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An upper bound for the total sum of the Baum-Bott indexes of a holomorphic foliation and the Poincaré Problem

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Abstract. An upper bound for the degree of an algebraic solution of a foliation in $\mathbb{CP}^2$ in terms of the degree of the foliation and some other local invariant is given. This result follows from the existence of an upper bound for the total sum of the Baum-Bott indexes of a holomorphic foliation in a compact algebraic surface.

Key words: holomorphic foliation, Indices of singularities, algebraic solutions.

1. Introduction

In [17] Poincaré established that, a differential equation of the first order and of the first degree in $\mathbb{C}^2$ is algebraically integrable as soon an upper bound for the possible degrees of the algebraic solutions of the equation in terms of the degree of the differential equation is established. In general it is not possible to find such upper bound as several simple examples show (see [9] or [21]). However the problem can be solved if some restriction to the singularities of the differential equation are given. In [9], for instance, a positive answer to this problem is given in the case that the singularities of the algebraic solution are ordinary nodes. Also, in [7], a positive answer is given in the case that the algebraic solution avoids the dicritical singularities of the differential equation. More generally, in [6] an answer is given if certain restrictions to the singularities of the foliation in the algebraic solution are given. From then several authors have been studied this problem in a more general setting [8], [1], [19].

The main purpose of this work is to show that an answer to the Poincaré problem can be given as a consequence of a more general fact: the existence of an upper bound for the total sum of the Baum-Bott indexes of the singularities of a foliation.

More precisely, let $\{a\}^+$ be the maximum between 0 and the real number a and denote the intersection number between two divisors, say D and

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C, by DC and by D^2 the self-intersection number of D. We shall prove:

Theorem 1.1 *Let $\mathcal{F}$ be a holomorphic foliation with isolated singularities in a compact algebraic surface X and let E be an effective divisor such that $S := \operatorname{Supp} E$ is an algebraic solution of $\mathcal{F}$. Assume that E has an irreducible component, say T, with $T^2 \geq 0$ and such that* $\mathrm{GSV}(\mathcal{F}, T) \leq ET - T^2$. *Then,*

$$\begin{aligned}&\mathrm{BB}(\mathcal{F})\\ &\leq S^2 + 2\,\mathrm{GSV}(\mathcal{F}, S) + \sum_{i=1}^{n}(\ell_i - 1)^2\big(S_i^2 + \{-\tilde{S}_i^{\,2}\}^+\big) + \Delta_\pi(\mathcal{F}, S)\\ &\quad + \sum_{i=1}^{n} 2(\ell_i - 1)\left\{\mathrm{GSV}(\mathcal{F}, S_i) + S_i^2 - ES_i + \sum_{j=i+1}^{n}(\ell_j - 1)S_jS_i\right\}^+,\end{aligned}$$

where $E = \ell_1 S_1 + \cdots + \ell_n S_n$ is the decomposition of E in its irreducible components. π is a map adapted to the pair $(\mathcal{F}, S)$ (Definition 3.3) *and $\tilde{S}_i$ are the strict transforms of S_i by π.*

Here and throughout this note $\mathrm{BB}(\mathcal{F})$ ($\mathrm{GSV}(\mathcal{F}, S)$, respectively) is the sum of the Baum-Bott's indexes (Gomez Mont-Seade-Verjovsky's indexes, resp.) of all singularities of $\mathcal{F}$ in X (in S, resp.). The symbol $\Delta_\pi(\mathcal{F}, S)$ stands for a positive integer that depends on the behavior of $\mathcal{F}$ along S (see Definition 3.2).

Any differential equation in $\mathbb{C}^2$ induces a foliation in $\mathbb{CP}^2$ and the problem raised by Poincaré in this context is to find an upper bound for the possible degrees of the algebraic solutions of the foliation in terms of the degree of the foliation (recall that the degree of a foliation in $\mathbb{CP}^2$ is defined as the number of tangencies of its leaves with a generic projective line (cf. [9])). From the above theorem we obtain,

Theorem 1.2 *Let $\mathcal{F}$ be a holomorphic foliation in $\mathbb{CP}^2$ of degree n and let S be an algebraic solution of $\mathcal{F}$ of degree k. Then, $k \leq n + 2 + \sqrt{\Delta_\pi(\mathcal{F}, S)}$.*

In particular we recover the upper bounds obtained in [9] and [7] (Corollaries 2.4 and 2.5).

This work is organized in the following form. In the next section we will recall the definition of Baum-Bott's indexes as well as Gomez Mont-Seade-Verjovsky's indexes and we will see how to obtain Theorem 1.2 from Theorem 1.1. In the last section we will prove Theorem 1.1.

2. The Poincaré's problem

This section is devoted to prove Theorem 1.2. Let us recall the definition of the Baum-Bott's indexes and Gomez Mont-Seade-Verjovsky's indexes.

Let X be a compact algebraic surface and $\mathcal{F}$ a holomorphic foliation with isolated singularities in X. Denote $\operatorname{Sing}(\mathcal{F})$ the set of singular points of $\mathcal{F}$.

2.1. Baum-Bott's index For each $p \in \operatorname{Sing}(\mathcal{F})$ denote by $\mathrm{BB}_p(\mathcal{F})$ the Baum-Bott index of $\mathcal{F}$ at p which is defined as

$$\mathrm{BB}_p(\mathcal{F}) := \left(\frac{1}{2\pi i}\right)^2 \int_\Gamma \frac{(\operatorname{Tr} J(x, y))^2}{P(x, y)Q(x, y)} dxdy,$$

where P and Q are holomorphic functions defined in a neighborhood of p with $\gcd(P, Q) = 1$ such that $\mathcal{F}$ is defined by the holomorphic 1-form

$$\omega = P(x, y)dy - Q(x, y)dx,$$

and where $J(x, y)$ is the Jacobian matrix of (P, Q),

$$\Gamma = \{(x, y) \mid |P(x, y)| = |Q(x, y)| = \epsilon\}$$

for sufficiently small number $\epsilon > 0$ and Γ is oriented so that the form $d(\arg P) \wedge d(\arg Q)$ is positive.

Let $N_{\mathcal{F}}$ be the normal bundle of $\mathcal{F}$ which is defined in the following way: let $\mathcal{U} = \{\mathcal{U}_i\}_{i\in I}$ be an open covering of X, if $\mathcal{F}$ is defined in each $\mathcal{U}_i$ by the holomorphic 1-form ω_i, there must exist $g_{ij} \in \mathcal{O}^*(\mathcal{U}_i \cap \mathcal{U}_j)$ such that $w_i = g_{ij} w_j$. Then $N_{\mathcal{F}}$ is the line bundle over X defined by the cocicle $\{g_{ij}\}_{ij} \in H^1(X, \mathcal{O}^*)$.

On the other hand, Baum-Bott formula asserts (cf. [2]),

$$\sum_{p\in\operatorname{Sing}(\mathcal{F})} \mathrm{BB}_p(\mathcal{F}) = c_1^2(TX - T_{\mathcal{F}}) \in H^4(X, \mathbb{Z}) \cong \mathbb{Z},$$

where $c_1^2(TX - T_{\mathcal{F}})$ denotes the first Chern number of the virtual bundle $TX - T_{\mathcal{F}}$, where TX and $T_{\mathcal{F}}$ denote the tangent bundles of X and $\mathcal{F}$, respectively. Note that $c_1(TX - T_{\mathcal{F}}) = c_1(N_{\mathcal{F}})$, since there is a relation $N_{\mathcal{F}}^* = K_X \otimes T_{\mathcal{F}}$, where $N_{\mathcal{F}}^*$ is the dual of $N_{\mathcal{F}}$ (cf. [3], [14] and [20, Ch. 5, Lemma 1.2]). Thus we can write,

(BB) $\mathrm{BB}(\mathcal{F}) := \sum_{p\in\operatorname{Sing}(\mathcal{F})} \mathrm{BB}_p(\mathcal{F}) = c_1^2(N_{\mathcal{F}})$.

2.2. Gomez Mont-Seade-Verjovsky's index Let S be an algebraic solution of $\mathcal{F}$ (that is, S is a reduced divisor in X such that $S \setminus \mathrm{Sing}(\mathcal{F})$ is a leaf of $\mathcal{F}$). Given a singularity $p \in \mathrm{Sing}(\mathcal{F}) \cap S$ in [12] it is introduced an index which is a kind of Poincaré-Hopf index for the restriction to S of the vector field that defines $\mathcal{F}$ in a neighborhood of p. We recall the algebraic formula for this index given in [11]: Let $p \in \mathrm{Sing}(\mathcal{F}) \cap S$ and f be a germ of holomorphic function that defines S in a neighborhood of p then,

$$\mathrm{GSV}_p(\mathcal{F}, S) = \dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^2,p}/\langle f, P, Q\rangle_p) - \dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^2,p}/\langle f, J_f\rangle_p),$$

where P and Q are as in 2.1 and J_f is the Jacobian ideal $J_f = (\partial_x f, \partial_y f)$.

This index has the particular importance that relates the restriction line bundle $N_{\mathcal{F}}|_S$ to the self-intersection number of S. In fact it was proved in [3] (and also in [15]) that

(GSV) $\quad \mathrm{GSV}(\mathcal{F}, S) := \sum_{p \in S \cap \mathrm{Sing}(\mathcal{F})} \mathrm{GSV}_p(\mathcal{F}, S) = c_1(N_{\mathcal{F}})S - S^2.$

Remark 2.3 In particular, in the case that $\mathcal{F}$ is a holomorphic foliation in $\mathbb{CP}^2$ of degree n it is well known that $N_{\mathcal{F}} = \mathcal{O}(n+2)$. Therefore, if T is an algebraic solution of degree t, by (GSV) we get $\mathrm{GSV}(\mathcal{F}, T) = (n+2-t)t$ and $\mathrm{BB}(\mathcal{F}) = (n+2)^2$ by (BB).

We are now able to prove Theorem 1.2 using Theorem 1.1.

Proof (*of* Theorem 1.2). Assume that $n+2 \leq k$. Then for any irreducible curve T in $\mathbb{CP}^2$ we have $\mathrm{GSV}(\mathcal{F}, T) \leq ET - T^2$ in view of Remark 2.3. Therefore, if we take $E := S$ in Theorem 1.1 and use Remark 2.3 we conclude that $(k-n-2)^2 \leq \Delta_\pi(\mathcal{F}, S)$. Hence the theorem follows. □

Theorem 1.2 must be compared with [6, Theorem 1] (see also [8]) where, from solving a problem of imposing singularities to a plane curve, it is obtained a bound for the degree of the curve. We observe that the terms that appear in the definition of Δ_π (Definition 3.2) are the same that the coefficients of the divisor R in [8]. Nevertheless we stress that in the above result we do not need to establish the condition of virtually passage of some effective divisor A by the cluster of points of the resolution of the curve as it is needed in [6] or [8] (see [8] or [6] for details). From the above theorem we recover the upper bounds obtained in [9] and [7].

Corollary 2.4 (Cerveau-Lins Neto, [9]) *Let $\mathcal{F}$ and S be as in* Theorem 1.2. *If S has only normal crossing singularities then, $k \leq n+2$.*

Recall that a singularity, q, is said non-dicritical if $\mathcal{F}$ has finite many separatrix through q.

Corollary 2.5 (Carnicer, [7]) *Let $\mathcal{F}$ and S be as in* Theorem 1.2. *If $\mathcal{F}$ has only non-dicritical singularities in S then, $k \leq n+2$.*

Proof. By [5, Theorem 3] we have the inequality $\mu_q + \varepsilon_q \leq \upsilon_q + 1$ at any infinitely near singularity of S, where μ_q, ε_q and υ_q are as in 3.1. Therefore $\Delta_\pi(\mathcal{F}, S) = 0$. □

Clearly Theorem 1.2 as well as the corollaries above hold (and with the same proof) if we replace $\mathbb{CP}^2$ with any compact algebraic surface X with Picard group $\mathbb{Z}$.
Note that in Theorem 1.2 we did not really have used Theorem 1.1 in all its strength; in fact in that result the ℓ_i's are always 1. We could have applied Theorem 1.1 in $\mathbb{CP}^2$ with general ℓ_i's to obtain other inequalities, however those inequalities do not seem to be of special interest. Nevertheless, for other algebraic surfaces, Theorem 1.1 gains interest if we realize that we can always obtain an upper bound for the GSV-indexes, as it is required in Theorem 1.1 if, for instance, the algebraic solution T has $T^2 > 0$ (just taking $E := \ell T$ with ℓ great enough). Also the required upper bound for the GSV-indexes holds if there exists another irreducible algebraic solution, say S, such that $S \cap T \neq \emptyset$ (just taking $E := T + \ell S$ with ℓ great enough).

2.6. We close this section with a result concerning the Tyurina number of a germ of plane curve singularity.

Given a germ of plane algebraic curve, say $(S;\, 0)$, its semi-universal deformation is a germ of a holomorphic deformation over $(\mathbb{C}^\tau; 0)$ such that (up to isomorphism) any deformation of $(S;\, 0)$, with parameter space $(C;\, 0)$, is obtained by pull-back via a holomorphic function from $(C;\, 0)$ to $(\mathbb{C}^\tau;\, 0)$. The number τ is called the Tyurina number of $(S;\, 0)$ and can be computed by the formula $\tau = \dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^2,0}/\langle f,\, J_f\rangle)$ where f is a germ of holomorphic function which defines $(S;\, 0)$ and J_f is the Jacobian ideal of f. Note that τ appears in the algebraic formula for GSV-index (see 2.2).

As was observed in [4] pag. 533 a lower bound for $\mathrm{GSV}(\mathcal{F},\, S)$ would readily imply an answer to the Poincaré problem. On the other hand, by the algebraic formula for GSV-index, to give an upper bound for the total Tyurina numbers is equivalent to give a lower bound for GSV. By Theorem

1.1 we can obtain an upper bound for the total Tyurina number of an algebraic curve in an algebraic surface (for related results in $\mathbb{CP}^2$ see [16, Theorem 3.2]).

Corollary 2.7 *Let $\mathcal{F}$ and X be as in* Theorem 1.1. *Let S be an algebraic solution of $\mathcal{F}$ and define $M := \sum_{p\in\mathrm{Sing}(\mathcal{F})\cap S}\dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^2,p}/\langle f, P, Q\rangle_p)$ with P, Q and f as in the algebraic formula for the GSV-index in* 2.2. *If any irreducible component of S has non negative self-intersection number then,*

$$\sum_{p\in\mathrm{Sing}(S)}\tau_p \le \max\left\{M-1,\, M+\frac{1}{2}\left(S^2-\mathrm{BB}(\mathcal{F})+\Delta_\pi(\mathcal{F}, S)\right)\right\},$$

where τ_p is the Tyurina number of (S, p).

Proof. Let us assume that $\sum_{p\in\mathrm{Sing}(S)}\tau_p \ge M$ then, by the algebraic formula for the GSV-index in 2.2, we see that $\mathrm{GSV}(\mathcal{F}, S)\le 0$. Thus $c_1(N_{\mathcal{F}})S \le S^2$ by (GSV). Hence $\sum_i(\mathrm{GSV}(\mathcal{F}, S_i)+S_i^2)\le S\cdot\sum_i S_i$ where the sum runs over all irreducible components, S_i, of S. Therefore there must exist an irreducible component of S, say T, such that $\mathrm{GSV}(\mathcal{F}, T)\le ST-T^2$. Hence $-\mathrm{GSV}(\mathcal{F}, S)\le 1/2(S^2-\mathrm{BB}(\mathcal{F})+\Delta_\pi(\mathcal{F}, S))$, by Theorem 1.1. Now the corollary follows by the algebraic formula for the GSV-index in 2.2. □

3. An upper bound for the total sum of the Baum-Bott indexes of a holomorphic foliation

As it was showed in the former section the result about the Poincaré's problem stated at the introduction is consequence of the existence of an upper bound for the total sum of the Baum-Bott indexes of $\mathcal{F}$. The aim of this section is to prove this upper bound.

Notation We shall use the standard notation in algebraic geometry. In particular, if D is a divisor in a compact algebraic surface X the symbol $\mathcal{O}_X(D)$ will denote the corresponding invertible sheaf induced by D in X and $\mathcal{O}_X(tD)$, or $\mathcal{O}(tD)$ if X is understood, will denote $\mathcal{O}_X(D)^{\otimes t}$. The intersection number between two divisors, say D and C, will be denoted by DC in particular D^2 will denote the self-intersection number of D. Let $\mathcal{L}$ be a holomorphic line bundle in X, if C is an effective divisor in X, then $\mathcal{O}_C(\mathcal{L})$ will denote the restriction $\mathcal{L}\otimes\mathcal{O}_C$. Also $h^0(\mathcal{L}) := \dim_{\mathbb{C}} H^0(X, \mathcal{L})$.

3.1. Infinitely near singularities Let $\pi\colon \tilde{X} = X_n \xrightarrow{\pi_n} X_{n-1} \to \cdots \xrightarrow{\pi_1} X_0 = X$ be a sequence of blow ups at points such that the first blow up is made at p and the blow up π_{k+1} is made at a point that lies in the exceptional curve of the first kind, say D_k, introduced in X_k by π_k. Let S ($\mathcal{F}$, respectively) be an algebraic curve (a foliation, resp.) in X, for each k and each $q \in X_k$ the symbol $\mu_{q,k}$ ($\upsilon_{q,k}$, resp.) or μ_q (υ_q, resp.) if X_k is understood will denotes the algebraic multiplicity in q of the strict transform of S ($\mathcal{F}$, resp.), say S_k, ($\mathcal{F}_k$, resp.) by $\pi_1 \circ \cdots \circ \pi_k$.
For each $q \in X_k$ set ε_q as the number of irreducible components of the exceptional divisor of $\pi_1 \circ \cdots \circ \pi_k$ which contains q and which are solutions of $\mathcal{F}_k$.

Definition 3.2 With the notations of 3.1. We call the irregularity of $\mathcal{F}$ along S to the positive integer $\Delta_\pi(\mathcal{F}, S) := \sum(\mu_q + \varepsilon_q - \upsilon_q - 1)^2$, where the sum runs over all infinitely near singularities of S, say q, with $\mu_q + \varepsilon_q \geq \upsilon_q + 2$.

Definition 3.3 Let $\pi\colon \tilde{X} \to X$ be a sequence of blow ups with center at infinitely near singularities of S such that the strict transform of S by π, say $\tilde{S}$, has the following properties:

1_ it is a disjoint union of smooth Riemann surfaces,

2_ all those irreducible components of the exceptional divisor of π which are algebraic solutions of the strict transform of $\mathcal{F}$ by π meets $\tilde{S}$ transversely.

A map π as above will be called a map adapted to the pair $(\mathcal{F}, S)$.

Definition 3.4 Let $\{C_i\}_{i=1,\dots,n}$ be an ordered set of n irreducible curves in X. For each $n-$tuple $\overline{\ell} := (\ell_1, \dots, \ell_n)$ of ordered non negative integers we define

$$\mathrm{vol}_{\overline{\ell}}(\mathcal{F}, C_1 + \cdots + C_n) := \limsup_{t \to +\infty} \frac{h^0([N^*_{\mathcal{F}} \otimes \mathcal{O}(\ell_1 C_1 + \cdots + \ell_n C_n)]^{\otimes t})}{t^2}.$$

Observe that in the definition of $\mathrm{vol}_{\overline{\ell}}(\mathcal{F}, *)$ we must take into account the order in which the irreducible curves, C_i, appears.

Proposition 3.5 *Let S be an algebraic solution of $\mathcal{F}$ and $\pi\colon \tilde{X} \to X$ a map adapted to the pair $(\mathcal{F}, S)$. If $S_1, \dots, S_n$ are the irreducible components of S then, for any n-tuple $\overline{\ell}$ of positive integers,*

$$\mathrm{vol}_{\overline{\ell}}(\mathcal{F},\, S_1+\cdots+S_n) \le \frac{1}{2}\Delta_\pi(\mathcal{F},\, S) + \frac{1}{2}\sum_{i=1}^{n}(\ell_i-1)^2\{-\tilde{S}_i^2\}^+$$
$$+\sum_{i=1}^{n}(\ell_i-1)\left\{-\,\mathrm{GSV}(\mathcal{F},\, S_i) - S_i^2 + ES_i - \sum_{j=i+1}^{n}(\ell_j-1)S_jS_i\right\}^+,$$

where $\tilde{S}_i$ are the strict transform of S_i by π and $E := \ell_1 S_1 + \cdots + \ell_n S_n$.

We postpone the proof of this result by a moment and we prove Theorem 1.1 using the above proposition.

Proof (*of* Theorem 1.1). We have $E = \ell_1 S_1 + \cdots + \ell_n S_n$ with ℓ_i positive integers and S_i irreducible. Since $\mathrm{GSV}(\mathcal{F},\, T) \le ET - T^2$ we obtain $c_1(N_{\mathcal{F}} \otimes \mathcal{O}(-E))T \le 0$ in view of formula (GSV) in 2.2. On the other hand, since $T^2 \ge 0$ the dimension $h^0(K_X \otimes (N_{\mathcal{F}} \otimes \mathcal{O}(-E))^{\otimes t})$ cannot grows like t^2, for otherwise we could write $m(N_{\mathcal{F}} \otimes \mathcal{O}(-E)) = H + L$ for some ample line bundle H, some effective divisor L and some $m \in \mathbb{N}$ (Kodaira's lemma, see [18, pg. 143]) hence $c_1(N_{\mathcal{F}} \otimes \mathcal{O}(-E))T > 0$, contradiction. Therefore, by Riemann-Roch formula and Serre duality ([13])

$$c_1^2(N_{\mathcal{F}}^* \otimes \mathcal{O}(E)) \le 2\,\mathrm{vol}_{\overline{\ell}}(\mathcal{F},\, S_1+\cdots+S_n) \quad \text{where } \ \overline{\ell} := (\ell_1, \ldots, \ell_n). \tag{1}$$

On the other hand a simple algebraic manipulation using Baum-Bott's formula (BB) and formula (GSV) shows that

$$c_1^2(N_{\mathcal{F}}^* \otimes \mathcal{O}(E)) = \mathrm{BB}(\mathcal{F}) + \sum_{i=1}^{n} 2(\ell_i-1)c_1(N_{\mathcal{F}}^* \otimes \mathcal{O}(E))S_i$$
$$+2\sum_{i<j}(\ell_i+\ell_j-\ell_i\ell_j)S_iS_j - \sum_{i=1}^{n} S_i^2 + 2\,\mathrm{GSV}(\mathcal{F},\, S_i) + (\ell_i-1)^2 S_i^2. \tag{2}$$

Now, by proposition 3.5, inequality (1), equality (2) and the fact that

$$\sum_{i=1}^{n}\mathrm{GSV}(\mathcal{F},\, S_i) = \mathrm{GSV}(\mathcal{F},\, S) + 2\sum_{i<j} S_iS_j,$$

where S is the curve whose irreducible components are the curves $S_1, \ldots, S_n$ (see (5.7) pag. 164 in [20]), the theorem follows. □

3.6. The rest of this section will be devoted to prove Proposition 3.5. The proof will follow as consequence of some technical results and Bogomolov-

Sommese's vanishing theorem ([10, Corollary 6.9 pag. 58]). The idea is to compare $vol_{\bar{\ell}}$ with $\mathrm{vol}_{\bar{1}}$, where $\bar{1}$ is the tuple with all entries equal to one, and to study its behaviour by blow ups. This is done in a series of lemmas.

We have the basic inequality:

Lemma 3.7 *Let C be a compact Riemann surface and $\mathcal{L}$ a line bundle over C. Then,*

$$h^0(\mathcal{L}) \leq 1 + \{\deg(\mathcal{L})\}^+,$$

where $\{a\}^+$ is the maximum between 0 and the number a.

Proof. We can assume that $h^0(\mathcal{L}) \geq 2$. Thinking $\mathcal{L}$ as a divisor in C, say E, we get that the complete linear system $|E|$ has dimension greater or equal to 1. Thus we can take an effective divisor, say D, linearly equivalent to E. Now, since D is effective, we have $h^0(\mathcal{L}) - 1 = \dim |D| \leq \deg(D) = \deg(\mathcal{L})$ (see exerc. 1.5 pag. 298 in [13]) and the lemma follows. □

The proof of Proposition 3.5 relies on the next technical lemma. For this we remind that the volume (also called degree) of a line bundle $\mathcal{L}$ over X is define by

$$\mathrm{vol}(\mathcal{L}) := \limsup_{t \to +\infty} \frac{h^0(\mathcal{L}^{\otimes t})}{t^2}.$$

Lemma 3.8 *Let $\mathrm{vol}(\mathcal{L})$ be as above, C an irreducible and reduced divisor in X and r an integer. If $\nu \colon \tilde{X} \to X$ is a sequence of blow ups such that the strict transform of C by ν, say $\tilde{C}$, is smooth then,*

$$\begin{aligned}
&\mathrm{vol}(\mathcal{L} \otimes \mathcal{O}(rC)) - \mathrm{vol}(\mathcal{L} \otimes \mathcal{O}(C)) \\
&\leq \sum_{k=2}^{r} \limsup_{t \to +\infty} \frac{1}{t^2} \sum_{j=t(k-1)+1}^{tk} \left\{ t(c_1(\mathcal{L})C + rC^2) + (j - tr)\tilde{C}^2 \right\}^+ \\
&\leq \left\{ (r-1)(c_1(\mathcal{L})C + rC^2) \right\}^+ + \frac{1}{2}(r-1)^2 \{-\tilde{C}^2\}^+.
\end{aligned}$$

Proof. We can assume that $r \geq 2$ since otherwise the inequalities are obvious. From the long exact sequences of cohomology induced by the short exact sequences of sheafs $0 \to \mathcal{L} \otimes \mathcal{O}_X(-(k+1)C) \to \mathcal{L} \otimes \mathcal{O}_X(-kC) \to \mathcal{L} \otimes \mathcal{O}_C(-kC) \to 0$, $k \in \mathbb{N}$ and induction we get

$$h^0\big((\mathcal{L} \otimes \mathcal{O}(rC))^{\otimes t}\big) - h^0\big((\mathcal{L} \otimes \mathcal{O}(C))^{\otimes t}\big)$$

$$\leq \sum_{k=2}^{r} \sum_{\jmath=t(k-1)+1}^{tk} h^0(\mathcal{L}^{\otimes t} \otimes \mathcal{O}_C(\jmath C)) \quad \text{for every} \quad t \in \mathbb{N}. \tag{3}$$

In order to compute the sum that appears in the right hand side of the above inequality we write $\mathcal{L}^{\otimes t} \otimes \mathcal{O}_C(\jmath C) = (\mathcal{L} \otimes \mathcal{O}_C(rC))^{\otimes t} \otimes \mathcal{O}_C((\jmath - tr)C)$ and for each $\jmath = t(k-1)+1, \ldots, tk$, we consider the line bundle with support in $\tilde{C}$ defined by $\mathcal{L}_\jmath := \mathcal{O}_{\tilde{C}}(\nu^*(\mathcal{L} \otimes \mathcal{O}(rC)))^{\otimes t} \otimes \mathcal{O}_{\tilde{C}}((\jmath - tr)\tilde{C})$ and the line bundle over X defined by $L_\jmath := (\mathcal{L} \otimes \mathcal{O}(rC))^{\otimes t} \otimes \mathcal{O}((\jmath - tr)C)$. Then,

$$h^0\big(\mathcal{O}_{\tilde{C}}(\nu^*(L_\jmath) \otimes \mathcal{O}(E))\big) = h^0(\mathcal{L}_\jmath), \tag{4}$$

where E is an effective divisor in $\tilde{X}$ which is contracted to points by ν. In fact $E := (tr - \jmath)D$ where D is the exceptional divisor of ν. On the other hand, since $\tilde{C}$ is a Riemann surface and $\mathcal{L}_\jmath$ has support in $\tilde{C}$, by the former lemma

$$h^0(\mathcal{L}_\jmath) \leq 1 + \{\deg(\mathcal{L}_\jmath)\}^+. \tag{5}$$

Meanwhile, we claim

$$h^0(\mathcal{O}_C(L_\jmath)) \leq h^0(\mathcal{O}_{\tilde{C}}(\nu^*(L_\jmath))). \tag{6}$$

In fact, tensoring the exact sequence $0 \to \mathcal{O}_C \to \nu_*(\mathcal{O}_{\tilde{C}}) \to I_Z \to 0$ (see exerc. 1.8 pag. 298 in [13]) by $L_\jmath$ and considering the induced long exact sequence of cohomology we have $h^0(\mathcal{O}_C(L_\jmath)) \leq h^0(\nu_*(\mathcal{O}_{\tilde{C}}) \otimes L_\jmath)$. Since $\nu_*(\mathcal{O}_{\tilde{C}} \otimes \nu^*(L_\jmath)) = \nu_*(\mathcal{O}_{\tilde{C}}) \otimes L_\jmath$ by projection formula (see exerc. 5.1 pag. 124 in [13]) the claim follows.

Therefore, since $\deg(\mathcal{L}_\jmath) = c_1(\mathcal{L}_\jmath)\tilde{C} = t(c_1(\mathcal{L})C + rC^2) + (\jmath - tr)\tilde{C}^2$, the first inequality stated in the lemma follows by (3), (4), (5) and (6).

The second inequality follows from the first one by observing that $\{\deg(\mathcal{L}_\jmath)\}^+ \leq t\{c_1(\mathcal{L})C + rC^2\}^+ + (tr - \jmath)\{-\tilde{C}^2\}^+$. □

In the next lemma $\mathrm{vol}_{\bar{\ell}}$ is related to $\mathrm{vol}_{\bar{1}}$ where $\bar{1}$ is the tuple with all entries equal to one.

Lemma 3.9 *Let $\mathcal{F}$, S and E be as in* Proposition 3.5. *Then,*

$$\begin{aligned}&\mathrm{vol}_{\bar{\ell}}(\mathcal{F}, S_1 + \cdots + S_n)\\ &\leq \mathrm{vol}_{\bar{1}}(\mathcal{F}, S_1 + \cdots + S_n) + \frac{1}{2}\sum_{i=1}^{n}(\ell_i - 1)^2\{-\tilde{S}_i^2\}^+\end{aligned}$$

$$+\sum_{i=1}^{n}(\ell_i-1)\left\{-\operatorname{GSV}(\mathcal{F},\, S_i)-S_i^2+ES_i-\sum_{j=i+1}^{n}(\ell_j-1)S_jS_i\right\}^+,$$

where $\tilde{S}_i$ *are any non singular models for* S_i.

Proof. In order to prove this lemma we must get rid of the coordinates greater than 1 in the n-tuple $\bar{\ell}$. To this end let us set $I=\{j \mid \ell_j\geq 2\}$. Without any loss of generality, by renumbering the indexes if necessary, we can (and shall) assume that $I=\{1,\, 2,\, \dots,\, m\}$. Now, for each $i=1,\, \dots,\, m$ define the line bundles over X by $\mathcal{L}_i := N^*_{\mathcal{F}}\otimes\mathcal{O}(\ell_1S_1+\cdots+\ell_{i-1}S_{i-1}+S_{i+1}+\cdots+S_n)$. Note that $\mathcal{L}_{i-1}\otimes\mathcal{O}(\ell_{i-1}S_{i-1})=\mathcal{L}_i\otimes\mathcal{O}(S_i)$ for $i=2,\, \dots,\, m$ and that $\operatorname{vol}_{\bar{\ell}}(\mathcal{F},\, S_1+\cdots+S_n)=\operatorname{vol}(\mathcal{L}_m\otimes\mathcal{O}(\ell_mS_m))$ and $\operatorname{vol}_{\bar{1}}(\mathcal{F},\, S_1+\cdots+S_n)=\operatorname{vol}(\mathcal{L}_1\otimes\mathcal{O}(S_1))$. On the other hand

$$c_1(\mathcal{L}_i)S_i+\ell_iS_i^2=-\operatorname{GSV}(\mathcal{F},\, S_i)-S_i^2+ES_i-\sum_{j=i+1}^{n}(\ell_j-1)S_jS_i$$

by formula (GSV) in 2.2. Therefore by Lemma 3.8 applied to each $\mathcal{L}_i$ with $r:=\ell_i$, $C:=S_i$, $\tilde{C}:=\tilde{S}_i$ and an inductive process the lemma follows. □

In the next result we study how $\operatorname{vol}_{\bar{1}}$ changes by blow ups. To this end we recall how the co-normal bundle of a foliation changes by blow ups.

Let $\pi\colon\tilde{X}\to X$ be a single blow up of X with center at p and let D be the exceptional curve of the first kind introduced by π. If $\tilde{\mathcal{F}}$ stands for the strict transform of $\mathcal{F}$ by π then,

$$N^*_{\tilde{\mathcal{F}}}=\pi^*(N^*_{\mathcal{F}})\otimes\mathcal{O}_{\tilde{X}}((\upsilon_p+\delta_p)D), \tag{7}$$

where υ_p is as in 3.1 and δ_p is defined as 0 if D is an algebraic solution of $\tilde{\mathcal{F}}$ and 1 otherwise (see [3, pag. 576]).

Lemma 3.10 *Let* $\mathcal{F}$, S, $\{S_i\}_i$ *and* $\bar{\ell}$ *be as in* Proposition 3.5 *and let* $\pi:\tilde{X}\to X$ *be a sequence of blow ups. Let* $\tilde{S}_i$ ($\tilde{\mathcal{F}}$, *resp.*) *be the strict transform of* S_i (*of* $\mathcal{F}$, *resp.*) *by* π. *If* D *is the reduced divisor in* $\tilde{X}$ *whose irreducible components are those irreducible components of the exceptional divisor of* π *which are algebraic solutions of* $\tilde{\mathcal{F}}$, *then*

$$\operatorname{vol}_{\bar{1}}(\mathcal{F},\, S_1+\cdots+S_n)\leq\operatorname{vol}_{\bar{1}}(\tilde{\mathcal{F}},\, \tilde{S}_1+\cdots+\tilde{S}_n+D)+\frac{1}{2}\Delta_\pi(\mathcal{F},\, S).$$

Proof. We will prove this inequality by induction in the number of blow ups made to obtain π. In order to simplify the notation, without any lost of generality, we will assume that $\mathcal{F}$ has only one singularity in S, say p. Now, we begin the induction process by assuming that π is a single blow up with center $p \in S$. Let D be the exceptional divisor of π then, using the identity (7) and the projection formula (see [13])

$$\mathrm{vol}_{\overline{1}}(\mathcal{F},\, S_1 + \cdots + S_n) = \mathrm{vol}_{\overline{u}_0}(\tilde{\mathcal{F}},\, \tilde{S}_1 + \cdots + \tilde{S}_n + D), \tag{8}$$

where $\overline{u}_0$ is the tuple with all entries equal to one except the last one which is equal to $\mu_p - \upsilon_p - \delta_p$, being μ_p (υ_p, resp.) the algebraic multiplicity of S (of $\mathcal{F}$, resp.) in p.

Now we will change the coefficient $\mu_p - \upsilon_p - \delta_p$ in the $(n+1)$-tuple $\overline{u}_0$ by the number $1 - \delta_p$. To this end we use Lemma 3.8 with $\mathcal{L} := N^*_{\tilde{\mathcal{F}}} \otimes \mathcal{O}(\tilde{S}_1 + \cdots + \tilde{S}_n) \otimes \mathcal{O}(-\delta_p D)$, $C := D$, $\tilde{C} := D$ and $r := \mu_p - \upsilon_p$ to conclude

$$\begin{aligned} &\mathrm{vol}_{\overline{u}_0}(\tilde{\mathcal{F}},\, \tilde{S}_1 + \cdots + \tilde{S}_n + D) \leq \mathrm{vol}_{\overline{u}_1}(\tilde{\mathcal{F}},\, \tilde{S}_1 + \cdots + \tilde{S}_n + D) \\ &+ \sum_{k=2}^{\mu_p - \upsilon_p} \limsup_{t \to +\infty} \frac{1}{t^2} \sum_{\jmath = t(k-1)+1}^{tk} \{t c_1(\mathcal{L}) D - \jmath\}^+, \end{aligned} \tag{9}$$

where $\overline{u}_1$ is the tuple with all entries equal to one except the last one which is equal to $1 - \delta_p$.

On the other hand, by the identity (7), $c_1(\mathcal{L})D = \mu_p - \upsilon_p$. Since $tk \geq \jmath \geq t(k-1)+1$ we have the inequality $tc_1(\mathcal{L})D - \jmath \geq 0$ for every $k \leq \mu_p - \upsilon_p$. Therefore the sum that appears in the right hand side of the inequality (9) is equal to $1/2(\mu_p - \upsilon_p - 1)^2$ if $\mu_p \geq \upsilon_p + 2$ and 0 otherwise. Thus, from (8) and (9) we get the proof of the lemma in the case that π is a single blow up.

To complete the induction process let us assume that π is a sequence of s blow ups. Let us decompose π as $\tilde{X} \xrightarrow{\hat{\pi}} Y \xrightarrow{\sigma} X$, where σ is a sequence of $s-1$ blow ups and $\hat{\pi}$ is a single blow up at a point, say $q \in Y$. Let D be the exceptional curve of the first kind introduced in $\tilde{X}$ by $\hat{\pi}$. Let us denote S_Y ($\mathcal{F}_Y$, resp.) the strict transform of S ($\mathcal{F}$, resp.) by σ and D_Y the algebraic curve in Y whose irreducible components are all those irreducible components of the exceptional divisor of σ which are algebraic solution of $\mathcal{F}_Y$. Now, by inductive hypothesis,

$$\mathrm{vol}_{\overline{1}}(\mathcal{F},\, S_1 + \cdots + S_n) \leq \mathrm{vol}_{\overline{1}}(\mathcal{F}_Y,\, S_Y + D_Y) + \frac{1}{2}\Delta_\sigma(\mathcal{F},\, S),$$

where, for simplicity of notation, we have used the symbols D_Y and S_Y in

$\mathrm{vol}_{\bar{1}}(\mathcal{F}_Y, S_Y+D_Y)$ instead of writing their decomposition in their irreducible components.

From the above inequality, by inductive hypothesis applied to $\hat{\pi}$, the divisor $S_Y + D_Y$ and the foliation $\mathcal{F}_Y$, we conclude

$$\mathrm{vol}_{\bar{1}}(\mathcal{F}_Y, S_Y + D_Y) \leq \mathrm{vol}_{\overline{u}_1}(\tilde{\mathcal{F}}, \tilde{S}_1 + \cdots + \tilde{S}_n + \tilde{D} + D) + \frac{1}{2}(\mu_q + \varepsilon_q - \upsilon_q - 1)^2,$$

where $\overline{u}_1$ is the tuple with all entries equal to one except the last one which is equal to $1 - \delta_q$ and where $\tilde{D}$ is the strict transform of D_Y by $\hat{\pi}$. Thus the induction process is completed and the lemma follows. □

Now, Proposition 3.5 is a consequence of the above results and Bogomolov-Sommese's vanishing Theorem.

Proof (*of* Proposition 3.5). By Lemma 3.9 is suffices to show that

$$\mathrm{vol}_{\bar{1}}(\mathcal{F}, S_1 + \cdots + S_n) \leq \frac{1}{2}\Delta_\pi(\mathcal{F}, S).$$

By Lemma 3.10, this inequality holds as soon we have

$$\mathrm{vol}_{\bar{1}}(\tilde{\mathcal{F}}, \tilde{S}_1 + \cdots + \tilde{S}_n + D) = 0.$$

But $\tilde{S}_1 + \cdots + \tilde{S}_n + D$ is an algebraic solution of $\tilde{\mathcal{F}}$ which implies that $N^*_{\tilde{\mathcal{F}}} \otimes \mathcal{O}(\tilde{S}_1 + \cdots + \tilde{S}_n + D)$ is a subsheaf of $\Omega^1(\ln(\tilde{S}_1 + \cdots + \tilde{S}_n + D))$. On the other hand, being π a map adapted to $(\mathcal{F}, S)$, $\tilde{S}_1 + \cdots + \tilde{S}_n + D$ is a reduced normal crossing divisor whence [10, Corollary 6.9] applies to get $\mathrm{vol}_{\bar{1}}(\tilde{\mathcal{F}}, \tilde{S}_1 + \cdots + \tilde{S}_n + D) = 0$. □

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