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On approximation of 2π-periodic functions in Hölder spaces

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Abstract. This note is connected with results given in papers [2-5]. We give two approximation theorems for 2π-periodic functions belonging to generalized Hölder spaces. We present also applications of these theorems.

Key words: Hölder space, approximation theorem, de la Vallée Poussin integral, Abel means.

1. Preliminaries

1.1. Let $L_{2\pi}^p$, $1 \le p \le \infty$, be the space of 2π-periodic real-valued functions, Lebesgue integrable with p-th power on $[-\pi, \pi]$ if $1 \le p < \infty$ and continuous on $R := (-\infty, +\infty)$ if $p = \infty$. Let the norm of f in $L_{2\pi}^p$ be defined by

$$\|f\|_p \equiv \|f(\,\cdot\,)\|_p := \begin{cases} \left(\displaystyle\int_{-\pi}^{\pi} |f(x)|^p\,dx\right)^{1/p} & \text{if} \quad 1 \le p < \infty, \\ \displaystyle\max_{|x|\le\pi} |f(x)| & \text{if} \quad p = \infty. \end{cases} \tag{1}$$

For $f \in L_{2\pi}^p$, we define as usual ([7]) the modulus of continuity $\omega_1(f, p; \cdot)$ and the modulus of smoothness $\omega_k(f, p; \cdot)$ of the order $2 \le k \in N := \{1, 2, \ldots\}$ by the formula

$$\omega_k(f, p; t) := \sup_{|h|\le t} \|\Delta_h^k f(\,\cdot\,)\|_p, \quad t \ge 0, \tag{2}$$

where

$$\Delta_h^1 f(x) := f(x+h) - f(x), \tag{3}$$
$$\Delta_h^k f(x) := \Delta_h^1\big(\Delta_h^{k-1} f(x)\big) \quad \text{if} \quad k \ge 2.$$

Hence, we have

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$$\Delta_h^k f(x) = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} f(x+jh).$$

It is known ([7]) that for every $f \in L_{2\pi}^p$, the function $\omega_k(f, p; \cdot)$ is positive and non-decreasing and

$$\lim_{t\to 0+} \omega_k(f, p; t) = 0. \tag{4}$$

Moreover,

$$\omega_k(f, p; t) \le 2\omega_{k-1}(f, p; t) \le \cdots \le 2^{k-1}\omega_1(f, p; t) \le 2^k \|f\|_p, \quad t \ge 0.$$

1.2. Let $k \in N$ be a fixed number. As in [4] we denote by Ω_k the set of all modulus - type functions i.e. Ω_k is the set of all functions ω defined on $R_0 := [0, +\infty)$ such that

a) $\omega(t) \ge 0$ for $t \in R_0$,
b) ω is increasing,
c) $\omega(t) \to \omega(0) = 0$ as $t \to 0+$,
d) $\omega(t)t^{-k}$ is monotonically decreasing.

Similarly as in [4], for a given $\omega \in \Omega_k$, $k \in N$, and $1 \le p \le \infty$ we define the generalized Hölder spaces $H_{2\pi}^{k,\omega,p}$ and $\widetilde{H}_{2\pi}^{k,\omega,p}$ as follows: the space $H_{2\pi}^{k,\omega,p}$ is the set of all functions $f \in L_{2\pi}^p$ for which

$$\|f\|_{k,\omega,p}^* \equiv \|f(\cdot)\|_{k,\omega,p}^* := \sup_{h>0} \frac{\|\Delta_h^k f(\cdot)\|_p}{\omega(h)} < +\infty \tag{5}$$

and the norm is defined by

$$\|f\|_{H^{k,\omega,p}} \equiv \|f(\cdot)\|_{H^{k,\omega,p}} := \|f\|_p + \|f\|_{k,\omega,p}^*. \tag{6}$$

It is easily verified that $f \in H_{2\pi}^{k,\omega,p}$ if and only if there exists a positive constant M depending only on f, p and k such that

$$\omega_k(f, p; t) \le M\omega(t) \quad \text{for} \quad t \ge 0.$$

The space $\widetilde{H}_{2\pi}^{k,\omega,p}$ is the set of all functions $f \in H_{2\pi}^{k,\omega,p}$ for which

$$\lim_{t\to 0+} \frac{\omega_k(f, p; t)}{\omega(t)} = 0 \tag{7}$$

and the norm is defined by (6).

If $\omega(t) = t^\alpha$ for $t \geq 0$ and for fixed $0 < \alpha \leq k$, $k \in N$, then $H_{2\pi}^{k,\omega,p}$ and $\widetilde{H}_{2\pi}^{k,\omega,p}$ are the classical Hölder-Lipschitz-Zygmund spaces.

If ω, $\mu \in \Omega_k$ and

$$\lambda(t) := \frac{\omega(t)}{\mu(t)}, \quad t > 0, \tag{8}$$

is a non-decreasing function, then

$$H_{2\pi}^{k,\omega,p} \subset H_{2\pi}^{k,\mu,p}, \quad \widetilde{H}_{2\pi}^{k,\omega,p} \subset \widetilde{H}_{2\pi}^{k,\mu,p}. \tag{9}$$

Moreover, for every $f \in H_{2\pi}^{k,\omega,p}$ we have

$$\omega_k(f, p; t) \leq \omega(t) \|f\|_{k,\omega,p}^*, \quad t \geq 0. \tag{10}$$

1.3. Let $I := [a, b) \subseteq R_0$ be a given interval. For functions $f \in L_{2\pi}^p$, $1 \leq p \leq \infty$, and for $r \in I$ we define the family of integral operators

$$A_r(f; x) := \int_{-\pi}^{\pi} f(t)\Phi_r(t - x)\, dt, \quad x \in R, \tag{11}$$

where $\Phi = \{\Phi_r : r \in I\}$ is a family of functions in $L_{2\pi}^\infty$ satisfying

$$\int_{-\pi}^{\pi} \Phi_r(t)\, dt = 1, \quad r \in I, \tag{12}$$

and there exists a positive constant M_Φ, depending on Φ and independent of $r \in I$ such that

$$\int_{-\pi}^{\pi} |\Phi_r(t)|\, dt \leq M_\Phi \quad \text{for} \quad r \in I. \tag{13}$$

Then $A_r(f)$ can be written in the form

$$A_r(f; x) := \int_{-\pi}^{\pi} f(x + t)\Phi_r(t)\, dt, \quad x \in R, \quad r \in I. \tag{14}$$

2. Main results

2.1. First we shall give some auxiliary results.

Lemma 1 *Let $\Phi_r(\cdot) \in \Phi$, $r \in I$. Then for any $f \in L_{2\pi}^p$, we have*

$$\|A_r(f; \cdot)\|_p \leq \|f(\cdot)\|_p \|\Phi_r(\cdot)\|_1 \leq M_\Phi \|f\|_p \quad \textit{for} \quad r \in I, \tag{15}$$

where M_Φ is the positive constant given in (13).

The inequality (15) *shows that A_r is a bounded linear operator from $L^p_{2\pi}$ into $L^p_{2\pi}$.*

Proof. If $p=\infty$, then (15) is clear. If $1\le p<\infty$, then (15) follows from Fubini's theorem and Hölder's inequality. □

Lemma 2 *Let $\Phi_r(\cdot)\in\Phi$, $r\in I$ and let $\omega\in\Omega_k$. Then for every $f\in H^{k,\omega,p}_{2\pi}$, we have*

$$\begin{aligned}\|A_r(f;\cdot)\|_{H^{k,\omega,p}} &\le \|f\|_{H^{k,\omega,p}}\|\Phi_r(\cdot)\|_1\\ &\le M_\Phi\|f\|_{H^{k,\omega,p}} \quad \text{for} \quad r\in I,\end{aligned} \tag{16}$$

where M_Φ is the positive constant as in (13). *From* (16) *it follows that A_r is a bounded linear operator from $H^{k,\omega,p}_{2\pi}$ into $H^{k,\omega,p}_{2\pi}$.*

Proof. By (6) and (5) we can write

$$\|A_r(f;\cdot)\|_{H^{k,\omega,p}} = \|A_r(f;\cdot)\|_p + \|A_r(f;\cdot)\|^*_{k,\omega,p}, \tag{17}$$

$$\|A_r(f;\cdot)\|^*_{k,\omega,p} = \sup_{h>0}\frac{\|\Delta^k_h A_r(f;\cdot)\|_p}{\omega(h)},$$

for every $f\in H^{k,\omega,p}_{2\pi}$ and $r\in I$. By (3) and (14) we get

$$\begin{aligned}\Delta^k_h A_r(f;x) &= \int_{-\pi}^{\pi}[\Delta^k_h f(x+t)]\Phi_r(t)\,dt\\ &= A_r(\Delta^k_h f;x), \quad x\in R, \quad h\in R_0.\end{aligned} \tag{18}$$

Now, applying Lemma 1, we obtain

$$\|\Delta^k_h A_r(f;\cdot)\|_p = \|A_r(\Delta^k_h f;\cdot)\|_p \le M_\Phi\|\Delta^k_h f(\cdot)\|_p, \tag{19}$$

and consequently

$$\|A_r(f;\cdot)\|^*_{k,\omega,p} \le M_\Phi\|f\|^*_{k,\omega,p}, \quad r\in I. \tag{20}$$

Using (15) and (20) to (17), we obtain (16). □

Lemma 3 *Suppose that Φ_r and ω be as in* Lemma 2. *Then, for every fixed $r\in I$, A_r is a bounded linear operator from $\widetilde{H}^{k,\omega,p}_{2\pi}$ into $\widetilde{H}^{k,\omega,p}_{2\pi}$.*

Proof. By (19), for all $f\in\widetilde{H}^{k,\omega,p}_{2\pi}$, we can write

$$0\le\omega_k(A_r(f),p;t)\le M_\Phi\omega_k(f,p;t), \quad t\ge 0, \quad r\in I,$$

which together with (7) implies $A_r(f;\cdot)\in\widetilde{H}^{k,\omega,p}_{2\pi}$. □

2.2. Now we shall prove two main theorems for the operators $A_r(f)$.

Theorem 1 *Assume that $s \in N$ is a fixed number and ω, $\mu \in \Omega_s$ are functions for which $\lambda(t) = \omega(t)/\mu(t)$ is non-decreasing for $t > 0$. Moreover assume that*

$$\|A_r(f;\,\cdot\,) - f(\,\cdot\,)\|_p \le M_1 \omega_s(f,\,p;\varphi(r)), \quad r \in I, \quad f \in L^p_{2\pi}, \tag{21}$$

with a given $M_1 = const. > 0$ and a given positive function $\varphi(\,\cdot\,)$, continuous and decreasing on $I = [a,\,b)$ and $\lim_{r\to b-}\varphi(r) = 0$.

Then there exists a positive constant M_2 depending only on Φ, s and $\mu(\varphi(a))$ such that for every $f \in H^{s,\,\omega,\,p}_{2\pi}$,

$$\|A_r(f;\,\cdot\,) - f(\,\cdot\,)\|_{H^{s,\,\mu,\,p}} \le M_2 \|f\|^*_{s,\,\omega,\,p} \lambda(\varphi(r)), \quad r \in I. \tag{22}$$

Proof. Denote

$$B_r(f;x) := A_r(f;x) - f(x), \quad x \in R, \quad r \in I, \tag{23}$$

for $f \in H^{s,\,\omega,\,p}_{2\pi}$. By our assumptions, we have $H^{s,\,\omega,\,p}_{2\pi} \subset H^{s,\,\mu,\,p}_{2\pi}$. Hence by Lemma 2 and (6), we can write

$$\|B_r(f;\,\cdot\,)\|_{H^{s,\,\mu,\,p}} = \|B_r(f;\,\cdot\,)\|_p + \|B_r(f;\,\cdot\,)\|^*_{s,\,\mu,\,p}, \tag{24}$$

for every $f \in H^{s,\,\omega,\,p}_{2\pi}$ and $r \in I$. Applying (21) and (10), we get

$$\begin{aligned}\|B_r(f;\,\cdot\,)\|_p &\le M_1 \omega_s(f,\,p;\varphi(r)) \le M_1 \|f\|^*_{s,\,\omega,\,p} \omega(\varphi(r)) \\ &\le M_1 \mu(\varphi(a)) \|f\|^*_{s,\,\omega,\,p} \lambda(\varphi(r))\end{aligned} \tag{25}$$

for $r \in I$. Moreover, we have

$$\begin{aligned}\|B_r(f;\,\cdot\,)\|^*_{s,\,\mu,\,p} &= \sup_{h>0} \frac{\|\Delta^s_h B_r(f;\,\cdot\,)\|_p}{\mu(h)} \\ &\le \left\{ \sup_{0<h\le\varphi(r)} + \sup_{h>\varphi(r)} \right\} \frac{\|\Delta^s_h B_r(f;\,\cdot\,)\|_p}{\mu(h)} := W_r + Z_r.\end{aligned} \tag{26}$$

By (23) and (19), it follows that

$$\begin{aligned}\|\Delta^s_h B_r(f;\,\cdot\,)\|_p &\le \|\Delta^s_h A_r(f;\,\cdot\,)\|_p + \|\Delta^s_h f(\,\cdot\,)\|_p \\ &\le (M_\Phi + 1) \|\Delta^s_h f(\,\cdot\,)\|_p\end{aligned}$$

and further we get

$$W_r \leq (M_\Phi + 1) \sup_{0<h\leq\varphi(r)} \frac{\|\Delta_h^s f(\cdot)\|_p}{\mu(h)} \tag{27}$$
$$\leq (M_\Phi + 1)\lambda(\varphi(r)) \sup_{0\leq h\leq\varphi(r)} \frac{\|\Delta_h^s f(\cdot)\|_p}{\omega(h)}$$
$$\leq (M_\Phi + 1)\lambda(\varphi(r))\|f\|^*_{s,\omega,p}, \quad r \in I.$$

Since $\|\Delta_h^s f(\cdot)\|_p \leq 2^s\|f\|_p$, we have by (21) and (10),

$$Z_r \leq 2^s \sup_{h>\varphi(r)} \frac{\|B_r(f;\cdot)\|_p}{\mu(h)} \leq M_1 2^s \frac{\omega_s(f,p;\varphi(r))}{\mu(\varphi(r))} \tag{28}$$
$$\leq M_1 2^s \|f\|^*_{s,\omega,p}\lambda(\varphi(r)), \quad r \in I.$$

Combining (24)-(28), we immediately obtain (22). □

Theorem 2 *Let assumptions of* Theorem 1 *be satisfied. Then for every* $f \in \widetilde{H}_{2\pi}^{s,\omega,p}$, *we have*

$$\|A_r(f;\cdot) - f(\cdot)\|_{H^{s,\mu,p}} = o(\lambda(\varphi(r))), \quad as \quad r \to b-. \tag{29}$$

Proof. Denoting $B_r(f;x)$ as in (23), by (9) and Lemma 3 we can write the formula (24). Applying (21), we get

$$0 \leq \|B_r(f;\cdot)\|_p \leq M_1\omega_s(f,p;\varphi(r))$$
$$\leq M_1\mu(\varphi(a))\lambda(\varphi(r))\frac{\omega_s(f,p;\varphi(r))}{\omega(\varphi(r))}, \quad r \in I,$$

which, by (7) and by $\varphi(r) \to 0+$ as $r \to b-$, implies

$$\|B_r(f;\cdot)\|_p = o(\lambda(\varphi(r))) \quad \text{as} \quad r \to b-. \tag{30}$$

Analogously as in the proof of Theorem 1 we can write the inequality (26) and similarly as in (27) and (28) we get

$$0 \leq W_r \leq (M_\Phi + 1)\lambda(\varphi(r)) \sup_{0<h\leq\varphi(r)} \frac{\omega_s(f,p;h)}{\omega(h)},$$
$$0 \leq Z_r \leq 2^s M_1\lambda(\varphi(r))\frac{\omega_s(f,p;\varphi(r))}{\omega(\varphi(r))}, \quad r \in I,$$

which, by (7) and the properties of $\varphi(\cdot)$ and $\lambda(\cdot)$, implies

$$W_r = o(\lambda(\varphi(r))), \quad Z_r = o(\lambda(\varphi(r))) \quad \text{as} \quad r \to b-.$$

Consequently we have

$$\|B_r(f;\,\cdot\,)\|^*_{s,\,\mu,\,p} = o(\lambda(\varphi(r))) \quad \text{as} \quad r \to b- \,. \tag{31}$$

Now the desired assertion (29) immediately follows from (24), (30) and (31). □

From the above theorems we derive the following two corollaries.

Corollary 1 *Let $\omega,\,\mu \in \Omega_k$ with a fixed $k \in N$ and let the function λ defined by (8) be increasing and $lim_{t\to 0+}\lambda(t) = 0$. Then for every $f \in H^{k,\,\omega,\,p}_{2\pi}$ satisfying the condition (21), we have*

$$\|A_r(f;\,\cdot\,) - f(\,\cdot\,)\|_{H^{k,\,\mu,\,p}} = o(1) \quad \textit{as} \quad r \to b- \,.$$

Corollary 2 *Let $\omega(t) = t^\alpha$, $\mu(t) = t^\beta$ for $t \ge 0$ and $0 < \beta < \alpha \le k$, where $k \in N$. Then for every $f \in H^{k,\,\omega,\,p}_{2\pi}$ satisfying the condition (21), we have*

$$\|A_r(f;\,\cdot\,) - f(\,\cdot\,)\|_{H^{k,\,\mu,\,p}} = O((\varphi(r))^{\alpha-\beta}) \quad \textit{as} \quad r \to b- \,.$$

Moreover,

$$\|A_r(f;\,\cdot\,) - f(\,\cdot\,)\|_{H^{k,\,\mu,\,p}} = o((\varphi(r))^{\alpha-\beta}) \quad \textit{as} \quad r \to b-,$$

for every $f \in \widetilde{H}^{k,\,\omega,\,p}_{2\pi}$ satisfying the condition (21).

3. Applications

In this section we shall consider two examples of operators of the type A_r defined by (11)-(13). Applying Theorem 1 and Theorem 2, we shall derive certain estimations for these operators.

3.1. First we consider the de la Vallée Poussin integral $V_n(f)$ ([3], [6]) of function $f \in L^p_{2\pi}$, $1 \le p \le \infty$,

$$V_n(f;x) := \frac{(2n)!!}{(2n-1)!!}\,\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\cos^{2n}\frac{t-x}{2}\,dt, \qquad x \in R, \quad n \in N, \tag{32}$$

where $(2n)!! = 2\cdot 4\cdot\dots\cdot(2n)$ and $(2n-1)!! = 1\cdot 3\cdot 5\cdot\dots\cdot(2n-1)$. It is known ([3]) that $V_n(f;\,\cdot\,)$ is a trigonomertic polynomial of the order at most n and

$$V_n(f;x)=\frac{1}{2}a_0(f) + \frac{(2n)!!}{2^{2n}(2n-1)!!}\sum_{k=1}^{n}\binom{2n}{n-k}\left[a_k(f)\cos kx + b_k(f)\sin kx\right],$$

where $a_k(f)$ and $b_k(f)$ are coefficients of the Fourier series of function $f \in L^p_{2\pi}$. Moreover, it is known ([3], [6]) that for every fixed $1 \le p \le \infty$ there exists a positive constant M_p depending only on p such that for every $f \in L^p_{2\pi}$,

$$\|V_n(f;\,\cdot\,) - f(\,\cdot\,)\|_p \le M_p\omega_2(f,\,p;1/\sqrt{n}), \quad n \in N.$$

Since

$$\int_{-\pi}^{\pi}\cos^{2n}\frac{t}{2}\,dt = 2\pi\frac{(2n-1)!!}{(2n)!!}, \quad n \in N,$$

we see that $V_n(f)$ is the operator of the type (11)-(13), with $r = n$ and $I \equiv N$.

Applying Theorem 1 and Theorem 2, we can formulate for $V_n(f)$ analogies of Corollary 1 and Corollary 2. Now $\varphi(n) = 1/\sqrt{n}$ for $n \in N$. In particular we have

Corollary 3 *If $\omega(t) = t^\alpha$ and $\mu(t) = t^\beta$ for $t \ge 0$ and $0 < \beta < \alpha \le 2$, then for every $f \in H^{2,\omega,p}_{2\pi}$, we have*

$$\|V_n(f;\,\cdot\,) - f(\,\cdot\,)\|_{H^{2,\mu,p}} = O(n^{(\beta-\alpha)/2}), \quad n \in N.$$

If $f \in \widetilde{H}^{2,\omega,p}_{2\pi}$, then

$$\|V_n(f;\,\cdot\,) - f(\,\cdot\,)\|_{H^{2,\omega,p}} = o(n^{(\beta-\alpha)/2}) \quad as \quad n \to \infty.$$

3.2. Let $f \in L^p_{2\pi}$ with a fixed $1 \le p \le \infty$ and let $S_k(f;\,\cdot\,)$ be the k-th partial sum of the Fourier series of f. In [1], we considered the following Abel means of the order $m \in N_0 := N \cup \{0\}$ of the Fourier series of f:

$$U_r(f,\,m;x) = (1-r)^{m+1}\sum_{k=0}^{\infty}\binom{m+k}{k}r^k S_k(f;x), \tag{33}$$

$x \in R$, $r \in [0,\,1)$. It is known ([1]) that $U_r(f,\,m)$ can be written in the integral form

$$U_r(f,m;x) = \int_{-\pi}^{\pi} f(x+t)K_r(m;t)\,dt,$$

where

$$K_r(m;t) = \frac{(1-r)^{m+1}}{\pi} \sum_{j=0}^{\infty} \binom{m+j}{j} r^j D_j(t),$$

$$D_0(t) = \frac{1}{2}, \quad D_j(t) = \frac{1}{2} + \cos t + \cos 2t + \cdots + \cos jt \quad \text{if} \quad j \geq 1.$$

Moreover it is known ([1]) that, for fixed $m \in N_0$ and $0 \leq r < 1$, the function $K_r(m;\,\cdot\,)$ belongs to the space $L^{\infty}_{2\pi}$ and satisfies conditions of the type (12) and (13). From the above we deduce that $U_r(f,\,m;\,\cdot\,)$ is an operator of the type (11)-(13), with $I = [0,\,1)$.

In [1] it was proved that for fixed $2 \leq m \in N$ and $1 \leq p \leq \infty$, there exists a positive constant $M_{m,p}$ depending only on m and p such that

$$\|U_r(f,\,m;\,\cdot\,) - f(\,\cdot\,)\|_p \leq M_{m,p}\omega_m(f,\,p;1-r) \tag{34}$$

for every $f \in L^p_{2\pi}$ and for all $r \in [0,\,1)$. Moreover if $m = 0,\,1$ and $0 < r_0 < 1$, then there exists a positive constant M_{p,r_0} such that for every $f \in L^p_{2\pi}$,

$$\begin{aligned} &\|U_r(f,\,m;\,\cdot\,) - f(\,\cdot\,)\|_p \\ &\leq M_{p,r_0} \begin{cases} \omega_2(f,\,p;1-r) & \text{if} \quad m = 1, \\ \omega_1(f,\,p;(1-r)|\ln(1-r)|) & \text{if} \quad m = 0, \end{cases} \end{aligned} \tag{35}$$

for all $r \in [r_0,\,1)$.

Applying Theorem 1, Theorem 2 and (34) we obtain

Corollary 4 *Suppose that $1 \leq p \leq \infty$, $2 \leq m \in N$, $\omega,\,\mu \in \Omega_m$ and $\lambda(t) = \omega(t)/\mu(t)$ is monotonically increasing function for $t > 0$. Then for the Abel means $U_r(f,\,m;\,\cdot\,)$ of the Fourier series of $f \in H^{m,\omega,p}_{2\pi}$, we have*

$$\|U_r(f,\,m;\,\cdot\,) - f(\,\cdot\,)\|_{H^{m,\mu,p}} = O(\lambda(1-r)) \quad \textit{for} \quad r \in [0,\,1). \tag{36}$$

If $f \in \widetilde{H}^{m,\omega,p}_{2\pi}$, then

$$\|U_r(f,\,m;\,\cdot\,) - f(\,\cdot\,)\|_{H^{m,\mu,p}} = o(\lambda(1-r)) \quad \textit{as} \quad r \to 1-. \tag{37}$$

Corollary 5 *Let $1 \leq p \leq \infty$ and $2 \leq m \in N$ and let $\omega(t) = t^{\alpha}$, $\mu(t) = t^{\beta}$ for $t \geq 0$ and $0 < \beta < \alpha \leq m$. Then for every $f \in H^{m,\omega,p}_{2\pi}$, we have*

$$\|U_r(f,\,m;\,\cdot\,) - f(\,\cdot\,)\|_{H^{m,\mu,p}} = O((1-r)^{\alpha-\beta}), \quad r \in [0,\,1). \tag{38}$$

If $f \in \widetilde{H}_{2\pi}^{m,\omega,p}$, *then*

$$\|U_r(f,\, m;\, \cdot\,) - f(\,\cdot\,)\|_{H^{m,\omega,p}} = o((1-r)^{\alpha-\beta}) \quad as \quad r \to 1- . \tag{39}$$

Applying (35), Theorem 1 and Theorem 2, we can formulate also analogies of (36)-(39) for the Abel means $U_r(f,\, m;\, \cdot\,)$ with $m = 0$ and $m = 1$.

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