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The structure of group C -algebras of some discrete solvable semi-direct products

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Abstract. We describe the algebraic structure of group C^*-algebras of some discrete solvable semi-direct products in terms of finite composition series, and show that some subquotients are decomposed into C^*-algebras of continuous fields with their fibers non-isomorphic to noncommutative tori. We also discuss some applications of these results.

Key words: group C*-algebra, discrete solvable group, stable rank.

0. Introduction

This paper is a continuation (to discrete cases) of the study on the algebraic structure of group C^*-algebras of either connected or disconnected Lie groups (cf. [16, 17] for the connected cases and [18, 19, 20] for the disconnected cases). Namely, we investigate the algebraic structure of group C^*-algebras of some discrete solvable semi-direct products. We first consider the case of some discrete nilpotent semi-direct products in both Sections 1 and 2. It is shown that their group C^*-algebras are decomposed into finite composition series, and their subquotients are decomposed into C^*-algebras of continuous fields whose fibers are non-isomorphic to noncommutative tori in general. We next consider the case of some discrete (non-nilpotent) solvable semi-direct products similarly. In particular, they include the discrete $ax + b$ groups and discrete Dixmier groups which are defined in Sections 3 and 4 respectively. The results of each section would be useful for the study on the algebraic structure of group C^*-algebras of more general discrete solvable groups. Also, the stable rank of group C^*-algebras of those discrete solvable semi-direct products can be estimated by using their algebraic structures (cf. [13-15, 17-24]). Furthermore, the primitive ideal spaces of those group C^*-algebras are determined by those of their subquotients.

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Notation Denote by $C^*(G)$ the group C^*-algebra of a discrete group G (cf. [3]). Denote by $C_0(X)$ the C^*-algebra of all continuous complex-valued functions on a locally compact Hausdorff space X vanishing at infinity, and let $C(X) = C_0(X)$ when X is compact. Let $\mathfrak{A} \rtimes_\alpha G$ be the C^*-crossed product of a C^*-algebra $\mathfrak{A}$ by G with α an action (cf. [11]). Let $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})$ be the C^*-algebra of a continuous field on X vanishing at infinity with C^*-algebras $\mathfrak{A}_t$ fibers (cf. [3], [8]). Set $\Gamma(\cdot) = \Gamma_0(\cdot)$ when X is compact. As a review for two applications mentioned above, we recall that for an exact sequence of C^*-algebras: $0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{I} \to 0$, we have

$$\max\{\mathrm{sr}(\mathfrak{I}),\, \mathrm{sr}(\mathfrak{A}/\mathfrak{I})\} \le \mathrm{sr}(\mathfrak{A}) \le \max\{\mathrm{sr}(\mathfrak{I}),\, \mathrm{sr}(\mathfrak{A}/\mathfrak{I}), \mathrm{csr}(\mathfrak{A}/\mathfrak{I})\},$$
$$\mathrm{csr}(\mathfrak{A}) \le \max\{\mathrm{csr}(\mathfrak{I}),\, \mathrm{csr}(\mathfrak{A}/\mathfrak{I})\},$$

where $\mathrm{sr}(\cdot)$, $\mathrm{csr}(\cdot)$ mean the stable rank and connected stable rank respectively [13], and the primitive ideal space of $\mathfrak{A}$ is identified with the union of all primitive ideals of $\mathfrak{I}$ and of $\mathfrak{A}/\mathfrak{I}$ by taking either $J \leftrightarrow J \cap \mathfrak{I}$ or $J \leftrightarrow J/\mathfrak{I}$ for a primitive ideal J of $\mathfrak{A}$ with either $J \not\supset \mathfrak{I}$ or $J \supset \mathfrak{I}$ respectively (cf. [3, Proposition 2.11.5]). On the other hand, for any continuous field C^*-algebra $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})$ [21],

$$\mathrm{sr}(\Gamma_0(X,\, \{\mathfrak{A}_t\}_{t\in X})) \le \sup_{t\in X} \mathrm{sr}(C_0(X,\, \mathfrak{A}_t)),$$
$$\mathrm{csr}(\Gamma_0(X,\, \{\mathfrak{A}_t\}_{t\in X})) \le \sup_{t\in X} \max\{\mathrm{sr}(C_0(X,\, \mathfrak{A}_t)), \mathrm{csr}(C_0(X,\, \mathfrak{A}_t))\},$$

where $C_0(X, \mathfrak{A}_t)$ is the C^*-algebra of all $\mathfrak{A}_t$-valued continuous functions on X vanishing at infinity (cf. [21, 22], [4]), and the primitive ideal space of $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})$ is regarded as a fiber space over X with fibers the primitive ideal spaces of $\{\mathfrak{A}_t\}_{t\in X}$ (cf. [8]). Moreover, it is known by [2] that any simple noncommutative torus has stable rank one. The method of [2] is applicable to some subquotients non-isomorphic to noncommutative tori given below. Recall that a noncommutative n-torus $\mathfrak{A}_\Theta$ is the (universal) C^*-algebra generated by unitaries $\{U_j\}_{j=1}^n$ with the relation $U_jU_k = e^{2\pi i\theta_{jk}}U_kU_j$ for $\theta_{jk} \in \mathbb{R}$ $(1 \le j, k \le n)$ and $\Theta = (\theta_{jk})_{j,k=1}^n$ a skew adjoint $n \times n$ matrix with $\theta_{jj} = 0$ $(1 \le j \le n)$. In particular, let $\mathfrak{A}_\theta$ denote a noncommutative 2-torus, that is, a rotation algebra.

1. Certain discrete nilpotent semi-direct products by $\mathbb{Z}$

First define $N_{n,1}$ ($n \geq 1$) to be the discrete nilpotent semi-direct products $\mathbb{Z}^n \rtimes_\alpha \mathbb{Z}$, where the action α of $\mathbb{Z}$ on $\mathbb{Z}^n$ is defined by the multiplication of the matrix:

$$\alpha_1 = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = (t_{ij})_{i,j=1}^n \in \mathrm{GL}_n(\mathbb{Z}).$$

Then $N_{1,1} = \mathbb{Z}^2$, and the discrete Heisenberg group is a special case of $N_{2,1}$ with $t_{12} = 1$. Note that the groups $N_{n,1}$ are n-step nilpotent in general since the subgroups $\mathbb{Z}^k \times (\Pi^{n-k}\{0\})$ ($1 \leq k \leq n$) of $\mathbb{Z}^n$ are α-invariant and their k-th components of $\mathbb{Z}^k$ are fixed under α.

Let $C^*(N_{n,1})$ be the group C^*-algebra of $N_{n,1} = \mathbb{Z}^n \rtimes_\alpha \mathbb{Z}$. By the Fourier transform, it is obtained that

$$C^*(N_{n,1}) \cong C^*(\mathbb{Z}^n) \rtimes_\alpha \mathbb{Z} \cong C(\mathbb{T}^n) \rtimes_{\hat{\alpha}} \mathbb{Z}$$

where the action $\hat{\alpha}$ is defined by the duality $\langle \alpha_t(s)|z\rangle = \langle s|\hat{\alpha}_t(z)\rangle$ for $s \in \mathbb{Z}^n$, $z = (z_j) \in \mathbb{T}^n$, and $\alpha_t = (\alpha_1)^t$ (t-times multiple of α_1). Specifically,

$$\hat{\alpha}_1(z) = (z_1,\, z_1^{t_{12}} z_2,\, z_1^{t_{13}} z_2^{t_{23}} z_3,\, \ldots,\, z_1^{t_{1n}} z_2^{t_{2n}} \cdots z_{n-1}^{t_{(n-1)n}} z_n).$$

Since $\{1\} \times \mathbb{T}^{n-1}$ is invariant under $\hat{\alpha}$, the following exact sequence is obtained:

$$\begin{aligned} 0 \to C_0((\mathbb{T} \setminus \{1\}) \times \mathbb{T}^{n-1}) \rtimes_{\hat{\alpha}} \mathbb{Z} \to C(\mathbb{T}^n) \rtimes_{\hat{\alpha}} \mathbb{Z} \\ \to C(\mathbb{T}^{n-1}) \rtimes_{\hat{\alpha}} \mathbb{Z} \to 0. \end{aligned}$$

Moreover, it follows that

$$\begin{aligned} C_0((\mathbb{T} \setminus \{1\}) \times \mathbb{T}^{n-1}) \rtimes_{\hat{\alpha}} \mathbb{Z} \\ \cong \Gamma_0(\mathbb{T} \setminus \{1\}, \{C(\mathbb{T}^{n-1}) \rtimes_{\hat{\alpha}, z_1} \mathbb{Z}\}_{z_1 \in \mathbb{T}\setminus\{1\}}) \end{aligned}$$

where the fibers $C(\mathbb{T}^{n-1}) \rtimes_{\hat{\alpha}, z_1} \mathbb{Z}$ correspond to the restrictions of $\hat{\alpha}$ to $\{z_1\} \times \mathbb{T}^{n-1}$ for $z_1 \in \mathbb{T} \setminus \{1\}$ (cf. [7, Theorem 4]). The following decomposition is obtained inductively:

$$\begin{aligned} 0 \to C_0((\mathbb{T} \setminus \{1\}) \times \mathbb{T}^{k-1}) \rtimes_{\hat{\alpha}} \mathbb{Z} \to C(\mathbb{T}^k) \rtimes_{\hat{\alpha}} \mathbb{Z} \\ \to C(\mathbb{T}^{k-1}) \rtimes_{\hat{\alpha}} \mathbb{Z} \to 0 \end{aligned}$$

for $2 \leq k \leq n-1$, and

$$C_0((\mathbb{T} \setminus \{1\}) \times \mathbb{T}^{k-1}) \rtimes_{\hat{\alpha}} \mathbb{Z} \cong \Gamma_0(\mathbb{T} \setminus \{1\}, \{C(\mathbb{T}^{k-1}) \rtimes_{\hat{\alpha}, z_{n-k+1}} \mathbb{Z}\}_{z_{n-k+1} \in \mathbb{T} \setminus \{1\}}),$$

and $C(\mathbb{T}) \rtimes_{\hat{\alpha}} \mathbb{Z} \cong C(\mathbb{T}^2)$. Note that the fibers $C(\mathbb{T}) \rtimes_{\hat{\alpha}, z_{n-1}} \mathbb{Z}$ are noncommutative 2-tori since $\hat{\alpha}_1(z_n) = z_{n-1}^{t_{(n-1)n}} z_n$, and if they are simple, they are AT-algebras, i.e. inductive limits of finite direct sums of matrix algebras over $C(\mathbb{T})$ [5]. The fibers $C(\mathbb{T}^{k-1}) \rtimes_{\hat{\alpha}, z_{n-k+1}} \mathbb{Z}$ $(k \geq 3)$ are not noncommutative tori if $t_{ij} \neq 0$ for some $n-k+2 \leq i < j \leq n$. If the fibers are simple, they are crossed products by minimal diffeomorphisms on $\mathbb{T}^{k-1}$ $(k \geq 3)$ so that they are approximately subhomogeneous, i.e. inductive limits of subhomogeneous algebras (cf. [9]). This remarkable fact is helpful for computing their stable rank.

To sum up we obtain

Theorem 1.1 *Let $N_{n,1} = \mathbb{Z}^n \rtimes_\alpha \mathbb{Z}$ as above. Then $C^*(N_{n,1})$ has the following finite composition series $\{\mathfrak{I}_j\}_{j=1}^n$ with $\mathfrak{I}_0 = \{0\}$: $\mathfrak{I}_n/\mathfrak{I}_{n-1} \cong C(\mathbb{T}^2)$, and*

$$\mathfrak{I}_{n-k+1}/\mathfrak{I}_{n-k} \cong \Gamma_0(\mathbb{T} \setminus \{1\}, \{C(\mathbb{T}^{k-1}) \rtimes_{\hat{\alpha}, z_{n-k+1}} \mathbb{Z}\}_{z_{n-k+1} \in \mathbb{T} \setminus \{1\}}) \quad \text{for } 2 \leq k \leq n.$$

Moreover, the fibers $C(\mathbb{T}^{k-1}) \rtimes_{\hat{\alpha}, z_{n-k+1}} \mathbb{Z}$ $(k \geq 3)$ are not noncommutative tori if $t_{ij} \neq 0$ for some $n-k+2 \leq i < j \leq n$.

Proof. Under the above situation, the following exact sequence is obtained:

$$0 \to \mathfrak{I}_{n-1} \to C(\mathbb{T}^n) \rtimes_{\hat{\alpha}} \mathbb{Z} \to C(\mathbb{T}) \rtimes_{\hat{\alpha}} \mathbb{Z} \to 0$$

where $\mathfrak{I}_{n-1} = C_0(\mathbb{T}^n \setminus \mathbb{T}) \rtimes_{\hat{\alpha}} \mathbb{Z}$. Moreover, the following exact sequence is also obtained:

$$0 \to \mathfrak{I}_{n-2} \to \mathfrak{I}_{n-1} \to C_0((\mathbb{T} \setminus \{1\}) \times \mathbb{T}) \rtimes_{\hat{\alpha}} \mathbb{Z} \to 0$$

where $\mathfrak{I}_{n-2} = C_0(\mathbb{T}^n \setminus (\mathbb{T} \sqcup ((\mathbb{T} \setminus \{1\}) \times \mathbb{T}))) \rtimes_{\hat{\alpha}} \mathbb{Z}$. Inductively, the following exact sequences are obtained:

$$0 \to \mathfrak{I}_{n-k} \to \mathfrak{I}_{n-k+1} \to C_0((\mathbb{T} \setminus \{1\}) \times \mathbb{T}^{k-1}) \rtimes_{\hat{\alpha}} \mathbb{Z} \to 0$$

for $2 \leq k \leq n$. □

Remark 1.2 Simple quotients of the C^*-algebras of compactly generated, locally compact 2-step nilpotent groups are isomorphic to tensor products of noncommutative tori and the C^*-algebra of compact operators on either a finite or an infinite dimensional Hilbert space ([12]). The fibers $C(\mathbb{T}^{k-1}) \rtimes_{\hat{\alpha}, z_{n-k+1}} \mathbb{Z}$ $(k \geq 3)$ can be simple, but not be noncommutative tori.

Next define $N_{n,m}$ to be the discrete nilpotent semi-direct products $\mathbb{Z}^n \rtimes_\alpha \mathbb{Z}^m$, where the action α of $\mathbb{Z}^m$ on $\mathbb{Z}^n$ is defined by the multiplication of the matrices as follows:

$$\alpha_{(1)_{k=1}^m} = \alpha_{1_1} \cdots \alpha_{1_m}, \quad \alpha_{1_k} = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = (t_{ij}^{(k)})_{i,j=1}^n \in \mathrm{GL}_n(\mathbb{Z})$$

for $1_k \in \mathbb{Z}^m$ with $1_k = (0, \ldots, 0, 1, 0, \ldots, 0)$ (only k-th component nonzero). Note that the groups $N_{n,m}$ are n-step nilpotent in general. It is obtained by the same way as Theorem 1.1 that

Theorem 1.3 *Let $N_{n,m} = \mathbb{Z}^n \rtimes_\alpha \mathbb{Z}^m$ as above. Then $C^*(N_{n,m})$ has the following finite composition series $\{\mathfrak{I}_j\}_{j=1}^n$: $\mathfrak{I}_0 = \{0\}$, $\mathfrak{I}_n/\mathfrak{I}_{n-1} \cong C(\mathbb{T}^{1+m})$, and*

$$\mathfrak{I}_{n-k+1}/\mathfrak{I}_{n-k} \cong \Gamma_0(\mathbb{T} \setminus \{1\}, \{C(\mathbb{T}^{k-1}) \rtimes_{\hat{\alpha}, z_{n-k+1}} \mathbb{Z}^m\}_{z_{n-k+1} \in \mathbb{T} \setminus \{1\}}) \quad \text{for } 2 \leq k \leq n.$$

Moreover, the fibers $C(\mathbb{T}^{k-1}) \rtimes_{\hat{\alpha}, z_{n-k+1}} \mathbb{Z}^m$ $(k \geq 3)$ are not noncommutative tori if $t_{ij}^{(k)} \neq 0$ for some $n-k+2 \leq i < j \leq n$ and $1 \leq k \leq m$.

Remark 1.4 Note that the fibers $C(\mathbb{T}) \rtimes_{\hat{\alpha}, z_{n-1}} \mathbb{Z}^m$ are noncommutative $(m+1)$-tori since $\hat{\alpha}_{1_k}(z_n) = z_{n-1}^{t(k,n)} z_n$ with $t(k,n) = t_{(n-1)n}^{(k)}$ $(1 \leq k \leq m)$ (a multi-rotational action for a fixed z_{n-1}), and they are isomorphic to $C(\mathbb{T}^m) \rtimes \mathbb{Z}$ by considering their generating unitaries. If these fibers are simple, they are AT-algebras by [6, 7], and so they have stable rank one.

Remark 1.5 If the action α is the diagonal sum: $\alpha_{(1)_{k=1}^m} = \alpha_{1_1} \oplus \cdots \oplus \alpha_{1_m}$ of $\alpha_{1_k} \in \mathrm{GL}_{n_k}(\mathbb{Z})$ on a direct product $\mathbb{Z}^n = \Pi_{k=1}^m \mathbb{Z}^{n_k}$ where $n = \sum_{k=1}^m n_k$, then $C^*(N_{n,m}) \cong (\otimes_{k=1}^m C(\mathbb{T}^{n_k})) \rtimes_{\hat{\alpha}} \mathbb{Z}^m$ is isomorphic to the tensor product $\otimes_{k=1}^m (C(\mathbb{T}^{n_k}) \rtimes_{\hat{\alpha}_k} \mathbb{Z})$.

2. Certain discrete nilpotent semi-direct products by $H_3^{\mathbb{Z}}$

Next consider the structure of the group C^*-algebra of the semi-direct product $L_7^{\mathbb{Z}} = (\mathbb{Z}^2 \times \mathbb{Z}^2) \rtimes_{(\alpha,\beta)} H_3^{\mathbb{Z}}$ where $H_3^{\mathbb{Z}}$ is the discrete Heisenberg group of rank 3 consisting of the following matrices:

$$\begin{pmatrix} 1 & n & l \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_3(\mathbb{Z}), \quad l, m, n \in \mathbb{Z}$$

and α_m, $\beta_n \in \mathrm{GL}_2(\mathbb{Z})$ for $(l, m, n) \in H_3^{\mathbb{Z}}$, and

$$\alpha_1 = \beta_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}).$$

Note that $H_3^{\mathbb{Z}} \cong \mathbb{Z}^2 \rtimes_\gamma \mathbb{Z}$ where $\gamma_n(l, m) = (l + nm, m)$, and the groups $L_7^{\mathbb{Z}}$, $H_3^{\mathbb{Z}}$ are 2-step nilpotent. Then

$$C^*(L_7^{\mathbb{Z}}) \cong C^*(\mathbb{Z}^2 \times \mathbb{Z}^2) \rtimes_{(\alpha,\beta)} H_3^{\mathbb{Z}} \cong C(\mathbb{T}^2 \times \mathbb{T}^2) \rtimes_{(\hat{\alpha},\hat{\beta})} H_3^{\mathbb{Z}}$$

where $\hat{\alpha}_g(z_1, z_2) = (z_1, z_1^m z_2)$ and $\hat{\beta}_g(w_1, w_2) = (w_1, w_1^n w_2)$ for $(z_1, z_2, w_1, w_2) \in \mathbb{T}^4$ and $g = (l, m, n) \in H_3^{\mathbb{Z}}$, and $C^*(H_3^{\mathbb{Z}}) \cong C^*(\mathbb{Z}^2) \rtimes_\gamma \mathbb{Z} \cong C(\mathbb{T}^2) \rtimes_{\hat{\gamma}} \mathbb{Z}$ where $\hat{\gamma}_n(p, q) = (p, p^n q)$ for $(p, q) \in \mathbb{T}^2$. Moreover, it follows that

$$\begin{aligned} 0 \to C(X_1) \rtimes_{(\hat{\alpha},\hat{\beta})} H_3^{\mathbb{Z}} &\to C(\mathbb{T}^2 \times \mathbb{T}^2) \rtimes_{(\hat{\alpha},\hat{\beta})} H_3^{\mathbb{Z}} \\ &\to C((\{1\} \times \mathbb{T}) \times (\{1\} \times \mathbb{T}))) \rtimes_{(\hat{\alpha},\hat{\beta})} H_3^{\mathbb{Z}} \to 0 \end{aligned}$$

and $C((\{1\} \times \mathbb{T}) \times (\{1\} \times \mathbb{T}))) \rtimes_{(\hat{\alpha},\hat{\beta})} H_3^{\mathbb{Z}} \cong C(\mathbb{T}^2) \otimes C^*(H_3^{\mathbb{Z}})$, and X_1 is the complement of $(\{1\} \times \mathbb{T})^2$ in $\mathbb{T}^4$. Moreover, the ideal of the above exact sequence has the following decomposition:

$$\begin{aligned} 0 \to C(X_2) \rtimes_{(\hat{\alpha},\hat{\beta})} H_3^{\mathbb{Z}} &\to C(X_1) \rtimes_{(\hat{\alpha},\hat{\beta})} H_3^{\mathbb{Z}} \\ &\to C_0(((\mathbb{T} \setminus \{1\}) \times \mathbb{T}) \times (\{1\} \times \mathbb{T})) \rtimes_{(\hat{\alpha},\hat{\beta})} H_3^{\mathbb{Z}} \to 0 \end{aligned}$$

and $C_0(((\mathbb{T}\setminus\{1\}) \times \mathbb{T}) \times (\{1\} \times \mathbb{T})) \rtimes_{(\hat{\alpha},\hat{\beta})} H_3^{\mathbb{Z}} \cong C(\mathbb{T}) \otimes C_0(((\mathbb{T}\setminus\{1\}) \times \mathbb{T})) \rtimes_{\hat{\alpha}} H_3^{\mathbb{Z}}$ (Case A), where X_2 is the complement of $((\mathbb{T} \setminus \{1\}) \times \mathbb{T}) \times (\{1\} \times \mathbb{T})$ in X_1. Moreover, the following exact sequence is obtained:

$$\begin{aligned} 0 \to C(X_3) \rtimes_{(\hat{\alpha},\hat{\beta})} H_3^{\mathbb{Z}} &\to C(X_2) \rtimes_{(\hat{\alpha},\hat{\beta})} H_3^{\mathbb{Z}} \\ &\to C_0((\{1\} \times \mathbb{T}) \times ((\mathbb{T} \setminus \{1\}) \times \mathbb{T})) \rtimes_{(\hat{\alpha},\hat{\beta})} H_3^{\mathbb{Z}} \to 0 \end{aligned}$$

and $C_0((\{1\} \times \mathbb{T}) \times ((\mathbb{T} \setminus \{1\}) \times \mathbb{T})) \rtimes_{(\hat{\alpha},\hat{\beta})} H_3^{\mathbb{Z}} \cong C(\mathbb{T}) \otimes C_0((\mathbb{T} \setminus \{1\}) \times \mathbb{T})) \rtimes_{\hat{\beta}} H_3^{\mathbb{Z}}$ (Case B), where $X_3 = ((\mathbb{T} \setminus \{1\}) \times \mathbb{T}) \times ((\mathbb{T} \setminus \{1\}) \times \mathbb{T})$ and

$$\begin{aligned} &C(X_3) \rtimes_{(\hat{\alpha},\hat{\beta})} H_3^{\mathbb{Z}} \\ &= C_0(((\mathbb{T} \setminus \{1\}) \times \mathbb{T}) \times ((\mathbb{T} \setminus \{1\}) \times \mathbb{T})) \rtimes_{(\hat{\alpha},\hat{\beta})} H_3^{\mathbb{Z}} \qquad \text{(Case C).} \end{aligned}$$

Case A: For a further analysis of $C_0(((\mathbb{T} \setminus \{1\}) \times \mathbb{T})) \rtimes_{\hat{\alpha}} H_3^{\mathbb{Z}}$, note that

$$C_0(((\mathbb{T} \setminus \{1\}) \times \mathbb{T})) \rtimes_{\hat{\alpha}} H_3^{\mathbb{Z}} \cong \Gamma_0(\mathbb{T} \setminus \{1\}, \{C(\mathbb{T}) \rtimes_{z,\hat{\alpha}} H_3^{\mathbb{Z}}\}_{z \in \mathbb{T} \setminus \{1\}}),$$

where $(z, \hat{\alpha})$ corresponds to $\{z\} \times \mathbb{T}$, and the fibers have the following isomorphisms:

$$\begin{aligned} C(\mathbb{T}) \rtimes_{z,\hat{\alpha}} H_3^{\mathbb{Z}} &\cong C(\mathbb{T}) \rtimes_{z,\hat{\alpha}} (\mathbb{Z}^2 \rtimes_\gamma \mathbb{Z}) \\ &\cong ((C(\mathbb{T}) \rtimes_{z,\hat{\alpha}} \mathbb{Z}) \rtimes \mathbb{Z}) \rtimes_\gamma \mathbb{Z} \cong (\mathfrak{A}_{\theta_z} \otimes C(\mathbb{T})) \rtimes_{\hat{\gamma}} \mathbb{Z}, \end{aligned}$$

where $H_3^{\mathbb{Z}} \cong \mathbb{Z}^2 \rtimes_\gamma \mathbb{Z}$ as above, $z = z_1 = e^{2\pi i \theta_z}$ and $\mathfrak{A}_{\theta_z}$ is the rotation algebra corresponding to θ_z. Moreover, it follows that $(\mathfrak{A}_{\theta_z} \otimes C(\mathbb{T})) \rtimes_{\hat{\gamma}} \mathbb{Z} \cong \Gamma(\mathbb{T}, \{\mathfrak{A}_{\theta_z} \rtimes_{p,\hat{\gamma}} \mathbb{Z}\}_{p \in \mathbb{T}})$ where the actions $(p, \hat{\gamma})$ of the fibers correspond to the restrictions to $\mathfrak{A}_{\theta_z} \otimes C(\{p\})$.

Case B: For a further analysis of $C_0(((\mathbb{T} \setminus \{1\}) \times \mathbb{T})) \rtimes_{\hat{\beta}} H_3^{\mathbb{Z}}$, it is obtained that

$$C_0(((\mathbb{T} \setminus \{1\}) \times \mathbb{T})) \rtimes_{\hat{\beta}} H_3^{\mathbb{Z}} \cong \Gamma_0(\mathbb{T} \setminus \{1\}, \{C(\mathbb{T}) \rtimes_{w,\hat{\beta}} H_3^{\mathbb{Z}}\}_{w \in \mathbb{T} \setminus \{1\}}),$$

where $(w, \hat{\beta})$ corresponds to $\{w\} \times \mathbb{T}$, and the fibers have the following isomorphisms:

$$\begin{aligned} C(\mathbb{T}) \rtimes_{w,\hat{\beta}} H_3^{\mathbb{Z}} &\cong C(\mathbb{T}) \rtimes_{w,\hat{\beta}} (\mathbb{Z}^2 \rtimes_\gamma \mathbb{Z}) \cong (C(\mathbb{T}) \otimes C(\mathbb{T}^2)) \rtimes_{(w,\hat{\beta},\hat{\gamma})} \mathbb{Z} \\ &\cong \Gamma(\mathbb{T}, \{C(\mathbb{T}^2) \rtimes_{(w,\hat{\beta}),(p,\hat{\gamma})} \mathbb{Z}\}_{p \in \mathbb{T}}), \end{aligned}$$

where the actions $(p, \hat{\gamma})$ correspond to the restrictions to $\{p\} \times \mathbb{T}$ in $\mathbb{T} \times \{p\} \times \mathbb{T}$.

Case C: For a further analysis for $C_0(((\mathbb{T} \setminus \{1\}) \times \mathbb{T}) \times ((\mathbb{T} \setminus \{1\}) \times \mathbb{T})) \rtimes_{(\hat{\alpha},\hat{\beta})} H_3^{\mathbb{Z}}$,

$$\begin{aligned} &C_0(((\mathbb{T} \setminus \{1\}) \times \mathbb{T}) \times ((\mathbb{T} \setminus \{1\}) \times \mathbb{T})) \rtimes_{(\hat{\alpha},\hat{\beta})} H_3^{\mathbb{Z}} \\ &\cong \Gamma_0((\mathbb{T} \setminus \{1\}) \times (\mathbb{T} \setminus \{1\}), \{C(\mathbb{T}^2) \rtimes_{z,w,\hat{\alpha},\hat{\beta}} H_3^{\mathbb{Z}}\}_{(z,w) \in (\mathbb{T} \setminus \{1\})^2}), \end{aligned}$$

where the actions $(z, w, \hat{\alpha}, \hat{\beta})$ correspond to the restrictions to $\{z\} \times \mathbb{T} \times \{w\} \times \mathbb{T}$. Moreover, the fibers have the following isomorphisms:

$$C(\mathbb{T}^2) \rtimes_{z,w,\hat{\alpha},\hat{\beta}} H_3^{\mathbb{Z}} \cong C(\mathbb{T}^2) \rtimes (\mathbb{Z}^2 \rtimes_\gamma \mathbb{Z}) \cong (\mathfrak{A}_{\theta_z} \otimes C(\mathbb{T}^2)) \rtimes_{(w,\hat{\beta}),\hat{\gamma}} \mathbb{Z}$$
$$\cong \Gamma(\mathbb{T}, \{(\mathfrak{A}_{\theta_z} \otimes C(\mathbb{T})) \rtimes_{(w,\hat{\beta}),(p,\hat{\gamma})} \mathbb{Z}\}_{p\in\mathbb{T}}).$$

Summing up the above argument, it is obtained that

Theorem 2.1 *The group C^*-algebra $C^*(L_7^{\mathbb{Z}}) = C^*(\mathbb{Z}^4 \rtimes_{(\alpha,\beta)} H_3^{\mathbb{Z}})$ has the following finite composition series $\{\mathfrak{K}_j\}_{j=1}^4$: $\mathfrak{K}_4/\mathfrak{K}_3 \cong C(\mathbb{T}^2) \otimes C^*(H_3^{\mathbb{Z}})$, and*

$$\begin{cases} \mathfrak{K}_3/\mathfrak{K}_2 \cong C(\mathbb{T}) \otimes \Gamma_0(\mathbb{T} \setminus \{1\}, \{C(\mathbb{T}) \rtimes_{z,\hat{\alpha}} H_3^{\mathbb{Z}}\}_{z\in\mathbb{T}\setminus\{1\}}), \\ \mathfrak{K}_2/\mathfrak{K}_1 \cong C(\mathbb{T}) \otimes \Gamma_0(\mathbb{T} \setminus \{1\}, \{C(\mathbb{T}) \rtimes_{w,\hat{\beta}} H_3^{\mathbb{Z}}\}_{w\in\mathbb{T}\setminus\{1\}}), \\ \mathfrak{K}_1 \cong \Gamma_0((\mathbb{T} \setminus \{1\})^2, \{C(\mathbb{T}^2) \rtimes_{z,w,\hat{\alpha},\hat{\beta}} H_3^{\mathbb{Z}}\}_{(z,w)\in(\mathbb{T}\setminus\{1\})^2}). \end{cases}$$

Moreover, it follows that

$$\begin{cases} C(\mathbb{T}) \rtimes_{z,\hat{\alpha}} H_3^{\mathbb{Z}} \cong \Gamma(\mathbb{T}, \{\mathfrak{A}_{\theta_z} \rtimes_{p,\hat{\gamma}} \mathbb{Z}\}_{p\in\mathbb{T}}), \\ C(\mathbb{T}) \rtimes_{w,\hat{\beta}} H_3^{\mathbb{Z}} \cong \Gamma(\mathbb{T}, \{C(\mathbb{T}^2) \rtimes_{(w,\hat{\beta}),(p,\hat{\gamma})} \mathbb{Z}\}_{p\in\mathbb{T}}), \\ C(\mathbb{T}^2) \rtimes_{z,w,\hat{\alpha},\hat{\beta}} H_3^{\mathbb{Z}} \cong \Gamma(\mathbb{T}, \{(\mathfrak{A}_{\theta_z} \otimes C(\mathbb{T})) \rtimes_{(w,\hat{\beta}),(p,\hat{\gamma})} \mathbb{Z}\}_{p\in\mathbb{T}}) \end{cases}$$

where $p \in \mathbb{T}$ corresponds to the dual of $l \in (\mathbb{Z}, 0, 0)$ in $H_3^{\mathbb{Z}} = \mathbb{Z}^2 \rtimes_\gamma \mathbb{Z}$, and $\mathfrak{A}_{\theta_z}$ is the rotation algebra $C(\mathbb{T}) \rtimes_{z,\hat{\alpha}} \mathbb{Z}$ with $z = e^{2\pi i\theta_z}$.

Remark 2.2 The group C^*-algebra $C^*(H_3^{\mathbb{Z}})$ is regarded as the C^*-algebra of a continuous field on $\mathbb{T}$, i.e. $C^*(H_3^{\mathbb{Z}}) \cong C(\mathbb{T}^2) \rtimes_{\hat{\gamma}} \mathbb{Z} \cong \Gamma(\mathbb{T}, \{\mathfrak{A}_{\theta_p}\}_{p\in\mathbb{T}})$ with $\mathfrak{A}_{\theta_p} = C(\mathbb{T}) \rtimes_{(p,\hat{\gamma})} \mathbb{Z}$ and $p = e^{2\pi i\theta_p}$. Note that $\mathfrak{K}_1$ as a C^*-algebra of continuous fields above has no local triviality over $(\mathbb{T} \setminus \{1\})^2$, so that it has no meaningful composition series. Also, all the fibers $\mathfrak{A}_{\theta_z} \rtimes_{p,\hat{\gamma}} \mathbb{Z}$, $C(\mathbb{T}^2) \rtimes_{(w,\hat{\beta}),(p,\hat{\gamma})} \mathbb{Z}$ and $(\mathfrak{A}_{\theta_z} \otimes C(\mathbb{T})) \rtimes_{(w,\hat{\beta}),(p,\hat{\gamma})} \mathbb{Z}$ are noncommutative tori since they are generated by the following unitaries respectively (cf. [19]):

$$\begin{cases} U_1, U_2, U_3: \quad U_1U_2 = zU_2U_1, \; U_2U_3 = pU_3U_2, \\ U_1, U_2, U_3: \quad U_3U_1 = wU_1U_3, \; U_2U_3 = pU_3U_2, \\ U_1, U_2, U_3, U_4: U_1U_2 = zU_2U_1, \; U_2U_4 = wU_4U_2, \quad U_3U_4 = pU_4U_3. \end{cases}$$

On the other hand, the center Z of $L_7^{\mathbb{Z}}$ consists of all elements $((s, 0), (t, 0), (l, 0, 0)) \in (\mathbb{Z}^2 \times \mathbb{Z}^2) \rtimes H_3^{\mathbb{Z}}$. Thus $Z \cong \mathbb{Z}^3$ and $\hat{Z} \cong \mathbb{T}^3$. By [8, Theorem 4], $C^*(L_7^{\mathbb{Z}})$ is isomorphic to the C^*-algebra of a continuous field on $\mathbb{T}^3$, i.e. $\Gamma(\mathbb{T}^3, \{\mathfrak{B}_u\}_{u\in\mathbb{T}^3})$ with $\mathfrak{B}_u$ certain fibers. However, this decomposition is not the same as ours, and the fibers $\mathfrak{B}_u$ are just given by $(\mathfrak{A}_{\theta_z}\otimes C(\mathbb{T}))\rtimes_{(w,\hat{\beta}),(p,\hat{\gamma})}\mathbb{Z}$ for $(z, w, p) = u \in \mathbb{T}^3$ by using our analysis.

Similarly, we consider a generalization of Theorem 2.1 in what follows. Let $H_{2n+1}^{\mathbb{Z}}$ be the generalized discrete Heisenberg group of rank $(2n+1)$ consisting of the following $(n+2)\times(n+2)$ matrices:

$$\begin{pmatrix} 1 & (n_j) & l \\ 0 & 1_n & m^t \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_{n+2}(\mathbb{Z}), \quad (n_j),\, m = (m_j) \in \mathbb{Z}^n, \quad l \in \mathbb{Z},$$

where m^t means the transpose of m, and 1_n is the $n\times n$ identity matrix. Let $L_{6n+1}^{\mathbb{Z}} = \mathbb{Z}^{4n}\rtimes_\alpha H_{2n+1}^{\mathbb{Z}}$ with the action $\alpha = (\alpha^1, \ldots, \alpha^{2n})$ such that $\alpha_{n_j}^j, \alpha_{m_j}^{n+j} \in \mathrm{GL}_2(\mathbb{Z})$ for $(l, (m_j)_{j=1}^n, (n_j)_{j=1}^n) \in H_{2n+1}^{\mathbb{Z}}$ and

$$\alpha_1^1 = \cdots = \alpha_1^{2n} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}).$$

Note that $H_{2n+1}^{\mathbb{Z}} \cong \mathbb{Z}^{n+1}\rtimes_\gamma \mathbb{Z}^n$ where $\gamma_{(n_j)}(l, m) = (l + \sum_{j=1}^n n_j m_j, m)$ for $(n_j), m \in \mathbb{Z}^n$ and $l \in \mathbb{Z}$, and the groups $L_{6n+1}^{\mathbb{Z}}, H_{2n+1}^{\mathbb{Z}}$ are 2-step nilpotent. Then $C^*(\mathbb{Z}^{4n}\rtimes H_{2n+1}^{\mathbb{Z}}) \cong C(\mathbb{T}^{4n})\rtimes_{\hat{\alpha}} H_{2n+1}^{\mathbb{Z}}$, and $C^*(H_{2n+1}^{\mathbb{Z}}) \cong C^*(\mathbb{Z}^{n+1})\rtimes_\gamma \mathbb{Z}^n \cong C(\mathbb{T}^{n+1})\rtimes_{\hat{\gamma}}\mathbb{Z}^n$ where $\hat{\gamma}_{(n_j)}(p, (q_j)_{j=1}^n) = (p, (p^{n_j}q_j)_{j=1}^n) \in \mathbb{T}^{n+1}$ for $(n_j)\in\mathbb{Z}^n$.

Theorem 2.3 *Let $L_{6n+1}^{\mathbb{Z}} = \mathbb{Z}^{4n}\rtimes_\alpha H_{2n+1}^{\mathbb{Z}}$ as above. Then the group C^*-algebra $C^*(L_{6n+1}^{\mathbb{Z}})$ has the following finite composition series $\{\mathfrak{K}_j\}_{j=1}^{2n+1}$: $\mathfrak{K}_0 = \{0\}$,*

$$\begin{cases} \mathfrak{K}_{2n+1}/\mathfrak{K}_{2n} \cong C(\mathbb{T}^{2n})\otimes C^*(H_{2n+1}^{\mathbb{Z}}), \\ \mathfrak{K}_{2n-j+1}/\mathfrak{K}_{2n-j} \cong \bigoplus^{\binom{2n}{j}} C(\mathbb{T}^{2n-j}) \\ \qquad\qquad \otimes\, \Gamma_0((\mathbb{T}\setminus\{1\})^j, \{C(\mathbb{T}^j)\rtimes_{z,\hat{\alpha}} H_{2n+1}^{\mathbb{Z}}\}_{z\in(\mathbb{T}\setminus\{1\})^j}) \end{cases}$$

for $1 \le j \le 2n$, where $\binom{2n}{j}$ is the combination ${}_{2n}\mathrm{C}_j$. Moreover, it follows that

$$\begin{aligned} C(\mathbb{T}^j)\rtimes_{z,\hat{\alpha}} H_{2n+1}^{\mathbb{Z}} \cong \Gamma(\mathbb{T}, \{&(\otimes^{k_0}\mathfrak{A}_{\theta_p})\otimes(\otimes^{k_1}\mathfrak{A}_{\theta_{z(s)}}\rtimes_{p,\hat{\gamma}}\mathbb{Z}) \\ &\otimes(\otimes^{k_2}C(\mathbb{T}^2)\rtimes_{\hat{\alpha},(p,\hat{\gamma})}\mathbb{Z}) \\ &\otimes(\otimes^{k_3}[(\mathfrak{A}_{\theta_{z(s)}}\otimes C(\mathbb{T}))\rtimes_{\hat{\alpha},(p,\hat{\gamma})}\mathbb{Z}])\}_{p\in\mathbb{T}}) \end{aligned}$$

where $\mathfrak{A}_{\theta_p}$ and $\mathfrak{A}_{\theta_{z(s)}}$ are the rotation algebras corresponding to $p = e^{2\pi i\theta_p}$

and $z_{2s-1} = e^{2\pi i\theta_{z(s)}}$ $(1 \leq s \leq n)$ *respectively, and* $k_1 + k_2 + 2k_3 = j$ *and* $\sum_{l=0}^{3} k_l = n$ *with* $0 \leq k_1, k_2, 2k_3 \leq j$ *and* $0 \leq k_0 < n$*, and* $p \in \mathbb{T}$ *corresponds to the dual of* $l \in (\mathbb{Z}, 0, \ldots, 0)$ *in* $H_{2n+1}^{\mathbb{Z}} = \mathbb{Z}^{n+1} \rtimes_\gamma \mathbb{Z}^n$.

Proof. The C^*-algebras $\mathfrak{K}_j$ in the finite composition series $\{\mathfrak{K}_j\}_{j=1}^{2n+1}$ of $C^*(D_{6n+1}^{\mathbb{Z}})$ cited above are defined by $\mathfrak{K}_j = C_0(X_j) \rtimes_{\hat{\alpha}} H_{2n+1}^{\mathbb{Z}}$, where $\hat{\alpha}$ is defined by

$$\begin{aligned}&\hat{\alpha}_g((z_j, z_{j+1})_{j=1}^{2n-1}, (z_{2n+j}, z_{2n+j+1})_{j=1}^{2n-1})\\&= ((z_j, z_j^{n_j} z_{j+1})_{j=1}^{2n-1}, (z_{2n+j}, z_{2n+j}^{m_j} z_{2n+j+1})_{j=1}^{2n-1}) \in \mathbb{T}^{4n}\end{aligned}$$

for $g = (l, (m_j)_{j=1}^n, (n_j)_{j=1}^n) \in H_{2n+1}^{\mathbb{Z}}$, and $X_{2n+1} = \mathbb{T}^{4n}$, and $X_{2n+1} \setminus X_{2n} = (\{1\} \times \mathbb{T})^{2n}$ is a $\hat{\alpha}$-fixed closed subspace of X_{2n+1} so that

$$\mathfrak{K}_{2n+1}/\mathfrak{K}_{2n} \cong C((\{1\} \times \mathbb{T})^{2n}) \rtimes_{\hat{\alpha}} H_{2n+1}^{\mathbb{Z}} \cong C(\mathbb{T}^{2n}) \otimes C^*(H_{2n+1}^{\mathbb{Z}}),$$

and

$$X_j \setminus X_{j-1} = \bigsqcup_{\binom{2n}{2n-j+1}} ((\mathbb{T} \setminus \{1\}) \times \mathbb{T})^{2n-j+1} \times (\{1\} \times \mathbb{T})^{j-1}$$

for $1 \leq j \leq 2n$, where the combination $\binom{2n}{2n-j+1}$ corresponds to choosing $\hat{\alpha}$-invariant subspaces of X_j which are homeomorphic to $((\mathbb{T}\setminus\{1\})\times\mathbb{T})^{2n-j+1} \times (\{1\}\times\mathbb{T})^{j-1}$ (that is, the product spaces of $(2n-j+1)$-copies of $(\mathbb{T}\setminus\{1\})\times\mathbb{T}$ and $(j-1)$-copies of $\{1\} \times \mathbb{T}$ in $\mathbb{T}^{4n} = (\mathbb{T}^2)^{2n}$). Thus,

$$\begin{aligned}\mathfrak{K}_j/\mathfrak{K}_{j-1} &\cong C_0(X_j \setminus X_{j-1}) \rtimes_{\hat{\alpha}} H_{2n+1}^{\mathbb{Z}}\\&\cong \bigoplus^{\binom{2n}{2n-j+1}} C_0(((\mathbb{T} \setminus \{1\}) \times \mathbb{T})^{2n-j+1}\\&\qquad \times (\{1\} \times \mathbb{T})^{j-1}) \rtimes_{\hat{\alpha}} H_{2n+1}^{\mathbb{Z}}\end{aligned}$$

with $\hat{\alpha} = (\hat{\alpha}^1, \ldots, \hat{\alpha}^{2n})$. Since $\hat{\alpha}^j$ for $1 \leq j \leq 2n$ are defined as above (cf. the actions $\hat{\alpha}$, $\hat{\beta}$ in Theorem 2.1), it is deduced that

$$\begin{aligned}&C_0(((\mathbb{T} \setminus \{1\}) \times \mathbb{T})^{2n-j+1} \times (\{1\} \times \mathbb{T})^{j-1}) \rtimes_{\hat{\alpha}} H_{2n+1}^{\mathbb{Z}}\\&\cong C(\mathbb{T}^{j-1})\\&\quad \otimes \Gamma_0((\mathbb{T} \setminus \{1\})^{2n-j+1}, \{C(\mathbb{T}^{2n-j+1}) \rtimes_{z,\hat{\alpha}} H_{2n+1}^{\mathbb{Z}}\}_{z\in(\mathbb{T}\setminus\{1\})^{2n-j+1}}).\end{aligned}$$

Moreover, replacing $2n - j - 1$ with j, it follows that for $1 \leq j \leq 2n$,

$$
\begin{aligned}
C(\mathbb{T}^j) \rtimes_{z,\hat{\alpha}} H_{2n+1}^{\mathbb{Z}} &\cong C(\mathbb{T}^j) \rtimes_{z,\hat{\alpha}} (\mathbb{Z}^{n+1} \rtimes_\gamma \mathbb{Z}^n) \\
&\cong \Gamma(\mathbb{T}, \{(C(\mathbb{T}^j) \rtimes_{z,\hat{\alpha}} \mathbb{Z}^n) \rtimes_{\hat{\alpha},(p,\hat{\gamma})} \mathbb{Z}^n\}_{p\in\mathbb{T}}),
\end{aligned}
$$

where the action $(p, \hat{\gamma})$ corresponds to the restriction of $\hat{\gamma}$ to $\{p\} \times \mathbb{T}^n$. Furthermore, the space $\mathbb{T}^j$ is decomposed into $\mathbb{T}^{k_1} \times \mathbb{T}^{k_2} \times \Pi^{k_3}\mathbb{T}^2$, and the actions $\hat{\alpha}^s$, $\hat{\alpha}^{n+s}$, and $(\hat{\alpha}^s, \hat{\alpha}^{n+s})$ for some $1 \le s \le n$ act on each direct factor of $\mathbb{T}^{k_1}$, $\mathbb{T}^{k_2}$ and $\Pi^{k_3}\mathbb{T}^2$ respectively. Then it is obtained that

$$
\begin{aligned}
&(C(\mathbb{T}^j) \rtimes_{z,\hat{\alpha}} \mathbb{Z}^n) \rtimes_{\hat{\alpha},(p,\hat{\gamma})} \mathbb{Z}^n \\
&\cong (\otimes^{k_0} C(\mathbb{T}) \rtimes_{p,\hat{\gamma}} \mathbb{Z}) \otimes (\otimes^{k_1}(C(\mathbb{T}) \rtimes_{\hat{\alpha}^s} \mathbb{Z}) \rtimes_{p,\hat{\gamma}} \mathbb{Z}) \\
&\qquad \otimes (\otimes^{k_2}(C(\mathbb{T}) \otimes C^*(\mathbb{Z})) \rtimes_{\hat{\alpha}^{n+s},(p,\hat{\gamma})} \mathbb{Z}) \\
&\qquad \otimes (\otimes^{k_3}(C(\mathbb{T}^2) \rtimes_{\hat{\alpha}^s} \mathbb{Z}) \rtimes_{\hat{\alpha}^{n+s},(p,\hat{\gamma})} \mathbb{Z}) \\
&\cong (\otimes^{k_0}\mathfrak{A}_{\theta_p}) \otimes (\otimes^{k_1}\mathfrak{A}_{\theta_{z(s-1)}} \rtimes_{p,\hat{\gamma}} \mathbb{Z}) \\
&\qquad \otimes (\otimes^{k_2} C(\mathbb{T}^2) \rtimes_{\hat{\alpha}^{n+s},(p,\hat{\gamma})} \mathbb{Z}) \\
&\qquad \otimes (\otimes^{k_3}[(\mathfrak{A}_{\theta_{z(s-1)}} \otimes C(\mathbb{T})) \rtimes_{\hat{\alpha}^{n+s},(p,\hat{\gamma})} \mathbb{Z}]),
\end{aligned}
$$

where $\mathfrak{A}_{\theta_p}$ and $\mathfrak{A}_{\theta_{z(s-1)}}$ are the rotation algebras corresponding to $p = e^{2\pi i\theta_p}$ and $z_{2(s-1)-1} = e^{2\pi i\theta_{z(s-1)}}$ $(1 \le s \le n)$ respectively. □

Remark 2.4 The group C^*-algebra $C^*(H_{2n+1}^{\mathbb{Z}})$ is regarded as a C^*-algebra of continuous fields on $\mathbb{T}$, i.e.

$$C^*(H_{2n+1}^{\mathbb{Z}}) \cong C(\mathbb{T}^{n+1}) \rtimes_{\hat{\gamma}} \mathbb{Z}^n \cong \Gamma(\mathbb{T}, \{\otimes^n \mathfrak{A}_{\theta_z}\}_{z\in\mathbb{T}})$$

where $C(\{z\} \times \mathbb{T}^n) \rtimes_{\hat{\gamma}} \mathbb{Z}^n \cong \otimes^n(C(\mathbb{T}) \rtimes_{z,\hat{\gamma}} \mathbb{Z})$ and $C(\mathbb{T}) \rtimes_{z,\hat{\gamma}} \mathbb{Z} = \mathfrak{A}_{\theta_z}$ the rotation algebra corresponding to $z = e^{2\pi i\theta_z}$. In the above decomposition of $C(\mathbb{T}^j) \rtimes_{z,\hat{\alpha}} H_{2n+1}^{\mathbb{Z}}$ into the continuous field on $\mathbb{T}$, its fibers are tensor products of noncommutative tori, so that they are also noncommutative tori. See [18, 19, 20] for the results on the stable rank of group C^*-algebras of some disconnected Lie groups, related with the structures of Theorems 2.1 and 2.3.

3. The C -algebras of the discrete $ax + b$ groups

We first consider discrete solvable groups of the form $\mathbb{Z} \rtimes_\alpha \mathbb{Z}$ with α nontrivial. Since $\mathrm{Aut}(\mathbb{Z}) = \{\pm\,\mathrm{id}\}$ where id is the identity automorphism of $\mathbb{Z}$, we assume that $\alpha_1 = -\,\mathrm{id}$. Let $\Gamma = \mathbb{Z} \rtimes_\alpha \mathbb{Z}$. Note the following quotient:

$$\Gamma = \mathbb{Z} \rtimes_\alpha \mathbb{Z} \ni (s,\, t) \mapsto \begin{pmatrix} e^{\pi i t} & s \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}).$$

Therefore, we say that Γ is the (extended) discrete $ax + b$ group.

Theorem 3.1 *Let Γ be the discrete $ax + b$ group defined above. Then $C^*(\Gamma)$ has the following finite composition series $\{\mathfrak{I}_j\}_{j=1}^3$: $\mathfrak{I}_3/\mathfrak{I}_2 \cong C(\mathbb{T}) \oplus C(\mathbb{T})$, and*

$$\mathfrak{I}_2/\mathfrak{I}_1 \cong C_0(\mathbb{R}) \otimes M_2(\mathbb{C}), \quad \textit{and} \quad \mathfrak{I}_1 \cong C_0(\mathbb{R}^2) \otimes M_2(\mathbb{C}).$$

Proof. Note that $C^*(\Gamma) \cong C(\mathbb{T}) \rtimes_{\hat{\alpha}} \mathbb{Z}$, where $\hat{\alpha}$ is the reflection on $\mathbb{T}$. Since $\pm 1 \in \mathbb{T}$ is fixed under $\hat{\alpha}$, the following exact sequence is obtained:

$$0 \to C_0(\mathbb{T} \setminus \{\pm 1\}) \rtimes \mathbb{Z} \to C(\mathbb{T}) \rtimes_{\hat{\alpha}} \mathbb{Z} \to \oplus^2 C^*(\mathbb{Z}) \to 0$$

with $C^*(\mathbb{Z}) \cong C(\mathbb{T})$. Since $\hat{\alpha}^2 = \mathrm{id}$ on $\mathbb{T} \setminus \{\pm 1\}$, the above ideal has the following decomposition:

$$\begin{aligned} 0 \to C_0(\mathbb{R}) \otimes (C_0(\mathbb{T} \setminus \{\pm 1\}) \rtimes \mathbb{Z}_2 &\to C_0(\mathbb{T} \setminus \{\pm 1\}) \rtimes \mathbb{Z} \\ &\to C_0(\mathbb{T} \setminus \{\pm 1\}) \rtimes \mathbb{Z}_2 \to 0. \end{aligned}$$

In fact, $C_0(\mathbb{T} \setminus \{\pm 1\}) \rtimes \mathbb{Z}$ is regarded as the mapping torus M_β of the dual action β of $\mathbb{Z}_2$ on $C_0(\mathbb{T} \setminus \{\pm 1\}) \rtimes_{\hat{\alpha}} \mathbb{Z}_2 \equiv \mathfrak{Q}$, that is, $M_\beta = \{f\colon [0,\, 1] \to \mathfrak{Q} \mid$ continuous and $f(1) = \beta_1(f(0))\}$ (cf. cite[p. 179]25), where β is trivial on $C_0(\mathbb{T} \setminus \{\pm 1\})$ and acts on $\mathbb{Z}_2$ by $\beta_l(t) = \langle t,\, l \rangle t$ for $t \in \mathbb{Z}_2$ and $l \in \hat{\mathbb{Z}}_2 \cong \mathbb{Z}_2$. Moreover, it is obtained that

$$\begin{aligned} C_0(\mathbb{T} \setminus \{\pm 1\}) \rtimes \mathbb{Z}_2 &\cong C_0((0,\, \pi)) \otimes (C(\{\pm i\}) \rtimes \mathbb{Z}_2) \\ &\cong C_0((0,\, \pi)) \otimes (\mathbb{C}^2 \rtimes \mathbb{Z}_2) \end{aligned}$$

and $\mathbb{C}^2 \rtimes \mathbb{Z}_2 \cong M_2(\mathbb{C})$, where the first isomorphism is deduced from the identifications: $\mathbb{T} \setminus \{\pm 1\} \ni z = e^{i\lambda} \leftrightarrow i\lambda \in i(0,\, \pi) \sqcup i(\pi,\, 2\pi)$ and $i(\pi,\, 2\pi) \approx (-i)(0,\, \pi)$ (homeomorphic). Note that $(0,\, \pi)$ is homeomorphic to $\mathbb{R}$. □

Remark 3.2 Note that $M_2(\mathbb{C}) \cong C^*(\mathbb{Z}_2) \rtimes \mathbb{Z}_2 \cong C^*(\mathbb{Z}_2 \rtimes \mathbb{Z}_2)$ with the action of $\mathbb{Z}_2$ the left multiplication on $\mathbb{Z}_2$. On the other hand, we can show that $\mathrm{sr}(C^*(\Gamma)) = 2$ and $\mathrm{csr}(C^*(\Gamma)) = 2$ as explained in the introduction using [13, Theorem 6.1], [15, p. 381].

Next define the generalized (extended) discrete $ax + b$ groups Γ_{n+1} to be the groups with the quotient map to the following $(n+1) \times (n+1)$

matrices:

$$\Gamma_{n+1} \ni (s_1, \ldots, s_n, t) \mapsto \begin{pmatrix} e^{\pi i t} & 0 & \cdots & 0 & s_1 \\ 0 & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & e^{\pi i t} & s_n \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \in \mathrm{GL}_{n+1}(\mathbb{Z})$$

for $t,\ s_j \in \mathbb{Z}$ $(1 \le j \le n)$. Then $\Gamma_{n+1} = \mathbb{Z}^n \rtimes_\alpha \mathbb{Z}$.

Theorem 3.3 *Let Γ_{n+1} be the generalized discrete $ax + b$ group defined above. Then $C^*(\Gamma_{n+1})$ has the following finite composition series $\{\mathfrak{J}_j\}_{j=1}^{n+1}$: $\mathfrak{J}_0 = \{0\}$, $\mathfrak{J}_{n+1}/\mathfrak{J}_n \cong \oplus^{2^n} C(\mathbb{T})$, and*

$$\mathfrak{J}_j/\mathfrak{J}_{j-1} \cong \bigoplus^{\binom{n}{n-j+1}} C_0((\mathbb{T} \setminus \{\pm 1\})^{n-j+1}) \rtimes \mathbb{Z},$$

for $1 \le j \le n$. Moreover, it is obtained by putting $Z_j = (\mathbb{T} \setminus \{\pm 1\})^{n-j+1}$ that

$$0 \to C_0(\mathbb{R}^{n-j+1}) \otimes (\oplus^{n-j+1} M_2(\mathbb{C})) \to C_0(Z_j) \rtimes \mathbb{Z} \\ \to \oplus^{n-j+1} M_2(\mathbb{C}) \to 0.$$

Proof. Note that $C^*(\Gamma_{n+1}) \cong C(\mathbb{T}^n) \rtimes_{\hat{\alpha}} \mathbb{Z}$. Since the points $(\pm 1, \ldots, \pm 1) \in \mathbb{T}^n$ are fixed under $\hat{\alpha}$, the following exact sequence is obtained:

$$0 \to C_0(\mathbb{T}^n \setminus \{(\pm 1, \ldots, \pm 1)\}) \rtimes \mathbb{Z} \to C(\mathbb{T}) \rtimes_{\hat{\alpha}} \mathbb{Z} \to \oplus^{2^n} C(\mathbb{T}) \to 0.$$

Put $Y_{n+1} \equiv \mathbb{T}^n \setminus \{(\pm 1, \ldots, \pm 1)\}$. Then $C_0(Y_{n+1}) \rtimes \mathbb{Z}$ has the following finite composition series $\{\mathfrak{J}_j\}_{j=1}^n$: $\mathfrak{J}_0 = \{0\}$, $\mathfrak{J}_j = C_0(Y_j) \rtimes \mathbb{Z}$ and

$$\mathfrak{J}_j/\mathfrak{J}_{j-1} \cong \bigoplus^{\binom{n}{n-j+1}} C_0((\mathbb{T} \setminus \{\pm 1\})^{n-j+1}) \rtimes \mathbb{Z}.$$

Put $Z_j \equiv (\mathbb{T} \setminus \{\pm 1\})^{n-j+1}$. Since $\hat{\alpha}^2 = \mathrm{id}$ on Z_j, each direct factor of the above subquotients has the following decomposition:

$$0 \to C_0(\mathbb{R}) \otimes (C_0(Z_j) \rtimes \mathbb{Z}_2) \to C_0(Z_j) \rtimes \mathbb{Z} \to C_0(Z_j) \rtimes \mathbb{Z}_2 \to 0$$

by the same way as in the proof of Theorem 3.1. Moreover, it follows that

$$C_0(Z_j) \rtimes \mathbb{Z}_2 \cong C_0(\mathbb{R}^{n-j+1}) \otimes (C(\Pi^{n-j+1}\{\pm i\}) \rtimes \mathbb{Z}_2)$$

since $\mathbb{T} \setminus \{\pm 1\} \approx i(0, \pi) \sqcup i(\pi, 2\pi) \approx i(0, \pi) \sqcup (-i)(0, \pi)$ and $(0, \pi) \approx \mathbb{R}$ (homeomorphic), and $C(\Pi^{n-j+1}\{\pm i\}) \rtimes \mathbb{Z}_2 \cong \mathbb{C}^{2(n-j+1)} \rtimes \mathbb{Z}_2 \cong \oplus^{n-j+1} M_2(\mathbb{C})$ since $\Pi^{n-j+1}\{\pm i\}$ is decomposed into the disjoint union of the orbits of its points. □

Remark 3.4 It can be shown as explained in the introduction that

$$\begin{aligned}\mathrm{sr}(C_0(\mathbb{R}^n) \otimes M_2(\mathbb{C})) &= \{[n/2]/2\} + 1 \\ &\le \mathrm{sr}(C^*(\Gamma_{n+1})) \le \{[(n+1)/2]/2\} + 1\end{aligned}$$

and $\mathrm{csr}(C^*(\Gamma_{n+1})) \le \{[(n+1)/2]/2\} + 1$, where $[x]$ means the maximum integer $\le x$, and $\{x\}$ means the least integer $\ge x$ ([13, Theorem 6.1], [14, Theorem 4.7], [10]). Compare this situation with some previous results on the stable rank of group C^*-algebras of connected or disconnected Lie groups ([17-20] and [23, 24]).

Next define the generalized (extended) discrete Mautner groups $M_{2n}^{\mathbb{Z}}$ to be the groups with the quotient map to the following $(n+1) \times (n+1)$ matrices:

$$M_{2n}^{\mathbb{Z}} \ni (s_1, \ldots, s_n, t_1, \ldots, t_n) \mapsto \begin{pmatrix} e^{\pi i t_1} & 0 & \cdots & 0 & s_1 \\ 0 & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & e^{\pi i t_n} & s_n \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \in \mathrm{GL}_{n+1}(\mathbb{Z})$$

for $t_j, s_j \in \mathbb{Z}$ $(1 \le j \le n)$ (See [1] or [18] for another definition of the discrete Mautner group (cf. [20])). Then $M_{2n}^{\mathbb{Z}} = \mathbb{Z}^n \rtimes_\alpha \mathbb{Z}^n$.

Theorem 3.5 *Let $M_{2n}^{\mathbb{Z}}$ be the generalized discrete Mautner group defined above. Then $C^*(M_{2n}^{\mathbb{Z}})$ has the following finite composition series $\{\mathfrak{I}_j\}_{j=1}^{3^n}$: $\mathfrak{I}_0 = \{0\}$,*

$$\mathfrak{I}_j/\mathfrak{I}_{j-1} \cong \mathfrak{F}_{1_j} \otimes \cdots \otimes \mathfrak{F}_{n_j}$$

for $1 \le l_j \le 3$ and $l_{j-1} \le l_j$ for $1 \le l \le n$, and $\mathfrak{F}_3 \cong C(\mathbb{T}) \oplus C(\mathbb{T})$, and

$$\mathfrak{F}_2 \cong C_0(\mathbb{R}) \otimes M_2(\mathbb{C}), \quad \textit{and} \quad \mathfrak{F}_1 \cong C_0(\mathbb{R}^2) \otimes M_2(\mathbb{C}).$$

Proof. Note that $M_{2n}^{\mathbb{Z}} \cong \Pi^n(\mathbb{Z} \rtimes_\alpha \mathbb{Z}) \cong \Pi^n\Gamma$, where Γ is the discrete $ax+b$ group. Thus $C^*(M_{2n}^{\mathbb{Z}}) \cong \otimes^n C^*(\Gamma)$. Therefore, the finite composition series in the statement is obtained from Theorem 3.1. □

Remark 3.6 It can be shown that

$$\mathrm{sr}(C(\mathbb{T}^n)) = \left[\frac{n}{2}\right] + 1 \leq \mathrm{sr}(C^*(M_{2n}^{\mathbb{Z}})) \leq \left[\frac{n+1}{2}\right] + 1$$

and $\mathrm{csr}(C^*(M_{2n}^{\mathbb{Z}})) \leq [(n+1)/2] + 1$ (cf. Remark 3.4).

4. Certain discrete solvable semi-direct products by $H_3^{\mathbb{Z}}$

Let $\Delta_5 = (\mathbb{Z} \times \mathbb{Z}) \rtimes_{(\alpha,\beta)} H_3^{\mathbb{Z}}$, where $\alpha_m = e^{\pi i m}$ and $\beta_n = e^{\pi i n}$ for $(l, m, n) \in H_3^{\mathbb{Z}}$. Then $C^*(\Delta_5) \cong C(\mathbb{T}^2) \rtimes_{(\hat\alpha,\hat\beta)} H_3^{\mathbb{Z}}$ with $\hat\alpha, \hat\beta$ reflections on each direct factor $\mathbb{T}$ of $\mathbb{T}^2$. Recall that $H_3^{\mathbb{Z}} \cong \mathbb{Z}^2 \rtimes_\gamma \mathbb{Z}$ and $C^*(H_3^{\mathbb{Z}}) \cong C(\mathbb{T}^2) \rtimes_{\hat\gamma} \mathbb{Z}$ as before Theorem 2.1.

Theorem 4.1 *Let Δ_5 be the discrete solvable group defined above. Then $C^*(\Delta_5)$ has the following finite composition series*

$$\{\mathfrak{D}_j\}_{j=1}^3 : \mathfrak{D}_3/\mathfrak{D}_2 \cong \oplus^4 C^*(H_3^{\mathbb{Z}}),$$

and

$$\begin{cases} \mathfrak{D}_2/\mathfrak{D}_1 \cong [C_0(\mathbb{T}\setminus\{\pm1\}) \rtimes_{\hat\alpha} H_3^{\mathbb{Z}}] \oplus [C_0(\mathbb{T}\setminus\{\pm1\}) \rtimes_{\hat\beta} H_3^{\mathbb{Z}}], \\ \mathfrak{D}_1 \cong C_0((\mathbb{T}\setminus\{\pm1\})^2) \rtimes_{(\hat\alpha,\hat\beta)} H_3^{\mathbb{Z}}. \end{cases}$$

Moreover, it is obtained that

$$\begin{cases} C_0(\mathbb{T}\setminus\{\pm1\}) \rtimes_{\hat\alpha} H_3^{\mathbb{Z}} \\ \qquad \cong \Gamma(\mathbb{T}, \{C_0(\mathbb{R}) \otimes (\mathbb{C}^2 \otimes C(\mathbb{T})) \rtimes_{\hat\alpha,(\bar z,\hat\gamma)} \mathbb{Z}\}_{z\in\mathbb{T}}), \\ C_0(\mathbb{T}\setminus\{\pm1\}) \rtimes_{\hat\beta} H_3^{\mathbb{Z}} \\ \qquad \cong \Gamma(\mathbb{T}, \{C_0(\mathbb{R}) \otimes (\mathbb{C}^2 \otimes C(\mathbb{T})) \rtimes_{\hat\beta,(w,\hat\gamma)} \mathbb{Z}\}_{w\in\mathbb{T}}), \\ C_0((\mathbb{T}\setminus\{\pm1\})^2) \rtimes_{(\hat\alpha,\hat\beta)} H_3^{\mathbb{Z}} \\ \qquad \cong \Gamma(\mathbb{T}, \{C_0(\mathbb{R}^2) \otimes ((\mathbb{C}^2 \rtimes_{\hat\alpha} \mathbb{Z}) \otimes \mathbb{C}^2) \rtimes_{(z,\hat\gamma)\otimes\hat\beta} \mathbb{Z}\}_{z\in\mathbb{T}}), \end{cases}$$

where $\mathbb{C}^2 = C(\{\pm i\})$, and the actions $(z, \hat\gamma)$ correspond to the restrictions to $\{z\} \times \mathbb{T}$ in $\mathbb{T}^2$ of $C(\mathbb{T}^2) \rtimes_{\hat\gamma} \mathbb{Z} \cong C^*(H_3^{\mathbb{Z}})$.

Proof. Since the points $(\pm 1, \pm 1) \in \mathbb{T}^2$ are fixed under $(\hat{\alpha}, \hat{\beta})$, it is deduced that

$$0 \to C_0(\mathbb{T}^2 \setminus \{(\pm 1, \pm 1)\}) \rtimes H_3^{\mathbb{Z}} \to C(\mathbb{T}^2) \rtimes_{(\hat{\alpha}, \hat{\beta})} H_3^{\mathbb{Z}} \to \oplus^4 C^*(H_3^{\mathbb{Z}}) \to 0.$$

Moreover, the ideal has the following decomposition:

$$(\mathrm{E})\colon 0 \to C_0((\mathbb{T} \setminus \{\pm 1\})^2) \rtimes H_3^{\mathbb{Z}} \to C_0(\mathbb{T}^2 \setminus \{(\pm 1, \pm 1)\}) \rtimes H_3^{\mathbb{Z}} \to \oplus^2 C_0(\mathbb{T} \setminus \{\pm 1\}) \rtimes H_3^{\mathbb{Z}} \to 0.$$

Case 1_1: One of the two direct factors of the quotient of (E) has the following isomorphisms:

$$\begin{aligned} C_0(\mathbb{T} \setminus \{\pm 1\}) \rtimes_{\hat{\alpha}} H_3^{\mathbb{Z}} &\cong C_0(\mathbb{T} \setminus \{\pm 1\}) \rtimes_{\hat{\alpha}} (\mathbb{Z}^2 \rtimes_\gamma \mathbb{Z}) \cong \\ ((C_0(\mathbb{T} \setminus \{\pm 1\}) \rtimes_{\hat{\alpha}} \mathbb{Z}) \otimes C(\mathbb{T})) \rtimes \mathbb{Z} \\ &\cong \Gamma(\mathbb{T}, \{(C_0(\mathbb{T} \setminus \{\pm 1\}) \rtimes_{\hat{\alpha}} \mathbb{Z}) \rtimes_{z, \hat{\gamma}} \mathbb{Z}\}_{z \in \mathbb{T}}), \end{aligned}$$

where the actions $(z, \hat{\gamma})$ correspond to the restrictions to $\{z\} \times \mathbb{T}$ in $\mathbb{T}^2$ of $C(\mathbb{T}^2) \rtimes_{\hat{\gamma}} \mathbb{Z} \cong C^*(H_3^{\mathbb{Z}})$. Moreover, it follows that

$$\begin{aligned} (C_0(\mathbb{T} \setminus \{\pm 1\}) \rtimes_{\hat{\alpha}} \mathbb{Z}) \rtimes_{z, \hat{\gamma}} \mathbb{Z} &\cong C_0((\mathbb{T} \setminus \{\pm 1\}) \times \mathbb{T}) \rtimes_{\hat{\alpha}, (\bar{z}, \hat{\gamma})} \mathbb{Z} \\ &\cong C_0(\mathbb{R}) \otimes (\mathbb{C}^2 \otimes C(\mathbb{T})) \rtimes_{\hat{\alpha}, (\bar{z}, \hat{\gamma})} \mathbb{Z}, \end{aligned}$$

where the first isomorphism is obtained by exchanging the actions $\hat{\alpha}$ and $(z, \hat{\gamma})$, and the second one uses the identifications $\mathbb{T} \setminus \{\pm 1\} \approx i(0, \pi) \sqcup i(\pi, 2\pi) \approx i(0, \pi) \sqcup (-i)(0, \pi)$ and $(0, \pi) \approx \mathbb{R}$ (homeomorphic), and $\mathbb{C}^2 = C(\{\pm i\})$.

Case 1_2: The other of the two direct factors of the quotient of (E) has the following isomorphisms:

$$\begin{aligned} C_0(\mathbb{T} \setminus \{\pm 1\}) \rtimes_{\hat{\beta}} H_3^{\mathbb{Z}} &\cong C_0(\mathbb{T} \setminus \{\pm 1\}) \rtimes_{\hat{\beta}} (\mathbb{Z}^2 \rtimes_\gamma \mathbb{Z}) \\ &\cong (C_0(\mathbb{T} \setminus \{\pm 1\}) \otimes C(\mathbb{T}^2)) \rtimes_{\hat{\beta}, \hat{\gamma}} \mathbb{Z} \\ &\cong \Gamma(\mathbb{T}, \{C_0((\mathbb{T} \setminus \{\pm 1\}) \times \mathbb{T}) \rtimes_{\hat{\beta}, (w, \hat{\gamma})} \mathbb{Z}\}_{w \in \mathbb{T}}), \end{aligned}$$

where $(w, \hat{\gamma})$ means the same as $(z, \hat{\gamma})$ above. Moreover, it follows that $C_0((\mathbb{T} \setminus \{\pm 1\}) \times \mathbb{T}) \rtimes_{\hat{\beta}, (w, \hat{\gamma})} \mathbb{Z} \cong C_0(\mathbb{R}) \otimes (\mathbb{C}^2 \otimes C(\mathbb{T})) \rtimes_{\hat{\beta}, (w, \hat{\gamma})} \mathbb{Z}$.

Case 2: The ideal of (E) has the following isomorphisms:

$$C_0((\mathbb{T}\setminus\{\pm 1\})^2) \rtimes_{(\hat{\alpha},\hat{\beta})} H_3^{\mathbb{Z}} \cong C_0((\mathbb{T}\setminus\{\pm 1\})^2) \rtimes_{(\hat{\alpha},\hat{\beta})} (\mathbb{Z}^2 \rtimes_\gamma \mathbb{Z})$$
$$\cong ((C_0(\mathbb{T}\setminus\{\pm 1\}) \rtimes_{\hat{\alpha}} \mathbb{Z}) \otimes C((\mathbb{T}\setminus\{\pm 1\}) \times \mathbb{T})) \rtimes_{\hat{\beta},\hat{\gamma}} \mathbb{Z}$$
$$\cong \Gamma(\mathbb{T}, \{((C_0(\mathbb{T}\setminus\{\pm 1\}) \rtimes_{\hat{\alpha}} \mathbb{Z}) \otimes C_0(\mathbb{T}\setminus\{\pm 1\})) \rtimes_{(z,\hat{\gamma})\otimes\hat{\beta}} \mathbb{Z}\}_{z\in\mathbb{T}}).$$

Moreover, it is obtained that

$$((C_0(\mathbb{T}\setminus\{\pm 1\}) \rtimes_{\hat{\alpha}} \mathbb{Z}) \otimes C_0(\mathbb{T}\setminus\{\pm 1\})) \rtimes_{(z,\hat{\gamma})\otimes\hat{\beta}} \mathbb{Z}$$
$$\cong C_0(\mathbb{R}^2) \otimes ((\mathbb{C}^2 \rtimes_{\hat{\alpha}} \mathbb{Z}) \otimes \mathbb{C}^2) \rtimes_{(z,\hat{\gamma})\otimes\hat{\beta}} \mathbb{Z},$$

where we use the same identification of $\mathbb{T}\setminus\{\pm 1\}$ as above. □

Remark 4.2 The C^*-algebras $(\mathbb{C}^2 \otimes C(\mathbb{T})) \rtimes_{\hat{\alpha},(\bar{z},\hat{\gamma})} \mathbb{Z}$, $(\mathbb{C}^2 \otimes C(\mathbb{T})) \rtimes_{\hat{\beta},(w,\hat{\gamma})} \mathbb{Z}$ and $((\mathbb{C}^2 \rtimes_{\hat{\alpha}} \mathbb{Z}) \otimes \mathbb{C}^2) \rtimes_{(z,\hat{\gamma})\otimes\hat{\beta}} \mathbb{Z}$ are not noncommutative tori. In fact, $(\mathbb{C}^2 \otimes C(\mathbb{T})) \rtimes_{\hat{\alpha},(\bar{z},\hat{\gamma})} \mathbb{Z} \cong (C(\mathbb{T}) \oplus C(\mathbb{T})) \rtimes_{\lambda^{\alpha,\bar{z}}} \mathbb{Z}$, where $\lambda_m^{\alpha,\bar{z}}(U_1) = z^m U_2$ and $\lambda_m^{\alpha,\bar{z}}(U_2) = z^m U_1$ for $(l, m, n) \in H_3^{\mathbb{Z}}$ and $(U_1, 0), (0, U_2) \in C(\mathbb{T}) \oplus C(\mathbb{T})$ the canonical generators of the direct factors $C(\mathbb{T})$. Also, it is able to consider the actions of the other algebras through the similar isomorphisms explicitly. Those algebras might be new, but it could be shown that if those algebras are non-rational, i.e. z and w irrational (rotations), they are approximately divisible by using the methods of [2]. Thus those simple algebras have stable rank one.

Next, let $\Delta_{4n+1} = \mathbb{Z}^{2n} \rtimes_{(\alpha,\beta)} H_{2n+1}^{\mathbb{Z}}$, where $\alpha = (\alpha^1, \ldots, \alpha^n)$, $\beta = (\beta^1, \ldots, \beta^n)$ with $\alpha_{m_j}^j = e^{\pi i m_j}$ and $\beta_{n_j}^j = e^{\pi i n_j}$ for $(l, (m_j)_{j=1}^n, (n_j)_{j=1}^n) \in H_{2n+1}^{\mathbb{Z}}$. Then $C^*(\Delta_{4n+1}) \cong C(\mathbb{T}^{2n}) \rtimes_{(\hat{\alpha},\hat{\beta})} H_{2n+1}^{\mathbb{Z}}$ with $\hat{\alpha}$, $\hat{\beta}$ reflections on each direct factor $\mathbb{T}$ of $\mathbb{T}^n \times \{0_n\}$ and $\{0_n\} \times \mathbb{T}^n$ respectively. Recall that $H_{2n+1}^{\mathbb{Z}} \cong \mathbb{Z}^{n+1} \rtimes_\gamma \mathbb{Z}^n$ and $C^*(H_{2n+1}^{\mathbb{Z}}) \cong C(\mathbb{T}^{n+1}) \rtimes_{\hat{\gamma}} \mathbb{Z}^n$ as before Theorem 2.3. Then it is obtained similarly as Theorem 4.1 that

Theorem 4.3 *Let Δ_{4n+1} be the discrete solvable group defined above. Then the group C^*-algebra $C^*(\Delta_{4n+1})$ has the following finite composition series $\{\mathfrak{D}_j\}_{j=1}^{2n+1}$: $\mathfrak{D}_0 = \{0\}$,*

$$\begin{cases} \mathfrak{D}_{2n+1}/\mathfrak{D}_{2n} \cong \oplus^{2^{2n}} C^*(H_{2n+1}^{\mathbb{Z}}), \\ \mathfrak{D}_j/\mathfrak{D}_{j-1} \cong \bigoplus^{\binom{2n}{2n-j+1}} C_0((\mathbb{T}\setminus\{\pm 1\})^{2n-j+1}) \rtimes_{(\hat{\alpha},\hat{\beta})} H_{2n+1}^{\mathbb{Z}} \end{cases}$$

for $1 \leq j \leq 2n$. Moreover, it is obtained by putting $Z_j = (\mathbb{T} \setminus \{\pm 1\})^{2n-j+1}$ that

$$\begin{aligned} & C_0(Z_j) \rtimes_{(\hat{\alpha},\hat{\beta})} H^{\mathbb{Z}}_{2n+1} \\ & \cong \Gamma\big(\mathbb{T}, \{C_0(\mathbb{R}^{2n-j+1}) \otimes (\otimes^{k_0} \mathfrak{A}_{\theta_z}) \\ & \qquad \otimes (\otimes^{k_1} (\mathbb{C}^2 \otimes C(\mathbb{T})) \rtimes_{\hat{\alpha},(\bar{z},\hat{\gamma})} \mathbb{Z}) \\ & \qquad \otimes (\otimes^{k_2} (\mathbb{C}^2 \otimes C(\mathbb{T})) \rtimes_{\hat{\beta},(z,\hat{\gamma})} \mathbb{Z}) \\ & \qquad \otimes (\otimes^{k_3} ((\mathbb{C}^2 \rtimes_{\hat{\alpha}} \mathbb{Z}) \otimes \mathbb{C}^2) \rtimes_{(z,\hat{\gamma})\otimes\hat{\beta}} \mathbb{Z}\}_{z\in\mathbb{T}}\big), \end{aligned}$$

where $\mathfrak{A}_{\theta_z}$ are the rotation algebras corresponding to $z = e^{2\pi i \theta_z}$, and the actions $(z, \hat{\gamma})$ correspond to the restrictions to $\{z\} \times \mathbb{T}^n$ of $C(\mathbb{T}^{n+1}) \rtimes_{\hat{\gamma}} \mathbb{Z}^n \cong C^*(H^{\mathbb{Z}}_{2n+1})$, and $k_1 + k_2 + 2k_3 = j$ and $\sum_{l=0}^{3} k_l = n$ with $0 \leq k_1, k_2, 2k_3 \leq j$ and $0 \leq k_0 < n$.

Next let $D^{\mathbb{Z}}_7 = (\mathbb{Z}^2 \times \mathbb{Z}^2) \rtimes_{(\alpha,\beta)} H^{\mathbb{Z}}_3$, where $\alpha_m = e^{\pi i m} \oplus e^{\pi i m}$ on $\mathbb{Z}^2$ and $\beta_n = e^{\pi i n} \oplus e^{\pi i n}$ on $\mathbb{Z}^2$. We say that $D^{\mathbb{Z}}_7$ is the discrete Dixmier group of rank 7 (cf. [19] for the disconnected Dixmier group). Then it follows that

Theorem 4.4 *Let $D^{\mathbb{Z}}_7$ be the discrete Dixmier group of rank 7. Then $C^*(D^{\mathbb{Z}}_7)$ has the following finite composition series*

$$\{\mathfrak{L}_j\}_{j=1}^{9} \colon \mathfrak{L}_9/\mathfrak{L}_8 \cong \oplus^{2^4} C^*(H^{\mathbb{Z}}_3),$$

and

$$\begin{cases} \mathfrak{L}_8/\mathfrak{L}_7 \cong \oplus^{2^2+2^2} C_0(\mathbb{T} \setminus \{\pm 1\}) \rtimes_{\hat{\alpha}} H^{\mathbb{Z}}_3, \\ \mathfrak{L}_7/\mathfrak{L}_6 \cong \oplus^{2^2+2^2} C_0(\mathbb{T} \setminus \{\pm 1\}) \rtimes_{\hat{\beta}} H^{\mathbb{Z}}_3, \\ \mathfrak{L}_6/\mathfrak{L}_5 \cong \oplus^4 C_0((\mathbb{T} \setminus \{\pm 1\})^2) \rtimes_{\hat{\alpha}} H^{\mathbb{Z}}_3, \\ \mathfrak{L}_5/\mathfrak{L}_4 \cong \oplus^4 C_0((\mathbb{T} \setminus \{\pm 1\})^2) \rtimes_{\hat{\beta}} H^{\mathbb{Z}}_3, \\ \mathfrak{L}_4/\mathfrak{L}_3 \cong \oplus^4 C_0((\mathbb{T} \setminus \{\pm 1\}) \times (\mathbb{T} \setminus \{\pm 1\})) \rtimes_{(\hat{\alpha},\hat{\beta})} H^{\mathbb{Z}}_3, \\ \mathfrak{L}_3/\mathfrak{L}_2 \cong \oplus^2 C_0((\mathbb{T} \setminus \{\pm 1\})^2 \times (\mathbb{T} \setminus \{\pm 1\})) \rtimes_{(\hat{\alpha},\hat{\beta})} H^{\mathbb{Z}}_3, \\ \mathfrak{L}_2/\mathfrak{L}_1 \cong \oplus^2 C_0((\mathbb{T} \setminus \{\pm 1\}) \times (\mathbb{T} \setminus \{\pm 1\})^2) \rtimes_{(\hat{\alpha},\hat{\beta})} H^{\mathbb{Z}}_3, \\ \mathfrak{L}_1 \cong C_0((\mathbb{T} \setminus \{\pm 1\})^2 \times (\mathbb{T} \setminus \{\pm 1\})^2) \rtimes_{(\hat{\alpha},\hat{\beta})} H^{\mathbb{Z}}_3. \end{cases}$$

Moreover, it is obtained that

$$\begin{cases} C_0((\mathbb{T}\setminus\{\pm1\})^2) \rtimes_{\hat{\alpha}} H_3^{\mathbb{Z}} \\ \qquad \cong \Gamma(\mathbb{T},\, \{C_0(\mathbb{R}^2)\otimes(\mathbb{C}^4\otimes C(\mathbb{T}))\rtimes_{(\bar{z},\hat{\gamma})}\mathbb{Z}\}_{z\in\mathbb{T}}), \\ C_0((\mathbb{T}\setminus\{\pm1\})^2) \rtimes_{\hat{\beta}} H_3^{\mathbb{Z}} \\ \qquad \cong \Gamma(\mathbb{T},\, \{C_0(\mathbb{R}^2)\otimes(\mathbb{C}^4\otimes C(\mathbb{T}))\rtimes_{(w,\hat{\gamma})}\mathbb{Z}\}_{w\in\mathbb{T}}), \\ C_0((\mathbb{T}\setminus\{\pm1\})^2\times(\mathbb{T}\setminus\{\pm1\})) \rtimes_{(\hat{\alpha},\hat{\beta})} H_3^{\mathbb{Z}} \\ \qquad \cong \Gamma(\mathbb{T},\{C_0(\mathbb{R}^3)\otimes((\mathbb{C}^4\rtimes_{\hat{\alpha}}\mathbb{Z})\otimes\mathbb{C}^2)\rtimes_{(z,\hat{\gamma})\otimes\hat{\beta}}\mathbb{Z}\}_{z\in\mathbb{Z}}), \\ C_0((\mathbb{T}\setminus\{\pm1\})\times(\mathbb{T}\setminus\{\pm1\})^2) \rtimes_{(\hat{\alpha},\hat{\beta})} H_3^{\mathbb{Z}} \\ \qquad \cong \Gamma(\mathbb{T},\, \{C_0(\mathbb{R}^3)\otimes((\mathbb{C}^2\rtimes_{\hat{\alpha}}\mathbb{Z})\otimes\mathbb{C}^4)\rtimes_{(z,\hat{\gamma})\otimes\hat{\beta}}\mathbb{Z}\}_{z\in\mathbb{Z}}), \\ C_0((\mathbb{T}\setminus\{\pm1\})^2\times(\mathbb{T}\setminus\{\pm1\})^2) \rtimes_{(\hat{\alpha},\hat{\beta})} H_3^{\mathbb{Z}} \\ \qquad \cong \Gamma(\mathbb{T},\, \{C_0(\mathbb{R}^4)\otimes((\mathbb{C}^4\rtimes_{\hat{\alpha}}\mathbb{Z})\otimes\mathbb{C}^4)\rtimes_{(z,\hat{\gamma})\otimes\hat{\beta}}\mathbb{Z}\}_{z\in\mathbb{Z}}), \end{cases}$$

where $\mathbb{C}^4 = C(\{\pm i,\, \pm i\})$, *and the actions* $(z,\, \hat{\gamma})$ *correspond to the restrictions to* $\{z\}\times\mathbb{T}$ *of* $C(\mathbb{T}^2)\rtimes_{\hat{\gamma}}\mathbb{Z}\cong C^*(H_3^{\mathbb{Z}})$.

Remark 4.5 Compare the algebraic structure of $C^*(D_7^{\mathbb{Z}})$ cited above with that of $C^*(L_7^{\mathbb{Z}})$ in Theorem 2.1. We can also define $D_{6n+1}^{\mathbb{Z}}$ by the same way as $L_{6n+1}^{\mathbb{Z}}$, and construct a finite composition series of $C^*(D_{6n+1}^{\mathbb{Z}})$ as given in Theorems 2.3 and 4.4, but omit the details.

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