamma^k - \gamma$ where $b' = (f'_\alpha)_{\alpha \in A}$, as well as the definition of $B_{b'}$ (see (2)), we obtain, via the identification

$$\mathbb{R}^a \ni x^k \equiv \sum_\alpha x_{\alpha k} u_\alpha \in P,$$

the equalities

$$\begin{aligned}
\int_T u_\beta g_k \, dt &= \int_T u_\beta \Big(f_k - \sum_{\alpha \in A} x_{\alpha k} f'_\alpha \Big) dt \\
&= \int_T u_\beta f_k \, dt - \sum_{\alpha \in A} x_{\alpha k} \int_T u_\beta f'_\alpha \, dt \\
&= \gamma_\beta^k - (B_{b'} x^k)_\beta = \gamma_\beta \quad (\beta \in A).
\end{aligned}$$

Thus each g_k is a representing density of γ.

Let us prove that g_k is nonnegative, if $k \geq 1$ is sufficiently large. Since $E'_\alpha \subset E_{\alpha r}$ and $E_{\alpha r} \subset B(t(\alpha), r)$, then E'_α, $\alpha \in A$ are disjoint. Hence on every set E'_α we have $g_k = f_k - x_{\alpha k} f'_\alpha$. Now $f_k \to f$ as $k \to \infty$ uniformly on E'_α with $f(t) > l/4$, $t \in E'_\alpha$ (see (5)), while $x_{\alpha k} \to 0$ for any $\alpha \in A$. Note also that f'_α are bounded. Then for sufficiently large k the differences $f_k - x_{\alpha k} f'_\alpha$ become positive on E'_α, respectively. Thus $g_k \geq 0$ on each E'_α. Outside $\cup_{\alpha \in A} E'_\alpha$, we have $g_k = f_k \geq 0$. Thus by fixing a sufficiently large $k_0 \geq 1$ we obtain nonnegative representing densities g_k, $k \geq k_0$ for γ on T. Since each f'_α vanishes outside E'_α, then all such g_k are continuous on T_r.

We prove now that for sufficiently large k the nonnegative representing density g_k is strictly positive on some nonempty open subset of T_r. For $\alpha := 0$ we have in particular $\int_T g_k \, dt = \gamma_0$. Now $f_k \to f$ in $L^1(T)$, the functions $f'_\alpha \in L^1(T)$ and $x_{\alpha k} \to 0$. Then by using the estimate (8) we obtain

$$\int_{\cup_{\alpha \in A} B(t(\alpha), r)} g_k \, dt = \int_{\cup_{\alpha \in A} B(t(\alpha), r)} \left(f_k - \sum_{\alpha \in A} x_{\alpha k} f'_\alpha \right) dt \to \int_{\cup_{\alpha \in A} B(t(\alpha), r)} f \, dt < \gamma_0.$$

By comparing the estimate from above with the equality

$$\int_{T_r} g_k \, dt + \int_{\cup_{\alpha \in A} B(t(\alpha), r)} g_k \, dt = \int_T g_k \, dt = \gamma_0,$$

we note that if $k \geq k_0$ is sufficiently large, g_k has nonzero integral on T_r. Since g_k is continuous on T_r, it must be strictly positive on some open subset of $\operatorname{int} T_r = T'$, because $m(\partial T_r) = 0$ (see the definition of T_r). Now take $f_* := g_k$ and Proposition 5 is proved. □

Theorem 6 *Let $\mu \geq 0$ be a measure on T whose absolutely continuous part is nonzero. Then there exist some nonnegative functions $f_{*1} \in C_0^\infty(\operatorname{int} T)$ and $f_{*2} \in C^\infty(\mathbb{R}^n)$ with $\inf_T f_{*2} > 0$ having the same moments with respect to the functions u_α ($\alpha \in A$):*

$$\int_T u_\alpha \, d\mu = \int_T u_\alpha f_{*1} \, dt = \int_T u_\alpha f_{*2} \, dt \quad (\alpha \in A).$$

Proof. Let $\mu = f \, dt + \nu$ be the Lebesgue decomposition of μ. That is, ν is a singular measure and $f \in L^1(T) \setminus \{0\}$. Let γ_α (resp. λ_α) denote the moments of $f dt$ (resp. of ν). Set $\gamma = (\gamma_\alpha)_\alpha$ (resp. $\lambda = (\lambda_\alpha)_\alpha$). Let

$\Gamma = \gamma + \lambda$. By Proposition 5, we may assume that f is continuous on (the neighbourhood of) some closed ball $T' \subset T$ such that $f|_{T'} > 0$. Namely, replace to this aim f (resp. μ) by f_* (resp. $f_* m + \nu$). By applying Lemma 3 to f on T' now, we obtain a basis $b = (\delta_{t(\alpha)})_{\alpha \in A}$ of P^* such that $t(\alpha) \in \operatorname{int} T'$ for every $\alpha \in A$. Obviously all $t(\alpha)$ are continuity points of f with $c_\alpha := c_f(t(\alpha)) = f(t(\alpha)) \in (0, +\infty)$.

To a large extent, we follow the proof of Proposition 5. Let $l = \min_\alpha c_\alpha$, take r_α as in (4) and define $E_{\alpha r}$ by (5). Again, $e_{\alpha r} := m(E_{\alpha r}) > 0$ by the same arguments. Take $f_{\alpha r} = e_{\alpha r}^{-1} h_{\alpha r}$ with $h_{\alpha r} =$ the characteristic function of $E_{\alpha r}$. Set $b_r = (f_{\alpha r})_\alpha$. As in the previous case we can fix $r > 0$ such that the balls $B(t(\alpha), r)$ be disjoint. By taking r sufficiently small, we can moreover assume all $B(t(\alpha), r)$ included in T'. Since f in continuous on T', in the present case we have in addition that $E_{\alpha r}$ are open, see (5). Thus $f_{\alpha r}$ are now multiples of the characteristic functions of some open sets. Then we can replace them respectively by some sufficiently close (in $L^1(T)$, and so in P^*) nonnegative test functions $f_\alpha \in C_0^\infty(T \setminus \partial T)$, such that the operator B_b defined by (2) via the basis $b := (f_\alpha)_{\alpha \in A}$ be still invertible. To this aim, use Lemma 2 and the continuity of the function $\det \tau$. Thus we can assume that $b = (f_\alpha)_{\alpha \in A}$ consists of some fixed functions $f_\alpha \in C_0^\infty(E_{\alpha r})$. By extending them with 0 outside the support, we get $f_\alpha \in C_0^\infty(T \setminus \partial T)$.

By Lemma 4, (2), there exists a sequence of nonnegative functions $f_k \in C_0^\infty(\operatorname{int} T)$ such that $f_k \to f$ in $L^1(T)$ and moreover we have $f_{k+1}(t) < f_k(t)$ for all $t \in T'$ and $k \geq 1$. By Dini's lemma, the convergence of the sequence $(f_k)_k$ is also uniform on T'. In order to find f_{*1}, we can proceed by considering this sequence $f_k \in C_0^\infty(T \setminus \partial T)$. To find f_{*2}, we replace f_k by $f_k + 1/k$, which still gives a sequence (of positive functions, now) that is uniformly convergent to f on T'. In both cases, $f_k \to f$ uniformly on $\cup_{\alpha \in A} E_{\alpha r}$ $(\subset T')$.

Also, by Lemma 4, (3) there exists a sequence of nonnegative functions $\varphi_k \in C_0^\infty(\operatorname{int} T)$ such that $\varphi_k \to \nu$ weakly* in $C(T)^*$. Let γ_α^k denote the moments of f_k respectively, and set $\gamma^k := (\gamma_\alpha^k)_{\alpha \in A}$. Also, let λ_α^k be the moments of φ_k, and set $\lambda^k := (\lambda_\alpha^k)_\alpha$. Let $\Gamma^k = \gamma^k + \lambda^k$ ($\in \mathbb{R}^a$ that is identified with P). Let

$$x^k := B_b^{-1}(\Gamma^k - \Gamma)$$

and set $x^k = (x_{\alpha k})_{\alpha \in A}$.

For every $k \geq 1$, we define

$$g_k := f_k + \varphi_k - \sum_{\alpha\in A} x_{\alpha k} f_\alpha.$$

As in the previous proof, one shows that each g_k is a representing density for γ. We check now that $g_k \geq 0$ if k is sufficiently big. Again, the convergence $f_k \to f$ in $L^1(T)$ compels $\gamma^k \to \gamma$. Moreover, the weak–* convergence $\varphi_k \to \nu$ insures that $\lambda^k \to \lambda$. Then $\Gamma^k \to \Gamma$, whence $x^k \to 0$, and so all $x_{\alpha k} \to 0$ as $k \to \infty$. Moreover f_α are bounded, and hence we have the uniform convergence

$$\sum_{\alpha\in A} x_{\alpha k} f_\alpha \to 0 \quad \text{as} \quad k \to \infty.$$

Also, the convergence of f_k to f ($> l/4$ on each $E_{\alpha r}$, see (5)) is uniform on $E_{\alpha r}$. Therefore all

$$g_k \mid E_{\alpha r} = f_k + \varphi_k - x_{\alpha k} f_\alpha \geq f_k - x_{\alpha k} f_\alpha > 0 \tag{10}$$

for sufficiently large k, while

$$g_k \mid T \setminus (\cup_{\alpha\in A} E_{\alpha r}) = f_k + \varphi_k \geq 0 \tag{11}$$

(respectively > 0 when we choose to replace f_k by $f_k + 1/k$). Hence for large $k \geq 1$ we can take f_{*1} (resp. f_{*2}) to be $g_k + \varphi_k \geq 0$ (resp. > 0). Thus Theorem 6 is proved. □

Theorem 7 *If $A \subset \mathbb{Z}_+^n$ and the set $\gamma = (\gamma_\alpha)_{\alpha\in A}$ has nonnegative representing densities on T with respect to the monomials $u_\alpha(t) = t^\alpha$, then there exists a polynomial $p(t) = \sum_{\alpha\in A} x_\alpha t^\alpha$ such that*

$$\int_T t^\alpha p_+(t)\, dt = \gamma_\alpha \quad (\alpha \in A),$$

where $p_+(t) := \max\{p(t), 0\}$, $t \in T$. Moreover the numbers $x_\alpha \in \mathbb{R}$, $\alpha \in A$ are uniquely determined (if $\gamma \neq 0$).

Proof. Let H be the real Hilbert space of all real functions $u \in L^2(T)$. Define the real functionals F and F_α on H by $F(u) := 2^{-1} \int_T u^2\, dt$ and

$$F_\alpha(u) := \int_T u_\alpha u\, dt - \gamma_\alpha \quad (u \in H,\ \alpha \in A).$$

Note that F_α are continuous and linear (modulo some constants), while $F = F(u) = 2^{-1}\langle u|u\rangle_H$ is continuously Fréchét differentiable since $\langle\cdot|\cdot\rangle_H$

is bilinear and continuous. Consider also the set $Q := \{u \in H \mid u \geq 0 \text{ a.e.}\}$. Define

$$S := \{u \in H \mid u \in Q;\ F_\alpha(u) = 0,\ \alpha \in A\}.$$

By Theorem 6, the set γ has at least one representig density $f \in C_0^\infty(T\backslash\partial T)$ (in particular, square–integrable); thus $f \in S$. Therefore S is a nonempty closed convex subset of H. Hence there exists $s \in H$ such that $\|s\|_H = \min\{\|u\|_H; u \in S\}$. Now in order to minimize F $(= 2^{-1}\| \cdot \|_H^2)$ on S, we apply a version of the method of the Lagrange multipliers for conditionned extremum in infinite–dimensional cones. There exist some real numbers x_α $(\alpha \in A)$ such that the Fréchét differential $L_s := G'(s) : H \to \mathbb{R}$ of the map G defined by

$$G(u) := F(u) - \sum_{\alpha\in A} x_\alpha F_\alpha(u) \quad (u \in H)$$

in the point s satisfies the inequalities

$$L_s s \leq L_s v \quad \text{for all} \quad v \in Q, \tag{12}$$

see for instance [8]. Now a simple computation using the formula of the Gâteaux differential

$$L_s v = G'(s)v = \lim_{\varepsilon\to 0} \varepsilon^{-1}(G(s+\varepsilon v) - G(s))$$

shows that the continuous linear functional L_s is given by

$$L_s v := \int_T sv\,dt - \sum_{\alpha\in A} x_\alpha \int_T u_\alpha v\,dt \quad (v \in H)$$

Therefore, by (12) we have the inequalities

$$\int_T (s - \sum_{\alpha\in A} x_\alpha u_\alpha)(s-v)\,dt \leq 0 \quad \text{for all} \quad v \in Q.$$

Then set $p := \sum_{\alpha\in A} x_\alpha u_\alpha$ and take $v := p_+$ $(\in Q)$ in the previous estimates. Hence

$$\int_T (s-p)(s-p_+)\,dt \leq 0.$$

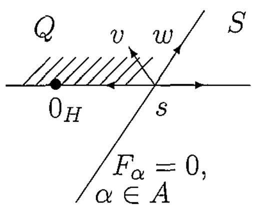
But the integral from above can be written also as the sum

$$\int_{\{t\in T|p(t)<0\}}(s-p)s\,dt+\int_{\{t\in T|p(t)\geq 0\}}(s-p)^2\,dt$$

which is nonnegative, since both integrals are ≥ 0 (note that $s \geq 0$). Therefore they must be null. It follows $s(t) = p(t)$ for almost all t such that $p(t) \geq 0$, while $s(t) = 0$ for almost all t with $p(t) < 0$. That is, $s = p_+$.

If $\gamma \neq 0$, then any representing density must be nonzero. In particular $p_+ \not\equiv 0$, and so the coefficients x_α, $\alpha \in A$ of p are uniquely determined by p_+, namely by the restriction of p to the open set $\{t \in T; p(t) > 0\} \neq \emptyset$. Now p_+ is the solution s of our minimization problem, and so it is uniquely determined by γ, due to the strict convexity of the norm. Thus Theorem 7 is proved. □

Remark To have an intuition on the relevance of (12), let H be represented below by the plane, Q by the upper half–plane, the set $\cap_{\alpha\in A}[F_\alpha = 0]$ by the oblique line and $(2F)^{1/2}$ by the distance to the origin 0_H. Then (12) is equivalent to $G'(s)(v-s) \geq 0$. This means that $G := F - \sum_\alpha x_\alpha F_\alpha$ increases locally along any direction given by some vector $v - s$ starting from s into Q. In particular, the restriction F of G to S increases in the directions starting from s along each vector $v := w \in S$. Hence s is a point of local minimum for F on S. On convexity reasons, it turns then to be a global minimum point.



$$S = \cap_\alpha[F_\alpha = 0] \cap Q$$

$$(F - \textstyle\sum_\alpha x_\alpha F_\alpha)'(s)(v-s) \geq 0,\ v \in Q$$

$$F'(s)(w-s) \geq 0,\ w \in S$$

As expected, almost word–for–word versions of these arguments lead to similar statements in other cases, like the trigonometric moments problems on the n–dimensional torus, for instance. To cover a larger class of spaces, we consider the hypothesys below.

Hypothesys *Let M be an n–dimensional manifold of class C^k ($k \geq 1$ or $k = \infty$). Let $T \subset M$ be a compact subset with nonempty interior. Assume the topological boundary of T negligible in M. Let u_α ($\alpha \in A$) be a finite set of functions of class C^1 on a neighbourhood of T.*

To fix the notation and for the sake of completness, we remind in this context a few definitions. Again, let the space $C(T)$ of all continuous functions on T be endowed with the sup norm. As usually, a measure $\mu \in C(T)^*$ is called *absolutely continuous* if it is locally absolutely continuous with respect to the Lebesgue measure. That is, we can cover T with open domains $U \subset M$ of local charts $\chi \colon U \to V$, where V is open in $\mathbb{R}^n$, such that each measure $\mu|_{T\cap U} \circ \chi^{-1}$ be of the form $\phi\, dx$ where $\phi \in L^1(V)$ is supported on $\chi(T \cap U)$. Here $\mu|_{T\cap U} \circ \chi^{-1}(B) = \mu(T \cap \chi^{-1}(B))$ for any Borel subset $B \subset V$. Also, $dx = dx_1 \cdots dx_n$ stands for the Lebesgue measure on the domain V of the real parameters $x_1, \ldots, x_n$. For any continuous function u compactly supported in U we have the "change of variables" formula $\mu(u|_T) = (\mu \circ \chi^{-1})(u \circ \chi^{-1})$. For an arbitrary $u \in C(T)$, set as usually $\mu(u) = \sum_i \mu(\psi_i u)$ where $(\psi_i)_i$ is a partition of unity subordinate to an open cover $(U_i)_i$ of T etc.

An absolutely continuous measure μ is said to be $(k-1)$–*smooth* (resp. *positive*) on (an open neighbourhood of) T if it can be locally represented as before by densities ϕ that are of class C^{k-1} (resp. that are positive).

Equivalently, by using a partition of unity we can fix any measure m on M whose localizations $m \circ \chi^{-1}$ by charts $\chi \colon U \to V$ are equivalent to the corresponding Lebesgue measures dx on $V \subset \mathbb{R}^n$ via some densities $\rho = \rho_\chi(x)$ of class C^{k-1}. More precisely, we have $m \circ \chi^{-1} = \rho\, dx$ and there are some positive constants $c = c(\chi)$, $C = C(\chi)$ such that $c \leq \rho \leq C$ on V. Then μ is $(k-1)$–smooth (resp. positive) on T iff $\mu|_T = fm$ for some function f that is of class C^{k-1} on T (resp. that is positive on T). These properties of μ prove to be well–defined, namely independent of the choice of one of the (equivalent) Lebesgue–type measures m on T etc. Note to this aim that they are local and (locally) preserved by maps of the form $\phi \mapsto (\phi \circ h)|\det h'|$ where $h = \chi_1 \circ \chi_2^{-1}$ is a changement of charts. If the measure μ has a density $f > 0$ on the compact set T, we write $\mu > 0$ on T. Within the hypothesys from above, we have the following.

Theorem 8 *Let μ be an absolutely continuous nonnegative measure on T. Then there exists a $(k-1)$–smooth absolutely continuous measure $\mu_* > 0$ on T such that $\mu_*(u_\alpha) = \mu(u_\alpha)$ for all $\alpha \in A$.*

Proof. Endow M with a metric compatible with the topology, via some embedding of M into a large euclidian space, for instance. Cover the compact subset T with a finite set of open domains U_i of charts $\chi_i \colon U \to \mathbb{R}^n$

$(i = 1, \ldots, p)$. Fix a constant $\delta > 0$ that is strictly less than the Lebesgue number of this cover. That is, for any subset $D \subset T$ of diameter $\leq \delta$ there exists at least an index i such that $D \subset U_i$. Cover again T, now by a finite number of open sets $D_j \subset M$ $(j = 1, \ldots, q)$ of diameter $\leq \delta$, each of them having compact closure $\overline{D}_j$ (of diameter $\leq \delta$) and C^1–regular (and hence negligible) boundary ∂D_j. For every $j = \overline{1, q}$ choose one index $i = i_j \in \{1, \ldots, p\}$ such that $\overline{D}_j \subset U_{i_j}$. For every $i = \overline{1, p}$ let T_i be the union of all $\overline{D}_j$ such that $i_j = i$. Thus $T_i \subset U_i$. We have then an open covering $\cup_i \operatorname{int} T_i$ of the set T. Take in the neighbourhood of T a partition of unity $(\psi_i)_i$ subordinated to the cover $(\operatorname{int} T_i)_i$. Define the function $u_{\alpha i} := (\psi_i u_\alpha) \circ \chi_i^{-1}$ on the image by χ_i of the domain of u_α $(\alpha \in A)$. Then every function $u_{\alpha i}$ is of class C^1 on a neighborhood of the compact subset $K_i := \chi_i(T_i \cap T)$ of $\mathbb{R}^n$. Also, the boundary of K_i is contained in the set $\chi_i(\partial T \cup (\cup_j \partial D_j))$, and hence it is negligible. Set $\mu_i = \mu \circ \chi_i^{-1} | K_i$, which is an absolutely continuous measure on $\mathbb{R}^n$. For each index i such that the measure $\mu_i \neq 0$ (and so $\operatorname{int} K_i \neq \emptyset$), we can apply Theorem 6 for μ_i on K_i with respect to the functions $u_{\alpha i}$ $(\alpha \in A)$. Hence for every i there exists a smooth absolutey continuous measure $\mu_{*i} > 0$ on K_i such that

$$\mu_i(u_{\alpha i}) = \mu_{*i}(u_{\alpha i}) \quad (\alpha \in A).$$

Then we have the smooth absolutely continuous measures $\mu_{*i} \circ \chi_i > 0$ on $T_i \cap T$. In the case $\mu_i = 0$, we simply set $\mu_{*i} = 0$. Define the measure μ_* on T by

$$\mu_* = \sum_i \psi_i(\mu_{*i} \circ \chi_i),$$

namley by

$$\begin{aligned} \mu_*(u) &= \int_T u \, d\mu_* \\ &:= \sum_i \int_{T_i \cap T} u \psi_i \, d(\mu_{*i} \circ \chi_i) \quad (u \in C(T)). \end{aligned}$$

The continuity of μ_* on $C(T)$ follows from the form of the right–hand side above. Also, $\mu_* > 0$ by using $\sum_i \psi_i = 1$. The following equalities hold

$$\mu(u_\alpha) = \int_T u_\alpha \, d\mu = \int_T \sum_i \psi_i u_\alpha \, d\mu$$

$$=\sum_i \int_{T_i\cap T} \psi_i u_\alpha \, d\mu = \sum_i \int_{K_i} u_{\alpha i} \, d\mu_i$$
$$=\sum_i \int_{K_i} u_{\alpha i} \, d\mu_{*i} = \sum_i \int_{T_i\cap T} \psi_i u_\alpha \, d(\mu_{*i} \circ \chi_i)$$
$$=\int_T u_\alpha \, d\mu_* = \mu_*(u_\alpha),$$

and so Theorem 8 is proved. □

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