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Germs of Engel structures along 3-manifolds

(Dedicated to Professor Hajime Sato on his sixtieth birthday)

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Abstract. Engel structures along an embedded 3-manifold in a 4-manifold are studied in this article. It is shown that, among the Engel structures having the same oriented even-contact structure around the given embedded closed oriented 3-manifold as derived distributions with the induced orientations, the germ of Engel structure along the 3-manifold is determined by the singular line field on the 3-manifold traced by the Engel structure.

Key words: Engel structures, even-contact structures.

1. Introduction

A tangent distribution of rank 2, maximally non-integrable, on a 4-dimensional manifold is called an *Engel structure*. Engel structures have a special property: all Engel structures are locally equivalent (see [BCG3]). Such a phenomenon occurs only for line fields, contact structures, even-contact structures, and Engel structures among generic regular distributions (see [VG], [Mon1]). Therefore studying global properties of Engel structures is important. We study Engel structures along an embedded 3-manifold, and investigate the conditions under which germs of Engel structures are determined up to isotopy.

Although an Engel structure has similar properties to a contact structure, they have some differences. A contact structure is also defined as a maximally non-integrable hyperplane field on an odd-dimensional manifold. Contact structures on a closed manifold have global stability for deformations: any two contact structures on a closed manifold which are homotopic among contact structures are isotopic (Gray's stability Theorem [Gr]). However, it is known that the corresponding statement for Engel structures does not hold (see [Ge1]). A homotopy of Engel structures is not always represented by an isotopy. An obstruction for the isotopy is considered as follows. An Engel structure D defines a line field $L(D^2) \subset$

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D, which is called an *Engel line field* (see Section 2 for the definition). If an Engel structure is deformed, the Engel line field for the structure is also deformed in general. However, line fields on a closed manifold have continuous moduli. Therefore, the deformed Engel line field is not always diffeomorphic to the original one. As for the global stability of Engel structures, A. Golubev showed a stability of Engel structures in the case when an Engel line field is fixed (see [Go]). E. Giroux showed in [Gi] that the germs of contact structures along an embedded closed orientable surface in a 3-manifold are determined by the characteristic foliations, constructed from the restrictions of the contact structures, on the surfaces. However, in order to determine the germ of an Engel structure along a hypersurface, we need more conditions other than the restriction of the Engel structure on the hypersurface.

We study a sufficient condition to determine the germ of an Engel structure along a hypersurface. In order to state the result of this paper, we need the following two notions. Let M be an embedded closed orientable 3-manifold in an orientable 4-manifold W, and D an Engel structure defined in a neighborhood of $M \subset W$. Then the Engel structure D traces on the 3-manifold M a singular line field $\mathcal{F}(M, D)$. It is defined as a subsheaf of the sheaf Ω^1_M of germs of C^∞ differential 1-forms on M generated by restriction to M of annihilators of D. (see Section 2 for details). We call it a *restricted line field* of D on M. We set $D^2 := D + [D, D]$. This distribution D^2 is an even-contact structure by the definition of an Engel structure. It is called the *derived even-contact structure* from D. Moreover, this distribution D^2 has an orientation induced from the Engel structure D (see Section 2 for details). We assume, from now on, that the derived even-contact structure D^2 is thus oriented. We say two derived even-contact structures coincide if they coincide as oriented even-contact structures. (See Section 2 for the precise definitions of the italic terms above.) Now, the main result of this article is the following.

Theorem *Let D_0 and D_1 be Engel structures defined in a neighborhood of an embedded closed orientable* 3*-manifold M in a orientable* 4*-dimensional manifold W. We suppose that* (i) *D_0 and D_1 trace the same line field on M in a strict sense, and that* (ii) *D_0 and D_1 have the same derived even-contact structure: namely,*

(i) $\mathcal{F}(M, D_0) = \mathcal{F}(M, D_1)$,

(ii) ${D_0}^2 = {D_1}^2$.

Then there exists an isotopy φ_s of local diffeomorphisms defined in a neighborhood of M which satisfies the following properties for any $s \in [0, 1]$:

- $\varphi_0 = \mathrm{id}, \quad \varphi_{1*}(D_0) = D_1$,
- $\varphi_s(M) = M$,
- $\mathcal{F}(M, \varphi_{s*}(D_0)) = \mathcal{F}(M, D_0)$,
- $\varphi_{s*}({D_0}^2) = {D_0}^2$.

When the Engel line field L is transverse to the 3-manifold M, the derived even-contact structure D^2 induces on M a contact structure as $D^2 \cap TM$. In such a case, R. Montgomery [Mon2] showed a similar result. In this article, since we do not assume the transversality of an Engel line field L and a 3-manifold M, the derived even-contact structure D^2 does not necessarily induce a contact structure on M. If the non-contact locus is not of measure zero, for example, it is difficult to deduce something about Engel structures around the locus only from the conditions on the 3-manifold M. Condition (ii) of Theorem guarantees the rigidity of Engel structures.

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2. Preliminaries

We begin with the definition of an Engel structure. Let W be a 4-dimensional manifold. A tangent distribution D is a subbundle of the tangent bundle TW, and is also regarded as a locally smooth sheaf $\mathcal{D}$ of vector fields. Let $[\mathcal{D}, \mathcal{D}]$ denote the sheaf generated by all Lie brackets $[X, Y]$ of vector fields $X, Y \in \mathcal{D}$. We define derived sheaves as $\mathcal{D}^2 := \mathcal{D} + [\mathcal{D}, \mathcal{D}]$ and $\mathcal{D}^3 := \mathcal{D}^2 + [\mathcal{D}, \mathcal{D}^2]$. Then the derived distributions of D are defined pointwise as the linear subspace of T_pW, $p \in W$, as follows:

$$D_p^2 = \{X_p \in T_pW \mid X \in \mathcal{D}_p^2\}, \quad D_p^3 = \{X_p \in T_pW \mid X \in \mathcal{D}_p^3\},$$

where $\mathcal{D}_p^i$ are the stalks at $p \in W$. Now, the notion of Engel structure is defined as follows.

Definition An *Engel structure* on a 4-dimensional manifold W is a distribution D of rank 2 which satisfies

$$\dim D_p^2 = 3, \quad \dim D_p^3 = 4,$$

at any point $p \in W$.

We next see that for an Engel structure D there is an unique line field $L(D^2)$ associated with D. Let D be an Engel structure on W. The distribution D^2 of rank 3 turns out to be an *even-contact structure*, that is, a distribution of corank 1 on the 4-manifold W defined locally by a 1-form θ with the property that $\theta \wedge d\theta$ vanishes nowhere. For an even-contact structure E on W, there is a unique subdistribution $L(E)$ of E of rank 1 determined by the relation $[\mathcal{L}(E), \mathcal{E}] \subset \mathcal{E}$, where $\mathcal{L}(E)$ and $\mathcal{E}$ are corresponding sheaves. It turns out to be the Cauchy characteristic space, in terms of differential systems (see [BCG3]). Hence, for an even-contact structure $E = D^2$, there is an unique line field $L(D^2)$. The line field $L(D^2)$ is called the *Engel line field* of D (see [KMS], [Ge1]). The Engel line field is explained in terms of framings of Engel structures as follows. Now that D is an Engel structure on W, we have $(D^2/D)_p \cong \mathbb{R}$, $(D^3/D^2)_p \cong \mathbb{R}$, at each $p \in W$. We define a mapping $\psi\colon D \otimes (D^2/D) \to (D^3/D^2)$ by $X \otimes Y \mapsto [X, Y]$. This ψ induces a mapping $\Psi\colon D \to \mathrm{Hom}(D^2/D, D^3/D^2)$ as $X \mapsto \psi(X, \cdot)$. The Engel line filed $L(D)$ is nothing but the kernel of Ψ. An Engel line field $L(D^2)$ has a remarkable property that any embedded 3-manifold transverse to $L(D^2)$ has a contact structure (see [Ge1], [Mon2]).

Let us observe an orientation of D^2 induced from D itself, and an orientation of "twisting" of an Engel structure. It is a key point of the proof of Theorem. We observe them by reviewing typical examples of Engel structures. It is sufficient to observe local models since all Engel structures are equivalent locally.

The first example is the standard Engel structure D_{st} on $\mathbb{R}^4$ given by the following pair of 1-forms as the kernel,

$$\beta_0 = dy - z \cdot dx, \quad \alpha_0 = dz - w \cdot dx. \tag{1}$$

In fact, this is nothing but the space of 2-jets of function $y(x)$ of one variable with $z = dy/dx$ and $w = d^2y/dx^2$. By easy calculations, we obtain that the derived even-contact structure ${D_{st}}^2$ and the Engel line filed $L({D_{st}}^2)$ are given as ${D_{st}}^2 = \ker \beta_0$, $L({D_{st}}^2) = \ker(\beta_0 \wedge d\beta_0)$ respectively.

We now observe a description of an Engel structure by a pair of 1-forms. It is known that an ordered pair (α, β) of differential 1-forms defines an Engel structure $D = \{\alpha = 0,\ \beta = 0\}$ as the kernel, if the pair satisfies the following conditions (see [Ge1]):

(1) $\beta \wedge \alpha \wedge d\alpha$ vanishes nowhere,
(2) $\alpha \wedge \beta \wedge d\beta \equiv 0$,
(3) $\beta \wedge d\beta$ is a 3-form vanishing nowhere.
Such an ordered pair (α, β) is called an *Engel pair* of 1-forms. For an Engel pair (α, β) of 1-forms, it turns out that β determines the derived even-contact structure $D^2 = \{\beta = 0\}$, and that the 3-form $\beta \wedge d\beta$ defines the Engel line field $L(D^2)$ as its kernel: $L(D^2) = \ker(\beta \wedge d\beta)$. For an Engel structure $D = \{\alpha = 0, \beta = 0\}$, choices of the 1-forms α, β of Engel pairs are flexible. If (α, β) is an Engel pair of 1-forms for an Engel structure D, then $(g \cdot \alpha + h \cdot \beta, f \cdot \beta)$ is also an Engel pair for D, for any smooth functions $f \neq 0$, $g \neq 0$, and h. We note here that an Engel structure D determines the orientation of its derived even-contact structure $D^2 = \{\beta = 0\}$ by $\alpha \wedge d\alpha$. Note that it is independent of the choice of α. We assume, from now on, that the derived even-contact structure is thus oriented. We say that two derived even-contact structure ${D_0}^2$, ${D_1}^2$ coincide if they coincide as even-contact structures oriented as above.

The second example is constructed from a given contact structure according to the prolongation procedure as follows (see [Mon2]). Let ξ be a contact structure on a 3-manifold M. First we construct a 4-dimensional manifold $\mathbb{P}(\xi)$ from ξ by fibrewise projectivizations,

$$\mathbb{P}(\xi) := \bigcup_{p \in M} \mathbb{P}(\xi_p),$$

where $\mathbb{P}(\xi_p) \cong \mathbb{R}\mathrm{P}^1$ is the projectivization of the contact plane ξ_p. A point of $\mathbb{P}(\xi_p)$ can be regarded as a line l in the contact plane ξ_p through the origin. Let $\pi\colon \mathbb{P}(\xi) \to M$ be its projection. The 4-manifold $\mathbb{P}(\xi)$ is endowed with a 2-plane field $D(\xi)$ induced naturally as follows. A point $q = (p, l) \in \mathbb{P}(\xi)$ is regarded as a pair of a point $p \in M$ and a tangent line $l \subset \xi_p \subset T_pM$. Then, set $D(\xi)_q := (d\pi)^{-1} l$. Thus we obtain a 2-plane field $D(\xi)$ on $\mathbb{P}(\xi)$. We call this $(\mathbb{P}(\xi), D(\xi))$ the *prolongation* of a contact structure ξ. It is known that the prolongation $(\mathbb{P}(\xi), D(\xi))$ is an Engel manifold (see [Mon2]). We can see this construction locally as follows. Let β be a contact 1-form defining a contact structure ξ. The contact form β is locally isomorphic to the standard 1-form $dy - z dx$ according to the Darboux theorem. With respect to the coordinates (x, y, z), a line on a contact plane ξ_p is represented by the slope $w = dz/dx$ of the projection to (x, z)-plane, except for the vertical. We note here that the sign of the slope depends on the orientation of the contact plane $\xi_p = \{\beta_p = 0\}$ defined by $d\beta_p$. In other words, w is the local coordinate of $\mathbb{P}(\xi_p) \cong \mathbb{R}\mathrm{P}^1$. Thus, the coordinates (x, y, z, w) give a local coordinate system of $\mathbb{P}(\xi)$. From the construction, the distribution $D(\xi)$ is defined locally by the pair of 1-forms $dy - z \cdot dx$ and $dz - w \cdot dx$. The pair represents the standard Engel structure (see equations (1)).

There is another prolonged distribution $\tilde{D}(\xi)$ on $\mathbb{P}(\xi)$. It is defined as $\tilde{D}(\xi)_q := (d\pi)^{-1}(-l)$, where $-l$ corresponds to a line on ξ_p symmetric to l for an axis of local coordinates of ξ_p. The distribution $\tilde{D}(\xi)$ is defined locally by $\beta = dy - z \cdot dx$ and $\tilde{\alpha} = dz + w \cdot dx$ with respect to the local coordinates (x, y, z, w) of $\mathbb{P}(\xi)$ as above. Two Engel structures $D(\xi)$ and $\tilde{D}(\xi)$ have the same derived even-contact structure $D^2 := D(\xi)^2 = \tilde{D}(\xi)^2$ defined by $\{\beta = 0\}$ if we ignore the orientations. However, $D(\xi)$ and $\tilde{D}(\xi)$ give different orientations on the even-contact structure D^2 since $\alpha \wedge d\alpha|_{\{\beta=0\}} = -\tilde{\alpha} \wedge d\tilde{\alpha}|_{\{\beta=0\}}$.

A geometric meaning of this orientation on D^2 is explained as follows. We observe behaviors of $D(\xi)$ and $\tilde{D}(\xi)$ when we move along $L := L(D(\xi)^2) = L(\tilde{D}(\xi)^2)$. Let $\varphi_t\colon \mathbb{P}(\xi) \to \mathbb{P}(\xi)$ be the isotopy generated by $(\partial/\partial w) \in L$. The isotopy φ_t preserves the even-contact structure $D^2 = \{\beta = 0\}$, since $L_{\partial/\partial w}(dy - z \cdot dx) = 0$. Let $l_t := d\pi(D(\xi)_{\varphi_t(q)})$, $\tilde{l}_t := d\pi(\tilde{D}(\xi)_{\varphi_t(q)})$ be lines on ξ_p, where $\pi\colon \mathbb{P}(\xi) \to M$ is a projection along L, $q = (p, l) \in \mathbb{P}(\xi)$. Then, l_t (resp., $\tilde{l}_t$) continues twisting positively (resp., negatively) as t increases with respect to the orientation on ξ_p given by $d\beta$, since $D(\xi)$ (resp., $\tilde{D}(\xi)$) is defined as $(d\pi)^{-1}l$ (resp., $(d\pi)^{-1}(-l)$). In other words, Engel structures with the same derived even-contact structure is considered locally as horizontal line fields to the contact structures induced from the even-contact structure, which continue twisting in one direction. The orientation of the even-contact structure implies the direction of the twisting.

Before we go to the proof, we need one more definition. We define a line filed with singularities on an embedded 3-manifold $M \subset W$ traced by an Engel structure D on W. We define it in terms of sheaves of annihilators. Let Ω^1_M be the sheaf of germs of C^∞ differential 1-forms on M. It is considered as an $\mathcal{E}_M$-module, where $\mathcal{E}_M$ is a sheaf of germs of C^∞ functions on M. We define a singular foliation $\mathcal{F}(M, D)$ on M as a submodule of Ω^1_M generated by pull-backs by the inclusion mapping $\iota\colon M \hookrightarrow W$ of generator ω of a sheaf $\mathcal{D}^\perp$ of annihilating bundle $D^\perp$ of D:

$$\mathcal{F}(M, D) = \langle \iota^* \omega \mid \omega \in \mathcal{D}^{\perp} \rangle_{\mathcal{E}_M}.$$

We call it the *restricted line field* of D on M.

3. Proof of Theorem

In this section, we prove Theorem in two steps. First, we construct a path D_s, $s \in [0, 1]$, among Engel structures between the given two Engel structures D_0, D_1. We can construct the path D_s so that it has a constant derived even-contact structure ${D_s}^2 \equiv E$, and a constant Engel line field $L = L(E)$ consequently. Next, we construct an isotopy realizing the path D_s above along the constant Engel line field L. We finally check that the isotopy thus obtained actually preserves the embedded 3-manifold M.

3.1. Deformation of Engel pairs

In order to prove Theorem, we may consider only a small neighborhood of $M \times \{0\}$ in $M \times \mathbb{R}$. There exists a diffeomorphism $\Phi \colon M \times \mathbb{R} \to W$ onto a neighborhood of $M \subset W$ which maps $M \times \{0\}$ to M identically and realize a tubular neighborhood of $M \subset W$, since M and W are orientable. Then we may regard germs of Engel structures along $M \subset W$ as those on $M \times \mathbb{R}$ defined along $M \times \{0\} =: M_0$. Let (α_0, β_0), (α_1, β_1) be Engel pairs of 1-forms defined on a neighborhood of a point $x \in M_0$ in $M \times \mathbb{R}$, which define locally D_0, D_1 respectively. These 1-forms can be written in the following split forms,

$$\alpha_i = \theta_i^t + u_i^t \cdot dt, \quad \beta_i = \omega_i^t + v_i^t \cdot dt, \quad (i = 0, 1), \tag{2}$$

where θ_i^t, ω_i^t are 1-forms, u_i^t, v_i^t are functions defined on an open subset of $M = M \times \{t\}$, and t is a coordinate on $\mathbb{R}$.

With these split forms, the conditions of Theorem mean the following. Let us recall that the derived even-contact structures depend only on the second 1-forms of Engel pairs. It follows from condition (ii) of Theorem that $\beta_0 = f \cdot \beta_1$ for some function f vanishing nowhere. We may assume $\beta_0 = \beta_1 =: \beta$, that is, $\omega_0^0 = \omega_1^0 =: \omega^0$ and $v_0^0 = v_1^0 =: v^0$, by replacing β_1 with $f \cdot \beta_1$, on account of the argument about choices of 1-forms of Engel pairs in Section 2. Note that both 4-forms $\beta_i \wedge \alpha_i \wedge d\alpha_i$, $i = 0, 1$, induce the same orientation because of the orientation of ${D_0}^2 = {D_1}^2$. In addition, we may assume $\theta_0^0 = \theta_1^0 =: \theta^0$, by condition (i) of Theorem. It is explained as follows. First, we show that there exists a matrix $G = (g_{uv})$, u, $v = 1, 2$, where g_{uv} are functions on a neighborhood $U \subset M_0$ of a point $x_0 \in M_0$, which satisfies

- $G(x) = (g_{uv}(x)) \in \mathrm{GL}(2, \mathbb{R})$, for any $x \in U$,
- $g_{21} = 0$, $g_{22} = 1$,
- $\begin{pmatrix}\theta_0^0 \\ \omega^0\end{pmatrix} = G\begin{pmatrix}\theta_1^0 \\ \omega^0\end{pmatrix} = \begin{pmatrix} g_{11}\cdot\theta_1^0 + g_{12}\cdot\omega^0 \\ \omega^0\end{pmatrix}$.

We show this in a similar way as in [Ma]. Note that $\theta_0^0 = \iota^*\alpha_0$, $\theta_1^0 = \iota^*\alpha_1$, and $\omega^0 = \iota^*\beta$, where $\iota\colon M \hookrightarrow M \times \mathbb{R}$ is an inclusion mapping. Since $\mathcal{F}(M, D_i)$ is generated by $\iota^*\alpha_i = \theta_i^0$ and $\iota^*\beta = \omega^0$, there exist (2, 2)-matrices $A = (a_{uv})$, $B = (b_{uv})$, a_{uv}, $b_{uv} \in \mathcal{E}_U$, u, $v = 1, 2$, which satisfies $a_{21} = b_{21} = 0$, $a_{22} = b_{22} = 1$, and

$$\begin{pmatrix}\theta_1^0 \\ \omega^0\end{pmatrix} = A\begin{pmatrix}\theta_0^0 \\ \omega^0\end{pmatrix}, \quad \begin{pmatrix}\theta_0^0 \\ \omega^0\end{pmatrix} = B\begin{pmatrix}\theta_1^0 \\ \omega^0\end{pmatrix}. \tag{3}$$

For each $x \in M$, $A(x) = (a_{uv}(x))$ and $B(x) = (b_{uv}(x))$ are (2, 2)-matrices with elements $a_{uv}(x)$, $b_{uv}(x) \in \mathbb{R}$. Then, there exists a (2, 2)-matrix $C(x) = (c_{uv}(x))$, $c_{uv}(x) \in \mathbb{R}$, which satisfies that $C(x)(I - A(x)B(x)) + B(x)$ is regular. In fact, we can take $C(x)$ as follows. At $x \in U$ where $\det B(x) \neq 0$, we take $C(x) = (0)$. It is trivial that $C(x)(I - A(x)B(x)) + B(x) = B(x)$ is regular. At $x \in U$ where $\det B(x) = b_{11} = 0$, we take $C(x) = (c_{ij})$, where $c_{11} = c \neq 0$ is constant and other entries are zeros. Then we have

$$C(x)(I - A(x)B(x)) + B(x) = \begin{pmatrix} c & b_{12} - c(a_{11}b_{12} + a_{12}) \\ 0 & 1 \end{pmatrix}.$$

It is regular. Then we obtain a matrix $C = (c_{uv})$, where c_{uv} are constant mappings defined on a neighborhood of $x \in M$ which values are $c_{uv}(x)$. Set $G := C(I - AB) + B$. It is regular from the observation above. From equations (3), it follows that

$$\begin{pmatrix}\theta_1^0 \\ \omega^0\end{pmatrix} = A\begin{pmatrix}\theta_0^0 \\ \omega^0\end{pmatrix} = AB\begin{pmatrix}\theta_1^0 \\ \omega^0\end{pmatrix}.$$

Therefore, we obtain

$$G\begin{pmatrix}\theta_1^0 \\ \omega^0\end{pmatrix} = \{C(I - AB) + B\}\begin{pmatrix}\theta_1^0 \\ \omega^0\end{pmatrix} = B\begin{pmatrix}\theta_1^0 \\ \omega^0\end{pmatrix} = \begin{pmatrix}\theta_0^0 \\ \omega^0\end{pmatrix}.$$

Thus, we conclude that G is the required matrix. Note that $g_{11} \neq 0$ because $\det G \neq 0$. Namely, we have $\theta_0^0 = f\cdot\theta_1^0 + g\cdot\omega^0$ for some functions $f \neq 0$, and g. According to the argument in Section 2 about choices of Engel pairs of 1-forms, we may take $\tilde{f}\cdot\alpha_1+\tilde{g}\cdot\beta =: \tilde{\alpha}_1$ instead of α_1 from the beginning, where $\tilde{f}$, $\tilde{g}$ are some extension of f, g to $U\times\mathbb{R}$. Note that $\iota^*\tilde{\alpha}_1 = \theta_1^0$. Therefore, we may assume $\theta_0^0=\theta_1^0$. Even after this replacement of Engel pairs, the 4-forms $\beta_i\wedge\alpha_i\wedge d\alpha_i$, $i=0,1$, have the same sign.

We then construct a homotopy of Engel pairs of 1-forms (α_s,β_s), $s\in[0,1]$, between (α_0,β_0) and (α_1,β_1). We set two families of 1-forms as follows,

$$\begin{aligned}\alpha_s :=&(1-s)\cdot\alpha_0+s\cdot\alpha_1\\ =&\big\{(1-s)\cdot\theta_0^t+s\cdot\theta_1^t\big\}+\big\{(1-s)\cdot u_0^t+s\cdot u_1^t\big\}\cdot dt\\ =:&\ \theta_s^t+u_s^t\cdot dt,\\ \beta_s :=&\beta =: \omega^t+v^t\cdot dt.\end{aligned}$$

We show that (α_s,β_s) is an Engel pair for each $s\in[0,1]$ sufficiently near M_0. We check conditions (1)–(3) of the definition of Engel pairs for the family (α_s,β_s) in a small neighborhood of M_0. We remark that both pairs (α_i,β_i), $i=0,1$, satisfy conditions (1)–(3), and both of the 4-forms $\beta_i\wedge\alpha_i\wedge d\alpha_i$ in condition (1) are positive for the underlying manifold.

Condition (1) We calculate the family $\beta_s\wedge\alpha_s\wedge d\alpha_s$ of 4-forms along a neighborhood $U\subset M_0$. First, we obtain by a simple calculation,

$$\begin{aligned}(\beta_s\wedge\alpha_s\wedge d\alpha_s)|_U&=(\omega^0+v^0\cdot dt)\wedge(\theta^0+u_s^0\cdot dt)\wedge d(\theta_s^t+u_s^t\cdot dt)|_U\\ &=\big\{\omega^0\wedge\theta^0+(u_s^0\cdot\omega^0-v^0\cdot\theta^0)\wedge dt\big\}\wedge\left\{d\theta^0+\left(du_s^0-\left.\frac{\partial\theta_s^t}{\partial t}\right|_{t=0}\right)\wedge dt\right\}\\ &=\left\{\omega^0\wedge\theta^0\wedge\left(du_s^0-\left.\frac{\partial\theta_s^t}{\partial t}\right|_{t=0}\right)+u_s^0\cdot\omega^0\wedge d\theta^0-v^0\cdot\theta^0\wedge d\theta^0\right\}\wedge dt.\end{aligned}$$

The first two terms depend on the parameter $s\in[0,1]$ linearly, and the last term is independent of s. Therefore we have,

$$\begin{aligned}&(\beta_s\wedge\alpha_s\wedge d\alpha_s)|_U\\ &=\left[(1-s)\cdot\left\{\omega_0\wedge\theta_0\wedge\left(du_0^0-\left.\frac{\partial\theta_0^t}{\partial t}\right|_{t=0}\right)+\left(u_0^0\cdot\omega^0-v^0\cdot\theta^0\right)\wedge d\theta^0\right\}\right.\\ &\qquad\left.+s\cdot\left\{\omega^0\wedge\theta^0\wedge\left(du_1^0-\left.\frac{\partial\theta_1^t}{\partial t}\right|_{t=0}\right)+\left(u_1^0\cdot\omega^0-v^0\cdot\theta^0\right)\wedge d\theta^0\right\}\right]\wedge dt\\ &=(1-s)\cdot(\beta_0\wedge\alpha_0\wedge d\alpha_0)|_U+s\cdot(\beta_1\wedge\alpha_1\wedge d\alpha_1)|_U\end{aligned}$$

Due to the assumption on Engel pairs of 1-forms, both $(\beta_i \wedge \alpha_i \wedge d\alpha_i)|_U$, $i = 0, 1$, are positive for the orientation of the underlying manifold. Then we deduce that $(\alpha_s \wedge \beta_s \wedge d\alpha_s)|_U$ are positive for any $s \in [0, 1]$. Thus $\beta_s \wedge \alpha_s \wedge d\alpha_s$ is positive sufficiently near M_0 for any $s \in [0, 1]$.

Condition (2) As $\beta_0 = \beta_1 = \beta$ near M_0, we have $\alpha_s \wedge \beta_s \wedge d\beta_s = (1 - s) \cdot \alpha_0 \wedge \beta \wedge d\beta + s \cdot \alpha_1 \wedge \beta \wedge d\beta = 0$.

Condition (3) Similarly, $\beta_s \wedge d\beta_s = \beta_0 \wedge d\beta_0 = \beta_1 \wedge d\beta_1 \neq 0$.

Thus, (α_s, β_s) is a family of Engel pairs of 1-forms defined on a neighborhood $V \subset M \times \mathbb{R}$ of a point $x_0 \in M_0$. Let D_s denote the family of Engel structures realized by (α_s, β_s). Then D_s is a path of Engel structures between D_0 and D_1.

3.2. Construction of isotopies

In this subsection we show the existence of families φ_s of local diffeomorphisms along $M \subset W$ representing the family D_s, $s \in [0, 1]$, of Engel structures obtained above. According to the argument in Section 2 about a geometric meaning of Engel structures, each D_s corresponds to horizontal line field twisting in one direction when we go along the Engel line field $L \equiv L({D_s}^2)$. Since all D_s twist in the same direction, we have only to adjust the speed. Therefore, we construct the family φ_s as a flow along the Engel line field L. In the proof, we first construct families of local diffeomorphisms φ_s. In fact, we construct locally families of vector fields X_s which define φ_s by using pairs of 1-forms α_s, β_s which determine locally a family D_s of Engel structures. And then, we obtain a family of global diffeomorphisms by constructing a family of global vector fields from families of local ones using a partition of unity method.

First, we determine the equations which we solve. Let α_s, β be 1-forms which determine the distributions D_s locally as above. A family φ_s of local diffeomorphisms satisfies the condition $(\varphi_s)_* D_0 = D_s$ if it satisfies the following equations:

$$\begin{cases} {\varphi_s}^*(f_s \cdot \beta) = \beta, \\ {\varphi_s}^*(g_s \cdot \alpha_s + h_s \cdot \beta) = \alpha_0, \end{cases} \tag{4}$$

for some families f_s, g_s of functions vanishing nowhere, and family h_s of functions. In order to show the existence of a family φ_s of local diffeomorphisms which satisfies equations (4), it is sufficient to show the solvability of the following equations

$$L_{X_s}\beta + a_s\beta = 0, \tag{5}$$

$$L_{X_s}\alpha_s + \frac{d\alpha_s}{ds} + b_s\alpha_s + c_s\beta = 0, \tag{6}$$

with respect to families X_s of vector fields, and a_s, b_s, c_s of functions. These are the equations we solve. It is explained as follows. Suppose that equations (5), (6) have a solution (X_s, a_s, b_s, c_s). Let φ_s be a family of diffeomorphisms generated from X_s as follows: $(d\varphi_s/ds)(p) = X_s(\varphi_s(p))$, $\varphi_0(p) = p$, where p is a point in a neighborhood of $M \subset W$. It is clear that equation (5) implies the first equation of equations (4). Therefore, it remains to prove that φ_t satisfies the second equation of equations (4). In order to show this, it suffices to show

$$\big(Q_s({\varphi_s}^*\alpha_s) - \alpha_0\big) \wedge \beta = 0, \quad \text{for any} \quad s \in [0,\, 1], \tag{7}$$

holds for some family Q_s of functions vanishing nowhere which satisfies $Q_0 = 1$. We put $\omega_s := Q_s({\varphi_s}^*\alpha_s) - \alpha_0$. In order to show equation (7), it is sufficient to show $(d\omega_s/ds) \wedge \beta = 0$ since $\omega_0 = Q_0({\varphi_0}^*\alpha_0) - \alpha_0 = 0$. By a simple calculation, we obtain

$$\begin{aligned}\frac{d\omega_s}{ds} &= Q_s\left\{\frac{dq_s}{ds}({\varphi_s}^*\alpha_s) + {\varphi_s}^*\left(L_{X_s}\alpha_s + \frac{d\alpha_s}{ds}\right)\right\}\\ &= Q_s{\varphi_s}^*\left\{\left(\frac{dq_s}{ds}\circ{\varphi_s}^{-1}\right)\alpha_s + L_{X_s}\alpha_s + \frac{d\alpha_s}{ds}\right\},\end{aligned}$$

where q_s is a family of functions which satisfies $Q_s = \exp(q_s)$. Then $(d\omega_s/ds) \wedge \beta = 0$ is rewritten as

$$\left(\tilde{q}_s\alpha_s + L_{X_s}\alpha_s + \frac{d\alpha_s}{ds}\right) \wedge \beta = 0, \tag{8}$$

where $\tilde{q}_s = (dq_s/ds) \circ {\varphi_s}^{-1}$. Since X_s is a solution to equation (6), equation (8) holds with $\tilde{q}_s = b_s$. Thus we have proved that equation (5) and (6) imply equations (4).

Now it remains to show the solvability of equations (5) and (6). As we remarked at the beginning of this subsection, we suppose X_s is tangent to the Engel line field $L \equiv L({D_s}^2) \subset D_s$. It is clear that if X_s is tangent to $L = \ker\beta \wedge d\beta$, it satisfies equation (5) for some family a_s of functions vanishing nowhere. We show the solvability of equation (6). Let Z be a nowhere vanishing vector field which is tangent to L, γ a 1-form which satisfies $\gamma(Z)=1$, and δ_s a family of 1-forms which satisfies $\alpha_s \wedge \beta \wedge \gamma \wedge \delta_s \neq 0$. Restricting equation (6) to D_s, we obtain

$$X_s \lrcorner d\alpha_s|_{D_s} + \left.\frac{d\alpha_s}{ds}\right|_{D_s} = 0. \tag{9}$$

Evaluating equation (9) with Z, we obtain $(d\alpha_s/ds)(Z)=0$. Then we obtain $X_s \lrcorner d\alpha_s|_{D_s} + F_s\delta_s|_{D_s} = 0$ for some family F_s of functions. According to the condition (i) $\beta \wedge \alpha_s \wedge d\alpha_s \neq 0$ of Engel pairs of 1-forms, we have $d\alpha_s|_{D_s} = H_s\gamma \wedge \delta_s|_{D_s}$ for some family H_s of functions. Then we obtain a solution $X_s = (-F_s/H_s)Z$ to equation (6) for some families b_s, c_s of functions.

Next, we make the local solutions global. Since M is closed, there exists a locally finite open covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of a neighborhood of $M \subset W$, each of that open set U_λ intersects M and has an Engel pair $(\alpha_s^\lambda, \beta^\lambda)$ of 1-forms determining D_s. On each U_λ, we have a solution X_s^λ to equations (5), (6). To make them global on a neighborhood of $M \subset W$, we need a partition of unity $\{f_\lambda\}_{\lambda\in\Lambda}$ subordinate to $\{U_\lambda\}_{\lambda\in\Lambda}$. Then we obtain a family $X_s := \sum_{\lambda\in\Lambda} f_\lambda X_s^\lambda$ of vector fields defined on a neighborhood of $M \subset W$. X_s satisfies equations (5), (6) at any $x \in M$ locally in W. Since M is closed, X_s defines a family φ_s of global diffeomorphisms.

Last of all, we show that the family φ_s of diffeomorphisms thus obtained preserves M and $\mathcal{F}(M, D_0)$. The isotopy φ_s, generated by X_s, preserves the Engel line field L and the derived even-contact structure ${D_0}^2$, since it preserves, up to multiplication by functions vanishing nowhere, the second 1-form β of Engel pair. Furthermore, since $\iota^*\alpha_s \equiv \iota^*\alpha_0 = \theta^0$, where $\iota\colon M \hookrightarrow M\times\mathbb{R}$ is an inclusion, we have $(\partial\alpha_s/\partial s)(\eta)=0$ for any vector field η on M. On account of equation (9), we have $d\alpha_s(X_s, \xi)=0$ for any vector field ξ on M tangent to D_s. Since $d\alpha_s$ is non-degenerate on D_s, we conclude that X_s is tangent to M. Therefore φ_s preserves M. Recall that $\mathcal{F}(M, \varphi_{s*}(D_0)) = \mathcal{F}(M, D_s)$ is a subsheaf of the sheaf Ω_M^1 of germs of C^∞ differential 1-forms on M generated by $\iota^*\alpha_s = \theta^0$ and $\iota^*\beta = \omega^0$. Therefore it coincides with $\mathcal{F}(M, D_0)$, which is generated by $\iota^*\alpha_0 = \theta^0$ and $\iota^*\beta = \omega^0$. This completes the proof of Theorem. □

We remark that $\varphi_s|_M$ is not necessarily an identity. If the Engel line field L is transverse to M, then we can take $\varphi_s|_M$ to be an identity.

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