Hokkaido Mathematical Journal Vol. 33 (2004) p. 539–549

On the excesses of sequences of complex exponentials

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(Received August 26, 2002; Revised April 24, 2003)

Abstract. We derive an equation for the excess in $L^2(-\pi, \pi)$ of a sequence of complex exponentials $\{e^{i\lambda_n t}\}_{-\infty}^{\infty}$ with $|\lambda_n - n| \leq \Delta < \infty$, $\forall n$, consider examples, and study the stability of the excess under small perturbations of λ_n.

Key words: L^2-completeness, excess.

1. Introduction

Let $\Lambda = \{\lambda_n\}_{n=-\infty}^{\infty}$ be a sequence of complex numbers and

$$E(\Lambda) = \{e^{i\lambda_n t}\}_{-\infty}^{\infty}.$$

The system $E(\Lambda)$ is said to be *closed* in $L^2(a, b)$ if there is no nontrivial function $f \in L^2(a, b)$ orthogonal to all functions in $E(\Lambda)$. The system $E(\Lambda)$ is said to be *complete* in $L^2(a, b)$ if each function $f \in L^2(a, b)$ can be approximated by the functions in $E(\Lambda)$ in the $L^2(a, b)$-norm.
Otherwise, the system is called *incomplete* in $L^2(a, b)$. It is known [9, Section 11] that in L^2 these two properties are equivalent.

The system is called *minimal* in $L^2(a, b)$ if each element of the system lies outside the closed linear span of the others. A closed minimal system is called *exact*. A closed system that remains closed if $r \geq 0$ of its terms are removed, but fails to be closed after removing $r + 1$ terms, is said to have the *excess* $r = \operatorname{exc}(E(\Lambda)) = r(\Lambda)$. If a non-closed system becomes exact after adding $-r$, $r < 0$, elements, the system is said to have the *deficiency* (negative excess) r. Thus, the system is exact if and only if its excess $r = 0$. If the system remains closed after removing any finite number of its terms, then the excess $r = \infty$, and if the system cannot be made closed by adding any finite number of elements, then the excess $r = -\infty$.

Since "The terminology of the subject is not uniform..." (R. Young [12, p. 16, footnote]), it should be mentioned that in terminology we follow [9].

An extensive literature devoted to the study of closed, minimal, etc.,

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2000 Mathematics Subject Classification : 42C30, 30D15.

systems of complex exponentials can be traced back to the well-known monographs of R. Paley and N. Wiener [9] and N. Levinson [7] and the works mentioned there. Extensive lists of references can be found in the survey of R. Redheffer [10] and the monograph of R. Young [12]. Our work was inspired by recent papers of N. Fujii, A. Nakamura, and R. Redheffer [2], and A. Nakamura [8].

Since any interval (a, b) can be reduced to $(-\pi, \pi)$ by a linear change of variable, we can consider the latter interval without loss of generality. As in the sources cited, we consider only sequences Λ with all different λ_n, which are bounded shifts of the integers, that is,

$$\lambda_n = n + d_n, \quad |d_n| \le \Delta,\ n = 0,\ \pm 1,\ \pm 2,\ \dots, \tag{1}$$

where $\{d_n\}$ is a bounded sequence of complex numbers and the constant $\Delta < \infty$. Throughout the paper we use these 'standard' assumptions without repeating them.

Under the condition (1) the function

$$f_\Lambda(z) = \lim_{N\to\infty} \prod_{n=-N}^{N} \left(1 - \frac{z}{\lambda_n}\right) \tag{2}$$

is an entire function of the exponential type π whose properties are intrinsically connected with the closeness properties of the sequence Λ. If $\lambda_0 = 0$, (2) should be replaced by $\lim_{N\to\infty} z \prod_{n=-N,\, n\neq 0}^{N} (1 - z/\lambda_n)$. In particular [9], the system $E(\Lambda)$ is closed in $L^2(-\pi, \pi)$ if and only if $f_\Lambda \notin L^2(-\infty, \infty)$. The results in the references above are stated either in terms of the function f_Λ or in terms of the counting function of its zeros.

Let us denote the sequence of shifts by $D = \{d_n\}_{n=-\infty}^{\infty}$. To study properties of functions like (2), M. Krein and B. Levin [4], see also [5, Appendix VI] or [6, Section 22.2], introduced a special functional

$$\begin{aligned} L_{\omega,h}(\Lambda) &= \sum_{n=-\infty}^{\infty} \{d_{n+\omega} - d_n\} \frac{n}{n^2 + h^2} \\ &= \lim_{N\to\infty} \sum_{n=-N}^{N} \{d_{n+\omega} - d_n\} \frac{n}{n^2 + h^2}, \end{aligned}$$

where h is a fixed non-zero real number and ω is an integer parameter. We interchangeably use the symbols $L_{\omega,h}(\Lambda)$ and $L_{\omega,h}(D)$ as synonyms.

Later on the present author [3] considered a modified functional

$$L_\omega(D) = \sum_{n=-\infty,\, n\neq 0}^{\infty} \frac{d_{n+\omega} - d_n}{n}$$

and used it to study certain properties of the functions (2) and the completeness properties of the system $E(\Lambda)$. In this note we use the functional $L_\omega(D)$ to extend some results of [2] and [8] and, in particular, to calculate the excesses of some systems of complex exponentials.

As it can be immediately seen from the exposition, many of our results can be carried over to a more general setting, for the weighted spaces $L^p(-\pi, \pi)$ with $0 < p \leq \infty$, however, we do not pursue this subject here. $[x]$ everywhere stands for the integer part of x.

2. Main results

Our results are based on the following proposition which gives an exact control over the change in the functional $L_\omega(\Lambda)$ after removing one term from (or adding to) the sequence Λ.

Lemma 1 *Let the sequence* Λ *be as above and* $\Lambda_1 = \Lambda \setminus \{\lambda_{n_1}\}$ *with any fixed* $n_1 \in Z$. *Let* $D_1 = D \setminus \{d_{n_1}\}$. *Then*

$$L_\omega(D) - L_\omega(D_1) = \ln|\omega| + O(1), \quad \omega \to \pm\infty,$$

that is, the deletion of a term from the sequence Λ *leads to asymptotical decrease of* $L_\omega(D)$ *by* $\ln|\omega|$. *So that, the deletion of* k *terms decreases the functional* L_ω *by* $k\ln|\omega|$, *that is, if* $\Lambda_k = \Lambda \setminus \{\lambda_{n_1}, \lambda_{n_2}, \ldots, \lambda_{n_k}\}$, *then*

$$L_\omega(D) - L_\omega(D_k) = k\ln|\omega| + O(1), \quad \omega \to \pm\infty.$$

Hereafter, $O(1)$ stands for a quantity, depending maybe on Δ in (1), k above, or other parameters, and uniformly bounded as $\omega \to \pm\infty$.

Proof. It is enough to reassign each shift d_n with $n > n_1$ to the subscript $n - 1$, that is, to consider a new sequence of shifts $D_1 = \{\varphi_n\}$ with $\varphi_n = d_n$ for $n < n_1$ and $\varphi_n = 1 + d_{n+1}$ for $n \geq n_1$. Without loss of generality we can assume $n_1 = 0$. If $\omega > 0$, then

$$L_\omega(D) = \lim_{N\to\infty} \left(\sum_{n=-N+\omega,\, n\neq\omega}^{N+\omega} \frac{d_n}{n-\omega} - \sum_{n=-N,\, n\neq 0}^{N} \frac{d_n}{n} \right)$$

$$
\begin{aligned}
&= \lim_{N\to\infty}\Bigg(\sum_{n=-N}^{-\omega-1} \frac{\varphi_{\omega+n}}{n} + \sum_{n=-\omega,\, n\neq 0}^{N} \frac{\varphi_{\omega+n-1}-1}{n} \\
&\qquad - \sum_{n=-N}^{-1} \frac{\varphi_n}{n} - \sum_{n=1}^{N} \frac{\varphi_{n-1}-1}{n} \Bigg) + O(1) \\
&= \lim_{N\to\infty}\Bigg(\sum_{n=-N}^{-\omega-1} \frac{\varphi_{\omega+n}}{n} + \sum_{n=-\omega-1,\, n\neq -1}^{N-1} \frac{\varphi_{\omega+n}}{n+1} \\
&\qquad - \sum_{n=-\omega}^{-1} \frac{1}{n} - \sum_{n=-N}^{-1} \frac{\varphi_n}{n} - \sum_{n=0}^{N-1} \frac{\varphi_n}{n+1} \Bigg) + O(1) \\
&= L_\omega(D_1) + \ln|\omega| + O(1), \quad \omega \to \pm\infty.
\end{aligned}
$$

If $\omega < 0$, calculations are similar. □

We need also the following proposition [3], which involves only the real parts $\Re d_n$ of d_n-it is well known that their imaginary parts in no way affect the results under consideration as soon as d_n are uniformly bounded.

Lemma 2 *Under our assumptions, the system $E(\Lambda)$ is complete in $L^2(-\pi, \pi)$ if and only if*

$$\sum_{\omega=-\infty}^{\infty} \exp\{2L_\omega(\Re D)\} = \infty,$$

where $\Re D = \{\Re d_n\}_{n=-\infty}^{\infty}$.

The following statement immediately follows from these lemmas.

Theorem 1 *Under our standard assumptions, the system $E(\Lambda)$ has a finite (positive or negative) excess $\mathrm{exc}(E(\Lambda)) = r$ in $L^2(-\pi, \pi)$ if and only if*

$$\sum_{\omega=-\infty,\, \omega\neq 0}^{\infty} \frac{1}{|\omega|^{2r}} \exp\{2L_\omega(\Re D)\} = \infty \tag{3}$$

and

$$\sum_{\omega=-\infty,\, \omega\neq 0}^{\infty} \frac{1}{|\omega|^{2r+2}} \exp\{2L_\omega(\Re D)\} < \infty, \tag{4}$$

that is, the excess $\operatorname{exc}(E(\Lambda))$ *is the largest integer* $r = r(\Lambda)$ *such that the series* (3) *diverges.*

Corollary *If* $L_\omega(\Re D) = O(1)$ *as* $\omega \to \pm\infty$, *then* $\operatorname{exc}(E(\Lambda)) = 0$.

Theorem 1 is analogues to [10, Theorem 22], which determines the excess r by the following relations

$$\int_0^\infty \frac{|f_\Lambda(x)|^2\,dx}{(1+x^2)^r} = \infty \quad \text{and} \quad \int_0^\infty \frac{|f_\Lambda(x)|^2\,dx}{(1+x^2)^{r+1}} < \infty.$$

However, in (3)-(4) the excess is determined straightforwardly in terms of the sequence Λ or D rather than in terms of its associated function f_Λ.

Let us also notice that the functional $L_\omega(D)$ can be rewritten [3] as

$$L_\omega(D) = -\sum_{n=1}^{|\omega|} \frac{d_n - d_{-n}}{n} + \sum_{n=1}^{|\omega|} \frac{d_{\omega+n} - d_{\omega-n}}{n} + O(1), \quad \omega \to \pm\infty,$$

leading to the following modification of Theorem 1, which is sometimes easier to use in calculations.

Theorem 2 *In addition to our standard assumptions, let be*

$$\Re\left(\sum_{n=1}^{|\omega|} \frac{d_{\omega+n} - d_{\omega-n}}{n}\right) = O(1), \quad \omega \to \pm\infty.$$

Then the system $E(\Lambda)$ *has a finite (positive or negative) excess* $r = r(\Lambda)$ *in* $L^2(-\pi, \pi)$ *if and only if*

$$\sum_{\omega=-\infty,\,\omega\neq 0}^{\infty} \frac{1}{|\omega|^{2r}} \exp\left\{-2\sum_{n=1}^{|\omega|} \frac{\Re d_n - \Re d_{-n}}{n}\right\} = \infty \tag{5}$$

and

$$\sum_{\omega=-\infty,\,\omega\neq 0}^{\infty} \frac{1}{|\omega|^{2r+2}} \exp\left\{-2\sum_{n=1}^{|\omega|} \frac{\Re d_n - \Re d_{-n}}{n}\right\} < \infty. \tag{6}$$

For symmetric sequences, $d_{-n} = -d_n$, (5)-(6) reduce to

$$\sum_{\omega=1}^{\infty} \frac{1}{\omega^{2r}} \exp\left\{-4\sum_{n=1}^{\omega} \frac{\Re d_n}{n}\right\} = \infty,$$

and

$$\sum_{\omega=1}^{\infty} \frac{1}{\omega^{2r+2}} \exp\left\{-4\sum_{n=1}^{\omega} \frac{\Re d_n}{n}\right\} < \infty.$$

Example 1 First, we consider a classical example of a sequence with constant symmetric shifts $\lambda_n = n + d_n$, $d_n = \alpha \times \operatorname{sign}(n)$, where α is a real number and $\operatorname{sign}(\cdot)$ is the sign of a number. The assumptions of Theorem 2 are readily verified and the conditions (5)-(6) become

$$\sum_{\omega=1}^{\infty} \frac{1}{\omega^{2r+4\alpha}} = \infty \quad \text{and} \quad \sum_{\omega=1}^{\infty} \frac{1}{\omega^{2r+4\alpha+2}} < \infty.$$

Thus

$$r = r(\Lambda) = \left[\frac{1}{2} - 2\alpha\right]. \tag{7}$$

In particular, if $-1 \le \alpha \le 1$, (7) gives Theorem 1.1 from [8]. If $-1/4 < \alpha \le 1/4$, we get the known example of a system with the zero excess.

Next, we consider the stability of the excess (7) under small perturbations of α. The borderline case occurs when $1/2 - 2\alpha$ is integer. Denote $1/2 - 2\alpha = r(\Lambda) = k_0$, that is, $\alpha = (1 - 2k_0)/4$, and consider a sequence $\Lambda = \{\lambda_n\}_{n=-\infty}^{\infty}$ with

$$\lambda_n = n + \frac{1 - 2k_0}{4} \operatorname{sign}(n) + \beta_n,$$

where a complex-valued sequence $B = \{\beta_n\}_{n=-\infty}^{\infty}$ satisfies the condition $|\beta_n| \to 0$ as $n \to \pm\infty$. A natural question arises for what shifts β_n does the system $E(\Lambda)$ have the same excess $r = k_0$ as that of the unperturbed sequence $\{n + \big((1 - 2k_0)/4\big) \operatorname{sign}(n)\}$?

Theorem 3 *The system $E(\Lambda)$ with $\lambda_n = n + \big((1 - 2k_0)/4\big) \operatorname{sign}(n) + \beta_n$ has the excess $r(\Lambda) = k_0$ if and only if*

$$\sum_{\omega=-\infty,\, \omega\neq 0}^{\infty} \frac{1}{|\omega|} \exp\{2L_\omega(\Re\beta)\} = \infty,$$

but

$$\sum_{\omega=-\infty,\, \omega\neq 0}^{\infty} \frac{1}{|\omega|^3} \exp\{2L_\omega(\Re\beta)\} < \infty.$$

Proof. Calculations similar to those in Lemma 1 deduce

$$L_\omega(\Lambda) = L_\omega(B) + \left(k_0 - \frac{1}{2}\right)\ln|\omega| + O(1),$$

where $L_\omega(B) = \sum_{n=-\infty,\, n\neq 0}^{\infty}(\beta_{\omega+n} - \beta_n)/n$, and since the excess of the unperturbed sequence is k_0, Theorem 3 follows from Theorem 1. □

Example 2 Consider again $E(\Lambda)$ with $\lambda_n = n + \big((1-2k_0)/4\big)\operatorname{sign}(n) + \beta_n$; we know that if $\beta_n = 0$, then $r(\Lambda) = k_0$. Now, for $|n| \geq 2$ let be

$$\beta_n = \frac{b \times \operatorname{sign}(n)}{\ln^\gamma |n|}$$

and $\beta_0 = \beta_{\pm 1} = 0$, where $b > 0$ is a constant. Using Theorem 2, we immediately deduce that the system has the excess $r(\Lambda) = k_0$ whenever $\gamma > 1$. In particular, the system is complete if $k_0 \geq 0$ and $\gamma > 1$. However, if $\gamma = 1$, then

$$\left|\sum_{n=1}^{|\omega|}\frac{\beta_{\omega+n} - \beta_{\omega-n}}{n}\right| = O(1), \quad \omega \to \pm\infty. \tag{8}$$

So that,

$$L_\omega(B) = -2b\sum_{n=2}^{|\omega|}\frac{1}{n\ln n} + O(1) = -2b\ln\ln|\omega| + O(1),$$

and we arrive at the series

$$\sum_{\omega=-\infty,\,\omega\neq 0}^{\infty}\frac{1}{|\omega|}\exp\{2L_\omega(B)\} = 2\sum_{\omega=2}^{\infty}\frac{1}{|\omega|(\ln|\omega|)^{4b}} + O(1),$$

diverging if and only if $b \leq 1/4$. That is, this system has the excess $r(\Lambda) = k_0$ if $b \leq 1/4$, however, $r(\Lambda) = k_0 - 1$ if $b > 1/4$. In particular, if $k_0 = 0$, we get the case $p = 2$ of [11, Theorem 3].

If $0 < \gamma < 1$, (8) still holds true. However, now

$$\exp\{2L_\omega(B)\} \approx \exp\left\{-4b\sum_{2}^{|\omega|}\frac{1}{n\ln^\gamma n}\right\} \approx |\omega|^{-4b/\{(1-\gamma)(\ln|\omega|)^\gamma\}},$$

which implies that the system $E(\Lambda)$ with $\beta_n = b \times \operatorname{sign}(n) \times \ln^{-\gamma}|n|$ and

$0 < \gamma < 1$ has the excess $r(\Lambda) = k_0 - 1$ by Theorem 3.

Let $\Phi = \{\varphi_n\}_{n=-\infty}^{\infty}$. N. Fujii, A. Nakamura, and R. Redheffer [2] proved that if the sequence $\Lambda = \{\lambda_n\}_{n=-\infty}^{\infty}$ is defined by $\lambda_n = \varphi_n + a$ for $n \geq 0$ and $\lambda_n = \varphi_n - b$ for $n < 0$, where $a \geq 0$, $b \geq 0$, then $r(\Lambda) = \operatorname{exc}(E(\Lambda)) \leq r(\Phi) = \operatorname{exc}(E(\Phi))$. Our method allows to find $r(\Lambda)$ exactly.

Theorem 4 *If the sequence Φ satisfies the conditions of* Theorem 2 *and the sequence Λ is as in the preceding paragraph, then*

$$r(\Lambda) = \begin{cases} [r(\Phi) - (a+b)] + 1 & \textit{if} \quad a+b \textit{ is not integer} \\ r(\Phi) - (a+b) & \textit{if} \quad a+b \textit{ is integer.} \end{cases}$$

Proof. It follows from similar calculations that

$$L_\omega(\Re\Lambda) = L_\omega(\Re\Phi) - (a+b)\ln|\omega| + O(1),$$

and the conclusion follows. □

3. More general example

A. Nakamura [8] considered the sequence Λ with $\lambda_n = n + d_n$,

$$d_n = \begin{cases} 0 & \text{if } n = 0 \\ \alpha & \text{if } n = k \cdot l > 0 \\ \beta & \text{if } n = k \cdot l - j > 0, \quad 1 \leq j \leq k-1 \\ -d_{-n} & \text{if } n < 0, \end{cases}$$

where $-2 < \alpha$, $-1 < \beta$, and the integer $k \geq 2$, and calculated the excess of the corresponding sequence of exponentials, which is

$$r(\Lambda) = \left[\frac{1}{2} - \frac{2}{k}(\alpha + (k-1)\beta)\right]. \tag{9}$$

In this section we calculate the excess of a more general system of exponentials. Namely, we split all natural numbers into k infinite arithmetic progressions

$$N_j = \{j,\, j+k,\, j+2k,\, \ldots\}, \quad j = 1, 2, \ldots, k,$$

where $k \geq 1$ is a fixed natural number, and consider the system $E(\Lambda)$ with

$\lambda_n = n + d_n$, where

$$d_n = \begin{cases} 0 & \text{if } n = 0 \\ \alpha_j & \text{if } n \in N_j, \quad j = 1, 2, \ldots, k \\ -d_{-n} & \text{if } n < 0 \end{cases} \tag{10}$$

with any real α_j.

Theorem 5 *Under the conditions above*

$$r(\Lambda) = \left[\frac{1}{2} - \frac{2}{k}\sum_{j=1}^{k} \alpha_j\right]. \tag{11}$$

When $\alpha_1 = \alpha$ and $\alpha_2 = \cdots = \alpha_k = \beta$, (11) becomes (9), that is, we get [8, Theorem 2.1]. If $k = 1$, (11) implies (7).

Proof. By virtue of the same transformation as in Lemma 1, if the sequence D satisfies (10), then (we consider here $\omega > 0$, if $\omega < 0$, calculations are similar)

$$\begin{aligned} L_\omega(D) =& 2\omega^2 \sum_{j=1}^{k} \alpha_j \left\{ \sum_{l=0,\, j+kl \neq \pm\omega}^{\infty} \frac{1}{(j+kl)((j+kl)^2 - \omega^2)} \right\} \\ =& \frac{1}{k} \sum_{j=1}^{k} \alpha_j \left\{ \sum_{l=1}^{\infty} \left\{ \left(\frac{1}{l + (j-\omega)/k + i} - \frac{1}{l} \right) \right.\right. \\ & \left.\left. -2\left(\frac{1}{l + j/k} - \frac{1}{l} \right) + \left(\frac{1}{l + (j+\omega)/k} - \frac{1}{l} \right) \right\} \right\} + O(1). \end{aligned}$$

The imaginary unit i has been introduced in the first denominator in the last sum to avoid possible singularities at $\omega = j + kl$; this affects only a $O(1)$ term. We can sum up these series using the Euler ψ function, that is, the logarithmic derivative of the Γ function,

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{l=1}^{\infty} \frac{z}{l(l+z)}.$$

Thus,

$$L_\omega(D) = \frac{1}{k} \sum_{j=1}^{k} \alpha_j \left\{ 2\psi\left(\frac{j}{k}\right) - \psi\left(\frac{\omega + j}{k}\right) - \psi\left(\frac{j - \omega}{k} + i\right) \right\} + O(1)$$

$$=-\frac{1}{k}\sum_{j=1}^{k}\alpha_j\left\{\psi\left(\frac{\omega+j}{k}\right)+\psi\left(\frac{j-\omega}{k}+i\right)\right\}+O(1).$$

It is known [1, Section 1.7.1 (11)] that

$$\psi(-z)=\psi(z)+\pi\cot(\pi z)+\frac{1}{z}.$$

If $\Im z=-\pi$, then $0<\text{const}\le|\cot z|\le\text{const}<\infty$ uniformly in $\Re z$, whence

$$\psi\left(\frac{j-\omega}{k}+i\right)=\psi\left(\frac{\omega-j}{k}-i\right)+O(1),$$

and finally,

$$L_\omega(D)=-\frac{1}{k}\sum_{j=1}^{k}\alpha_j\left\{\psi\left(\frac{\omega+j}{k}\right)+\psi\left(\frac{\omega-j}{k}-i\right)\right\}+O(1).$$

By making use of the asymptotic formula [1, Section 1.7.2(27)]

$$\psi(z)=\ln z+O\left(\frac{1}{|z|}\right),\quad \text{where } |\arg z|<\pi,$$

we conclude that

$$L_\omega(D)=-\frac{2}{k}\left(\sum_{j=1}^{k}\alpha_j\right)\ln|\omega|+O(1)$$

and Theorem 5 follows from Theorem 1. □

Acknowledgment The author is thankful to the anonymous referee for the very thorough reading of the manuscript, in particular, for correcting an error in the initial formulation of Theorem 4.

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