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Singular integrals associated to homogeneous mappings with rough kernels

H. AL-QASSEM, A. AL-SALMAN and Y. PAN

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Abstract. In this paper, we study the L^p mapping properties of singular integral operators related to homogeneous mappings with kernels belonging to certain block spaces. An example is presented to show that our condition on the kernel is nearly optimal.

Key words: singular integrals, oscillatory integrals, Fourier transform, L^p boundedness, rough kernels, block spaces.

1. Introduction and results

Let $\mathbf{R}^n$, $n \geq 2$ be the n-dimensional Euclidean space and $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbf{R}^n$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(\cdot)$. Let $\Omega(x')|x|^{-n}$ be a homogeneous function of degree $-n$, and satisfy the cancellation condition

$$\int_{\mathbf{S}^{n-1}} \Omega(x')\, d\sigma(x') = 0, \tag{1.1}$$

where $x' = x/|x| \in \mathbf{S}^{n-1}$ for any $x \neq 0$.

The Calderón-Zygmund singular integral operator T_Ω is defined by

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbf{R}^n} \Omega(y')|y|^{-n} f(x-y)\, dy \tag{1.2}$$

and the corresponding maximal truncated singular integral operator T_Ω^* by

$$T_\Omega^* f(x) = \sup_{\varepsilon>0} \left| \int_{|y|>\varepsilon} \Omega(y')|y|^{-n} f(x-y)\, dy \right| \tag{1.3}$$

where $y' = y/|y|$ and $f \in \mathcal{S}(\mathbf{R}^n)$.

In their celebrated paper [CZ], Calderón and Zygmund introduced the method of rotation and showed that the operators T_Ω and T_Ω^* are bounded on L^p for $1 < p < \infty$ if $\Omega \in L\log^+ L(\mathbf{S}^{n-1})$. Furthermore, in the same paper [CZ], it was shown that the condition $\Omega \in L\log^+ L(\mathbf{S}^{n-1})$ is essentially the

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weakest possible size condition on Ω for the L^p boundedness of T_Ω and T_Ω^* to hold. Subsequently, the result of Calderón-Zygmund was improved by Connett ([Co]) and Coifman-Weiss ([CW]) who proved independently that the L^p boundedness of T_Ω and T_Ω^* continue to hold if $\Omega \in H^1(\mathbf{S}^{n-1})$. Here, $H^1(\mathbf{S}^{n-1})$ denotes the Hardy space on the unit sphere $\mathbf{S}^{n-1}$ in the sense of Coifman and Weiss [CW] and it contains $L\log^+ L(\mathbf{S}^{n-1})$ as a proper subspace.

For a suitable mapping $\Psi\colon \mathbf{R}^n \to \mathbf{R}^m$, define the singular integral operator $T_{\Omega,\Psi}$ and its truncated maximal operator $T^*_{\Omega,\Psi}$ by

$$T_{\Omega,\Psi}f(x)=\text{p.v.}\int_{\mathbf{R}^n} f(x-\Psi(y))\frac{\Omega(y')}{|y|^n}\,dy \tag{1.4}$$

$$T^*_{\Omega,\Psi}f(x)=\sup_{\varepsilon>0}\left|\int_{|y|>\varepsilon} f(x-\Psi(y))\frac{\Omega(y')}{|y|^n}\,dy\right| \tag{1.5}$$

for $f \in \mathcal{S}(\mathbf{R}^m)$.

Clearly, by specializing into the case $m = n$, $\Psi = I = \mathrm{id}_{\mathbf{R}^n\to\mathbf{R}^n}$, one obtains the classical Calderón-Zygmund operators $T_{\Omega,I} = T_\Omega$ and $T^*_{\Omega,I} = T^*_\Omega$.

For $d = (d_1, \ldots, d_m) \in \mathbf{R}^m$, define the family of dilations $\{\delta_t\}_{t>0}$ on $\mathbf{R}^m$ by

$$\delta_t(x_1, \ldots, x_m) = (t^{d_1}x_1, \ldots, t^{d_m}x_m).$$

We say that a mapping $\Psi\colon \mathbf{R}^n \to \mathbf{R}^m$ is homogeneous of degree d if

$$\Psi(tx) = \delta_t(\Psi(x))$$

holds for all $x \in \mathbf{R}^n\backslash\{0\}$ and $t > 0$.

Very recently, Leslie Cheng studied the L^p boundedness of singular integrals related to homogeneous mappings with $\Omega \in H^1(\mathbf{S}^{n-1})$. The following is the main result in [Ch].

Theorem A *Let $T_{\Omega,\Psi}$ and Ω be given as in* (1.1) *and* (1.4). *Let $\Psi\colon \mathbf{R}^n \to \mathbf{R}^m$ be a homogeneous mapping of degree $d = (d_1, \ldots, d_m)$ with $d_j \neq 0$ for $1 \le j \le m$. Suppose that $\Omega \in H^1(\mathbf{S}^{n-1})$ and $\Psi \mid \mathbf{S}^{n-1}$ is real-analytic. Then for $1 < p < \infty$ there exists a constant $C_p > 0$ such that*

$$\|T_{\Omega,\Psi}(f)\|_{L^p(\mathbf{R}^m)} \le C_p\|\Omega\|_{H^1(\mathbf{S}^{n-1})}\|f\|_{L^p(\mathbf{R}^m)}$$

for any $f \in L^p(\mathbf{R}^m)$.

Theorem A was first proved by Fan-Guo-Pan in [FGP] in the special case $m = n+1$, $\Psi(y) = (y, \phi(y))$, $\phi|\mathbf{S}^{n-1}$ is real-analytic and $d = (1, \dots 1, h)$ with $h \neq 0$.

On the other hand, Jiang and Lu introduced a special class of block spaces $B_q^{\kappa, v}(\mathbf{S}^{n-1})$ with respect to the study of the mapping properties of singular integral operators T_Ω (see [LTW]). In fact, they obtained the following L^2 boundedness result.

Theorem B ([LTW]) *Let Ω, T_Ω and T_Ω^* be given as in* (1.1)-(1.3). *Then we have*

(i) *if $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1})$, T_Ω is a bounded operator on $L^2(\mathbf{R}^n)$;*

(ii) *if $\Omega \in B_q^{0,1}(\mathbf{S}^{n-1})$, T_Ω^* is a bounded operator on $L^2(\mathbf{R}^n)$.*

It is noteworthy that the L^p boundedness of the operators T_Ω and T_Ω^* were known to hold for all $p \in (1, \infty)$ under the condition $B_q^{0,0}(\mathbf{S}^{n-1})$ (see for example, [AqP, AqAs, AlH, AlHF]).

The definition of block space $B_q^{\kappa, v}(\mathbf{S}^{n-1})$ will be recalled in Section 2.

The main purpose of this paper is to establish the L^p boundedness of the more general class of operators $T_{\Omega,\Psi}$ and $T_{\Omega,\Psi}^*$ under the condition $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1})$ and under the same conditions on Ψ as stated in Theorem A. Furthermore, we shall show that the condition imposed on $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1})$ is nearly optimal. In fact, at the end of this paper we present an example which shows that the L^2 boundedness of T_Ω may fail despite having $\Omega \in B_q^{0,v}(\mathbf{S}^{n-1})$ for any $-1 < v < 0$.

Our main theorem is the following:

Theorem C *Let $T_{\Omega,\Psi}$, $T_{\Omega,\Psi}^*$ and Ω be given as in* (1.1) *and* (1.4)-(1.5). *Let $\Psi\colon \mathbf{R}^n \to \mathbf{R}^m$ be a homogeneousmapping of degree $d = (d_1, \dots, d_m)$ with $d_j \neq 0$ for $1 \leq j \leq m$. Suppose that $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1})$ with $q > 1$ and $\Psi|\mathbf{S}^{n-1}$ is real-analytic. Then for, $1 < p < \infty$, there exists a constant $C_p > 0$ such that*

$$\|T_{\Omega,\Psi}(f)\|_{L^p(\mathbf{R}^m)} \leq C_p \|\Omega\|_{B_q^{0,0}(\mathbf{S}^{n-1})} \|f\|_{L^p(\mathbf{R}^m)}; \tag{1.6}$$

$$\|T_{\Omega,\Psi}^*(f)\|_{L^p(\mathbf{R}^m)} \leq C_p \|\Omega\|_{B_q^{0,0}(\mathbf{S}^{n-1})} \|f\|_{L^p(\mathbf{R}^m)} \tag{1.7}$$

for any $f \in L^p(\mathbf{R}^m)$.

As a consequence of Theorem C one can easily obtain the following L^p boundedness result of the oscillatory singular integral operator S_λ defined by

$$S_\lambda f(x) = \text{p.v.} \int_{\mathbf{R}^n} e^{i\lambda\cdot\Psi(x-y)} K(x-y) f(y)\, dy,$$

where $\Psi\colon \mathbf{R}^n \to \mathbf{R}^m$ is a mapping and $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbf{R}^m$. In fact, we have the following:

Theorem D *Let $K(x) = \Omega(x)|x|^{-n}$ where $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1})$. Let $\Psi\colon \mathbf{R}^n \to \mathbf{R}^m$ be a homogeneous mapping of degree $d = (d_1, \ldots, d_m)$ with $d_j \neq 0$ for $1 \leq j \leq m$. Then the operator S_λ is bounded from $L^p(\mathbf{R}^n)$ to itself for $1 < p < \infty$. The bound for the operator norm is independent of $\lambda_1, \ldots, \lambda_m$.*

Throughout the rest of the paper the letter C will stand for a positive constant but not necessarily the same one in each occurrence.

2. Some Definitions

We start by giving the following definition.

Definition 2.1 (1) For $x'_0 \in \mathbf{S}^{n-1}$ and $0 < \theta_0 \leq 2$, the set

$$B(x'_0, \theta_0) = \{x' \in \mathbf{S}^{n-1}\colon |x' - x'_0| < \theta_0\}$$

is called a cap on $\mathbf{S}^{n-1}$.
(2) For $1 < q \leq \infty$, a measurable function b is called a q-block on $\mathbf{S}^{n-1}$ if b is a function supported on some cap $I = B(x'_0, \theta_0)$ with $\|b\|_{L^q} \leq |I|^{-1/q'}$ where $|I| = \sigma(I)$ and $1/q + 1/q' = 1$.
(3) $B_q^{\kappa,\upsilon}(\mathbf{S}^{n-1}) = \{\Omega \in L^1(\mathbf{S}^{n-1})\colon \Omega = \sum_{\mu=1}^{\infty} c_\mu b_\mu$ where each c_μ is a complex number; each b_μ is a q-block supported on a cap I_μ on $\mathbf{S}^{n-1}$; and $M_q^{\kappa,\upsilon}(\{c_k\}, \{I_k\}) = \sum_{\mu=1}^{\infty} |c_\mu|(1 + \phi_{\kappa,\upsilon}(|I_\mu|)) < \infty\}$, where

$$\phi_{\kappa,\upsilon}(t) = \chi_{(0,1)}(t) \int_t^1 u^{-1-\kappa} \log^{\upsilon}(u^{-1})\, du. \tag{2.1}$$

The definition of $B_q^{\kappa,\upsilon}([a, b])$, $a, b \in \mathbf{R}$ will be the same as that of $B_q^{\kappa,\upsilon}(\mathbf{S}^{n-1})$ except for minor modifications. One observes that

$$\phi_{\kappa,\upsilon}(t) \sim t^{-\kappa} \log^{\upsilon}(t^{-1}) \quad \text{as } t \to 0 \text{ for } \kappa > 0,\ \upsilon \in \mathbf{R},$$

and

$$\phi_{0,\upsilon}(t) \sim \log^{\upsilon+1}(t^{-1}) \quad \text{as } t \to 0 \text{ for } \upsilon > -1.$$

The following properties of $B_q^{\kappa,\upsilon}$ can be found in [KS]:

(i) $B_q^{\kappa,\upsilon_2} \subset B_q^{\kappa,\upsilon_1}$ if $\upsilon_2 > \upsilon_1 > -1$ and $\kappa \geq 0$; (2.2)

(ii) $B_q^{\kappa_2,\upsilon_2} \subset B_q^{\kappa_1,\upsilon_1}$ if $\upsilon_1, \upsilon_2 > -1$ and $0 \leq \kappa_1 < \kappa_2$; (2.3)

(iii) $B_{q_2}^{\kappa,\upsilon} \subset B_{q_1}^{\kappa,\upsilon}$ if $1 < q_1 < q_2$; (2.4)

(iv) $L^q(\mathbf{S}^{n-1}) \subset B_q^{\kappa,\upsilon}(\mathbf{S}^{n-1})$ for $\upsilon > -1$ and $\kappa \geq 0$. (2.5)

In their investigations of block spaces, Keitoku and Sato showed in [KS] that these spaces enjoy the following properties:

Lemma 2.2 (i) *If $1 < p \leq q \leq \infty$, then for $\kappa > 1/p'$ we have*

$$B_q^{\kappa,\upsilon}(\mathbf{S}^{n-1}) \subseteq L^p(\mathbf{S}^{n-1}) \quad \text{for any } \upsilon > -1;$$

(ii)

$$B_q^{\kappa,\upsilon}(\mathbf{S}^{n-1}) = L^q(\mathbf{S}^{n-1}) \quad \text{if and only if } \kappa \geq 1/q' \text{ and } \upsilon \geq 0;$$

(iii) *for any $\upsilon > -1$, we have*

$$\bigcup_{q>1} B_q^{0,\upsilon}(\mathbf{S}^{n-1}) \supsetneqq \bigcup_{q>1} L^q(\mathbf{S}^{n-1}).$$

Definition 2.3 For a suitable mapping $\Psi\colon \mathbf{R}^n \backslash \{0\} \to \mathbf{R}^m$, $\rho \in [2, \infty)$, and a suitable function $\tilde{b}(\cdot)$ on $\mathbf{S}^{n-1}$ we define the measures $\{\lambda_{\tilde{b},\Psi,k,\rho} \colon k \in \mathbf{Z}\}$ and the corresponding maximal operator $\lambda^*_{\tilde{b},\Psi,\rho}$ on $\mathbf{R}^m$ by

$$\int_{\mathbf{R}^m} f \, d\lambda_{\tilde{b},\Psi,k,\rho} = \int_{\rho^k \leq |y| < \rho^{k+1}} f(\Psi(y)) \frac{\tilde{b}(y')}{|y|^n} \, dy$$

$$\lambda^*_{\tilde{b},\Psi,\rho} f(x) = \sup_{k \in \mathbf{Z}} \left| |\lambda_{\tilde{b},\Psi,k,\rho}| * f(x) \right|.$$

3. Some Technical Lemmas

We shall begin by recalling the following two lemmas due to Ricci and Stein.

Lemma 3.1 ([RS2]) *Let $\gamma(t) = (a_1 t^{q_1}, \ldots, a_s t^{q_s})$ where $a_j, q_j \in \mathbf{R}$ for $1 \leq j \leq s$. Let $\mathcal{M}_\gamma$ be the maximal operator defined on $\mathbf{R}^s$ by*

$$\mathcal{M}_\gamma f(x) = \sup_{R>0} \frac{1}{R} \left| \int_0^R f(x - \gamma(t)) \, dt \right|$$

for $x \in \mathbf{R}^s$. *Then, for* $1 < p \leq \infty$, *there exists a constant* $C_p > 0$ *such that*

$$\|\mathcal{M}_\gamma f\|_{L^p(\mathbf{R}^s)} \leq C_p \|f\|_{L^p(\mathbf{R}^s)}$$

for all f *in* $L^p(\mathbf{R}^s)$. *The constant* C_p *is independent of* a_j *for all* $1 \leq j \leq s$.

Let $\Phi\colon \mathbf{R}^+ \to \mathbf{R}$ be a generalized polynomial defined by

$$\Phi(t) = t^{a_1} + \mu_2 t^{a_2} + \cdots + \mu_m t^{a_m} \tag{3.1}$$

where $\mu_2, \ldots, \mu_m$ are real parameters and $a_1, \ldots, a_m$ are real numbers.

Lemma 3.2 ([RS1]) *Let* $\psi \in C^1[0, 1]$ *and* Φ *be given by* (3.1) *with* $a_1, \ldots,$ a_m *are distinct positive (not necessarily integers) exponents. If*

$$I(\lambda) = \int_\alpha^\beta e^{i\lambda\Phi(t)}\psi(t)dt,$$

then

$$|I(\lambda)| \leq C|\lambda|^{-\varepsilon}\left\{\sup_{\alpha\leq t\leq\beta}|\psi(t)| + \int_\alpha^\beta |\psi'(t)|\,dt\right\},$$

where $\lambda \in \mathbf{R}\backslash\{0\}$, $\varepsilon = \min\{1/a_1, 1/m\}$ *and* C *does not depend on* $\mu_2, \ldots,$ μ_m *as long as* $0 \leq \alpha < \beta \leq 1$.

By Lemma 3.2 and the change of variable $t \to 1/t$ we immediately get the following:

Lemma 3.3 *Let* $\psi \in C^1[1, 2]$ *and* Φ *be given by* (3.1) *with* $a_1, \ldots, a_m$ *are distinct negative (not necessarily integers) exponents. If*

$$I(\lambda) = \int_\alpha^\beta e^{i\lambda\Phi(t)}\psi(t)\,dt, \quad 1 \leq \alpha < \beta \leq 2,$$

then

$$|I(\lambda)| \leq C|\lambda|^{-\delta}\left\{\sup_{\alpha\leq t\leq\beta}|\varphi(t)| + \int_\alpha^\beta |\varphi'(t)|\,dt\right\}, \quad \lambda \neq 0,$$

where $\lambda \in \mathbf{R}\backslash\{0\}$, $\delta = \min\{-1/a_1, 1/m\}$, $\varphi(t) = t^{-2}\psi(1/t)$ *and* C *does not depend on* $\mu_2, \ldots, \mu_m$.

By an argument which is similar to the proof of Lemma 3 in [RS1] we get the following:

Lemma 3.4 *Let $\psi \in C^1([1/2, 1])$ and*

$$\Lambda(t) = t^{a_1} + \mu_2 t^{a_2} + \cdots + \mu_k t^{a_k} + \mu_{k+1} t^{-a_{k+1}} + \cdots + \mu_m t^{-a_m}$$

where $\mu_2, \ldots, \mu_m$ are real parameters and $a_1, \ldots, a_m$ are distinct positive exponents. Let

$$I(\lambda) = \int_\alpha^\beta e^{i\lambda\Lambda(t)} \psi(t)\, dt,$$

$\lambda \in \mathbf{R}\backslash\{0\}$ and $1/2 < \alpha < \beta \leq 1$. Then

$$|I(\lambda)| \leq C|\lambda|^{-\varepsilon} \left\{ \sup_{\alpha \leq t \leq \beta} |\psi(t)| + \int_\alpha^\beta |\psi'(t)|\, dt \right\},$$

with $\varepsilon = \min\{1/a_1, 1/m\}$, where C does not depend on $\mu_2, \ldots, \mu_m$ and λ.

By using Malgrange Preparation Theorem ([Ho]), the compactness of $\mathbf{S}^{n-1}$ and $\mathbf{S}^{m-1}$, and the arguments in the proof of Theorem 3.1 in [FGP], we get the following:

Lemma 3.5 *Let $n, m \in \mathbf{N}$ and $F\colon \mathbf{S}^{n-1} \times S^{m-1} \to \mathbf{R}$ be such that for each $\eta \in S^{m-1}$, $F(\,\cdot\,, \eta)$ is a nonconstant real-valued analytic function on $\mathbf{S}^{n-1}$. Then there exist positive constants δ and A such that for each $\eta \in S^{m-1}$, there exist open subsets $U_1, \ldots, U_{l(\eta)}$ of $\mathbf{S}^{n-1}$ which cover $\mathbf{S}^{n-1}$ such that*

$$\sup_{y \in U} \int_U |F(x, \eta) - F(y, \eta)|^{-\delta}\, d\sigma(x) \leq A \tag{3.2}$$

for $U \in \{U_1, \ldots, U_{l(\eta)}\}$.

The following result follows directly from Lemmas 3.3-3.5 in [AqP] which is an extension of a result of Duoandikoetxea and Rubio de Francia in [DR] (see also [FP]).

Lemma 3.6 *Let $N \in \mathbf{N}$ and $\{\sigma_k^{(l)}\colon k \in \mathbf{Z}, 0 \leq l \leq N\}$ be a family of Borel measures on $\mathbf{R}^n$ with $\sigma_k^{(N)} = 0$ for every $k \in \mathbf{Z}$. Let $\{a_l\colon 0 \leq l \leq N-1\} \subseteq [2, \infty)$, $\{m_l\colon 0 \leq l \leq N-1\} \subseteq \mathbf{N}$, $\{\alpha_l\colon 0 \leq l \leq N-1\} \subseteq \mathbf{R}^+$, and let $L_l\colon \mathbf{R}^n \to \mathbf{R}^{m_l}$ be linear transformations for $0 \leq l \leq N-1$. Suppose that for all $k \in \mathbf{Z}$, $0 \leq l \leq N-1$, for all $\xi \in \mathbf{R}^n$ and for some $C > 0$, $A > 1$ we have*

(i) $\|\sigma_k^{(l)}\| \leq CA$;

(ii) $|\hat{\sigma}_k^{(l)}(\xi)| \leq CA|a_l^{kA}L_l(\xi)|^{-\alpha_l/A}$;
(iii) $|\hat{\sigma}_k^{(l)}(\xi) - \hat{\sigma}_k^{(l+1)}(\xi)| \leq CA|a_l^{kA}L_l(\xi)|^{\alpha_l/A}$.

Assume that

$$\left\|\sigma^{*(l)}f\right\|_p \leq C_pA\|f\|_p \tag{3.3}$$

for $1 < p < \infty$ *and for every* $f \in L^p(\mathbf{R}^n)$ *where* $\sigma^{*(l)}(f) = \sup_{k\in\mathbf{Z}}\left||\sigma_k^{(l)}| * f\right|$, $0 \leq l \leq N-1$.

Then for every $1 < p < \infty$ *there exists a constant* $C_p > 0$ *which is independent of the linear transformations* $\{L_l\}$ *such that*

$$\left\|\sum_{k\in\mathbf{Z}}\sigma_k^{(0)} * f\right\|_p \leq C_pA\|f\|_p; \tag{3.4}$$

$$\left\|\sup_{k\in\mathbf{Z}}\left|\sum_{j=k}^{\infty}\sigma_j^{(0)} * f(x)\right|\right\|_p \leq C_pA\|f\|_p \tag{3.5}$$

for every $f \in L^p(\mathbf{R}^n)$.

Lemma 3.7 *Let* $m \in \mathbf{N}$, *let* $\tilde{b}(\,\cdot\,)$ *be a function on* $\mathbf{S}^{n-1}$ *satisfying the following conditions*: (i) $\|\tilde{b}\|_{L^q(\mathbf{S}^{n-1})} \leq |I|^{-1/q'}$ *for some* $q > 1$ *and for some cap* I *on* $\mathbf{S}^{n-1}$; (ii) $\|\tilde{b}\|_{L^1(\mathbf{S}^{n-1})} \leq 1$. *Let* $\Psi\colon \mathbf{R}^n \to \mathbf{R}^m$ *be a homogeneous mapping of degree* $d = (d_1, \ldots, d_m)$ *with* $d_j > 0$ *for* $1 \leq j \leq m$. *Assume that* $\Psi|\mathbf{S}^{n-1}$ *is real-analytic and that there are* $s_1, \tilde{s}_1 \in N$ *such that* $s_1 \leq \tilde{s}_1 \leq m$, $\{j\colon 1 \leq j \leq m$ *and* $d_j = d_1\} = \{1, \ldots, \tilde{s}_1\}$ *and* $\{\Psi_1, \ldots, \Psi_{s_1}\}$ *forms a basis for* $\operatorname{span}\{\Psi_1, \ldots, \Psi_{\tilde{s}_1}\}$. *Then there exist* α, $C > 0$ *and a linear transformation* $L\colon \mathbf{R}^{\tilde{s}_1} \to \mathbf{R}^{s_1}$ *such that*

$$|\hat{\lambda}_{\tilde{b},\Psi,k,\rho}(\xi)| \leq C(\log|I|^{-1})\left|\rho^{kd_1}L(\Pi_{\tilde{s}_1}\xi)\right|^{-\alpha/\log(|I|^{-1})} \tag{3.6}$$

if $\rho = 2^{\log(|I|^{-1})}$ *and* $|I| < e^{-1}$, *whereas*

$$\left|\hat{\lambda}_{\tilde{b},\Psi,k,\rho}(\xi)\right| \leq C\left|\rho^{kd_1}L(\Pi_{\tilde{s}_1}\xi)\right|^{-\alpha} \quad \text{if } \rho = 2 \text{ and } |I| \geq e^{-1} \tag{3.7}$$

for all $\xi = (\xi_1, \ldots, \xi_m) \in \mathbf{R}^m$ *where* $\Pi_{\tilde{s}_1}\xi = (\xi_1, \ldots, \xi_{\tilde{s}_1})$.

Proof. We shall only prove (3.6) and the proof of (3.7) will be easier. Let $\xi = (\xi_1, \ldots, \xi_m) \in \mathbf{R}^m$ be arbitrary but fixed. By assumption there exists a linear transformation $L = (L_1, \ldots, L_{s_1})\colon \mathbf{R}^{\tilde{s}_1} \to \mathbf{R}^{s_1}$ such that

$$\sum_{j=1}^{\tilde{s}_1} \xi_j \Psi_j(y) = \sum_{j=1}^{s_1} L_j(\Pi_{\tilde{s}_1}\xi)\Psi_j(y).$$

By the definition of $\lambda_{\tilde{b},\,\Psi,\,k,\,\rho}$ we have

$$\hat{\lambda}_{\tilde{b},\,\Psi,\,k,\,\rho}(\xi) = \int_{\mathbf{S}^{n-1}} \tilde{b}(y) \int_{1/\rho}^{1} e^{-iH_{\xi,\,k}(t,\,y)} \frac{dt}{t}\, d\sigma(y)$$

where

$$H_{\xi,\,k}(t,\,y) = \Bigg(\sum_{j=1}^{\tilde{s}_1} \xi_j \Psi_j(y)\Bigg) t^{d_1} \rho^{(k+1)d_1} + \cdots + \xi_m \Psi_m(y) t^{d_m} \rho^{(k+1)d_m}.$$

For $L(\Pi_{\tilde{s}_1}\xi) = (L_1(\Pi_{\tilde{s}_1}\xi),\, \ldots,\, L_{s_1}(\Pi_{\tilde{s}_1}\xi)) \neq 0$, write

$$\sum_{j=1}^{\tilde{s}_1} \xi_j \Psi_j(y) = |L(\Pi_{\tilde{s}_1}\xi)| F(y,\,\eta)$$

where $\eta = L(\Pi_{\tilde{s}_1}\xi)/|L(\Pi_{\tilde{s}_1}\xi)| \in \mathbf{S}^{s_1-1}$ and

$$F(y,\,\eta) = \eta \cdot (\Psi_1(y),\, \ldots,\, \Psi_{s_1}(y)).$$

We need to consider two cases:

Case 1: $F(y,\,\eta)$ is a nonzero constant function.

In this case, by Lemma 3.2,

$$\big|\hat{\lambda}_{\tilde{b},\,\Psi,\,k,\,\rho}(\xi)\big| \le C\|\tilde{b}\|_{L^1(\mathbf{S}^{n-1})} |L(\Pi_{\tilde{s}_1}\xi)\rho^{(k+1)d_1}|^{-\varepsilon} \tag{3.8}$$

where $\varepsilon = \min\{1/d_1,\, 1/m\}$. By combining (3.8) with the trivial estimate $|\hat{\lambda}_{\tilde{b},\,\Psi,\,k,\,\rho}(\xi)| \le (\log 2)\log(|I|^{-1})$ we get

$$\big|\hat{\lambda}_{\tilde{b},\,\Psi,\,k,\,\rho}(\xi)\big| \le C\log(|I|^{-1}) |L(\Pi_{\tilde{s}_1}\xi)\rho^{kd_1}|^{-\varepsilon/\log(|I|^{-1})}.$$

Case 2: $F(y,\,\eta)$ is a non-constant function.

Let A, δ, $U_1,\, \ldots,\, U_{l(\eta)}$ be as in Lemma 3.5. Construct in the usual way a smooth partition of unity

$$\sum_{U \in \{U_1,\ldots,U_{l(\eta)}\}} h_U(y) \equiv 1 \quad \text{for } y \in \mathbf{S}^{n-1}$$

with $\operatorname{supp}(h_U) \subseteq U$. Then

$$\hat{\lambda}_{\tilde{b},\Psi,k,\rho}(\xi) = \sum_{U\in\{U_1,\dots,U_{l(\eta)}\}} I_U(\xi) \tag{3.9}$$

where

$$I_U(\xi) = \int_{1/\rho}^1 \int_{\mathbf{S}^{n-1}} \tilde{b}(y) e^{-iG_k(\xi,t,y,\eta)} h_U(y)\, d\sigma(y) \frac{dt}{t}$$

and

$$G_k(\xi, t, y, \eta) = |L(\Pi_{\tilde{s}_1}\xi)| F(y, \eta) t^{d_1} \rho^{(k+1)d_1} + \cdots + \xi_m \Psi_m(y) t^{d_m} \rho^{(k+1)d_m}.$$

Now

$$|I_U(\xi)|^2 \le C \log(|I|^{-1}) \times \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} \int_{1/\rho}^1 e^{-i\mathcal{F}_{k,\xi}(x,y,t)} h_U(y) h_U(x) \tilde{b}(y) \overline{\tilde{b}(x)} \frac{dt}{t}\, d\sigma(y) d\sigma(x),$$

where

$$\mathcal{F}_{k,\xi}(x, y, t) = (F(y, \eta) - F(x, \eta)) |L(\Pi_{\tilde{s}_1}\xi)| t^{d_1} \rho^{(k+1)d_1} + \cdots + t^{d_m} \rho^{(k+1)d_m} \xi_m (\Psi_m(y) - \Psi_m(x)).$$

Let

$$\Gamma_{k,\xi}(x, y) = \int_{1/\rho}^1 e^{-i\mathcal{F}_{k,\xi}(x,y,t)} \frac{dt}{t}.$$

By Lemma 3.2 we have

$$|\Gamma_{k,\xi}(x, y)| \le C |L(\Pi_{\tilde{s}_1}\xi) \rho^{(k+1)d_1} (F(y, \eta) - F(x, \eta))|^{-\varepsilon} \quad \text{with} \quad \varepsilon = \min\left\{\frac{1}{d_1}, \frac{1}{m}\right\}.$$

Let $\delta^* = \min\{\varepsilon, \delta\}$. Since $|\Gamma k, \xi(x, y)| \le (\log 2) \log(|I|^{-1})$ we immediately get that

$$|\Gamma_{k,\xi}(x, y)| \le C \log(|I|^{-1}) |L(\Pi_{\tilde{s}_1}\xi) \rho^{(k+1)d_1} (F(y, \eta) - F(x, \eta))|^{-\delta^*/q'}.$$

Thus

$$|I_U(\xi)| \le C|L(\Pi_{\tilde{s}_1}\xi)\rho^{(k+1)d_1}|^{-\delta^*/2q'}\|\tilde{b}\|_q \times \left\{\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} \frac{(h_U(y)h_U(x))^{q'}}{|F(y,\eta)-F(x,\eta)|^{\delta^*}}\,d\sigma(y)d\sigma(x)\right\}^{1/2q'}. \quad (3.10)$$

Hence by (3.9)-(3.10), Lemma 3.5 and the assumption (i) on $\tilde{b}$ we have

$$\left|\hat{\lambda}_{\tilde{b},\Psi,k,\rho}(\xi)\right| \le C|I|^{-1/q'}|L(\Pi_{\tilde{s}_1}\xi)\rho^{(k+1)d_1}|^{-\delta^*/2q'}. \quad (3.11)$$

By interpolation between this estimate and the trivial estimate

$$\left|\hat{\lambda}_{\tilde{b},\Psi,k,\rho}(\xi)\right| \le C\log(|I|^{-1})$$

we

$$\left|\hat{\lambda}_{\tilde{b},\Psi,k,\rho}(\xi)\right| \le C(\log|I|^{-1})|\rho^{kd_1}L(\Pi_{\tilde{s}_1}\xi)|^{-\delta^*/\{2q'\log(|I|^{-1})\}}. \quad (3.12)$$

This completes the proof of our lemma. □

4. Proof of Theorem C

By assumption Ω can be written as $\Omega = \sum_{\mu=1}^{\infty} c_\mu b_\mu$ where $c_\mu \in \mathbf{C}$, b_μ is a q-block with support on a cap I_μ on $\mathbf{S}^{n-1}$ and

$$M_q^{0,0}(\{c_\mu\}) = \sum_{\mu=1}^{\infty} |c_\mu|\big(1+(\log|I_\mu|^{-1})\big) < \infty. \quad (4.1)$$

Also, by assumption $\Psi = (\Psi_1, \ldots, \Psi_m)\colon \mathbf{R}^n \to \mathbf{R}^m$ is a homogeneous mapping of degree $d = (d_1, \ldots, d_m)$ with $d_j \ne 0$ for $1 \le j \le m$ and $\Psi|\mathbf{S}^{n-1}$ is real-analytic. In view of Lemmas 3.2-3.4, we shall only prove our theorem for the case $d_1, \ldots, d_m > 0$. The argument for the case that some or all of the d_j's are negative is similar and requires only minor modifications. Also, by a simple reordering of the mappings $\Psi_1, \ldots, \Psi_m$ we may assume that there are $s_1, \tilde{s}_1 \in \mathbf{N}$ such that $s_1 \le \tilde{s}_1 \le m$, $\{j\colon 1 \le j \le m \text{ and } d_j = d_1\} = \{1, \ldots, \tilde{s}_1\}$ and $\{\Psi_1, \ldots, \Psi_{s_1}\}$ forms a basis for span$\{\Psi_1, \ldots, \Psi_{\tilde{s}_1}\}$.

To each block function $b_\mu(\cdot)$, let $\tilde{b}_\mu(\cdot)$ be a function defined by

$$\tilde{b}_\mu(x) = b_\mu(x) - \int_{\mathbf{S}^{n-1}} b_\mu(u)\,d\sigma(u). \quad (4.2)$$

Then one can easily verify that $\tilde{b}_\mu$ enjoys the following properties:

$$\int_{\mathbf{S}^{n-1}} \tilde{b}_\mu(u)\, d\sigma(u) = 0, \tag{4.3}$$

$$\|\tilde{b}_\mu\|_{L^q} \leq 2|I_\mu|^{-1/q'}, \tag{4.4}$$

$$\|\tilde{b}_\mu\|_{L^1} \leq 2. \tag{4.5}$$

Let $A = \{\mu \in \mathbf{N}\colon |I_\mu| \geq e^{-1}\}$ and $B = \{\mu \in \mathbf{N}\colon |I_\mu| < e^{-1}\}$. For $\mu \in \mathbf{N}$, we set

$$\rho_\mu = \begin{cases} 2 & , \text{ if } \mu \in A \\ 2^{\log(|I_\mu|^{-1})} & , \text{ if } \mu \in B. \end{cases}$$

Using the assumption that Ω has the mean zero property (1.1), and the definition of $\tilde{b}_\mu$, we deduce that Ω can be written as

$$\Omega = \sum_{\mu=1}^{\infty} c_\mu \tilde{b}_\mu$$

which in turn gives

$$T_{\Omega,\,\Psi}(f) = \sum_{\mu=1}^{\infty} c_\mu T_{\tilde{b}_\mu,\,\Psi}(f) \tag{4.6}$$

$$T^*_{\Omega,\,\Psi}(f) \leq \sum_{\mu=1} |c_\mu| T^*_{\tilde{b}_\mu,\,\Psi}(f). \tag{4.7}$$

Let $\Gamma_0 = \Psi$, $\Gamma_1 = (0, \ldots, 0, \Psi_{\tilde{s}_1+1}, \ldots, \Psi_m)$, $L_0(\xi) = L(\Pi_{\tilde{s}_1}\xi)$ for $\xi \in \mathbf{R}^m$, and $\lambda^{(l)}_{\tilde{b},\,k,\,\rho_\mu} = \lambda_{\tilde{b},\,\Gamma_l,\,k,\,\rho_\mu}$ for $l = 0, 1$. By invoking (4.4)-(4.5) and Lemma 3.7 we get

$$\left|\lambda^{(l)}_{\tilde{b},\,k,\,\rho_\mu}(\xi)\right| \leq C\delta_\mu \quad \text{for } l = 0, 1; \tag{4.8}$$

$$\left|\hat{\lambda}^{(0)}_{\tilde{b},\,k,\,\rho_\mu}(\xi)\right| \leq C\delta_\mu \left|\rho_\mu^{kd_1} L_0(\xi)\right|^{-\alpha_0/\delta_\mu} \tag{4.9}$$

where

$$\delta_\mu = \begin{cases} 1 & , \text{ if } \mu \in A \\ \log(|I_\mu|^{-1}) & , \text{ if } \mu \in B. \end{cases}$$

Also,

$$\left|\hat{\lambda}^{(0)}_{\tilde{b},\,k,\,\rho_\mu}(\xi) - \hat{\lambda}^{(1)}_{\tilde{b},\,k,\,\rho_\mu}(\xi)\right| \leq C\rho_\mu |\rho_\mu^{(k+1)d_1} L_0(\xi)|.$$

By using the inequality $|\hat{\lambda}^{(0)}_{\tilde{b},k,\rho_\mu}(\xi)-\hat{\lambda}^{(1)}_{\tilde{b},k,\rho_\mu}(\xi)| \leq C\delta_\mu$ if necessary we obtain

$$|\hat{\lambda}^{(0)}_{\tilde{b},k,\rho_\mu}(\xi) - \hat{\lambda}^{(1)}_{\tilde{b},k,\rho_\mu}(\xi)| \leq C\delta_\mu |\rho_\mu^{kd_1} L_0(\xi)|^{\alpha_0/\delta_\mu}. \tag{4.10}$$

Similarly, by using (4.3)-(4.5) we can find additional mappings $\Gamma_2, \ldots, \Gamma_N$ from $\mathbf{R}^n \backslash \{0\}$ to $\mathbf{R}^m$, $\{\alpha_l \colon 1 \leq l \leq N-1\} \subset (0,\infty)$, appropriate linear transformations $\{L_l \colon 1 \leq l \leq N-1\}$ and a finite family of measures $\{\lambda^{(l)}_{\tilde{b},k,\rho_\mu} \colon 2 \leq l \leq N\}$ with the following properties:

$$\Gamma_N = (0,\ldots,0), \quad \lambda^{(l)}_{\tilde{b},k,\rho_\mu} = \lambda_{\tilde{b},\Gamma_l,k,\rho_\mu} \quad \text{for } 2 \leq l \leq N \tag{4.11}$$

$$\lambda^{(N)}_{\tilde{b},k,\rho_\mu} = 0, \quad |\lambda^{(l)}_{\tilde{b},k,\rho_\mu}(\xi)| \leq C\delta_\mu; \quad \text{for } 2 \leq l \leq N-1 \tag{4.12}$$

$$|\hat{\lambda}^{(l)}_{\tilde{b},k,\rho_\mu}(\xi)| \leq C\delta_\mu |\rho_\mu^{kd_l} L_l(\xi)|^{-\alpha_l/\delta_\mu}, \quad \text{for } 2 \leq l \leq N-1 \tag{4.13}$$

$$|\hat{\lambda}^{(l)}_{\tilde{b},k,\rho_\mu}(\xi) - \hat{\lambda}^{(l+1)}_{\tilde{b},k,\rho_\mu}(\xi)| \leq C\delta_\mu |\rho_\mu^{kd_l} L_l(\xi)|^{\alpha_l/\delta_\mu}, \quad \text{for } 2 \leq l \leq N-1. \tag{4.14}$$

By (4.5) and Lemma 3.1 we immediately get

$$\|\lambda^{(s)*}_{\tilde{b}_\mu,\rho_\mu}(f)\|_p \leq C_p \delta_\mu \|f\|_p \quad \text{for } p \in (1,\infty) \tag{4.15}$$

where

$$\lambda^{(s)*}_{\tilde{b}_\mu,\rho_\mu}(f) = \sup_{k\in\mathbf{Z}} \big| |\lambda^{(s)}_{\tilde{b},k,\rho_\mu}| * f \big| \quad \text{and } 0 \leq s \leq N-1.$$

By (4.8)-(4.15) and Lemma 3.6 we have

$$\big\|T_{\tilde{b}_\mu,\Psi} f\big\|_p = \Big\| \sum_{k\in\mathbf{Z}} \lambda^{(0)}_{\tilde{b}_\mu,k,\rho_\mu} * f \Big\|_p \leq C_p \delta_\mu \|f\|_p \tag{4.16}$$

and

$$\Big\| \sup_{k\in\mathbf{Z}} \Big| \sum_{j=k}^{\infty} \lambda^{(0)}_{\tilde{b}_\mu,j,\rho_\mu} * f \Big| \Big\|_p \leq C_p \delta_\mu \|f\|_p \tag{4.17}$$

for $p \in (1,\infty)$. Since

$$|T^*_{\tilde{b}_\mu,\Psi} f(x)| \leq \sup_{k\in\mathbf{Z}} \Big| \sum_{j=k}^{\infty} \lambda^{(0)}_{\tilde{b}_\mu,j,\rho_\mu} * f(x) \Big| + \lambda^{(0)*}_{\tilde{b}_\mu,\rho_\mu} f(x),$$

by (4.17) we obtain

$$\|T^*_{\tilde{b}_\mu,\Psi}\|_p \le C_p\delta_\mu\|f\|_p \tag{4.18}$$

for every $p \in (1, \infty)$. Hence (1.6)-(1.7) follow by (4.1), (4.6)-(4.7), (4.16) and (4.18). This concludes the proof of Theorem C. □

5. A Counterexample

In this section, we shall give an example to show that the L^2 boundedness of T_Ω may fail if the condition $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1})$ is replaced by the weaker condition $\Omega \in B_q^{0,v}(\mathbf{S}^{n-1})$ for any v, $-1 < v < 0$.

Notice that $\widehat{T_\Omega f}(\xi) = m(\xi)\hat{f}(\xi)$ where

$$m(\xi) = \int_0^\infty \int_{\mathbf{S}^{n-1}} \Omega(\theta)e^{-ir\xi\cdot\theta}\, d\sigma(\theta)\frac{dr}{r}.$$

It is well-known that

$$m(\xi) = \int_{\mathbf{S}^{n-1}} \Omega(\theta)\left[\frac{\pi i}{2}\mathrm{sgn}(\theta\cdot\xi') + \log(|\theta\cdot\xi'|^{-1})\right] d\sigma(\theta) \tag{5.1}$$

and the convolution operator T_Ω is a bounded operator from $L^2(\mathbf{R}^n)$ to itself if and only if the multiplier $m \in L^\infty(\mathbf{R}^n)$.

Before presenting our example, we shall need some notations and also we need to prove some simple results on block spaces.

Let $N_q^{0,v}(\Omega) = \inf\{M_q^{0,v}(\{c_k\}, \{I_k\})\colon \Omega = \sum_{k=1}^\infty c_k b_k$ and each b_k is a q-block function supported on a interval $I_k\}$.

Then we have the following lemma:

Lemma 5.1 *For any $v > -1$, $a, b \in \mathbf{R}$,*

(i) *$N_q^{0,v}$ is a norm on $B_q^{0,v}([a, b])$ and $(B_q^{0,v}([a, b]), N_q^{0,v})$ is a Banach space;*

(ii) *If $f \in B_q^{0,v}([a, b])$ and g is a measurable on $[a, b]$ with $|g| \le |f|$, then $g \in B_q^{0,v}([a, b])$ with*

$$N_q^{0,v}(g) \le N_q^{0,v}(f);$$

(iii) *Let I_1 and I_2 be two disjoint intervals in $[a, b]$ with $|I_1|, |I_2| < 1$ and $\alpha_1, \alpha_2 \in \mathbf{R}^+$. Then*

$$N_q^{0,v}(\alpha_1\chi_{I_1} + \alpha_2\chi_{I_2}) \ge N_q^{0,v}(\alpha_1\chi_{I_1}) + N_q^{0,v}(\alpha_2\chi_{I_2});$$

(iv) *Let I be an interval in $[a, b]$ with $|I| < 1$. Then*

$$N_q^{0,v}(\chi_I) \geq |I|(1 + \log^{v+1}(|I|^{-1})).$$

Proof. The proof of (i) is straightforward while the proof of (iii) follows from the same arguments as in the proof of (2.11) in [LTW]. Next, we turn to the proof of (iii). First, notice that if

$$\alpha_1\chi_{I_1} + \alpha_2\chi_{I_2} = \sum_{k=1}^{\infty} c_k b_k \tag{5.2}$$

where each b_k is a q-block function supported on a interval I_k in $[a, b]$ and $M_q^{0,v}(\{c_k\}, \{I_k\}) < \infty$, then

$$\alpha_1\chi_{I_1} + \alpha_2\chi_{I_2} = \sum_{k=1}^{\infty} c_k(\chi_{I_1 \cup I_2} b_k).$$

This immediately implies

$$\begin{aligned} &N_q^{0,v}(\alpha_1\chi_{I_1} + \alpha_2\chi_{I_2}) \\ &= \inf\Big\{ M_q^{0,v}(\{c_k\}, \{I_k\}) \colon \alpha_1\chi_{I_1} + \alpha_2\chi_{I_2} = \sum_{k=1}^{\infty} c_k b_k \\ &\qquad \text{and each } b_k \text{ is a } q\text{-block function supported} \\ &\qquad \text{on a interval } I_k \subset I_1 \cup I_2 \Big\}. \end{aligned}$$

Let $\varepsilon > 0$. Then

$$N_q^{0,v}(\alpha_1\chi_{I_1} + \alpha_2\chi_{I_2}) \geq M_q^{0,v}(\{c_k\}, \{I_k\}) - \varepsilon$$

for some sequences $\{c_k\}$, $\{I_k\}$ with

$$\alpha_1\chi_{I_1} + \alpha_2\chi_{I_2} = \sum_{k=1}^{\infty} c_k b_k$$

where each b_k is a q-block function supported on a interval $I_k \subset I_1 \cup I_2$ and $M_q^{0,v}(\{c_k\}, \{I_k\}) < \infty$. Since I_1 and I_2 are two disjoint intervals and I_k is an interval subset of $I_1 \cup I_2$ we have either $I_k \subset I_1$ or $I_k \subset I_2$. Let $A = \{k \in \mathbf{N} \colon I_k \subset I_1\}$ and $B = \{k \in \mathbf{N} \colon I_k \subset I_2\}$. By (5.2) and since I_1 and I_2 are disjoint we get

$$\alpha_1\chi_{I_1} = \sum_{k\in A}^{\infty} c_k b_k \quad \text{and} \quad \alpha_2\chi_{I_2} = \sum_{k\in B}^{\infty} c_k b_k. \tag{5.3}$$

Since $A \cup B = \mathbf{N}$, we have

$$\begin{aligned} N_q^{0,v}(\alpha_1\chi_{I_1}) + N_q^{0,v}(\alpha_2\chi_{I_2}) &\le M_q^{0,v}(\{c_k\}, \{I_k\}) \\ &\le N_q^{0,v}(\alpha_1\chi_{I_1} + \alpha_2\chi_{I_2}) + \varepsilon \end{aligned}$$

which in turn ends the proof of (iii).

Finally, we prove (iv). By the same argument as in the proof of (iii) we have $N_q^{0,v}(\chi_I) = \inf\{M_q^{0,v}(\{c_k\}, \{I_k\}) \colon \chi_I = \sum_{k=1}^{\infty} c_k b_k$ and each b_k is a q-block function supported on a interval $I_k \subset I\}$. Let $\varepsilon > 0$. Then

$$N_q^{0,v}(\chi_I) \ge M_q^{0,v}(\{c_k\}, \{I_k\}) - \varepsilon$$

for some sequences $\{c_k\}$, $\{I_k\}$ with $\chi_I = \sum_{k=1}^{\infty} c_k b_k$ where each b_k is a q-block function supported on a interval $I_k \subset I$ and $M_q^{0,v}(\{c_k\}, \{I_k\}) < \infty$. Since $\|b_k\|_{L^1} \le 1$, we have $|I| \le \sum_{k=1}^{\infty} |c_k|$. The last inequality along with the relation $I_k \subset I$ implies the desired inequality in (iv). This concludes the proof of the lemma. □

Let us now give our example. For simplicity, we shall present our example only in the case $n = 2$ where $\mathbf{S}^1$is identified with $[-1, 1]$. Let $q > 1$ be fixed and for $u \in [-1, 1]$,

$$\Omega(u) = \sum_{k=1}^{\infty} C_k b_k(u) \tag{5.4}$$

where

$$\begin{aligned} C_1 &= \sum_{k=2}^{\infty} \frac{k}{k^{q'}(k^{q'}+1)(\log k)^2}, \; b_1(u) = -\chi_{[-1,0]}(u), \\ C_k &= \frac{k|I_k|^{1/q'}}{(\log k)^2}, \; b_k(u) = |I_k|^{-1/q'}\chi_{I_k}(u) \\ &\qquad \text{and} \quad I_k = \left(\frac{1}{k^{q'}+1}, \frac{1}{k^{q'}}\right) \quad \text{for} \quad k \ge 2. \end{aligned}$$

Then Ω has the desired properties. In fact, Ω satisfies the following:

$$\int_{-1}^{1} \Omega(u)du = 0; \tag{5.5}$$

$$\Omega \in B_q^{0,v}([-1, 1]); \tag{5.6}$$

$$\int_0^1 |\Omega(u)\log(|u|^{-1})|\, du = \infty; \tag{5.7}$$

$$\int_{-1}^0 |\Omega(u)\log(|u|^{-1})|\, du < \infty; \tag{5.8}$$

$$\Omega \notin B_q^{0,0}([-1, 1]). \tag{5.9}$$

The proof of (5.5)-(5.8) is straightforward. However, the proof of (5.9) will rely heavily on Lemma 5.1. We first notice that each b_k is a q-block supported in an interval I_k. So we only need to show that $N_q^{0,0}(\Omega) = \infty$. To this end, by Lemma 5.1 we have for each m,

$$\begin{aligned} N_q^{0,0}(\Omega + C_1\chi_{[-1,0]}) &\geq \sum_{k=2}^{m} |C_k||I_k|^{-1/q'} N_q^{0,0}(\chi_{I_k}) \\ &\geq \sum_{k=2}^{m} |C_k||I_k|^{1/q}(1+\log(|I_k|^{-1})). \end{aligned}$$

Letting $m \to \infty$, we get $N_q^{0,0}(\Omega + C_1\chi_{[-1,0]}) = \infty$. Since, $N_q^{0,0}(C_1\chi_{[-1,0]}) < \infty$ we get $N_q^{0,0}(\Omega) = \infty$.

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H. Al-Qassem
Department of Mathematics
Yarmouk University
Irbid-Jordan
E-mail: husseink@yu.edu.jo

A. Al-Salman
Department of Mathematics
Yarmouk University
Irbid-Jordan
E-mail: alsalman@yu.edu.jo

Y. Pan
Department of Mathematics
University of Pittsburgh
Pittsburgh, PA 15260
U.S.A.
E-mail: yibiao@pitt.edu