Hokkaido Mathematical Journal Vol. 33 (2004) p. 637–645

L^p-boundedness of wavelet multipliers

Bolin MA and M. W. WONG

(Received October 23, 2002; Revised February 12, 2003)

Abstract. We give conditions on the symbol σ in $L^\infty(\mathbb{R}^n)$ to ensure that the corresponding wavelet multiplier is a bounded linear operator on $L^2(\mathbb{R}^n)$ and on $L^p(\mathbb{R}^n)$, $4/3 < p < 4$.

Key words: Fourier multipliers, wavelet multipliers, localization operators.

1. Wavelet Multipliers

Let $\sigma \in L^\infty(\mathbb{R}^n)$. Then we define the linear operator $T_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by

$$T_\sigma u = (\sigma\hat{u})^\vee, \quad u \in L^2(\mathbb{R}^n),$$

where $\hat{u}$ is the Fourier transform of u defined by

$$\hat{u}(\xi) = \lim_{R\to\infty} (2\pi)^{-n/2} \int_{|x|\le R} e^{-ix\cdot\xi} u(x)\, dx, \quad \xi \in \mathbb{R}^n,$$

the convergence is understood to take place in $L^2(\mathbb{R}^n)$ and $(\sigma\hat{u})^\vee$ is the inverse Fourier transform of $\sigma\hat{u}$. It is a consequence of Plancherel's theorem that $T_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bounded linear operator.

Let $\pi : \mathbb{R}^n \to U(L^2(\mathbb{R}^n))$ be the unitary representation of the additive group $\mathbb{R}^n$ on $L^2(\mathbb{R}^n)$ defined by

$$(\pi(\xi)u)(x) = e^{ix\cdot\xi} u(x), \quad x, \xi \in \mathbb{R}^n,$$

for all functions u in $L^2(\mathbb{R}^n)$, where $U(L^2(\mathbb{R}^n))$ is the group of all unitary operators on $L^2(\mathbb{R}^n)$. Let φ be any function in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_2 = 1$, where $\|\ \|_p$ denotes the norm in $L^p(\mathbb{R}^n)$ for $1 \le p \le \infty$. Then it has been proved in the paper [5] by He and Wong that

---

2000 Mathematics Subject Classification : 42B15, 47G10.

This research has been supported partially by National Natural Science Foundation of P. R. China under 19971010 and the Natural Sciences and Engineering Research Council of Canada under OGP0008562.

$$(\varphi u,\, \varphi v) = (2\pi)^{-n} \int_{\mathbb{R}^n} (u,\, \pi(\xi)\varphi)(\pi(\xi)\varphi,\, v)\, d\xi \tag{1.1}$$

for all functions u and v in the Schwartz space $\mathcal{S}$, where $(\ ,\ )$ is the inner product in $L^2(\mathbb{R}^n)$.

Let $\sigma \in L^1(\mathbb{R}^n) \cup L^\infty(\mathbb{R}^n)$. Then for all functions u in $\mathcal{S}$, we define $P_{\sigma,\varphi}u$ by

$$(P_{\sigma,\varphi}u,\, v) = (2\pi)^{-n} \int_{\mathbb{R}^n} \sigma(\xi)(u,\, \pi(\xi)\varphi)(\pi(\xi)\varphi,\, v)\, d\xi, \quad v \in \mathcal{S}. \tag{1.2}$$

Then it can be proved easily that $P_{\sigma,\varphi}u \in L^2(\mathbb{R}^n)$ for all u in $\mathcal{S}$ and $P_{\sigma,\varphi}$, initially defined on $\mathcal{S}$, can be extended to a bounded linear operator from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. In fact, we have the following theorem of He and Wong in [5], which is formulated as Theorem 19.6 in the book [10] by Wong.

Theorem 1.1 *Let $\sigma \in L^p(\mathbb{R}^n)$, $1 < p < \infty$. Then there exists a unique bounded linear operator $P_{\sigma,\varphi}\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ such that*

$$\|P_{\sigma,\varphi}\|_* \le (2\pi)^{-n/p} \|\varphi\|_\infty^{2/p'} \|\sigma\|_p,$$

where $\|\ \|_*$ is the norm in the C^*-algebra of all bounded linear operators from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$, and for all functions u and v in $L^2(\mathbb{R}^n)$, $(P_{\sigma,\varphi}u,\, v)$ is given by (1.2) for all simple functions σ on $\mathbb{R}^n$ for which the Lebesgue measure of the set $\{\xi \in \mathbb{R}^n\colon \sigma(\xi) \neq 0\}$ is finite.

Remark 1.2 Let $\sigma \in L^\infty(\mathbb{R}^n)$. Then it has been proved in the paper [5] by He and Wong that the bounded linear operators $P_{\sigma,\varphi}\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $\varphi T_\sigma \overline{\varphi}\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ are equal. Henceforth we denote $\varphi T_\sigma \overline{\varphi}$ by $P_{\sigma,\varphi}$. By (1.1) and (1.2), the bounded linear operator $P_{\sigma,\varphi}\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a variant of a localization operator first studied in [1], [2], [3] and [4], and extensively in the book [10] by Wong. Had the "admissible wavelet" φ in (1.2) been replaced by the function φ_0 on $\mathbb{R}^n$ given by $\varphi_0(x) = 1$ for all x in $\mathbb{R}^n$, we would have obtained $T_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ instead of $P_{\sigma,\varphi}\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. In other words, the bounded linear operator $P_{\sigma,\varphi}\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ would have been a Fourier multiplier. Since the function φ in the bounded linear operator $P_{\sigma,\varphi}\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ plays the role of the admissible wavelet in a localization operator, it is natural to call the bounded linear operator $P_{\sigma,\varphi}\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ a wavelet multiplier.

The results on wavelet multipliers hitherto obtained are on $L^2(\mathbb{R}^n)$. See, for instance, the works [3], [4], [5] and [10]. In the paper [9] by Wong, the

L^p-boundedness of localization operators associated to left regular representations is studied and the techniques therein can be employed to obtain similar results on the L^p-boundedness of wavelet multipliers for $1 \leq p \leq \infty$. The aim of this paper is to give another set of results on the L^p-boundedness of wavelet multipliers by reducing the problem to the corresponding one on Fourier multipliers.

Of pivotal importance in this paper is the notion of an admissible wavelet of the representation $\pi\colon \mathbb{R}^n \to U(L^2(\mathbb{R}^n))$. Following the book [10] by Wong, a function φ in $L^2(\mathbb{R}^n)$ satisfying $\|\varphi\|_2 = 1$ and

$$\int_{\mathbb{R}^n} |(\varphi, \pi(\xi)\varphi)|^2 \, d\xi < \infty$$

is said to be an admissible wavelet of $\pi\colon \mathbb{R}^n \to U(L^2(\mathbb{R}^n))$. For every admissible wavelet φ, we define the wavelet constant c_φ by

$$c_\varphi = \int_{\mathbb{R}^n} |(\varphi, \pi(\xi)\varphi)|^2 \, d\xi.$$

It can be found on page 111 of the book [10] by Wong that the set of admissible wavelets for $\pi\colon \mathbb{R}^n \to U(L^2(\mathbb{R}^n))$ consists of all functions φ in $L^2(\mathbb{R}^n) \cap L^4(\mathbb{R}^n)$ for which $\|\varphi\|_2 = 1$, and for every admissible wavelet φ,

$$c_\varphi = (2\pi)^n \|\varphi\|_4^4.$$

2. The Main Results

We state in this section the main results. They are proved in Section 4. A technical lemma that we need for the proofs of the main results is given in Section 3.

Theorem 2.1 *Let $\sigma \in L^\infty(\mathbb{R}^n)$ be such that there exist positive constants C, a_1 and a_2 for which*

$$|\sigma(\xi)| \leq C|\xi|^{-a_1}, \quad \xi \neq 0, \tag{2.1}$$

and

$$|(\nabla\sigma)(\xi)| \leq |\xi|^{-a_2}, \quad \xi \neq 0. \tag{2.2}$$

If $a_1 + a_2 > n$, then for every admissible wavelet φ of the representation $\pi\colon \mathbb{R}^n \to U(L^2(\mathbb{R}^n))$, the wavelet multiplier $P_{\sigma,\varphi}\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bounded linear operator.

Theorem 2.1 is an L^2-boundedness result. A genuine L^p-boundedness result is given in the following theorem.

Theorem 2.2 *Let $\sigma \in L^\infty(\mathbb{R}^n)$ be such that there exist positive constants C, a_1 and a_2 for which* (2.1) *and* (2.2) *are valid. If $a_1 \geq n/2$ and $a_1 + a_2 > n$, then for every admissible wavelet φ of the representation $\pi \colon \mathbb{R}^n \to U(L^2(\mathbb{R}^n))$ and $4/3 < p < 4$, the wavelet multiplier $P_{\sigma,\varphi} \colon L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a bounded linear operator.*

Remark 2.3 If $\varphi \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\sigma \in L^\infty(\mathbb{R}^n)$, then the L^p-boundedness of wavelet multipliers is an easy consequence of that of Fourier multipliers, which can be found in many references such as the paper [7] by Seeger and Sogge, and the book [8] by Stein.

3. A Lemma

The following lemma is motivated by a result in the paper [6] by Ma and Hu.

Lemma 3.1 *Let $\sigma \in L^\infty(\mathbb{R}^n)$ be such that there exist positive constants C, a_1 and a_2 for which* (2.1) *and* (2.2) *are valid. If $a_1 + a_2 > n$, then the Fourier multiplier T_σ is a bounded linear operator from $L^{4/3}(\mathbb{R}^n)$ into $L^4(\mathbb{R}^n)$.*

Proof. Let ψ_0 and ψ be functions in $C_0^\infty(\mathbb{R}^n)$ such that

$$\operatorname{supp}(\psi_0) \subseteq \{\xi \in \mathbb{R}^n \colon |\xi| \leq 1\},$$
$$\operatorname{supp}(\psi) \subseteq \left\{\xi \in \mathbb{R}^n \colon \frac{1}{2} \leq |\xi| \leq 2\right\}$$

and

$$\sum_{j=0}^{\infty} \psi_j(\xi) = 1, \quad \xi \in \mathbb{R}^n,$$

where

$$\psi_j(\xi) = \psi\left(\frac{\xi}{2^j}\right), \quad \xi \in \mathbb{R}^n.$$

For $j = 0, 1, 2, \ldots$, we let σ_j be the function on $\mathbb{R}^n$ defined by

$$\sigma_j(\xi) = \psi_j(\xi)\sigma(\xi), \quad \xi \in \mathbb{R}^n.$$

Then we get

$$\sigma(\xi) = \sum_{j=0}^{\infty} \sigma_j(\xi), \quad \xi \in \mathbb{R}^n.$$

If we denote by T_j the Fourier multiplier T_{σ_j}, then by Minkowski's inequality,

$$\|T_\sigma f\|_4 \leq \sum_{j=0}^{\infty} \|T_j f\|_4, \quad f \in \mathcal{S}.$$

So, it is sufficient to prove that there exist positive numbers C and ε such that

$$\|T_j f\|_4 \leq C 2^{-j\varepsilon} \|f\|_{4/3} \tag{3.1}$$

for all f in $\mathcal{S}$ and $j = 0, 1, 2, \ldots$. To prove (3.1), we let K_j be the kernel of T_j, *i.e.*, $\hat{K}_j = \sigma_j$ for $j = 0, 1, 2, \ldots$. For $j = 0$, (3.1) is trivially true. For $j \neq 0$, we write

$$K_j(x) = \sum_{l=0}^{\infty} K_j^l(x), \quad x \in \mathbb{R}^n,$$

where

$$K_j^l(x) = \psi\left(\frac{x}{2^l}\right) K_j(x), \quad x \in \mathbb{R}^n.$$

Then for $l \neq 0$, we can use (2.1) to obtain a positive constant C_1 such that

$$\begin{aligned} |K_j^l(x)| &= \left| \psi\left(\frac{x}{2^l}\right) (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma_j(\xi)\, d\xi \right| \\ &\leq C_1 2^{j(n-a_1)}, \quad x \in \mathbb{R}^n, \end{aligned}$$

which implies that

$$\|K_j^l * f\|_\infty \leq C_1 2^{j(n-a_1)} \|f\|_1. \tag{3.2}$$

On the other hand, we have

$$\|\widehat{K_j^l}\|_\infty \leq 2^{-ja_1}$$

and hence we get a positive constant C_2 such that

$$\|K_j^l * f\|_2 \leq C_2 2^{-ja_1} \|f\|_2. \tag{3.3}$$

Since ψ is supported away from the origin, it follows that for every multi-index α,

$$\int_{\mathbb{R}^n} x^\alpha \hat{\psi}(x)\, dx = 0$$

and

$$\int_{\mathbb{R}^n} |x|^{|\alpha|} |\hat{\psi}(x)|\, dx < \infty.$$

Hence for all ξ in $\mathbb{R}^n$, we get

$$\begin{aligned} |\widehat{K_j^l}(\xi)| &= 2^{ln}(2\pi)^{-n/2} \left| \int_{\mathbb{R}^n} \sigma_j(\xi - \eta)\hat{\psi}(2^l\eta)\, d\eta \right| \\ &= (2\pi)^{-n/2} \left| \int_{\mathbb{R}^n} \sigma_j(\xi - 2^{-l}\eta)\hat{\psi}(\eta)\, d\eta \right| \\ &= (2\pi)^{-n/2} \left| \int_{\mathbb{R}^n} \{\sigma_j(\xi) - 2^{-l}\eta \cdot (\nabla\sigma_j)(\xi')\}\hat{\psi}(\eta)\, d\eta \right|, \end{aligned}$$

where ξ' is a point on the line segment joining ξ and $\xi - 2^{-l}\eta$. Thus, there is a positive constant C_3 such that

$$|\widehat{K_j^l}(\xi)| \le C_3 2^{-ja_2} 2^{-l}, \quad \xi \in \mathbb{R}^n.$$

Hence, by Plancherel's theorem, we get

$$\|K_j^l * f\|_2 \le C_3 2^{-ja_2} 2^{-l} \|f\|_2, \quad f \in L^2(\mathbb{R}^n). \tag{3.4}$$

By (3.3), (3.4) and an interpolation, we get a positive constant C_4 such that

$$\|K_j^l * f\|_2 \le C_4 2^{-j(a_1 + t(a_2 - a_1))} 2^{-tl} \|f\|_2, \quad f \in L^2(\mathbb{R}^n), \tag{3.5}$$

where t is a positive number with $0 < t < 1$. Using an interpolation of (3.2) and (3.5), we get a positive constant C_5 such that

$$\|K_j^l * f\|_4 \le C_5 2^{-j(a_1 + t(a_2 - a_1)/2 - n/2)} 2^{-lt/2} \|f\|_{4/3}$$

for all simple functions f on $\mathbb{R}^n$ such that the Lebesgue measure of $\{x \in \mathbb{R}^n : f(x) \neq 0\}$ is finite. Thus, for all these simple functions f on $\mathbb{R}^n$, we obtain by summing over l a positive constant C_6 such that

$$\|T_j f\|_4 = \|K_j * f\|_4 \le C_6 2^{-j(a_1 + t(a_2 - a_1)/2 - n/2)} \|f\|_{4/3}.$$

Since t is an arbitrary number in $(0, 1)$, we can choose t close enough to 1 such that

$$a_1 + \frac{t(a_2 - a_1)}{2} = \frac{1}{2}(a_1 + (1-t)a_1 + ta_2) > \frac{n}{2}.$$

Therefore (3.1) is valid and the proof of Lemma 3.1 is complete. □

4. Proofs of the Main Results

To prove Theorem 2.1, let $f \in \mathcal{S}$. Then, by Hölder's inequality and Lemma 3.1, we get a positive constant C such that

$$\begin{aligned}\|P_{\sigma,\varphi}f\|_2 =& \|(\varphi T_\sigma \overline{\varphi})f\|_2 \leq \|\varphi\|_4 \|T_\sigma(\overline{\varphi}f)\|_4 \\ \leq& C\|\varphi\|_4 \|\overline{\varphi}f\|_{4/3} \leq C\|\varphi\|_4^2 \|f\|_2\end{aligned}$$

and the proof is complete.

To prove Theorem 2.2, we use the same notation as in the proof of Lemma 3.1. If $a_1 = n/2$, then we get a positive constant C such that

$$\begin{aligned}\|T_j f\|_2^2 =& \int_{\mathbb{R}^n} |(T_j f)(x)|^2 dx = \int_{\mathbb{R}^n} |\sigma_j(\xi)\hat{f}(\xi)|^2 d\xi \\ \leq& \|\hat{f}\|_\infty^2 \int_{\mathbb{R}^n} |\sigma_j(\xi)|^2 d\xi \leq C\|f\|_1^2 \int_{2^{j-1}}^{2^{j+1}} \frac{1}{r}\, dr = C \ln 4 \|f\|_1^2\end{aligned}$$

for all f in $\mathcal{S}$ and $j = 0, 1, 2, \ldots$. If $a_1 > n/2$, then we also get positive constants C and C' such that

$$\begin{aligned}\|T_j f\|_2^2 =& \int_{\mathbb{R}^n} |(T_j f)(x)|^2 dx = \int_{\mathbb{R}^n} |\sigma_j(\xi)\hat{f}(\xi)|^2 d\xi \\ \leq& \|\hat{f}\|_\infty^2 \int_{\mathbb{R}^n} |\sigma_j(\xi)|^2 d\xi \leq C\|f\|_1^2 \int_{2^{j-1}}^{2^{j+1}} r^{n-2a_1-1} dr \leq C'\|f\|_1^2\end{aligned}$$

for all f in $\mathcal{S}$ and $j = 0, 1, 2, \ldots$. From the above two cases, we get a positive constant C such that

$$\|T_j f\|_2 \leq C\|f\|_1, \quad f \in \mathcal{S}, \tag{4.1}$$

for $j = 0, 1, 2, \ldots$. By an interpolation of (3.1) and (4.1), there is a positive constant C such that

$$\|T_j f\|_q \leq C 2^{-j\delta} \|f\|_r, \quad f \in \mathcal{S},$$

for $j = 0, 1, 2, \ldots$, where $\delta = 4\varepsilon(r-1)/r$, and q and r satisfy

$$1 < r \leq \frac{4}{3}, \quad q = \frac{2r}{2-r}. \tag{4.2}$$

Summing over j, we obtain a positive constant C such that

$$\|T_\sigma f\|_q \leq C\|f\|_r, \quad f \in \mathcal{S}. \tag{4.3}$$

By (4.1) and a duality argument, we have

$$\|T_j f\|_\infty \leq C\|f\|_2, \quad f \in \mathcal{S}. \tag{4.4}$$

By an interpolation of (3.1) and (4.4), we get a positive constant C such that

$$\|T_j f\|_q \leq C 2^{-j\lambda}\|f\|_r, \quad f \in \mathcal{S},$$

where q and r satisfy

$$\frac{4}{3} \leq r < 2, \quad q = \frac{2r}{2-r}, \tag{4.5}$$

and $\lambda = 2\varepsilon(2-r)/r$. Summing over j, we get a positive constant C such that

$$\|T_\sigma f\|_q \leq C\|f\|_r, \quad f \in \mathcal{S}. \tag{4.6}$$

Now, let $f \in \mathcal{S}$. For $4/3 < p < 4$, let q_1 be such that $p/4 + 1/q_1 = 1$, *i.e.*, $q_1 = 4/(4-p)$. Then, using Hölder's inequality, (4.3), (4.6), $r = 4p/(4+p)$ and $q = pq_1 = 4p/(4-p)$, we get a positive constant C such that

$$\begin{aligned}\|P_{\sigma,\varphi} f\|_p &= \|(\varphi T_\sigma \overline{\varphi}) f\|_p \leq \|\varphi\|_4 \|T_\sigma(\overline{\varphi} f)\|_{pq_1} \\ &\leq C\|\varphi\|_4 \|\overline{\varphi} f\|_{4p/(4+p)} \leq C\|\varphi\|_4^2 \|f\|_p\end{aligned}$$

for all f in $\mathcal{S}$. This completes the proof of Theorem 2.2.

References

[1] Daubechies I., *Time-frequency localization operators: A geometric phase space approach*, IEEE Trans. Inform. Theory **34** (1988), 605–612.

[2] Daubechies I., *Ten Lectures on Wavelets*, SIAM, (1992).

[3] Du J. and Wong M.W., *Traces of localization operators*, C. R. Math. Rep. Acad. Sci. Canada **22** (2000), 92–96.

[4] He Z. and Wong M.W., *Localization operators associated to square integrable group representations*, Panamer. Math. J. **6** (1) (1996), 93–104.

[5] He Z. and Wong M.W., *Wavelet multipliers and signals*, J. Austral. Math. Soc. Ser. B **40** (1999), 437–446.

[6] Ma B. and Hu G., *Maximal operators associated with commutators of spherical means*, Tohoku Math. J. **50** (1998), 349–363.

[7] Seeger A. and Sogge C., *Bounds for eigenfunctions of differential operators*, Indiana Math. J. **38** (1989), 669–682.

[8] Stein E.M., *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[9] Wong M.W., *L^p-boundedness of localization operators associated to left regular representations*, Proc. Amer. Math. Soc. **130** (2002), 2911–2919.

[10] Wong M.W., *Wavelet Transforms and Localization Operators*, Birkhäuser, 2002.

B. Ma
Institute of Applied Mathematics
Hunan University
Changsha, Hunan 410082
P. R. China

M.W. Wong
Department of Mathematics and Statistics
York University
4700 Keele Street
Toronto, Ontario M3J 1P3
Canada