

Rigidity of the canonical isometric imbedding of the symplectic group $Sp(n)$

Yoshio AGAOKA and Eiji KANEDA

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Abstract. In this paper, we discuss the rigidity of $Sp(n)$ as a Riemannian submanifold of $M(n, n; \mathbb{H})$. We prove that the inclusion map \mathbf{f}_0 , which is called the canonical isometric imbedding of $Sp(n)$, is rigid in the following strongest sense: Any isometric immersion \mathbf{f}_1 of a connected open set $U(\subset Sp(n))$ into $\mathbf{R}^{4n^2}(\cong M(n, n; \mathbb{H}))$ coincides with \mathbf{f}_0 up to a euclidean transformation of \mathbf{R}^{4n^2} , i.e., there is a euclidean transformation a of \mathbf{R}^{4n^2} satisfying $\mathbf{f}_1 = a\mathbf{f}_0$ on U .

Key words: curvature invariant, isometric imbedding, rigidity, symplectic group.

Introduction

The subject of this paper is to prove the rigidity of the symplectic group $Sp(n)$ as a Riemannian submanifold of the space of matrices over the field of quaternion numbers.

Let $M(n, n; \mathbb{H})$ be the space of $n \times n$ -matrices over the field \mathbb{H} of quaternion numbers. Considering $M(n, n; \mathbb{H})$ as a real vector space, we define a bilinear form ν on $M(n, n; \mathbb{H})$ by setting

$$\nu(X, Y) = \operatorname{Re}(\operatorname{Trace}({}^t\bar{X}Y)), \quad X, Y \in M(n, n; \mathbb{H}).$$

It is easily seen that ν defines an inner product on $M(n, n; \mathbb{H})$. With this inner product ν we can regard $M(n, n; \mathbb{H})$ as the euclidean space \mathbb{R}^{4n^2} . The *symplectic group* $Sp(n)$ is given by a submanifold of $M(n, n; \mathbb{H})$ consisting of all matrices $g \in M(n, n; \mathbb{H})$ satisfying $g{}^t\bar{g} = {}^t\bar{g}g = I_n$, where I_n is the identity matrix of degree n . The induced metric on $Sp(n)$, which is denoted by the same symbol ν , is bi-invariant on $Sp(n)$. The inclusion map $\mathbf{f}_0: Sp(n) \rightarrow M(n, n; \mathbb{H}) \cong \mathbb{R}^{4n^2}$ gives an isometric imbedding of the Riemannian manifold $(Sp(n), \nu)$ into \mathbb{R}^{4n^2} and is called the *canonical isometric imbedding* of $Sp(n)$ into \mathbb{R}^{4n^2} (cf. Kobayashi [17]). In this paper we will discuss the rigidity of the canonical isometric imbedding \mathbf{f}_0 .

Let M be a Riemannian manifold and let \mathbf{f} be an isometric imbedding of

M into the euclidean space \mathbb{R}^N . By definition \mathbf{f} is called *strongly rigid* when \mathbf{f} is rigid even if we restrict \mathbf{f} to any connected open set of M , i.e., for any isometric immersion \mathbf{f}' of a connected open set $U(\subset M)$ into \mathbb{R}^N there exists a euclidean transformation a of \mathbb{R}^N satisfying $\mathbf{f}' = a\mathbf{f}$ on U . In [8] and [9] we showed that the canonical isometric imbeddings of the quaternion projective plane $P^2(\mathbb{H})$ and the Cayley projective plane $P^2(\mathbb{CAY})$ are strongly rigid.

Concerning the canonical isometric imbedding \mathbf{f}_0 of $Sp(n)$ into \mathbb{R}^{4n^2} , the following results are known:

- (1) In the case where $n = 1$, \mathbf{f}_0 is just the standard isometric imbedding of $S^3(\cong Sp(1))$ into \mathbb{R}^4 with radius 1, which is a typical example of isometric imbeddings with type number 3. Accordingly, by Allendoerfer [12] \mathbf{f}_0 is known to be strongly rigid.
- (2) By investigating the Gauss equation of $Sp(2)$ in codimension 6 (for the definition, see §2 below), Agaoka [1] showed that the set of solutions of the Gauss equation is composed of essentially one solution, i.e., any solution is equivalent to the second fundamental form of \mathbf{f}_0 . Utilizing this fact, Agaoka proved that \mathbf{f}_0 is strongly rigid when $n = 2$.
- (3) Kaneda [15] proved that $\mathbf{f}_0(n \geq 1)$ is globally rigid in the sense of Tanaka [19], i.e., if two differentiable maps $\mathbf{f}_i(i = 1, 2)$ of $Sp(n)$ into \mathbb{R}^{4n^2} lie both near to \mathbf{f}_0 with respect to C^3 -topology, and if they induce the same Riemannian metric on $Sp(n)$, then there is a euclidean transformation a of \mathbb{R}^{4n^2} such that $\mathbf{f}_2 = a\mathbf{f}_1$.
- (4) By determining the pseudo-nullity of $Sp(n)(n \geq 1)$, Agaoka-Kaneda [4] proved that \mathbb{R}^{4n^2} is the least dimensional euclidean space into which $Sp(n)$ can be locally isometrically immersed. (For the definition of the pseudo-nullity, see §1.) In other words, $Sp(n)(n \geq 1)$ cannot be isometrically immersed into \mathbb{R}^{4n^2-1} even locally.

In this paper, we will extend these results (1) ~ (4) in the following strongest sense:

Theorem 1 *Let \mathbf{f}_0 be the canonical isometric imbedding of the symplectic group $Sp(n)$ into the euclidean space \mathbb{R}^{4n^2} . Then \mathbf{f}_0 is strongly rigid, i.e., for any isometric immersion \mathbf{f} of a connected open set $U(\subset Sp(n))$ into \mathbb{R}^{4n^2} there is a euclidean transformation a of \mathbb{R}^{4n^2} satisfying $\mathbf{f} = a\mathbf{f}_0$ on U .*

It should be noted that $Sp(n)(n \geq 1)$ are the first examples such that the canonical isometric imbeddings of a series of Riemannian symmetric spaces parametrized by rank are strongly rigid. We note that Theorem 1

for the cases $n \geq 2$ cannot be proved by applying the theory of type number in [12]. In fact, the type number of the canonical isometric imbedding \mathbf{f}_0 of $Sp(n)$ is less than 2 in case $n \geq 2$ (precisely, see Remark 11 in §2). The method of our proof is quite similar to the methods adopted in [8] and [9]. We first make a preparatory study on pseudo-abelian subspaces of $\mathfrak{sp}(n)$, which is the Lie algebra of $Sp(n)$. Utilizing the knowledge about the pseudo-abelian subspaces of maximum dimension, we determine the set of all solutions of the Gauss equation of $Sp(n)$ in codimension $2n^2 - n (= 4n^2 - \dim Sp(n))$. Under this situation, it will be shown that the set of solutions is composed of essentially one solution, i.e., any solution is equivalent to the second fundamental form of \mathbf{f}_0 . Therefore by the theorem of coincidence (Theorem 5 of [8, pp. 335–336]) we can establish our rigidity theorem of $Sp(n)$ (Theorem 1).

Throughout this paper we will assume the differentiability of class C^∞ . For the notations of Lie algebras and Riemannian symmetric spaces, see Helgason [14]. For the quaternion numbers and the symplectic group $Sp(n)$, see Chevalley [13].

1. The pseudo-nullity of $Sp(n)$

In this section we study the pseudo-nullity of $Sp(n)$. We first recall the notion of a pseudo-abelian subspace (precisely, see [3]). Let G be a compact simple Lie group. Let \mathfrak{g} be the Lie algebra of G and \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} . A subspace $W \subset \mathfrak{g}$ is called *pseudo-abelian with respect to \mathfrak{h}* (or simply, *pseudo-abelian*) if it satisfies $[W, W] \subset \mathfrak{h}$. The maximum dimension of pseudo-abelian subspaces, which does not depend on the choice of a Cartan subalgebra \mathfrak{h} , is called the *pseudo-nullity* of G and is denoted by p_G . The pseudo-nullity of the symplectic group $Sp(n)$ has been already determined:

Theorem 2 (see [4]) *For the symplectic group $G = Sp(n)$ ($n \geq 1$), the pseudo-nullity is equal to $2n$, i.e., $p_{Sp(n)} = 2n$.*

In what follows we determine the pseudo-abelian subspace W of $\mathfrak{sp}(n)$ which attains the maximum dimension, i.e., $\dim W = p_{Sp(n)} = 2n$. First recall the field of quaternion numbers: Let \mathbb{R} be the field of real numbers. The field \mathbb{H} of quaternion numbers is an algebra over \mathbb{R} generated by the elements e^0, e^1, e^2 and e^3 satisfying

- (1) $e^0 e^i = e^i e^0 = e^i$ ($i = 0, 1, 2, 3$);
- (2) $(e^i)^2 = -e^0$ ($i = 1, 2, 3$);
- (3) For each permutation $\{i, j, k\}$ of $\{1, 2, 3\}$ it holds $e^i e^j = \varepsilon(ijk)e^k$, where $\varepsilon(ijk) = 1$ (resp. $\varepsilon(ijk) = -1$) if $\{i, j, k\}$ is an even (resp. odd) permutation.

From (1) we can see that e^0 is a unit element of \mathbb{H} . Let us simply express the element ae^0 ($a \in \mathbb{R}$) as a . In this meaning \mathbb{R} is contained in \mathbb{H} and forms a subfield of \mathbb{H} .

Let $f \in \mathbb{H}$. Then f may be written in the form $f = f_0 + \sum_{i=1}^3 f_i e^i$, where $f_0, f_1, f_2, f_3 \in \mathbb{R}$. As usual we define the real part and the conjugate of f as follows: $\operatorname{Re}(f) = f_0$; $\bar{f} = f_0 - \sum_{i=1}^3 f_i e^i$. Then we have $\operatorname{Re}(f) = \operatorname{Re}(\bar{f})$, $f\bar{f} = \bar{f}f = \sum_{i=0}^3 f_i^2$. Moreover:

$$\operatorname{Re}(fh) = \operatorname{Re}(hf), \quad \overline{f\bar{h}} = \bar{h}\bar{f}, \quad f, h \in \mathbb{H}.$$

Let $i = 1, 2$ or 3 . Define a subset \mathbb{C}^i of \mathbb{H} by $\mathbb{C}^i = \mathbb{R} + \mathbb{R}e^i$. It is easily seen that \mathbb{C}^i forms a subfield of \mathbb{H} and is isomorphic to the field \mathbb{C} of complex numbers. We also define a subset \mathbb{D}^i of \mathbb{H} by $\mathbb{D}^i = \mathbb{R}e^j + \mathbb{R}e^k$, where j and k are so chosen that $\{i, j, k\}$ is a permutation of $\{1, 2, 3\}$. Then it is clear that

$$\mathbb{C}^i \mathbb{D}^i = \mathbb{D}^i \mathbb{C}^i = \mathbb{D}^i; \quad \mathbb{D}^i \mathbb{D}^i = \mathbb{C}^i.$$

In the following we denote by $M(p, q; \mathbb{H})$ the space of $p \times q$ -matrices over \mathbb{H} . As stated in Introduction, the symplectic group $Sp(n)$ is considered as a submanifold of $M(n, n; \mathbb{H}) \cong \mathbb{R}^{4n^2}$. As usual, we identify the tangent space of $Sp(n)$ at the identity $I_n \in Sp(n)$ with the Lie algebra $\mathfrak{sp}(n)$, which is consisting of all matrices $X \in M(n, n; \mathbb{H})$ satisfying ${}^t \bar{X} = -X$. Let us denote by E_{st} ($1 \leq s, t \leq n$) the matrix of $M(n, n; \mathbb{H})$ such that the (s, t) -component is 1 and the others are 0. We define subspaces $\mathfrak{h}(n)^i$ and $\mathfrak{p}(n)^i$ of $\mathfrak{sp}(n)$ by

$$\mathfrak{h}(n)^i = \sum_{s=1}^n \mathbb{R}e^i E_{ss}; \quad \mathfrak{p}(n)^i = \sum_{s=1}^n \mathbb{D}^i E_{ss}.$$

As is well-known, $\mathfrak{h}(n)^i$ is a Cartan subalgebra of $\mathfrak{sp}(n)$. Moreover:

Proposition 3 *Let $i = 1, 2$ or 3 . Then, $\mathfrak{p}(n)^i$ is pseudo-abelian with respect to $\mathfrak{h}(n)^i$ with $\dim \mathfrak{p}(n)^i = p_{Sp(n)}$.*

Proof. It is clear that $\dim \mathfrak{p}(n)^i = 2n$. Let $X = \sum_s u_s E_{ss}$, $Y = \sum_s v_s E_{ss} \in \mathfrak{p}(n)^i$, where $u_s, v_s \in \mathbb{D}^i$. Then, since $E_{ss}E_{ss} = E_{ss}$ and $E_{ss}E_{s's'} = 0$ ($s \neq s'$), we have $[X, Y] = \sum_s (u_s v_s - v_s u_s) E_{ss}$. Since $u_s, v_s \in \mathbb{D}^i$, it follows that $u_s v_s, v_s u_s \in \mathbb{C}^i$ and $u_s v_s - v_s u_s \in \mathbb{R}e^i$. Hence $[X, Y] \in \mathfrak{h}(n)^i$, proving $[\mathfrak{p}(n)^i, \mathfrak{p}(n)^i] \subset \mathfrak{h}(n)^i$. \square

Further, the space $\mathfrak{p}(n)^i$ is the only pseudo-abelian subspace with respect to $\mathfrak{h}(n)^i$ of dimension $p_{Sp(n)}$. In fact, we have

Theorem 4 *Let $i = 1, 2$ or 3 . Let W be a pseudo-abelian subspace with respect to $\mathfrak{h}(n)^i$ satisfying $\dim W = p_{Sp(n)}$. Then $W = \mathfrak{p}(n)^i$.*

In the rest of this section we prove this theorem. Let $X = \sum_{st} \xi_{st} E_{st} \in M(n, n; \mathbb{H})$. We denote by $x_p = (\xi_{p1}, \dots, \xi_{pn}) \in M(1, n; \mathbb{H})$ the p -th row of X and by $x^q = {}^t(\xi_{1q}, \dots, \xi_{nq}) \in M(n, 1; \mathbb{H})$ the q -th column of X . Then we may write

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (x^1, \dots, x^n).$$

As is easily seen, $X \in \mathfrak{sp}(n)$ if and only if

$${}^t \bar{x}_p + x^p = 0 \quad (1 \leq p \leq n). \quad (1.1)$$

Let $X = (x^1, \dots, x^n)$, $Y = (y^1, \dots, y^n) \in \mathfrak{sp}(n)$. Then $[X, Y] \in \mathfrak{h}(n)^i$ if and only if the following conditions are satisfied:

$$(x^p, y^q) = (y^p, x^q) \quad (1 \leq p < q \leq n), \quad (1.2)$$

$$(x^r, y^r) \in \mathbb{C}^i \quad (1 \leq r \leq n), \quad (1.3)$$

where $(,)$ denotes the inner product of $M(n, 1; \mathbb{H})$ defined by $(\xi, \eta) = {}^t \bar{\xi} \eta$ for $\xi, \eta \in M(n, 1; \mathbb{H})$. Then we note the following formula:

$$\overline{(\xi, \eta)} = (\eta, \xi), \quad (\xi f, \eta) = \bar{f}(\xi, \eta), \quad (\xi, \eta f) = (\xi, \eta) f, \quad f \in \mathbb{H}. \quad (1.4)$$

Now we start the proof of Theorem 4 by induction on n . First consider the case $n = 1$. In a natural way we identify $M(1, 1; \mathbb{H})$ with \mathbb{H} . Then by (1.1) we know that $w = a_0 + \sum_{j=1}^3 a_j e^j \in \mathbb{H}$ belongs to $\mathfrak{sp}(1)$ if and only if $a_0 = 0$. Let W be a pseudo-abelian subspace of $\mathfrak{sp}(1)$ with respect to $\mathfrak{h}(1)^i$ with $\dim W = 2$. Suppose that $W \neq \mathbb{D}^i$. Take a basis $\{w, w'\}$ of W

such that $w \notin \mathbb{D}^i$, i.e., w is an element written in the form $w = \sum_{j=1}^3 a_j e^j$, where $a_i \neq 0$. By subtracting a scalar multiple of w from w' if necessary, we may assume that $w' \in \mathbb{D}^i$. Then we have $ww' = (\sum_{j \neq i} a_j e^j)w' + a_i e^i w'$, $(\sum_{j \neq i} a_j e^j)w' \in \mathbb{C}^i$ and $a_i e^i w' \in \mathbb{D}^i$. On the other hand, by (1.3) we have $ww' = -\bar{w}w' \in \mathbb{C}^i$. This is impossible because $a_i e^i w' \neq 0$. Hence we have $W = \mathbb{D}^i = \mathfrak{p}(1)^i$, showing that Theorem 4 is true when $n = 1$.

We now assume that $n \geq 2$ and Theorem 4 is true for any n' ($1 \leq n' < n$). For simplicity, we regard $\mathfrak{sp}(s)$ ($1 \leq s < n$) as a subalgebra of $\mathfrak{sp}(n)$ in the following manner:

$$\mathfrak{sp}(s) \ni X \longmapsto \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{sp}(n).$$

Let W be a pseudo-abelian subspace of $\mathfrak{sp}(n)$ with respect to $\mathfrak{h}(n)^i$. As in [4] we define an ascending chain of subspaces

$$0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_n = W$$

by setting $W_r = \mathfrak{sp}(r) \cap W$ ($1 \leq r \leq n$). (Note that the numbering of the above chain is the reverse order of that in [4, p. 79].) It is obvious that W_r is a pseudo-abelian subspace of $\mathfrak{sp}(r)$ with respect to $\mathfrak{h}(r)^i$. Put

$$C_r = \{x^r \in M(n, 1; \mathbb{H}) \mid (x^1, \dots, x^r, \overbrace{0, \dots, 0}^{n-r}) \in W_r\} \\ (r = 1, \dots, n).$$

Then it is clear that $C_r \cong W_r/W_{r-1}$ ($1 \leq r \leq n$) and $\dim W = c_1 + \cdots + c_n$, where we set $c_r = \dim C_r$ ($1 \leq r \leq n$). Moreover, by (1.2) and (1.3) we have

$$(C_p, C_q) = 0 \quad (1 \leq p < q \leq n), \quad (1.5)$$

$$(C_r, C_r) \subset \mathbb{C}^i \quad (1 \leq r \leq n). \quad (1.6)$$

The above equalities (1.5) and (1.6) will play decisive roles in the proof of Theorem 4.

By $C_r^{\mathbb{H}}$ ($1 \leq r \leq n$) we denote the right \mathbb{H} -subspace of $M(n, 1; \mathbb{H})$ generated by C_r . Set $k_r = \dim_{\mathbb{H}} C_r^{\mathbb{H}}$ ($1 \leq r \leq n$). Then, in view of (1.5) and (1.4) we have

$$(C_p^{\mathbb{H}}, C_q^{\mathbb{H}}) = 0 \quad (1 \leq p < q \leq n). \quad (1.7)$$

Utilizing (1.6) and (1.7), we have proved in [4] the following

Lemma 5 (see [4]) *Under the setting stated above the following (1) and (2) hold:*

- (1) $k_1 + \cdots + k_n \leq n$.
- (2) $c_r \leq 2k_r$ ($1 \leq r \leq n$).

In particular, if $\dim W = p_{Sp(n)}$ ($= 2n$), then $k_1 + \cdots + k_n = n$ and $c_r = 2k_r$ ($1 \leq r \leq n$).

In what follows we assume that W is a pseudo-abelian subspace with respect to $\mathfrak{h}(n)^i$ satisfying $\dim W = p_{Sp(n)}$. Let us define an \mathbb{R} -linear endomorphism $\xi \mapsto \tilde{\xi}$ of $M(n, 1; \mathbb{H})$ by setting $\tilde{\xi} = {}^t(\xi_1, \dots, \xi_{n-1}, 0)$ for $\xi = {}^t(\xi_1, \dots, \xi_n) \in M(n, 1; \mathbb{H})$. Let \widetilde{C}_n be the image of C_n by this endomorphism. We first prove

Lemma 6 $k_n \geq 1$ and $\dim_{\mathbb{H}} \widetilde{C}_n^{\mathbb{H}} \leq k_n - 1$.

Proof. Suppose that $k_n = 0$. Then we have $C_n = 0$ and hence $W = W_{n-1}$. Therefore, in a natural way W may be regarded as a pseudo-abelian subspace of $\mathfrak{sp}(n-1)$ with respect to $\mathfrak{h}(n-1)^i$. This implies $\dim W \leq p_{Sp(n-1)} = 2(n-1)$, contradicting the assumption $\dim W = 2n$. Consequently, we have $k_n \geq 1$. Let $\xi \in C_n$ and $\eta \in C_1 + \cdots + C_{n-1}$. Since η is written as $\eta = {}^t(\eta_1, \dots, \eta_{n-1}, 0)$, we have $(\tilde{\xi}, \eta) = (\xi, \eta) = 0$ (see (1.5)). Hence we have $(\widetilde{C}_n, C_1 + \cdots + C_{n-1}) = 0$. Viewing (1.4), we have $(\widetilde{C}_n^{\mathbb{H}}, C_1^{\mathbb{H}} + \cdots + C_{n-1}^{\mathbb{H}}) = 0$. Since both $\widetilde{C}_n^{\mathbb{H}}$ and $C_1^{\mathbb{H}} + \cdots + C_{n-1}^{\mathbb{H}}$ may be regarded as subspaces of $M(n-1, 1; \mathbb{H})$, we have $\dim_{\mathbb{H}} \widetilde{C}_n^{\mathbb{H}} \leq n-1 - (k_1 + \cdots + k_{n-1})$ (see (1.7)). Therefore by Lemma 5 we obtain $\dim_{\mathbb{H}} \widetilde{C}_n^{\mathbb{H}} \leq k_n - 1$. \square

Let C'_n be the subset of C_n consisting of all ${}^t(\xi_1, \dots, \xi_n) \in C_n$ such that the n -th component $\xi_n \in \mathbb{D}^i$, i.e., $C'_n = \{{}^t(\xi_1, \dots, \xi_n) \in C_n \mid \xi_n \in \mathbb{D}^i\}$. Clearly, C'_n is a subspace of C_n . We denote by \widetilde{C}'_n the image of C'_n by the endomorphism $\xi \mapsto \tilde{\xi}$. Then we can show

Lemma 7 $\dim C'_n \geq 2k_n - 1$ and $\dim \widetilde{C}'_n \leq 2(k_n - 1)$.

Proof. First we note that $\xi_n \in \mathbb{R}e^i + \mathbb{D}^i$ holds for any $\xi = {}^t(\xi_1, \dots, \xi_n) \in C_n$. Indeed, ξ_n is the (n, n) -component of a certain matrix $X \in \mathfrak{sp}(n)$ (recall the definition of C_n). Consequently, we have $\dim C'_n \geq \dim C_n - 1 = c_n - 1 = 2k_n - 1$.

We next prove the second inequality. Let $\xi = {}^t(\xi_1, \dots, \xi_n) \in C'_n$ and $\eta = {}^t(\eta_1, \dots, \eta_n) \in C'_n$. Then we easily have $(\tilde{\xi}, \tilde{\eta}) = (\xi, \eta) - \overline{\xi_n} \eta_n$. Since $(\xi, \eta) \in \mathbb{C}^i$ (see (1.6)) and $\overline{\xi_n} \eta_n \in \mathbb{D}^i \mathbb{D}^i = \mathbb{C}^i$, it follows that $(\tilde{\xi}, \tilde{\eta}) \in \mathbb{C}^i$. This proves $(\widetilde{C}'_n, \widetilde{C}'_n) \subset \mathbb{C}^i$. By this fact we can deduce that $\widetilde{C}'_n \cap \widetilde{C}'_n e^j = 0$ for any j ($= 1, 2, 3$) such that $j \neq i$. In fact, if there is an element $\tilde{\xi} \in \widetilde{C}'_n$ such that $\tilde{\xi} e^j \in \widetilde{C}'_n$, then we have $\mathbb{C}^i \ni (\tilde{\xi}, \tilde{\xi} e^j) = (\tilde{\xi}, \tilde{\xi}) e^j \in \mathbb{C}^i e^j = \mathbb{D}^i$. Since $\mathbb{C}^i \cap \mathbb{D}^i = 0$, it follows that $(\tilde{\xi}, \tilde{\xi}) = 0$, i.e., $\tilde{\xi} = 0$. Thus, we know that $\widetilde{C}'_n + \widetilde{C}'_n e^j$ ($\subset \widetilde{C}_n^{\mathbb{H}}$) is a direct sum if $j \neq i$. Consequently, we have $2 \dim \widetilde{C}'_n \leq 4 \dim_{\mathbb{H}} \widetilde{C}_n^{\mathbb{H}} \leq 4(k_n - 1)$, i.e., $\dim \widetilde{C}'_n \leq 2(k_n - 1)$ (see Lemma 6). This completes the proof of the lemma. \square

With the basis of Lemma 7 we can show

Lemma 8 *Let D_n be the kernel of the linear mapping $C_n \ni \xi \mapsto \tilde{\xi} \in \widetilde{C}_n$. Then:*

- (1) $D_n = \{{}^t(0, \dots, 0, w) \in M(n, 1; \mathbb{H}) \mid w \in \mathbb{D}^i\}$.
- (2) $\widetilde{C}_n \subset C_n$.
- (3) $C_n = D_n + \widetilde{C}_n$ (direct sum); $\dim \widetilde{C}_n = c_n - 2$.

Proof. By Lemma 7 we have $\dim C'_n - \dim \widetilde{C}'_n \geq 2k_n - 1 - 2(k_n - 1) > 0$. This implies that $D_n \cap C'_n \neq 0$. Let ξ be a non-trivial element of $D_n \cap C'_n$. Then, by the definitions of D_n and C'_n , we know that ξ may be written as $\xi = {}^t(0, \dots, 0, w)$, where $w \in \mathbb{D}^i$ ($w \neq 0$). Let $\eta = {}^t(\eta_1, \dots, \eta_n)$ be an arbitrary element of C_n . Then by (1.6) we have $(\xi, \eta) = \overline{w} \eta_n \in \mathbb{C}^i$. Hence we can easily show that $\eta_n \in \mathbb{D}^i$ (see the proof for the case $n = 1$). Accordingly, $\eta \in C'_n$ and hence $C'_n = C_n$. Therefore, we have

$$\dim D_n = \dim C_n - \dim \widetilde{C}_n = \dim C_n - \dim \widetilde{C}'_n \geq c_n - 2(k_n - 1) = 2.$$

On the other hand, since $D_n \subset C_n = C'_n$, we have $D_n \subset \{{}^t(0, \dots, 0, w) \mid w \in \mathbb{D}^i\}$ and hence $\dim D_n \leq \dim \mathbb{D}^i = 2$. This, together with the above inequality, proves $\dim D_n = 2$ and $D_n = \{{}^t(0, \dots, 0, w) \mid w \in \mathbb{D}^i\}$. Thus we obtain (1).

Let $\zeta = {}^t(\zeta_1, \dots, \zeta_n) \in M(n, 1; \mathbb{H})$ be an arbitrary element of C_n . Since $C_n = C'_n$, we have $\zeta_n \in \mathbb{D}^i$ and hence $\zeta' = {}^t(0, \dots, 0, \zeta_n) \in D_n \subset C_n$. Consequently, $\tilde{\zeta} = {}^t(\zeta_1, \dots, \zeta_{n-1}, 0) = \zeta - \zeta' \in C_n$, showing (2). The assertion (3) immediately follows from (1) and (2). \square

With these preparations we can show

Lemma 9 $\widetilde{C}_n = 0$. Accordingly, $C_n = D_n$.

Proof. We first prove

$$\widetilde{C}_n \cap \widetilde{C}_n e^i = 0. \quad (1.8)$$

Suppose that there is an element $\widetilde{\xi} = {}^t(\xi_1, \dots, \xi_{n-1}, 0) \in \widetilde{C}_n$ such that $\widetilde{\xi} e^i \in \widetilde{C}_n$. Note that $\widetilde{C}_n \subset C_n$ (see Lemma 8 (2)). By the definition of C_n we know that there are matrices X and $Y \in W$ written in the form

$$X = \begin{pmatrix} X' & \xi' \\ -{}^t\bar{\xi}' & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} Y' & \xi' e^i \\ e^i {}^t\bar{\xi}' & 0 \end{pmatrix},$$

where $X', Y' \in \mathfrak{sp}(n-1)$ and $\xi' = {}^t(\xi_1, \dots, \xi_{n-1}) \in M(n-1, 1; \mathbb{H})$. Take an integer j ($= 1, 2, 3$) such that $j \neq i$. Since ${}^t(0, \dots, 0, e^j) \in D_n \subset C_n$, we know that there is an element $Z \in W$ of the form

$$Z = \begin{pmatrix} Z' & 0 \\ 0 & e^j \end{pmatrix},$$

where $Z' \in \mathfrak{sp}(n-1)$. Since W is a pseudo-abelian with respect to $\mathfrak{h}(n)^i$, we have $[X, Z] \in \mathfrak{h}(n)^i$ and $[Y, Z] \in \mathfrak{h}(n)^i$. Hence by a direct calculation we can show

$$Z' \xi' = \xi' e^j; \quad Z'(\xi' e^i) = (\xi' e^i) e^j. \quad (1.9)$$

By the second equality of (1.9) we have $(Z' \xi') e^i = \xi' (e^i e^j) = -\xi' (e^j e^i) = -(\xi' e^j) e^i$ and hence $Z' \xi' = -\xi' e^j$. This, together with the first equality of (1.9), proves $Z' \xi' = \xi' e^j = 0$. Hence we have $\xi' = 0$, i.e., $\widetilde{\xi} = 0$. This implies (1.8). As a result of (1.8), the subspace $\widetilde{C}_n + \widetilde{C}_n e^i$ ($\subset \widetilde{C}_n^{\mathbb{H}}$) is a direct sum. Since $\dim \widetilde{C}_n = c_n - 2 = 2(k_n - 1)$ (see Lemma 8 (3) and Lemma 5), it follows that $\dim_{\mathbb{R}} \widetilde{C}_n^{\mathbb{H}} \geq 2 \dim \widetilde{C}_n = 4(k_n - 1)$. Hence we have $\dim_{\mathbb{H}} \widetilde{C}_n^{\mathbb{H}} = (1/4) \dim_{\mathbb{R}} \widetilde{C}_n^{\mathbb{H}} \geq k_n - 1$. On the other hand, we have $\dim_{\mathbb{H}} \widetilde{C}_n^{\mathbb{H}} \leq k_n - 1$ (see Lemma 6). Therefore, we obtain $\dim_{\mathbb{H}} \widetilde{C}_n^{\mathbb{H}} = k_n - 1$ and $\widetilde{C}_n^{\mathbb{H}} = \widetilde{C}_n + \widetilde{C}_n e^i$. More strongly, we can prove $\widetilde{C}_n = 0$. In fact, since $\widetilde{C}_n^{\mathbb{H}} = \widetilde{C}_n + \widetilde{C}_n e^i$, it follows that

$$(\widetilde{C}_n^{\mathbb{H}}, \widetilde{C}_n^{\mathbb{H}}) \subset (\widetilde{C}_n, \widetilde{C}_n) + (\widetilde{C}_n e^i, \widetilde{C}_n) + (\widetilde{C}_n, \widetilde{C}_n e^i) + (\widetilde{C}_n e^i, \widetilde{C}_n e^i).$$

If $\widetilde{C}_n \neq 0$, then it is easy to see that $(\widetilde{C}_n^{\mathbb{H}}, \widetilde{C}_n^{\mathbb{H}}) = \mathbb{H}$. However, the

right side of the above inclusion is contained in \mathbb{C}^i , because $(\widetilde{C}_n, \widetilde{C}_n) \subset (C_n, C_n) \subset \mathbb{C}^i$ (see Lemma 8 (2) and (1.6)), $(\widetilde{C}_n e^i, \widetilde{C}_n) \subset e^i \mathbb{C}^i = \mathbb{C}^i$, $(\widetilde{C}_n, \widetilde{C}_n e^i) \subset \mathbb{C}^i e^i = \mathbb{C}^i$ and $(\widetilde{C}_n e^i, \widetilde{C}_n e^i) \subset e^i \mathbb{C}^i e^i = \mathbb{C}^i$ (see (1.4)). This is a contradiction. Hence we have $\widetilde{C}_n = 0$. The equality $C_n = D_n$ now follows immediately. \square

Proof of Theorem 4. By Lemma 9 and Lemma 8 (3) we have $c_n = 2k_n = 2$. Hence, W_{n-1} , which is a pseudo-abelian subspace of $\mathfrak{sp}(n-1)$ with respect to $\mathfrak{h}(n-1)^i$, satisfies $\dim W_{n-1} = c_1 + \cdots + c_{n-1} = 2(n-1) = p_{Sp(n-1)}$. Therefore, by the hypothesis of our induction we know that $W_{n-1} = \mathfrak{p}(n-1)^i$. From this fact we can deduce $W = \mathfrak{p}(n)^i$. In fact, let X be an arbitrary element of W . Then X may be written as $X = \begin{pmatrix} X' & 0 \\ 0 & w \end{pmatrix}$, where $X' \in \mathfrak{sp}(n-1)$, $w \in \mathbb{D}^i$ (see Lemma 9 and Lemma 8 (1)). Since $[X, W_{n-1}] \subset \mathfrak{h}(n)^i$, it follows that $[X', \mathfrak{p}(n-1)^i] \subset \mathfrak{h}(n-1)^i$. Hence we have $X' \in \mathfrak{p}(n-1)^i$, because $\mathfrak{p}(n-1)^i$ is a maximal pseudo-abelian subspace of $\mathfrak{sp}(n-1)$ with respect to $\mathfrak{h}(n-1)^i$. Consequently, we have $X \in \mathfrak{p}(n)^i$ and $W = \mathfrak{p}(n)^i$, which completes the proof of Theorem 4. \square

2. The Gauss equation of $Sp(n)$

Let M be a Riemannian manifold. We denote by g the Riemannian metric of M and by R the Riemannian curvature tensor of type (1, 3) with respect to g . Let $x \in M$ and let $T_x(M)$ (resp. $T_x^*(M)$) be the tangent (resp. cotangent) vector space of M at x . Let r be a non-negative integer. We define a quadratic equation with respect to an unknown $\Psi \in S^2 T_x^*(M) \otimes \mathbb{R}^r$ by

$$\begin{aligned} & -g(R(X, Y)Z, W) \\ & = \langle \Psi(X, Z), \Psi(Y, W) \rangle - \langle \Psi(X, W), \Psi(Y, Z) \rangle, \end{aligned} \quad (2.1)$$

where $X, Y, Z, W \in T_x(M)$ and $\langle \cdot, \cdot \rangle$ is the standard inner product of \mathbb{R}^r . We call (2.1) the *Gauss equation* in codimension r at x . The set of solutions of (2.1) is called the *Gaussian variety* in codimension r at x and is denoted by $\mathcal{G}_x(M, \mathbb{R}^r)$.

Let $O(r)$ be the orthogonal group of \mathbb{R}^r . We define an action of $O(r)$ on $S^2 T_x^*(M) \otimes \mathbb{R}^r$ by

$$(\rho\Psi)(X, Y) = \rho(\Psi(X, Y)), \quad X, Y \in T_x(M), \rho \in O(r). \quad (2.2)$$

As is easily seen, if Ψ is a solution of (2.1), then $\rho\Psi$ is also a solution of (2.1) for any $\rho \in O(r)$. We say that $\mathcal{G}_x(M, \mathbb{R}^r)$ is *EOS* if $\mathcal{G}_x(M, \mathbb{R}^r) \neq \emptyset$ and if $\mathcal{G}_x(M, \mathbb{R}^r)$ is composed of essentially one solution, i.e., for any solutions Ψ_1 and $\Psi_2 \in \mathcal{G}_x(M, \mathbb{R}^r)$ there is an element $\rho \in O(r)$ such that $\Psi_2 = \rho\Psi_1$.

In the following we consider the case where M is the symplectic group $Sp(n)$ endowed with the bi-invariant metric ν , which is induced from the inclusion $Sp(n) \subset M(n, n; \mathbb{H})$. As usual we identify the tangent space of $Sp(n)$ at the identity I_n with the Lie algebra $\mathfrak{sp}(n)$. We denote by $(\ , \)$ the inner product of $\mathfrak{sp}(n)$ induced from ν at I_n . The curvature transformation $R_0(X, Y)$ ($X, Y \in \mathfrak{sp}(n)$) of $Sp(n)$ at I_n is given by $R_0(X, Y) = -(1/4) \text{ad}([X, Y])$ (see [14]). Hence at I_n the Gauss equation (2.1) is written as

$$\begin{aligned} & \frac{1}{4} ([X, Y], Z], W) \\ & = \langle \Psi(X, Z), \Psi(Y, W) \rangle - \langle \Psi(X, W), \Psi(Y, Z) \rangle, \end{aligned} \quad (2.3)$$

where $\Psi \in S^2(\mathfrak{sp}(n)^*) \otimes \mathbb{R}^r$ and $X, Y, Z, W \in \mathfrak{sp}(n)$. We simply denote by $\mathcal{G}(Sp(n), \mathbb{R}^r)$ the Gaussian variety in codimension r at I_n . The main aim of this and the subsequent sections is to prove

Theorem 10 *For any positive integer n the Gaussian variety $\mathcal{G}(Sp(n), \mathbb{R}^{2n^2-n})$ in codimension $2n^2 - n$ is EOS.*

By homogeneity, we know that the Gaussian variety $\mathcal{G}_x(Sp(n), \mathbb{R}^{2n^2-n})$ in codimension $2n^2 - n$ is *EOS* at each $x \in Sp(n)$. By this result we conclude that $Sp(n)$ is formally rigid in codimension $2n^2 - n$. (For the definition of formal rigidity, see [8].) Accordingly, by Theorem 5 of [8] we can establish the rigidity theorem of $Sp(n)$ (Theorem 1).

In the following we will prove Theorem 10 by induction on n . As we have stated in the introduction, if $n = 1$, then $Sp(1) \cong S^3$ and the canonical isometric imbedding f_0 is the inclusion map of the standard sphere S^3 with radius 1 into \mathbb{R}^4 . The second fundamental form Ψ_0 of f_0 at $\mathbf{x} \in S^3$ is given by $\Psi_0 = -\nu\mathbf{x}$. Hence f_0 is a typical example of an isometric imbedding with type number 3. By applying the theory of type number in [12] or by a direct calculation we know that any solution Ψ of the Gauss equation of S^3 in codimension 1 can be represented by $\Psi = \pm\Psi_0$. Therefore we get Theorem 10 for the case $n = 1$. For this reason we may assume $n \geq 2$ in the following discussion.

Remark 11 It should be noted that in case $n \geq 2$ the theory of type number in [12] is not applicable to the canonical isometric imbedding \mathbf{f}_0 of $Sp(n)$. In fact, for an isometric imbedding \mathbf{f} of a Riemannian manifold M into the euclidean space \mathbb{R}^m , the type number k of \mathbf{f} must satisfy the inequality $k \leq \dim M / (m - \dim M)$ (see [18] or [16]). Consequently, in the case of \mathbf{f}_0 we can easily show that $k < 2$ when $n \geq 2$.

Now let $\mathfrak{N}(n)$ be the subspace of $M(n, n; \mathbb{H})$ composed of all $X \in M(n, n; \mathbb{H})$ satisfying ${}^t\bar{X} = X$. Clearly, we have $\dim \mathfrak{N}(n) = 2n^2 - n$ and

$$M(n, n; \mathbb{H}) = \mathfrak{sp}(n) + \mathfrak{N}(n) \quad (\text{orthogonal direct sum}).$$

As is easily seen, $\mathfrak{N}(n)$ is the normal vector space of the canonical isometric imbedding \mathbf{f}_0 at I_n . The second fundamental form Ψ_0 of \mathbf{f}_0 at I_n is an element of $S^2(\mathfrak{sp}(n)^*) \otimes \mathfrak{N}(n)$ given by

$$\Psi_0(X, Y) = \frac{1}{2}(XY + YX), \quad X, Y \in \mathfrak{sp}(n) \quad (2.4)$$

(see [15, p. 370]). Under a natural identification $(\mathfrak{N}(n), \nu) \cong (\mathbb{R}^{2n^2-n}, \langle \cdot, \cdot \rangle)$ as euclidean vector spaces we can regard the unknown Ψ in the Gauss equation (2.3) in codimension $2n^2 - n$ as an element of $S^2(\mathfrak{sp}(n)^*) \otimes \mathfrak{N}(n)$. (In what follows, the inner product ν of $\mathfrak{N}(n)$ will be denoted by $\langle \cdot, \cdot \rangle$.) Therefore the Gaussian variety $\mathcal{G}(Sp(n), \mathbb{R}^{2n^2-n})$ may be considered as a subset of $S^2(\mathfrak{sp}(n)^*) \otimes \mathfrak{N}(n)$. In this meaning we write $\mathcal{G}(Sp(n), \mathbb{R}^{2n^2-n})$ as $\mathcal{G}(Sp(n), \mathfrak{N}(n))$. Then Ψ_0 may be considered as an element of $\mathcal{G}(Sp(n), \mathfrak{N}(n))$, which is called the *canonical solution* of the Gauss equation (2.3) in codimension $2n^2 - n$. Now Theorem 10 may be stated in the following way: Any solution $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$ of the Gauss equation (2.3) is equivalent to Ψ_0 , i.e., there is an element $\rho \in O(\mathfrak{N}(n))$ such that $\Psi = \rho\Psi_0$, where $O(\mathfrak{N}(n))$ stands for the orthogonal group of $\mathfrak{N}(n)$.

3. The space $\mathbf{K}_\Psi(X)$

In this section we assume that $n \geq 2$. Let $\Psi \in S^2(\mathfrak{sp}(n)^*) \otimes \mathfrak{N}(n)$ and let $X \in \mathfrak{sp}(n)$. We define a linear mapping $\Psi_X: \mathfrak{sp}(n) \rightarrow \mathfrak{N}(n)$ by setting $\Psi_X(Y) = \Psi(X, Y)$ ($Y \in \mathfrak{sp}(n)$). By $\mathbf{K}_\Psi(X)$ ($\subset \mathfrak{sp}(n)$) we denote the kernel of Ψ_X . In this section we investigate the kernel $\mathbf{K}_\Psi(X)$ for a solution Ψ of the Gauss equation (2.3), i.e., $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$. As in the case of $P^2(\mathbb{H})$ or $P^2(\mathbb{CAY})$, the knowledge about $\mathbf{K}_\Psi(X)$ will play an

important role to determine the solutions of the Gauss equation (2.3) (cf. [8] and [9]).

Let $X \in \mathfrak{sp}(n)$. By $C(X)$ we denote the centralizer of X in $\mathfrak{sp}(n)$. Then we have

Lemma 12 *Let $\Psi \in S^2(\mathfrak{sp}(n)^*) \otimes \mathfrak{N}(n)$ and $X \in \mathfrak{sp}(n)$. Then:*

- (1) $\dim \mathbf{K}_\Psi(X) \geq 2n$.
- (2) *If $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$, then $[\mathbf{K}_\Psi(X), \mathbf{K}_\Psi(X)] \subset C(X)$.*

Proof. Since

$$\dim \mathbf{K}_\Psi(X) \geq \dim Sp(n) - \dim \mathfrak{N}(n) = (2n^2 + n) - (2n^2 - n) = 2n,$$

we get (1). Assume that $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$. Then by (2.3) for each $Y \in \mathfrak{sp}(n)$ we have

$$\begin{aligned} ([[\mathbf{K}_\Psi(X), \mathbf{K}_\Psi(X)], X], Y) &\subset \langle \Psi(\mathbf{K}_\Psi(X), X), \Psi(\mathbf{K}_\Psi(X), Y) \rangle \\ &= 0. \end{aligned}$$

Consequently, we have $[[\mathbf{K}_\Psi(X), \mathbf{K}_\Psi(X)], X] = 0$. The assertion (2) immediately follows from this equality (cf. [10, Lemma 3]). \square

Let $X \in \mathfrak{sp}(n)$. Since $\mathfrak{sp}(n)$ is a compact simple Lie algebra, we know that $\dim C(X) \geq \text{rank}(\mathfrak{sp}(n)) = n$. We recall that an element $X \in \mathfrak{sp}(n)$ is called *regular* (resp. *singular*) if $\dim C(X) = n$ (resp. $\dim C(X) > n$).

Lemma 13 *Let $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$ and $H \in \mathfrak{h}(n)^i$ ($i = 1, 2, 3$). Then $\mathbf{K}_\Psi(H) \supset \mathfrak{p}(n)^i$. If H is regular, then the equality $\mathbf{K}_\Psi(H) = \mathfrak{p}(n)^i$ holds.*

Proof. Let $H \in \mathfrak{h}(n)^i$. Then by Lemma 12 (2) we have $[\mathbf{K}_\Psi(H), \mathbf{K}_\Psi(H)] \subset C(H)$. Assume that H is regular. Then, since $C(H) = \mathfrak{h}(n)^i$, we have $[\mathbf{K}_\Psi(H), \mathbf{K}_\Psi(H)] \subset \mathfrak{h}(n)^i$. This implies that $\mathbf{K}_\Psi(H)$ is a pseudo-abelian subspace with respect to $\mathfrak{h}(n)^i$. Therefore we have $\dim \mathbf{K}_\Psi(H) \leq p_{Sp(n)} = 2n$ (see Theorem 2). On the other hand, since $\dim \mathbf{K}_\Psi(H) \geq 2n$ (see Lemma 12 (1)), it follows that $\dim \mathbf{K}_\Psi(H) = 2n$. Hence $\mathbf{K}_\Psi(H) = \mathfrak{p}(n)^i$ (see Theorem 4). Let $H' \in \mathfrak{h}(n)^i$ be an arbitrary element. Note that regular elements are dense in $\mathfrak{h}(n)^i$ and, as we have shown, $\Psi(H, \mathfrak{p}(n)^i) = 0$ holds for any regular element $H \in \mathfrak{h}(n)^i$. Because of the continuity of Ψ we have $\Psi(H', \mathfrak{p}(n)^i) = 0$. This shows that $\mathbf{K}_\Psi(H') \supset \mathfrak{p}(n)^i$. \square

Let $\Psi \in S^2(\mathfrak{sp}(n)^*) \otimes \mathfrak{N}(n)$ and let $g \in Sp(n)$. We define an element

$\Psi^g \in S^2(\mathfrak{sp}(n)^*) \otimes \mathfrak{N}(n)$ by

$$(\Psi^g)(X, Y) = \Psi(\text{Ad}(g^{-1})X, \text{Ad}(g^{-1})Y), \quad X, Y \in \mathfrak{sp}(n). \quad (3.1)$$

Then we can easily see the following

Lemma 14 *Let $\Psi \in S^2(\mathfrak{sp}(n)^*) \otimes \mathfrak{N}(n)$ and let $g \in Sp(n)$. Then:*

- (1) $\mathbf{K}_{\Psi^g}(X) = \text{Ad}(g)\mathbf{K}_{\Psi}(\text{Ad}(g^{-1})X)$, $X \in \mathfrak{sp}(n)$.
- (2) $\Psi^g \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$ if and only if $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$.

Combining Lemma 13 with Lemma 14, we have

Proposition 15 *Let $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$, $X \in \mathfrak{sp}(n)$ and $g \in Sp(n)$. Assume that $\text{Ad}(g)X \in \mathfrak{h}(n)^i$ for some i ($= 1, 2, 3$). Then $\mathbf{K}_{\Psi}(X) \supset \text{Ad}(g^{-1})\mathfrak{p}(n)^i$. Further, if X is regular, then $\mathbf{K}_{\Psi}(X) = \text{Ad}(g^{-1})\mathfrak{p}(n)^i$.*

Proof. Note that $\Psi^g \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$ (see Lemma 14 (2)). Applying Lemma 13 to Ψ^g , we have $\mathbf{K}_{\Psi^g}(\text{Ad}(g)X) \supset \mathfrak{p}(n)^i$. Therefore by Lemma 14 (1) we have $\mathfrak{p}(n)^i \subset \mathbf{K}_{\Psi^g}(\text{Ad}(g)X) = \text{Ad}(g)\mathbf{K}_{\Psi}(X)$. Consequently, $\mathbf{K}_{\Psi}(X) \supset \text{Ad}(g^{-1})\mathfrak{p}(n)^i$. If X is regular, then $\text{Ad}(g)X$ is also regular. Accordingly, we have $\mathbf{K}_{\Psi^g}(\text{Ad}(g)X) = \mathfrak{p}(n)^i$ and hence $\mathbf{K}_{\Psi}(X) = \text{Ad}(g^{-1})\mathfrak{p}(n)^i$. \square

Remark 16 Let $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$. It is well-known that any element of $\mathfrak{sp}(n)$ is conjugate to an element of a Cartan subalgebra $\mathfrak{h}(n)^i$. Therefore, for a regular element $X \in \mathfrak{sp}(n)$ the space $\mathbf{K}_{\Psi}(X)$ is determined by Proposition 15. Here we note that if X is regular, then $\mathbf{K}_{\Psi}(X)$ does not depend on the choice of the solution $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$, i.e., $\mathbf{K}_{\Psi}(X) = \mathbf{K}_{\Psi'}(X)$ holds for any $\Psi, \Psi' \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$.

In the following discussion, we will determine $\mathbf{K}_{\Psi}(X)$ for singular elements $X \in \mathfrak{sp}(n)$ of special type. By Proposition 15 we immediately obtain

Proposition 17 *Let $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$. Let $i = 1, 2$ or 3 and $X \in \mathfrak{sp}(n)$. Denote by G_X^i the subset of $Sp(n)$ consisting of all $g \in Sp(n)$ such that $\text{Ad}(g)X \in \mathfrak{h}(n)^i$. Then:*

$$\mathbf{K}_{\Psi}(X) \supset \sum_{g \in G_X^i} \text{Ad}(g^{-1})\mathfrak{p}(n)^i. \quad (3.2)$$

Let a, b and i are integers satisfying $1 \leq a \neq b \leq n$, $1 \leq i \leq 3$. Define

elements H_a^i , P_{ab} and $Q_{ab}^i \in M(n, n; \mathbb{H})$ by

$$H_a^i = E_{aa}e^i; \quad P_{ab} = -P_{ba} = E_{ab} - E_{ba}; \quad Q_{ab}^i = Q_{ba}^i = (E_{ab} + E_{ba})e^i.$$

Then it is easily seen that H_a^i , P_{ab} , $Q_{ab}^i \in \mathfrak{sp}(n)$ and

$$\begin{aligned} (H_a^i, H_b^j) &= \delta_{ab}\delta_{ij}; & (H_a^i, P_{cd}) &= (H_a^i, Q_{cd}^j) = 0; \\ (P_{ab}, P_{cd}) &= 2(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}); & (P_{ab}, Q_{cd}^i) &= 0; \\ (Q_{ab}^i, Q_{cd}^j) &= 2(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})\delta_{ij}. \end{aligned} \quad (3.3)$$

Therefore the set $\{H_a^i \ (1 \leq a \leq n)\}$ forms an orthonormal basis of $\mathfrak{h}(n)^i$ ($1 \leq i \leq 3$) and the set $\{H_a^i \ (1 \leq a \leq n, 1 \leq i \leq 3), (1/\sqrt{2})P_{ab} \ (1 \leq a < b \leq n), (1/\sqrt{2})Q_{ab}^i \ (1 \leq a < b \leq n, 1 \leq i \leq 3)\}$ forms an orthonormal basis of $\mathfrak{sp}(n)$.

Let a, b and i are integers satisfying $1 \leq a \neq b \leq n, 1 \leq i \leq 3$. Define a subspace \mathfrak{s}_{ab}^i by $\mathfrak{s}_{ab}^i = \mathbb{R}(H_a^i - H_b^i) + \mathbb{R}P_{ab} + \mathbb{R}Q_{ab}^i$. By an easy calculation we have

$$\begin{aligned} [H_a^i - H_b^i, P_{ab}] &= 2Q_{ab}^i; & [H_a^i - H_b^i, Q_{ab}^i] &= -2P_{ab}; \\ [P_{ab}, Q_{ab}^i] &= 2(H_a^i - H_b^i). \end{aligned}$$

This indicates that \mathfrak{s}_{ab}^i forms a three-dimensional subalgebra of $\mathfrak{sp}(n)$ and is not abelian. Now we note the following lemma, which holds for any compact Lie algebra:

Lemma 18 *Let \mathfrak{s} be a three-dimensional subalgebra of a compact Lie algebra \mathfrak{g} . Assume that \mathfrak{s} is not abelian. Then, for any linearly independent elements $Z, Z' \in \mathfrak{s}$, there is an element $g \in \exp(\mathbb{R}[Z, Z']) \ (\subset \exp(\mathfrak{g}))$ such that $\text{Ad}(g)Z \in \mathbb{R}Z'$.*

Proof. Since \mathfrak{g} is compact, \mathfrak{s} is also a compact Lie algebra. Hence \mathfrak{s} may be represented by a direct sum of its center and its semi-simple part. Note that any simple Lie algebra is of dimension ≥ 3 . Under the assumption that \mathfrak{s} is not abelian and $\dim \mathfrak{s} = 3$, we know that the center of \mathfrak{s} is trivial and that \mathfrak{s} is simple. Hence, \mathfrak{s} is isomorphic to the simple Lie algebra $\mathfrak{su}(2)$.

Let B be an $\text{ad}(\mathfrak{g})$ -invariant inner product of \mathfrak{g} . Let $Z, Z' \in \mathfrak{s}$. If Z and Z' are linearly independent, then it follows that $[Z, Z'] \neq 0$, because $\text{rank}(\mathfrak{s}) = 1$. Set $\mathfrak{s}' = \mathbb{R}Z + \mathbb{R}Z'$. Then we have $B(\mathfrak{s}', \mathbb{R}[Z, Z']) = 0$, i.e., $\mathbb{R}[Z, Z']$ is the orthogonal complement of \mathfrak{s}' in \mathfrak{s} with respect to B . Indeed,

we have

$$\begin{aligned} B(Z, [Z, Z']) &= B([Z, Z], Z') = 0; \\ B(Z', [Z, Z']) &= -B([Z', Z'], Z) = 0. \end{aligned}$$

Similarly, we can prove $B(\text{ad}[Z, Z'](Z), [Z, Z']) = B(\text{ad}[Z, Z'](Z'), [Z, Z']) = 0$. This means that \mathfrak{s}' is invariant by $\text{ad}[Z, Z']$. Moreover, we have $\text{ad}([Z, Z'])Z'' \neq 0$ for any $Z'' \in \mathfrak{s}'$ with $Z'' \neq 0$. Therefore, $\text{Ad}(\exp(\mathbb{R}[Z, Z']))$ forms a non-trivial subgroup of rotations of \mathfrak{s}' with respect to B . From this fact the lemma follows immediately. \square

In the following, we say a subalgebra \mathfrak{s} of $\mathfrak{sp}(n)$ is *NAT* if \mathfrak{s} is non-abelian and $\dim \mathfrak{s} = 3$. As we have seen, $\mathfrak{s}_{ab}^i = \mathbb{R}(H_a^i - H_b^i) + \mathbb{R}P_{ab} + \mathbb{R}Q_{ab}^i$ is *NAT*. For non-zero elements X and $Y \in \mathfrak{sp}(n)$ we write $X \sim Y$ if there is an element $g \in Sp(n)$ such that $\text{Ad}(g)X \in \mathbb{R}Y$. Apparently, \sim defines an equivalence relation in $\mathfrak{sp}(n) \setminus \{0\}$. According to Lemma 18 if \mathfrak{s} is *NAT*, then $Z \sim Z'$ for any $Z, Z' \in \mathfrak{s} \setminus \{0\}$. For example, we have $(H_a^i - H_b^i) \sim P_{ab} \sim Q_{ab}^i$.

For simplicity in the following discussion we set $\mathbf{K}_0(X) = \mathbf{K}_{\Psi_0}(X)$. As in the previous section we regard $\mathfrak{sp}(s)$ ($0 \leq s < n$) as a subalgebra of $\mathfrak{sp}(n)$. Then by easy calculations we have

$$\begin{aligned} \mathbf{K}_0(H_n^i) &= \mathfrak{sp}(n-1) + \sum_{j \neq i} \mathbb{R}H_n^j; \\ \mathbf{K}_0(H_{n-1}^i + H_n^i) &= \mathfrak{sp}(n-2) + \sum_{j \neq i} \mathbb{R}H_{n-1}^j \\ &\quad + \sum_{j \neq i} \mathbb{R}H_n^j + \sum_{j \neq i} \mathbb{R}Q_{n-1,n}^j. \end{aligned} \tag{3.4}$$

Let Ψ be an arbitrary solution of the Gauss equation (2.3). By Remark 16 we know that $\mathbf{K}_{\Psi}(X) = \mathbf{K}_0(X)$ holds for a regular element $X \in \mathfrak{sp}(n)$. We now extend this relation to singular elements:

Proposition 19 *Let $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$. Then for each i ($= 1, 2, 3$) it holds:*

- (1) $\mathbf{K}_{\Psi}(H_n^i) = \mathbf{K}_0(H_n^i)$.
- (2) $\mathbf{K}_{\Psi}(H_{n-1}^i + H_n^i) = \mathbf{K}_0(H_{n-1}^i + H_n^i)$.

Proof. Let $Sp(n-1)$ be the analytic subgroup of $Sp(n)$ corresponding to the subalgebra $\mathfrak{sp}(n-1)$. Let $g \in Sp(n-1)$. Then it is easy to

see that $\text{Ad}(g)H_n^i = H_n^i$. Hence by Proposition 17 we have $\mathbf{K}_\Psi(H_n^i) \supset \sum_{g \in \text{Sp}(n-1)} \text{Ad}(g^{-1})\mathfrak{p}(n)^i$. Since $\mathfrak{h}(n-1)^j$ ($j \neq i$) is a Cartan subalgebra of $\mathfrak{sp}(n-1)$, any element of $\mathfrak{sp}(n-1)$ is conjugate to an element of $\mathfrak{h}(n-1)^j$ under the action of $\text{Sp}(n-1)$. Hence we have $\bigcup_{g \in \text{Sp}(n-1)} \text{Ad}(g^{-1})\mathfrak{h}(n-1)^j = \mathfrak{sp}(n-1)$. Since $\mathfrak{p}(n)^i \supset \mathfrak{h}(n-1)^j$, we have $\mathbf{K}_\Psi(H_n^i) \supset \mathfrak{sp}(n-1)$. This, together with $\mathbf{K}_\Psi(H_n^i) \supset \mathfrak{p}(n)^i$, shows $\mathbf{K}_\Psi(H_n^i) \supset \mathfrak{sp}(n-1) + \mathfrak{p}(n)^i = \mathbf{K}_0(H_n^i)$. We now show the equality $\mathbf{K}_\Psi(H_n^i) = \mathbf{K}_0(H_n^i)$. Take an element $X \in \mathbf{K}_\Psi(H_n^i) \cap \mathbf{K}_0(H_n^i)^\perp$, where $\mathbf{K}_0(H_n^i)^\perp$ is the orthogonal complement of $\mathbf{K}_0(H_n^i)$ in $\mathfrak{sp}(n)$. Then X can be expressed as

$$X = \begin{pmatrix} 0 & \xi \\ -t\bar{\xi} & ce^i \end{pmatrix}, \quad \xi \in M(n-1, 1; \mathbb{H}), c \in \mathbb{R}.$$

Take j, k ($= 1, 2, 3$) so that $\{i, j, k\}$ is an even permutation of $\{1, 2, 3\}$. Then since $X \in \mathbf{K}_\Psi(H_n^i)$ and $H_n^j \in \mathbf{K}_\Psi(H_n^i)$, we obtain by Lemma 12 the following

$$0 = [[X, H_n^j], H_n^i] = \begin{pmatrix} 0 & -\xi e^k \\ -e^k t\bar{\xi} & 4ce^j \end{pmatrix}.$$

Hence we have $\xi = 0$ and $c = 0$, i.e., $X = 0$. This proves $\mathbf{K}_\Psi(H_n^i) \cap \mathbf{K}_0(H_n^i)^\perp = 0$, i.e., $\mathbf{K}_\Psi(H_n^i) = \mathbf{K}_0(H_n^i)$.

Next we prove $\mathbf{K}_\Psi(H_{n-1}^i + H_n^i) = \mathbf{K}_0(H_{n-1}^i + H_n^i)$. As in the case of $\mathbf{K}_\Psi(H_n^i)$, we can easily show that $\mathbf{K}_\Psi(H_{n-1}^i + H_n^i) \supset \mathfrak{sp}(n-2) + \sum_{j \neq i} \mathbb{R}H_{n-1}^j + \sum_{j \neq i} \mathbb{R}H_n^j$. Take an element $Y \in \mathbf{K}_\Psi(H_{n-1}^i + H_n^i)$ such that $(Y, \mathfrak{sp}(n-2) + \sum_{j \neq i} \mathbb{R}H_{n-1}^j + \sum_{j \neq i} \mathbb{R}H_n^j) = 0$. Then Y can be expressed as

$$Y = \begin{pmatrix} 0 & \xi & \eta \\ -t\bar{\xi} & \alpha & \beta \\ -t\bar{\eta} & -\bar{\beta} & \gamma \end{pmatrix}, \quad \xi, \eta \in M(n-2, 1; \mathbb{H}), \alpha, \gamma \in \mathbb{R}e^i, \beta \in \mathbb{H}.$$

Take j, k ($= 1, 2, 3$) so that $\{i, j, k\}$ is an even permutation of $\{1, 2, 3\}$. Then by a direct calculation have

$$[[Y, H_{n-1}^j \pm H_n^j], H_{n-1}^i + H_n^i] = \begin{pmatrix} 0 & -\xi e^k & \mp \eta e^k \\ -e^k t\bar{\xi} & -4\alpha e^k & \beta'' \\ \mp e^k t\bar{\eta} & -\bar{\beta}'' & \mp 4\gamma e^k \end{pmatrix},$$

where $\beta' = \pm \beta e^j - e^j \beta$, $\beta'' = \beta' e^i - e^i \beta'$. (Note that $e^j \alpha = -\alpha e^j$, $e^j \gamma = -\gamma e^j$, $e^i \alpha = \alpha e^i$, $e^i \gamma = \gamma e^i$, because $\alpha, \gamma \in \mathbb{R}e^i$.) Since $Y \in \mathbf{K}_\Psi(H_{n-1}^i + H_n^i)$

H_n^i) and $H_{n-1}^j \pm H_n^j \in \mathbf{K}_\Psi(H_{n-1}^i + H_n^i)$, we have $[[Y, H_{n-1}^j \pm H_n^j], H_{n-1}^i + H_n^i] = 0$ (see Lemma 12). Hence we conclude that $\xi = \eta = 0$ and $\alpha = \gamma = 0$ and $\beta'' = 0$. From the equality $\beta'' = 0$, we immediately have $\beta' \in \mathbb{C}^i$. Further, from $\beta' \in \mathbb{C}^i$ we can easily conclude that $\beta \in \mathbb{D}^i$. Thus we have $Y \in \sum_{j \neq i} \mathbb{R}Q_{n-1,n}^j$ and hence $\mathbf{K}_\Psi(H_{n-1}^i + H_n^i) \subset \mathbf{K}_0(H_{n-1}^i + H_n^i)$.

To complete the proof of (2) we have to show $\mathbf{K}_\Psi(H_{n-1}^i + H_n^i) \supset \sum_{j \neq i} \mathbb{R}Q_{n-1,n}^j$. Take j ($1 \leq j \leq 3$) such that $j \neq i$. Since $\mathfrak{s}_{n-1,n}^j = \mathbb{R}(H_{n-1}^j - H_n^j) + \mathbb{R}P_{n-1,n} + \mathbb{R}Q_{n-1,n}^j$ is NAT, there is an element $g \in \exp(\mathbb{R}P_{n-1,n})$ such that $\text{Ad}(g)Q_{n-1,n}^j \in \mathbb{R}(H_{n-1}^j - H_n^j) (\subset \mathfrak{p}(n)^i)$ (see Lemma 18). Moreover, since $[P_{n-1,n}, H_{n-1}^i + H_n^i] = 0$, we have $\text{Ad}(g)(H_{n-1}^i + H_n^i) = H_{n-1}^i + H_n^i \in \mathfrak{h}(n)^i$, i.e., $g \in G_{(H_{n-1}^i + H_n^i)}^i$. Therefore, by Proposition 17 we have $Q_{n-1,n}^j \in \mathbf{K}_\Psi(H_{n-1}^i + H_n^i)$. Accordingly, it follows that $\mathbf{K}_\Psi(H_{n-1}^i + H_n^i) \supset \sum_{j \neq i} \mathbb{R}Q_{n-1,n}^j$, completing the proof of (2). \square

By \mathcal{S} we denote the subset of $\mathfrak{sp}(n)$ consisting of all non-zero elements $X \in \mathfrak{sp}(n)$ such that $X \sim H_n^i$ or $X \sim H_{n-1}^i + H_n^i$ for some i ($= 1, 2, 3$). We note that each element $X \in \mathcal{S}$ is a singular element of $\mathfrak{sp}(n)$, because H_n^i and $H_{n-1}^i + H_n^i$ are singular elements of $\mathfrak{sp}(n)$.

By use of Proposition 19 we can prove

Proposition 20 *Let $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$. Assume $X \in \mathcal{S}$. Then $\mathbf{K}_\Psi(X) = \mathbf{K}_0(X)$.*

Proof. Let $g \in Sp(n)$. Then we have Ψ^g and $\Psi_0^g \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$ (see Lemma 14 (2)). By applying Proposition 19 to Ψ^g and Ψ_0^g , we have

$$\begin{aligned} \mathbf{K}_{\Psi^g}(H_n^i) &= \mathbf{K}_0(H_n^i) = \mathbf{K}_{\Psi_0^g}(H_n^i); \\ \mathbf{K}_{\Psi^g}(H_{n-1}^i + H_n^i) &= \mathbf{K}_0(H_{n-1}^i + H_n^i) = \mathbf{K}_{\Psi_0^g}(H_{n-1}^i + H_n^i) \end{aligned}$$

for any i ($= 1, 2, 3$). Now assume that $X \in \mathcal{S}$ and that g is an element of $Sp(n)$ such that $\text{Ad}(g)X \in \mathbb{R}H_n^i$ or $\text{Ad}(g)X \in \mathbb{R}(H_{n-1}^i + H_n^i)$. Then by the above equalities we have $\mathbf{K}_{\Psi^g}(\text{Ad}(g)X) = \mathbf{K}_{\Psi_0^g}(\text{Ad}(g)X)$. (Note that $\mathbf{K}_\Psi(cZ) = \mathbf{K}_\Psi(Z)$ holds for any $\Psi \in S^2(\mathfrak{sp}(n)^*) \otimes \mathfrak{N}(n)$, $Z \in \mathfrak{sp}(n)$ and $c \in \mathbb{R}$ ($c \neq 0$).) On account of Lemma 14 (1) we have $\mathbf{K}_{\Psi^g}(\text{Ad}(g)X) = \text{Ad}(g)\mathbf{K}_\Psi(X)$ and $\mathbf{K}_{\Psi_0^g}(\text{Ad}(g)X) = \text{Ad}(g)\mathbf{K}_{\Psi_0}(X) = \text{Ad}(g)\mathbf{K}_0(X)$. Therefore $\mathbf{K}_\Psi(X) = \mathbf{K}_0(X)$ follows immediately. \square

As a consequence of Proposition 20 we can show

Proposition 21 *Let $i = 1, 2$ or 3 . Then*

- (1) $H_a^i \in \mathcal{S}$ ($1 \leq a \leq n$);
- (2) $H_a^i \pm H_b^i \in \mathcal{S}$ ($1 \leq a < b \leq n$);
- (3) $P_{ab} \in \mathcal{S}$, $Q_{ab}^i \in \mathcal{S}$ ($1 \leq a < b \leq n$).

Consequently, for any $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$ the following equalities hold:

$$\begin{aligned} \mathbf{K}_\Psi(H_a^i) &= \mathbf{K}_0(H_a^i); & \mathbf{K}_\Psi(H_a^i \pm H_b^i) &= \mathbf{K}_0(H_a^i \pm H_b^i); \\ \mathbf{K}_\Psi(P_{ab}) &= \mathbf{K}_0(P_{ab}); & \mathbf{K}_\Psi(Q_{ab}^i) &= \mathbf{K}_0(Q_{ab}^i). \end{aligned} \quad (3.5)$$

Proof. Let $i = 1, 2$ or 3 . It is easily shown that under the action of $Sp(n)$, H_a^i ($1 \leq a \leq n-1$) is conjugate to H_n^i . This implies that $H_a^i \in \mathcal{S}$ ($1 \leq a \leq n$). It is also known that $H_a^i + H_b^i$ ($1 \leq a < b \leq n$) (resp. $H_a^i - H_b^i$ ($1 \leq a < b \leq n$)) is conjugate to $H_{n-1}^i + H_n^i$ (resp. $H_{n-1}^i - H_n^i$). Let $\{i, j, k\}$ be a permutation of $\{1, 2, 3\}$. Then we easily have $[H_n^i, H_n^j] = 2\varepsilon(ijk)H_n^k$. This proves that $\mathfrak{s} = \sum_{i=1}^3 \mathbb{R}H_n^i$ is NAT. In view of the proof of Lemma 18 $\exp(\mathbb{R}H_n^k)$ acts on $\mathfrak{s}' = \mathbb{R}H_n^i + \mathbb{R}H_n^j$ as a non-trivial subgroup of rotations of \mathfrak{s}' . Hence, we can find an element $h \in \exp(\mathbb{R}H_n^k)$ such that $\text{Ad}(h)H_n^i = -H_n^i$. Since $[H_n^k, H_{n-1}^i] = 0$, we have $\text{Ad}(h)H_{n-1}^i = H_{n-1}^i$ and hence $\text{Ad}(h)(H_{n-1}^i - H_n^i) = H_{n-1}^i + H_n^i$. Therefore, we have $H_a^i \pm H_b^i \in \mathcal{S}$ ($1 \leq a < b \leq n$). As we have pointed out, $P_{ab} \sim Q_{ab}^i \sim (H_a^i - H_b^i)$. Since $H_a^i - H_b^i \in \mathcal{S}$, it follows that $P_{ab} \in \mathcal{S}$ and $Q_{ab}^i \in \mathcal{S}$. This completes the proof. \square

Remark 22 In the next section, after the proof of Theorem 10 we will know that $\mathbf{K}_\Psi(X) = \mathbf{K}_0(X)$ holds for any $X \in \mathfrak{sp}(n)$ (see Remark 36).

4. Solutions of the Gauss equation

In this section we will prove Theorem 10. We assume that $n \geq 2$ and that the Gaussian variety $\mathcal{G}(Sp(n'), \mathfrak{N}(n'))$ is EOS for any n' such that $n' < n$.

We now regard $\mathfrak{N}(n-1)$ as a subspace of $\mathfrak{N}(n)$ by the assignment

$$\mathfrak{N}(n-1) \ni Z \longmapsto \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{N}(n).$$

Let \mathfrak{M} be the orthogonal complement of $\mathfrak{N}(n-1)$ in $\mathfrak{N}(n)$. Then we easily have $\dim \mathfrak{M} = 4n - 3$ and

$$\mathfrak{M} = \mathbb{R}E_{nn} + \sum_{a=1}^{n-1} \left\{ \mathbb{R}(E_{an} + E_{na}) + \sum_{j=1}^3 \mathbb{R}(E_{an} - E_{na})e^j \right\}$$

(orthogonal direct sum).

As in the previous section, we denote by Ψ_0 the canonical solution (2.4). By a simple calculation we can easily verify that $\Psi_0(\mathfrak{sp}(n-1), \mathfrak{sp}(n-1)) = \mathfrak{N}(n-1)$ and $\mathfrak{M} = (\Psi_0)_{H_n^i}(\mathfrak{sp}(n))$ ($i = 1, 2, 3$). In a natural manner, the restriction $\Psi_0|_{\mathfrak{sp}(n-1)}$ of Ψ_0 to $\mathfrak{sp}(n-1)$ may be regarded as an element $\mathcal{G}(Sp(n-1), \mathfrak{N}(n-1))$. Therefore, by the hypothesis of our induction we have:

Lemma 23 *For any $\Psi' \in \mathcal{G}(Sp(n-1), \mathfrak{N}(n-1))$ there is an element $\rho' \in O(\mathfrak{N}(n-1))$ such that $\rho'\Psi' = \Psi_0|_{\mathfrak{sp}(n-1)}$.*

Let $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$. By $\mathbf{V}_\Psi(X)$ ($\subset \mathfrak{N}(n)$) we denote the image of $\mathfrak{sp}(n)$ by the map Ψ_X . We call Ψ a *normal solution* if Ψ satisfies:

(1) $\mathbf{V}_\Psi(H_n^i) = \mathfrak{M}$ ($i = 1, 2, 3$);

(2) $\Psi|_{\mathfrak{sp}(n-1)} = \Psi_0|_{\mathfrak{sp}(n-1)}$,

where $\Psi|_{\mathfrak{sp}(n-1)}$ means the restriction of Ψ to $\mathfrak{sp}(n-1)$. By $\mathcal{G}^0(Sp(n), \mathfrak{N}(n))$ we mean the subset of $\mathcal{G}(Sp(n), \mathfrak{N}(n))$ consisting of all normal solutions.

Proposition 24 *Let $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$. Then there is an element $\rho \in O(\mathfrak{N}(n))$ such that $\rho\Psi \in \mathcal{G}^0(Sp(n), \mathfrak{N}(n))$.*

Proof. Since $\dim \mathbf{K}_\Psi(H_n^i) = \dim \mathbf{K}_0(H_n^i)$ (see Proposition 19), we have $\dim \mathbf{V}_\Psi(H_n^i) = \dim \mathbf{V}_{\Psi_0}(H_n^i)$. Hence we have $\dim \mathbf{V}_\Psi(H_n^i) = \dim \mathfrak{M}$ for any i ($= 1, 2, 3$). Let $X, Y \in \mathfrak{sp}(n-1)$. Then by the Gauss equation (2.3) we get

$$\begin{aligned} & \frac{1}{4}([X, H_n^i], Y), Z) \\ &= \langle \Psi(X, Y), \Psi(H_n^i, Z) \rangle - \langle \Psi(X, Z), \Psi(H_n^i, Y) \rangle \end{aligned}$$

for any $Z \in \mathfrak{sp}(n)$ and i ($= 1, 2, 3$). Since $[X, H_n^i] = 0$ and $\mathbf{K}_\Psi(H_n^i) = \mathbf{K}_0(H_n^i) \supset \mathfrak{sp}(n-1)$ (see (3.4) and Proposition 19), we have $\Psi(H_n^i, Y) = 0$. Consequently, we have $\langle \Psi(X, Y), \Psi(H_n^i, Z) \rangle = 0$, which proves

$$\langle \Psi(\mathfrak{sp}(n-1), \mathfrak{sp}(n-1)), \mathbf{V}_\Psi(H_n^i) \rangle = 0. \quad (4.1)$$

Take an element $\rho_1 \in O(\mathfrak{N}(n))$ such that $\rho_1(\mathbf{V}_\Psi(H_n^1)) = \mathfrak{M}$. Then by (4.1) we have $(\rho_1\Psi)(\mathfrak{sp}(n-1), \mathfrak{sp}(n-1)) = \rho_1(\Psi(\mathfrak{sp}(n-1), \mathfrak{sp}(n-1))) \subset$

$\mathfrak{N}(n-1)$. Hence, in a natural manner, $(\rho_1\Psi)|_{\mathfrak{sp}(n-1)}$ may be regarded as an element of $\mathcal{G}(Sp(n-1), \mathfrak{N}(n-1))$. Hence there is an element $\rho'_2 \in O(\mathfrak{N}(n-1))$ such that $\rho'_2((\rho_1\Psi)|_{\mathfrak{sp}(n-1)}) = \Psi_0|_{\mathfrak{sp}(n-1)}$ (see Lemma 23). Take $\rho_2 \in O(\mathfrak{N}(n))$ such that $\rho_2|_{\mathfrak{M}} = \mathbf{1}_{\mathfrak{M}}$ and $\rho_2|_{\mathfrak{N}(n-1)} = \rho'_2$. Put $\rho = \rho_2\rho_1$. Then we have $\mathbf{V}_{\rho\Psi}(H_n^1) = \rho(\mathbf{V}_{\Psi}(H_n^1)) = \mathfrak{M}$ and $(\rho\Psi)|_{\mathfrak{sp}(n-1)} = \Psi_0|_{\mathfrak{sp}(n-1)}$. We finally prove $\mathbf{V}_{\rho\Psi}(H_n^i) = \mathfrak{M}$ ($i = 2, 3$). As is easily seen, we have $\Psi(\mathfrak{sp}(n-1), \mathfrak{sp}(n-1)) = \rho^{-1}(\mathfrak{N}(n-1))$. Hence by (4.1) we have $\mathbf{V}_{\Psi}(H_n^i) \subset \rho^{-1}(\mathfrak{M})$. Therefore, $\mathbf{V}_{\rho\Psi}(H_n^i) = \rho(\mathbf{V}_{\Psi}(H_n^i)) \subset \mathfrak{M}$. Since $\dim \mathbf{V}_{\rho\Psi}(H_n^i) = \dim \mathfrak{M}$, we have $\mathbf{V}_{\rho\Psi}(H_n^i) = \mathfrak{M}$, implying $\rho\Psi \in \mathcal{G}^0(Sp(n), \mathfrak{N}(n))$. This completes the proof. \square

By virtue of Proposition 24 to show Theorem 10 it suffices to prove that any element of $\mathcal{G}^0(Sp(n), \mathfrak{N}(n))$ is equivalent to Ψ_0 .

By \mathfrak{m} we denote the orthogonal complement of $\mathfrak{sp}(n-1)$ in $\mathfrak{sp}(n)$. For simplicity, we set $P_a = P_{an}$, $Q_a^i = Q_{an}^i$ and $H^i = H_n^i$ for integers a ($1 \leq a \leq n-1$) and i ($1 \leq i \leq 3$). Set

$$\mathfrak{m}_a = \mathbb{R}P_a + \sum_{i=1}^3 \mathbb{R}Q_a^i \quad (1 \leq a \leq n-1), \quad \mathfrak{m}_n = \sum_{i=1}^3 \mathbb{R}H^i.$$

Since $(\mathfrak{m}_a, \mathfrak{m}_b) = 0$ ($a \neq b$), we have

$$\mathfrak{m} = \sum_{a=1}^{n-1} \mathfrak{m}_a + \mathfrak{m}_n \quad (\text{orthogonal direct sum}).$$

Lemma 25 *Let $\Psi \in \mathcal{G}^0(Sp(n), \mathfrak{N}(n))$ and let $i = 1, 2$ or 3 . Then:*

$$\mathfrak{M} = \sum_{a=1}^{n-1} \Psi(H^i, \mathfrak{m}_a) + \mathbb{R}\Psi(H^i, H^i) \quad (\text{direct sum}).$$

Proof. Since $\mathbf{K}_{\Psi}(H^i) = \mathfrak{sp}(n-1) + \sum_{j \neq i} \mathbb{R}H^j$ and $\mathbf{V}_{\Psi}(H^i) = \Psi(H^i, \mathfrak{m}) = \mathfrak{M}$, we have the lemma. \square

In what follows we will observe the value $\Psi(X, Y)$ ($X, Y \in \mathfrak{sp}(n)$) for the following four cases:

- (I) $X \in \mathfrak{m}$ and $Y \in \mathfrak{sp}(n-1)$;
- (II) $X \in \mathfrak{m}_n$ and $Y \in \mathfrak{m}_n$;
- (III) $X \in \mathfrak{m}_a$ and $Y \in \mathfrak{m}_a$ ($1 \leq a \leq n-1$);
- (IV) $X \in \mathfrak{m}_n$ and $Y \in \mathfrak{m}_a$ ($1 \leq a \leq n-1$).

We first observe Case (I):

Proposition 26 *Let $\Psi \in \mathcal{G}^0(\mathfrak{Sp}(n), \mathfrak{N}(n))$. Then:*

- (1) $\Psi(\mathfrak{m}, \mathfrak{sp}(n-1)) \subset \mathfrak{M}$.
- (2) *Let $X, Y \in \mathfrak{m}$ and $Z \in \mathfrak{sp}(n-1)$. Then:*

$$\langle \Psi(X, Z), \Psi(H^i, Y) \rangle = \frac{1}{4} ([[X, Z], H^i], Y). \quad (4.2)$$

Proof. We first note that $\Psi(H^i, \mathfrak{sp}(n-1)) = 0$ ($1 \leq i \leq 3$), because $\mathbf{K}_\Psi(H^i) \supset \mathfrak{sp}(n-1)$. This proves $\Psi(\mathfrak{m}_n, \mathfrak{sp}(n-1)) = 0$. We now prove $\Psi(\mathfrak{m}_a, \mathfrak{sp}(n-1)) \subset \mathfrak{M}$ for any a ($1 \leq a \leq n-1$). To show this we prove

$$\Psi(P_a, \mathfrak{sp}(n-1)) \subset \mathfrak{M}; \quad \Psi(Q_a^i, \mathfrak{sp}(n-1)) \subset \mathfrak{M} \quad (i = 1, 2, 3). \quad (4.3)$$

Define an element $Z_0^i \in \mathfrak{sp}(n-1)$ ($1 \leq i \leq 3$) by $Z_0^i = (\sum_{s=1}^{n-1} sE_{ss})e^i$. Then it is well-known that Z_0^i is a regular element of $\mathfrak{sp}(n-1)$. Moreover, since $\Psi|_{\mathfrak{sp}(n-1)} = \Psi_0|_{\mathfrak{sp}(n-1)}$, it follows that $\Psi(Z_0^i, \mathfrak{sp}(n-1)) \subset \mathfrak{N}(n-1)$. Here we note that the equality $\Psi(Z_0^i, \mathfrak{sp}(n-1)) = \mathfrak{N}(n-1)$ holds. Indeed, since $\dim \mathbf{Ker}((\Psi_0)_{Z_0^i}|_{\mathfrak{sp}(n-1)}) = 2(n-1)$ (see Proposition 15), we have

$$\begin{aligned} \dim \Psi(Z_0^i, \mathfrak{sp}(n-1)) &= \dim \mathfrak{sp}(n-1) - \dim \mathbf{Ker}((\Psi_0)_{Z_0^i}|_{\mathfrak{sp}(n-1)}) \\ &= \dim \mathfrak{N}(n-1). \end{aligned}$$

Now let us set $W_a^i = Z_0^i - aH^i \in \mathfrak{sp}(n)$ ($1 \leq a \leq n-1$). By a direct calculation we can verify $\Psi_0(P_a, W_a^i) = \Psi_0(Q_a^i, W_a^i) = 0$. Hence by (3.5) we have $\Psi(P_a, W_a^i) = \Psi(Q_a^i, W_a^i) = 0$. Moreover, since $\Psi(H^i, \mathfrak{sp}(n-1)) = 0$, we have $\Psi(W_a^i, \mathfrak{sp}(n-1)) = \Psi(Z_0^i, \mathfrak{sp}(n-1)) = \mathfrak{N}(n-1)$. Let $Z, Z' \in \mathfrak{sp}(n-1)$. Then by the Gauss equation (2.3) we have

$$\begin{aligned} &\frac{1}{4} ([[W_a^i, Z], Z'], P_a) \\ &= \langle \Psi(W_a^i, Z'), \Psi(Z, P_a) \rangle - \langle \Psi(W_a^i, P_a), \Psi(Z, Z') \rangle, \quad (4.4) \end{aligned}$$

$$\begin{aligned} &\frac{1}{4} ([[W_a^i, Z], Z'], Q_a^i) \\ &= \langle \Psi(W_a^i, Z'), \Psi(Z, Q_a^i) \rangle - \langle \Psi(W_a^i, Q_a^i), \Psi(Z, Z') \rangle. \quad (4.5) \end{aligned}$$

Since $[H^i, Z] = 0$, we have $[[W_a^i, Z], Z'] = [[Z_0^i, Z], Z'] \in \mathfrak{sp}(n-1)$. Hence, the left sides of (4.4) and (4.5) vanish. Further, since $\Psi(P_a, W_a^i) = \Psi(Q_a^i, W_a^i) = 0$, we have $\langle \Psi(W_a^i, Z'), \Psi(Z, P_a) \rangle = \langle \Psi(W_a^i, Z'), \Psi(Z, Q_a^i) \rangle = 0$. Since Z and Z' are arbitrary elements of $\mathfrak{sp}(n-1)$ and since

$\Psi(W_a^i, \mathfrak{sp}(n-1)) = \mathfrak{N}(n-1)$, we have

$$\langle \mathfrak{N}(n-1), \Psi(\mathfrak{sp}(n-1), P_a) \rangle = \langle \mathfrak{N}(n-1), \Psi(\mathfrak{sp}(n-1), Q_a^i) \rangle = 0,$$

showing (4.3). Consequently, we have $\Psi(\mathfrak{m}_a, \mathfrak{sp}(n-1)) \subset \mathfrak{M}$, which completes the proof of (1).

Next we show (2). Let $X, Y \in \mathfrak{m}$ and $Z \in \mathfrak{sp}(n-1)$. Then by the Gauss equation (2.3) we have

$$\frac{1}{4}([X, H^i], Z], Y) = \langle \Psi(X, Z), \Psi(H^i, Y) \rangle - \langle \Psi(X, Y), \Psi(H^i, Z) \rangle.$$

Note that $\Psi(H^i, Z) = 0$ and $[Z, H^i] = 0$. The latter equality, together with the Jacobi identity, shows $[[X, H^i], Z] = [[X, Z], H^i]$. Thus we obtain (4.2). \square

Remark 27 Here we state a remark on the value $\Psi(X, Z)$ ($X \in \mathfrak{m}$, $Z \in \mathfrak{sp}(n-1)$). Note that the right side of (4.2) is an intrinsic quantity. Since $\Psi(H^i, \mathfrak{m}) = \mathfrak{M}$, we know that $\Psi(X, Z) \in \mathfrak{M}$ is uniquely determined if the values $\Psi(H^i, Y)$ ($Y \in \mathfrak{m}$) are given. Therefore, if $\Psi(H^i, Y) = \Psi_0(H^i, Y)$ holds for any $Y \in \mathfrak{m}$, then we may conclude that $\Psi(X, Z) = \Psi_0(X, Z)$ ($X \in \mathfrak{m}$, $Z \in \mathfrak{sp}(n-1)$). See Case (c) below in the proof of Theorem 10.

We next observe Case (II):

Proposition 28 *Let $\Psi \in \mathcal{G}^0(Sp(n), \mathfrak{N}(n))$. Then:*

- (1) $\Psi(H^1, H^1) = \Psi(H^2, H^2) = \Psi(H^3, H^3)$.
- (2) $\Psi(H^1, H^2) = \Psi(H^2, H^3) = \Psi(H^3, H^1) = 0$.
- (3) $\langle \Psi(H^i, H^i), \Psi(H^i, H^i) \rangle = 1$ ($1 \leq i \leq 3$).
- (4) $\langle \Psi(H^i, H^i), \Psi(H^i, \mathfrak{m}_a) \rangle = 0$ ($1 \leq i \leq 3, 1 \leq a \leq n-1$).

To prove the proposition we prepare

Lemma 29 *Let $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$. Let X and $Y \in \mathfrak{sp}(n)$. Assume:*

- (i) $\Psi_0(X, X) = \Psi_0(Y, Y)$.
- (ii) $X + Y \in \mathcal{S}$.

Then $\Psi(X, X) = \Psi(Y, Y)$.

Proof. By (i) we easily have $\Psi_0(X+Y, X-Y) = 0$, i.e., $X-Y \in \mathbf{K}_0(X+Y)$. Since $X+Y \in \mathcal{S}$, we have $\mathbf{K}_0(X+Y) = \mathbf{K}_\Psi(X+Y)$ (see Proposition 20). Consequently, it follows that $X-Y \in \mathbf{K}_\Psi(X+Y)$, i.e., $\Psi(X+Y, X-Y) = 0$. This implies $\Psi(X, X) = \Psi(Y, Y)$. \square

Proof of Proposition 28. Let $\{i, j, k\}$ be a permutation of $\{1, 2, 3\}$. As shown in the proof of Proposition 21, $\mathfrak{s} = \sum_{i=1}^3 \mathbb{R}H^i$ is NAT. Consequently, $H^i + H^j \in \mathcal{S}$, because $(H^i + H^j) \sim H^i$. On the other hand, it is easily checked that $\Psi_0(H^i, H^i) = \Psi_0(H^j, H^j) = -E_{nn}$. Hence by Lemma 29 we have $\Psi(H^i, H^i) = \Psi(H^j, H^j)$. Similarly, we have $\Psi(H^j, H^j) = \Psi(H^k, H^k)$, proving (1). The assertion (2) is clear from Lemma 13. Finally we prove (3) and (4). Let k be an integer such that $1 \leq k \leq 3$, $k \neq i$ and $X \in \mathfrak{sp}(n)$. Then by the Gauss equation (2.3) we have

$$\begin{aligned} & \frac{1}{4}([\![H^i, H^k], H^k], X) \\ &= \langle \Psi(H^i, H^k), \Psi(H^k, X) \rangle - \langle \Psi(H^i, X), \Psi(H^k, H^k) \rangle. \end{aligned}$$

By a simple calculation we have $[\![H^i, H^k], H^k] = -4H^i$. Moreover, by the results obtained in (1) and (2) we have $\Psi(H^i, H^k) = 0$ and $\Psi(H^k, H^k) = \Psi(H^i, H^i)$. Consequently, we have

$$\langle \Psi(H^i, X), \Psi(H^i, H^i) \rangle = (H^i, X).$$

Therefore, we obtain (3) and (4), because $(H^i, H^i) = 1$ and $(H^i, \mathfrak{m}_a) = 0$ (see (3.3)). \square

In Case (III) the value $\Psi(X, Y)$ ($X, Y \in \mathfrak{m}_a$) ($1 \leq a \leq n-1$) are determined by

Proposition 30 *Let $\Psi \in \mathcal{G}^0(\mathfrak{Sp}(n), \mathfrak{N}(n))$ and let a be an integer such that $1 \leq a \leq n-1$. Then:*

- (1) $\Psi(P_a, Q_a^i) = 0$ ($1 \leq i \leq 3$).
- (2) $\Psi(Q_a^i, Q_a^j) = 0$ ($1 \leq i \neq j \leq 3$).
- (3) $\Psi(P_a, P_a) = \Psi(Q_a^i, Q_a^i) = \Psi(H^i, H^i) + \Psi(H_a^i, H_a^i)$ ($1 \leq i \leq 3$).

Proof. Since $\Psi_0(P_a, Q_a^i) = 0$ and $\Psi_0(Q_a^i, Q_a^j) = 0$ ($i \neq j$), we obtain (1) and (2) (see (3.5)). We now prove (3). Since $\mathfrak{s}_{an}^i = \mathbb{R}(H^i - H_a^i) + \mathbb{R}P_a + \mathbb{R}Q_a^i$ is NAT, it follows that $Q_a^i + (H^i - H_a^i) \in \mathcal{S}$. Indeed, $Q_a^i + (H^i - H_a^i) \sim (H^i - H_a^i)$. By Lemma 29 we have $\Psi(Q_a^i, Q_a^i) = \Psi(H^i - H_a^i, H^i - H_a^i)$, because $\Psi_0(Q_a^i, Q_a^i) = \Psi_0(H^i - H_a^i, H^i - H_a^i) = -(E_{aa} + E_{nn})$. Since $H_a^i \in \mathfrak{sp}(n-1)$, we have $\Psi(H^i, H_a^i) = 0$. Consequently, $\Psi(Q_a^i, Q_a^i) = \Psi(H^i, H^i) + \Psi(H_a^i, H_a^i)$. Similarly, we can prove $\Psi(P_a, P_a) = \Psi(H^i, H^i) + \Psi(H_a^i, H_a^i)$. \square

Before proceeding to Case (IV) we extend Lemma 29 to the following

form:

Lemma 31 *Let $\Psi \in \mathcal{G}(Sp(n), \mathfrak{N}(n))$. Let X, X', Y and $Y' \in \mathfrak{sp}(n)$.*

Assume:

- (i) $\Psi_0(X, Y') = \Psi_0(Y, X') = 0$.
- (ii) $\Psi_0(X, X') = \Psi_0(Y, Y')$.
- (iii) $X \in \mathcal{S}, Y \in \mathcal{S}$ and $X + Y \in \mathcal{S}$.

Then $\Psi(X, X') = \Psi(Y, Y')$.

Proof. By (i) and (ii) we have $Y' \in \mathbf{K}_0(X)$, $X' \in \mathbf{K}_0(Y)$ and $\Psi_0(X + Y, X' - Y') = 0$. The last equality implies that $X' - Y' \in \mathbf{K}_0(X + Y)$. Hence by (iii) we have $Y' \in \mathbf{K}_\Psi(X)$, $X' \in \mathbf{K}_\Psi(Y)$ and $X' - Y' \in \mathbf{K}_\Psi(X + Y)$. Consequently, we have $\Psi(Y', X) = \Psi(X', Y) = \Psi(X + Y, X' - Y') = 0$. Hence $\Psi(X, X') = \Psi(Y, Y')$. \square

With this preparation we observe Case (IV).

Proposition 32 *Let $\Psi \in \mathcal{G}^0(Sp(n), \mathfrak{N}(n))$. Let a be an integer such that $1 \leq a \leq n - 1$. Then:*

- (1) $\Psi(H^1, Q_a^1) = \Psi(H^2, Q_a^2) = \Psi(H^3, Q_a^3)$.
- (2) $\Psi(H^i, Q_a^j) = -\varepsilon(ijk)\Psi(H^k, P_a)$, where $\{i, j, k\}$ is a permutation of $\{1, 2, 3\}$.
- (3) $\Psi(H^1, \mathfrak{m}_a) = \Psi(H^2, \mathfrak{m}_a) = \Psi(H^3, \mathfrak{m}_a)$.
- (4) For each i ($1 \leq i \leq 3$) the set $\{\sqrt{2}\Psi(H^i, P_a), \sqrt{2}\Psi(H^i, Q_a^j) \ (1 \leq j \leq 3)\}$ forms an orthonormal basis of $\Psi(H^i, \mathfrak{m}_a)$.

Proof. Let $\{i, j, k\}$ be a permutation of $\{1, 2, 3\}$. We note that the subspace $\mathfrak{s} = \mathbb{R}(H_a^i + H^i) + \mathbb{R}Q_a^j + \mathbb{R}Q_a^k$ forms a subalgebra of $\mathfrak{sp}(n)$ and is NAT. In fact, by simple calculations we have

$$\begin{aligned} [H_a^i + H^i, Q_a^j] &= 2\varepsilon(ijk)Q_a^k; & [H_a^i + H^i, Q_a^k] &= -2\varepsilon(ijk)Q_a^j; \\ [Q_a^j, Q_a^k] &= 2\varepsilon(ijk)(H_a^i + H^i). \end{aligned}$$

Hence we have $H_a^i + H^i + Q_a^j \in \mathcal{S}$ and $H_a^i + H^i + Q_a^k \in \mathcal{S}$, because $H_a^i + H^i + Q_a^j \sim H_a^i + H^i + Q_a^k \sim H_a^i + H^i \in \mathcal{S}$.

Now we prove (1). By direct calculations we can show $\Psi_0(H_a^1 + H^1, Q_a^1) = \Psi_0(H_a^2 + H^2, Q_a^2) = \Psi_0(H_a^3 + H^3, Q_a^3) = -(E_{an} + E_{na})$. Moreover we have $\Psi_0(H_a^i + H^i, H_a^j + H^j) = \Psi_0(Q_a^i, Q_a^j) = 0$ if $i \neq j$ (see Lemma 13 and Proposition 30). Therefore by Lemma 31 we have

$$\Psi(H_a^1 + H^1, Q_a^1) = \Psi(H_a^2 + H^2, Q_a^2) = \Psi(H_a^3 + H^3, Q_a^3). \quad (4.6)$$

Here we show $\Psi(H_a^1, Q_a^1) = \Psi(H_a^2, Q_a^2) = \Psi(H_a^3, Q_a^3)$. Let $i = 1, 2$ or 3 . Since $H_a^i \in \mathfrak{sp}(n-1)$ and $Q_a^i \in \mathfrak{m}$, it follows from Proposition 26 (1) that $\Psi(H_a^i, Q_a^i) \in \mathfrak{M}$. Moreover, by Proposition 26 (2) we have

$$\langle \Psi(Q_a^i, H_a^i), \Psi(H^1, Y) \rangle = \frac{1}{4}([Q_a^i, H_a^i], H^1, Y)$$

for any $Y \in \mathfrak{m}$. Since $[Q_a^i, H_a^i] = P_a$, the right side of the above equality does not depend on the choice of i . This implies that $\Psi(H_a^1, Q_a^1) = \Psi(H_a^2, Q_a^2) = \Psi(H_a^3, Q_a^3)$, because $\Psi(H^1, \mathfrak{m}) = \mathfrak{M}$. This, together with (4.6), proves (1).

We next prove (2). Let $\{i, j, k\}$ be a permutation of $\{1, 2, 3\}$. Then by direct calculations we have $\Psi_0(H_a^i - H^i, Q_a^j) = \varepsilon(ijk)\Psi_0(H_a^k + H^k, P_a) = \varepsilon(ijk)(E_{an} - E_{na})e^k$. Moreover, $\Psi_0(H_a^i - H^i, H_a^k + H^k) = \Psi_0(Q_a^j, P_a) = 0$ (see Lemma 13 and Proposition 30). Since $H_a^k + H^k + Q_a^j \in \mathcal{S}$, we obtain by Lemma 31 the following

$$\Psi(H_a^i - H^i, Q_a^j) = \varepsilon(ijk)\Psi(H_a^k + H^k, P_a). \quad (4.7)$$

Note that $H_a^i, H_a^k \in \mathfrak{sp}(n-1)$, $Q_a^j, P_a \in \mathfrak{m}$ and $[Q_a^j, H_a^i] = \varepsilon(ijk)[P_a, H_a^k] = -\varepsilon(ijk)Q_a^k$. As in the proof of (1) we have $\Psi(H_a^i, Q_a^j) = \varepsilon(ijk)\Psi(H_a^k, P_a)$. Accordingly, from (4.7) we have $\Psi(H^i, Q_a^j) = -\varepsilon(ijk)\Psi(H^k, P_a)$. This completes the proof of (2).

By (1) and (2) we have

$$\begin{aligned} \Psi(H^1, P_a) &= -\Psi(H^2, Q_a^3) = \Psi(H^3, Q_a^2); \\ \Psi(H^1, Q_a^1) &= \Psi(H^2, Q_a^2) = \Psi(H^3, Q_a^3); \\ \Psi(H^1, Q_a^2) &= -\Psi(H^2, Q_a^1) = -\Psi(H^3, P_a); \\ \Psi(H^1, Q_a^3) &= \Psi(H^2, P_a) = -\Psi(H^3, Q_a^1). \end{aligned} \quad (4.8)$$

By these equalities we clearly obtain (3).

Finally, we prove (4). Let X and Y are one of P_a and Q_a^j ($1 \leq j \leq 3$), i.e., $X, Y \in \{P_a, Q_a^j \mid 1 \leq j \leq 3\}$. By the Gauss equation (2.3) we have

$$\begin{aligned} &\frac{1}{4}([H^i, X], H^i, Y) \\ &= \langle \Psi(H^i, H^i), \Psi(X, Y) \rangle - \langle \Psi(H^i, Y), \Psi(X, H^i) \rangle. \end{aligned}$$

By direct calculations we can verify $[[H^i, X], H^i] = X$. Hence the left side of the above equality becomes $(1/4)(X, Y)$. First assume that $X = Y$. Then we have $\Psi(X, X) = \Psi(H^i, H^i) + \Psi(H_a^i, H_a^i)$ (see Proposition 30 (3)). Since $\langle \Psi(H^i, H^i), \Psi(H^i, H^i) \rangle = 1$ (see Proposition 28), $\Psi(H^i, H^i) \in \mathfrak{M}$

and $\Psi(H_a^i, H_a^i) \in \mathfrak{N}(n-1)$, we have

$$\begin{aligned} \langle \Psi(H^i, H^i), \Psi(X, X) \rangle &= \langle \Psi(H^i, H^i), \Psi(H^i, H^i) + \Psi(H_a^i, H_a^i) \rangle \\ &= 1. \end{aligned}$$

Since $(X, X) = 2$ (see (3.3)), we have $\langle \Psi(H^i, X), \Psi(H^i, X) \rangle = 1/2$. We next consider the case $X \neq Y$. Then we have $(X, Y) = 0$ and $\Psi(X, Y) = 0$ (see (3.3) and Proposition 30 (1), (2)). Hence it follows that $\langle \Psi(H^i, X), \Psi(H^i, Y) \rangle = 0$. This completes the proof of (4). \square

We are now in a position to prove Theorem 10.

Proof of Theorem 10. Let $\Psi \in \mathcal{G}^0(Sp(n), \mathfrak{N}(n))$. Set $\mathbf{H} = \Psi(H^1, H^1)$, $\mathbf{P}_a = \sqrt{2}\Psi(H^1, P_a)$ ($1 \leq a \leq n-1$), $\mathbf{Q}_a^i = \sqrt{2}\Psi(H^1, Q_a^i)$ ($1 \leq a \leq n-1$, $1 \leq i \leq 3$). Then we have

Lemma 33 *The set $\mathfrak{D} = \{\mathbf{H}, \mathbf{P}_a$ ($1 \leq a \leq n-1$), \mathbf{Q}_a^i ($1 \leq a \leq n-1$, $1 \leq i \leq 3$)}* forms an orthonormal basis of \mathfrak{M} .

Proof. By virtue of Proposition 28 (3), (4) and Proposition 32 (4) we have only to prove

$$\langle \Psi(H^1, \mathfrak{m}_a), \Psi(H^1, \mathfrak{m}_b) \rangle = 0 \quad (1 \leq a \neq b \leq n-1). \quad (4.9)$$

Let $X \in \mathfrak{m}_a$ and $Y \in \mathfrak{m}_b$. By the Gauss equation (2.3) we have

$$\begin{aligned} &\frac{1}{4}([H^1, X], H^2, Y) \\ &= \langle \Psi(H^1, H^2), \Psi(X, Y) \rangle - \langle \Psi(H^1, Y), \Psi(X, H^2) \rangle. \end{aligned}$$

As is easily seen, $[[H^1, X], H^2] \in \mathfrak{m}_a$. Hence the left side of the above equality vanishes. On the other hand, since $\Psi(H^1, H^2) = 0$ (see Proposition 28), it follows that $\langle \Psi(H^1, Y), \Psi(X, H^2) \rangle = 0$. This proves that $\langle \Psi(H^1, \mathfrak{m}_b), \Psi(H^2, \mathfrak{m}_a) \rangle = 0$. Therefore, we obtain (4.9), because $\Psi(H^2, \mathfrak{m}_a) = \Psi(H^1, \mathfrak{m}_a)$ (see Proposition 32 (3)). This completes the proof. \square

Let $\mathfrak{D}_0 = \{\mathbf{H}_0, (\mathbf{P}_a)_0$ ($1 \leq a \leq n-1$), $(\mathbf{Q}_a^i)_0$ ($1 \leq a \leq n-1$, $1 \leq i \leq 3$) $\}$ be the orthonormal basis of \mathfrak{M} corresponding to Ψ_0 , i.e., $\mathbf{H}_0 = \Psi_0(H^1, H^1)$, $(\mathbf{P}_a)_0 = \sqrt{2}\Psi_0(H^1, P_a)$ and $(\mathbf{Q}_a^i)_0 = \sqrt{2}\Psi_0(H^1, Q_a^i)$. Then, there is an orthogonal transformation ρ' of \mathfrak{M} such that $\mathbf{H}_0 = \rho'(\mathbf{H})$, $(\mathbf{P}_a)_0 = \rho'(\mathbf{P}_a)$ and $(\mathbf{Q}_a^i)_0 = \rho'(\mathbf{Q}_a^i)$. Extend ρ' to the orthogonal transformation ρ of $\mathfrak{N}(n)$

satisfying $\rho|_{\mathfrak{M}} = \rho'$ and $\rho|_{\mathfrak{N}(n-1)} = \mathbf{1}_{\mathfrak{N}(n-1)}$. Then, it is easy to see that $\rho\Psi \in \mathcal{G}^0(Sp(n), \mathfrak{N}(n))$. For simplicity, set $\Psi_1 = \rho\Psi$. In the following we will prove $\Psi_1 = \Psi_0$. In view of Lemma 25 and the decomposition $\mathfrak{sp}(n) = \mathfrak{m} + \mathfrak{sp}(n-1)$, we may conclude $\Psi_1 = \Psi_0$ if $\Psi_1(X, Y) = \Psi_0(X, Y)$ holds for any pairs X and Y listed in the following (a) \sim (e):

- (a) $X \in \mathfrak{sp}(n-1)$ and $Y \in \mathfrak{sp}(n-1)$;
- (b) $X \in \mathfrak{m}_n$ and $Y \in \mathfrak{m}$;
- (c) $X \in \mathfrak{m}$ and $Y \in \mathfrak{sp}(n-1)$;
- (d) $X \in \mathfrak{m}_a$ and $Y \in \mathfrak{m}_a$ ($1 \leq a \leq n-1$);
- (e) $X \in \mathfrak{m}_a$ and $Y \in \mathfrak{m}_b$ ($1 \leq a \neq b \leq n-1$).

Case (a): Let $X, Y \in \mathfrak{sp}(n-1)$. Since $\Psi(X, Y) = \Psi_0(X, Y) \in \mathfrak{N}(n-1)$ and $\rho|_{\mathfrak{N}(n-1)} = \mathbf{1}_{\mathfrak{N}(n-1)}$, we have $\Psi_1(X, Y) = \rho(\Psi(X, Y)) = \rho(\Psi_0(X, Y)) = \Psi_0(X, Y)$.

Case (b): By the very definition of ρ we have $\Psi_1(H^1, Y) = \Psi_0(H^1, Y)$ for $Y \in \sum_{a=1}^{n-1} \mathfrak{m}_a + \mathbb{R}H^1$. Applying Proposition 32 to both Ψ_1 and Ψ_0 , we have $\Psi_1(H^i, Y) = \Psi_0(H^i, Y)$ for $i = 2, 3$, $Y \in \sum_{a=1}^{n-1} \mathfrak{m}_a$ (see (1), (2) and (4.8)). Further, since $\Psi_1(H^1, H^1) = \Psi_0(H^1, H^1)$, we have $\Psi_1(H^i, H^j) = \Psi_0(H^i, H^j)$ ($1 \leq i, j \leq 3$) (see Proposition 28 (1), (2)). Thus we obtain $\Psi_1(X, Y) = \Psi_0(X, Y)$ for any $X \in \mathfrak{m}_n$ and $Y \in \sum_{a=1}^{n-1} \mathfrak{m}_a + \mathfrak{m}_n = \mathfrak{m}$.

Case (c): By Case (b) we have $\Psi_1(H^i, Y) = \Psi_0(H^i, Y)$ ($i = 1, 2, 3; Y \in \mathfrak{m}$). As we have remarked (see Remark 27), we obtain $\Psi_1(X, Y) = \Psi_0(X, Y)$ for $X \in \mathfrak{m}$, $Y \in \mathfrak{sp}(n-1)$.

Case (d): As seen in Case (b), we have $\Psi_1(H^i, H^i) = \Psi_0(H^i, H^i)$. Moreover, since $H_a^i \in \mathfrak{sp}(n-1)$, we have $\Psi_1(H_a^i, H_a^i) = \Psi_0(H_a^i, H_a^i)$ ($i = 1, 2, 3$). Hence by applying Proposition 30 to Ψ_1 and Ψ_0 , we easily have $\Psi_1(X, Y) = \Psi_0(X, Y)$ for $X, Y \in \mathfrak{m}_a$.

Case (e): We note that this case occurs when $n \geq 3$. We first show

Lemma 34 *Assume that $n \geq 3$. Let a and c be integers such that $1 \leq a \neq c \leq n-1$. Then $P_a \pm P_{ac} \in \mathcal{S}; Q_a^i \pm Q_{ac}^i \in \mathcal{S}$ ($i = 1, 2, 3$).*

Proof. By easy calculations we have

$$\begin{aligned} [H_c^i - H^i, P_a \pm P_{ac}] &= Q_a^i \mp Q_{ac}^i; \\ [H_c^i - H^i, Q_a^i \mp Q_{ac}^i] &= -(P_a \pm P_{ac}); \end{aligned}$$

$$[P_a \pm P_{ac}, Q_a^i \mp Q_{ac}^i] = 2(H_c^i - H^i).$$

Consequently, both the subspaces $\mathfrak{s}_+ = \mathbb{R}(H_c^i - H^i) + \mathbb{R}(P_a + P_{ac}) + \mathbb{R}(Q_a^i - Q_{ac}^i)$ and $\mathfrak{s}_- = \mathbb{R}(H_c^i - H^i) + \mathbb{R}(P_a - P_{ac}) + \mathbb{R}(Q_a^i + Q_{ac}^i)$ are NAT. Therefore, we have $P_a \pm P_{ac} \sim H_c^i - H^i \sim Q_a^i \pm Q_{ac}^i$. Since $H_c^i - H^i \in \mathcal{S}$, it follows that $P_a \pm P_{ac} \in \mathcal{S}$ and $Q_a^i \pm Q_{ac}^i \in \mathcal{S}$. \square

First assume $n \geq 4$. Let us consider the case $X = P_a$ and $Y = P_b$. Take an integer c ($1 \leq c \leq n-1$) such that $c \neq a$ and $c \neq b$. By easy calculations we have $\Psi_0(P_a, P_b) = \Psi_0(P_{ac}, P_{bc}) = -(1/2)(E_{ab} + E_{ba})$ and $\Psi_0(P_a, P_{bc}) = \Psi_0(P_{ac}, P_b) = 0$. Since P_a, P_{ac} and $P_a + P_{ac} \in \mathcal{S}$, it follows that $\Psi_1(P_a, P_b) = \Psi_1(P_{ac}, P_{bc})$ (see Lemma 31). Since $P_{ac}, P_{bc} \in \mathfrak{sp}(n-1)$, we have $\Psi_1(P_{ac}, P_{bc}) = \Psi_0(P_{ac}, P_{bc})$ (see the Case (a)). Hence we have $\Psi_1(P_a, P_b) = \Psi_0(P_a, P_b)$. In a similar manner we can prove $\Psi_1(P_a, Q_b^i) = \Psi_0(P_a, Q_b^i)$ ($i = 1, 2, 3$) and $\Psi_1(Q_a^i, Q_b^j) = \Psi_0(Q_a^i, Q_b^j)$ ($i, j = 1, 2, 3$). By these facts we obtain the equality $\Psi_1(X, Y) = \Psi_0(X, Y)$ ($X \in \mathfrak{m}_a, Y \in \mathfrak{m}_b$) when $n \geq 4$.

Next we assume $n = 3$. Apparently, the method used in the case $n \geq 4$ cannot be applied to this case. We prove

Lemma 35 *Assume that $n = 3$. Then $\Psi_1(\mathfrak{m}_1, \mathfrak{m}_2) \subset \mathfrak{N}(2)$.*

Proof. Set $\mathfrak{B}_a = \{P_a, Q_a^1, Q_a^2, Q_a^3\}$ ($a = 1, 2$). Let $X \in \mathfrak{B}_1$ and $Y \in \mathfrak{B}_2$. We first show

$$\langle \Psi_1(X, Y), \Psi_1(H^1, H^1) \rangle = \langle \Psi_1(X, Y), \Psi_1(H^1, \mathfrak{m}_1 + \mathfrak{m}_2) \rangle = 0. \quad (4.10)$$

If this is true, then we have $\Psi_1(X, Y) \in \mathfrak{N}(2)$, because $\mathfrak{M} = \mathbb{R}\Psi_1(H^1, H^1) + \Psi_1(H^1, \mathfrak{m}_1 + \mathfrak{m}_2)$ (see Lemma 25) and because $\mathfrak{N}(2)$ is the orthogonal complement of \mathfrak{M} in $\mathfrak{N}(3)$.

By the Gauss equation (2.3) we have

$$\begin{aligned} & \frac{1}{4}([H^1, X], H^1, Y) \\ &= \langle \Psi_1(H^1, H^1), \Psi_1(X, Y) \rangle - \langle \Psi_1(H^1, Y), \Psi_1(X, H^1) \rangle. \end{aligned}$$

As observed in the proof of Proposition 32, we have $[[H^1, X], H^1] = X$. Since $(X, Y) = 0$, the left side of the above equality vanishes. Moreover, in view of (4.9) we have $\langle \Psi_1(H^1, Y), \Psi_1(X, H^1) \rangle = 0$. Consequently, we have $\langle \Psi_1(X, Y), \Psi_1(H^1, H^1) \rangle = 0$. Let Z be an arbitrary element of \mathfrak{B}_1 .

Then by the Gauss equation (2.3) we have

$$\begin{aligned} & \frac{1}{4}([X, H^1], Y, Z) \\ &= \langle \Psi_1(X, Y), \Psi_1(H^1, Z) \rangle - \langle \Psi_1(X, Z), \Psi_1(H^1, Y) \rangle. \end{aligned}$$

Here we can easily verify that $[X, H^1], Y \in \mathfrak{sp}(2)$ and hence the left side of the above equality vanishes. By Proposition 30 (1), (2) we have $\Psi_1(X, Z) = 0$ if $X \neq Z$. Hence $\langle \Psi_1(X, Y), \Psi_1(H^1, Z) \rangle = 0$. On the other hand, if $X = Z$, then we have $\Psi_1(X, Z) = \Psi_1(X, X) = \Psi_1(H^1, H^1) + \Psi_1(H_1^1, H_1^1)$ (see Proposition 30). Hence by Proposition 28 (4) and the fact $\Psi_1(H_1^1, H_1^1) \in \mathfrak{N}(2)$ we have $\langle \Psi_1(X, Z), \Psi_1(H^1, Y) \rangle = 0$. Therefore, in this case, we also obtain $\langle \Psi_1(X, Y), \Psi_1(H^1, Z) \rangle = 0$. Since Z is an arbitrary element of \mathfrak{B}_1 , we have $\langle \Psi_1(X, Y), \Psi_1(H^1, \mathfrak{m}_1) \rangle = 0$. In a similar way we can prove $\langle \Psi_1(X, Y), \Psi_1(H^1, \mathfrak{m}_2) \rangle = 0$, showing (4.10). Accordingly, we get $\Psi_1(X, Y) \in \mathfrak{N}(2)$ and hence $\Psi_1(\mathfrak{m}_1, \mathfrak{m}_2) \subset \mathfrak{N}(2)$. \square

Now let $X \in \mathfrak{m}_1, Y \in \mathfrak{m}_2$. Take arbitrary elements $Z_1, Z_2 \in \mathfrak{sp}(2)$. Then by the Gauss equation (2.3) we have

$$\begin{aligned} & \frac{1}{4}([X, Z_1], Y, Z_2) \\ &= \langle \Psi_1(X, Y), \Psi_1(Z_1, Z_2) \rangle - \langle \Psi_1(X, Z_2), \Psi_1(Z_1, Y) \rangle. \end{aligned}$$

By the results of Case (a) and Case (c) we have $\Psi_1(Z_1, Z_2) = \Psi_0(Z_1, Z_2)$, $\Psi_1(X, Z_2) = \Psi_0(X, Z_2)$ and $\Psi_1(Y, Z_1) = \Psi_0(Y, Z_1)$. Therefore we have

$$\begin{aligned} & \langle \Psi_1(X, Y), \Psi_0(Z_1, Z_2) \rangle \\ &= \frac{1}{4}([X, Z_1], Y, Z_2) + \langle \Psi_0(X, Z_2), \Psi_0(Z_1, Y) \rangle. \end{aligned}$$

Since Ψ_0 is a solution of the Gauss equation (2.3), we have

$$\begin{aligned} & \langle \Psi_0(X, Y), \Psi_0(Z_1, Z_2) \rangle \\ &= \frac{1}{4}([X, Z_1], Y, Z_2) + \langle \Psi_0(X, Z_2), \Psi_0(Z_1, Y) \rangle. \end{aligned}$$

Hence, by subtraction, we have $\langle \Psi_1(X, Y) - \Psi_0(X, Y), \Psi_0(Z_1, Z_2) \rangle = 0$. Here we note that $\Psi_1(X, Y) - \Psi_0(X, Y) \in \mathfrak{N}(2)$. Indeed, we have $\Psi_1(X, Y) \in \mathfrak{N}(2)$ (see Lemma 35) and have $\Psi_0(X, Y) \in \mathfrak{N}(2)$ by a simple calculation. Since $\Psi_0(\mathfrak{sp}(2), \mathfrak{sp}(2)) = \mathfrak{N}(2)$, the above equality implies that

$\Psi_1(X, Y) - \Psi_0(X, Y) = 0$, i.e., $\Psi_1(X, Y) = \Psi_0(X, Y)$. This completes the proof of (e) in the case where $n = 3$.

Thus by the above case studies (a) \sim (e) we get $\Psi_1 = \Psi_0$, i.e., $\rho\Psi = \Psi_0$. This completes the proof of Theorem 10. \square

Remark 36 As seen in the above discussion, we have proved Theorem 10 by utilizing the equality $\mathbf{K}_\Psi(X) = \mathbf{K}_0(X)$ for regular elements X or for elements $X \in \mathcal{S}$. After we have established Theorem 10, we easily conclude that $\mathbf{K}_\Psi(X) = \mathbf{K}_0(X)$ holds for any element $X \in \mathfrak{sp}(n)$.

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Y. Agaoka
Faculty of Integrated Arts and Sciences
Hiroshima University
1-7-1 Kagamiyama, Higashi-Hiroshima City
Hiroshima, 739-8521 Japan
E-mail: agaoka@mis.hiroshima-u.ac.jp

E. Kaneda
Faculty of Foreign Studies
Osaka University of Foreign Studies
8-1-1 Aomadani-Higashi, Minoo City
Osaka, 562-8558 Japan
E-mail: kaneda@osaka-gaidai.ac.jp