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Local energy decay for some hyperbolic equations with initial data decaying slowly near infinity

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Abstract. A uniform local energy decay property is discussed to a linear hyperbolic equation with spatial variable coefficients. We shall deal with this equation in an exterior domain with a star-shaped complement. Our advantage is that we assume algebraic order weight restrictions as $|x| \to +\infty$ on the initial data in order to derive the uniform local energy decay, and its proof is quite simple.

Key words: hyperbolic equation, exterior mixed problem, weighted initial data, local energy decay.

1. Introduction

Let $\Omega \subset \mathbf{R}^N$ $(N \geq 2)$ be an exterior domain with a compact smooth boundary $\partial\Omega$ (in the case when $N = 1$, we take $\Omega = (0, +\infty)$). Without loss of generality, in the case when $N \geq 2$ we may assume $0 \notin \bar{\Omega}$ and $\partial\Omega \subset B_{\rho_0}(0) \equiv \{x \in R^N : |x| < \rho_0\}$ for some $\rho_0 > 0$, where $|\cdot|$ denotes the usual norm in $\mathbf{R}^N$ and $\bar{\Omega}$ represents the closure of Ω in $\mathbf{R}^N$.

In this paper, we are concerned with the following initial-boundary value problem

$$u_{tt}(t, x) - a(x)^2 \Delta u(t, x) = 0, \quad (t, x) \in (0, \infty) \times \Omega, \tag{1.1}$$

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \Omega, \tag{1.2}$$

$$u|_{\partial\Omega} = 0, \quad t \in (0, \infty), \tag{1.3}$$

where the given function $a \in C^1(\bar{\Omega}) \cap L^\infty(\Omega)$ satisfies

$$a(x) \geq a_m > 0, \quad x \in \Omega \tag{1.4}$$

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with some constant $a_m > 0$, and

$$u_{tt} = \frac{\partial^2 u}{\partial t^2}, \quad \Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}, \quad x = (x_1, \ldots, x_N).$$

In the case when $N \geq 2$ we shall impose a geometrical condition on the domain Ω:

(A-1) the obstacle $\mathbf{R}^N \setminus \bar{\Omega}$ is star-shaped relative to the origin.

Furthermore, we shall give some restrictions to $a(x)$:

(A-2) $x \cdot \nabla a(x) \leq 0$ for all $x \in \Omega$.

(A-3) There exists a constant $a_0 > 0$ ($a_0 \geq a_m$) such that $a(x) = a_0$ for $x \in \Omega$ satisfying $|x| > r_0$ with some constant $r_0 > \rho_0$.

(A-4) $x \cdot \nabla a(x) \leq (\gamma_0/2)a(x)$ for $x \in \Omega$ with some constant $0 \leq \gamma_0 < 1$.

Note that either condition (A-2) or (A-4) will be used in each statements below.

The purpose of this paper is to establish uniform local energy decay estimates to problem (1.1)-(1.4) under the assumptions (A-1)-(A-4). In this case we do not use any compactness assumptions of the support on the initial data. Instead we assume some weight restrictions of the algebraic decay order as $|x| \to +\infty$ on the initial data. Our advantage is that the method to derive the local energy decay estimate is quite simple as compared with the well-known cases, which were obtained in the framework of compactly supported solutions. We shall rely on so called the multiplier method, and do not use any spectral analysis and so on. A device due to Ikehata-Matsuyama [10] plays an important role to obtain the L^2-bound of solutions without compact supports, and in order to derive the energy estimate of solutions with spatial algebraic order weights, which forms an important part throughout this paper, the (modified) Todorova-Yordanov method [22] can be adopted. It is not so important how we derive the weighted energy estimate (see Lemma 3.3 below), but it is essential how we use it.

Historically speaking, under the assumption (A-1) Morawetz [15] derived the local energy decay estimate to problem (1.1)-(1.3) with $a(x) \equiv 1$ in the case when the initial data are compactly supported. After Morawetz those results were generalized to the more general (non-trapping) exterior domain cases in Morawetz-Ralston-Strauss [16]. From the viewpoint of the spectral analysis Vainberg [23] derived the precise local energy decay estimates, and various precise properties of the decay rate of the local energy were studied in Vodev [24] through also the spectral analysis. Melrose [14] and Shibata-Tsutsumi [21] also discussed the more precise decay rates under the non-trapping assumptions on the obstacle $\bar{\Omega}^c$ in the sense of Vainberg. The role of the local energy decay property in the scattering theory can be discussed in Lax-Phillips [12]. Zachmanoglou [25] directly generalized the Morawetz result [15] to the general hyperbolic equations including (1.1). It is important to cite the work by Bloom-Kazarinoff [2] that they allow time-varying coefficients, and only impose conditions on the asymptotic behavior as $|x| \to +\infty$ of the coefficients in the principal part of thier differential operators. In Liu [13] the local energy decay problem for a system of second-order hyperbolic equations with coefficients that also depend on both spatial and time variables is studied under more general conditions on the boundary including the class of star-shaped exterior domains. Bloom [1] investigated the reduced hyperbolic equations of the second order with spatial variables, which are asymptotically equal to the identity matrix as $|x| \to +\infty$. On the other hand, it should be mentioned that in the framework of H^2-solutions the local energy always decays logarithmically even if no geometrical assumptions on the obstacle $\bar{\Omega}^c$ are imposed and for related results we refer the reader to Burq [4] and Ikawa [6]. To the best of the authors' knowledge all these investigations seem to be done in the framework of compactly supported initial data. For these local energy decay results it is well-known that there is no uniform rate of local energy decay in the case where the obstacle is trapping (see Ralston [18, 19]). Especially, in [19] the same type equation as (1.1) is considered in detail.

On the other hand, recently Muraveĭ [17] announced (without proof) the local energy decay estimate like $O(t^{-1})$ as $t \to +\infty$ to the solutions of the problem (1.1)-(1.3) with $a(x) \equiv \mathbf{1}$ under the assumption (A-1), and it is worth while to be mentioned that the result in [17] can be derived without assuming the compactness of the support on initial data. Quite recently Ikehata [8] and Ikehata-Nishihara [11] have also removed the compactness assumption on the supports of the initial data, but they dealt with the case $a(x) \equiv \mathbf{1}$. In the case where the hyperbolic equation is represented by

$$u_{tt} - \nabla \cdot (K(x)\nabla u) = 0, \tag{1.5}$$

Ikehata [9] has established the local energy decay result to problem (1.5), (1.2) and (1.3) without assuming compactness of the supports on the initial data. Unfortunately the weight condition assumed on the initial data as

$|x| \to +\infty$ in [9] was exponential one. In the case of present problem with non-divergence form (1.1) one can weaken the exponential weight condition assumed in [9] to the algebraic one. As far as general hyperbolic operators are concerned, to the best of the authors' knowledge there seem to be no any other results, that dealt with the algebraic decay order weight condition on the initial data (as $|x| \to +\infty$).

Notation By $\|\cdot\|_p$ we mean the usual $L^p(\Omega)$-norm ($1 \le p \le +\infty$), and especially we set $\|\cdot\| = \|\cdot\|_2$. The L^2-inner product is defined by (as usual)

$$(f,\, g) = \int_\Omega f(x)g(x)dx \quad \text{for } f,\, g \in L^2(\Omega).$$

The total energy $E(t)$ associated with the equation (1.1) is defined by

$$E(t) = \frac{1}{2}\left\{\left\|\frac{1}{a}u_t(t,\,\cdot)\right\|^2 + \|\nabla u(t,\,\cdot)\|^2\right\}.$$

On the other hand, let $R > 0$ be an arbitrary real number. Then the local energy associated with the equation (1.1) is given by

$$E_R(t) = \frac{1}{2}\int_{\Omega(R)}\left\{\frac{1}{a(x)^2}|u_t(t,\,x)|^2 + |\nabla u(t,\,x)|^2\right\}dx,$$

where we set $\Omega(R) \equiv \Omega \cap B_R(0)$.

Now let us mention the well-posedness of problem (1.1)-(1.3) (cf. Brezis [3, Théorem X.14] or Ikawa [7, Theorem 2.25]).

Proposition 1.1 *Let $N \ge 1$. For each $[u_0,\, u_1] \in H_0^1(\Omega) \times L^2(\Omega)$, there exists a unique solution $u \in C([0,\,\infty); H_0^1(\Omega)) \cap C^1([0,\,\infty); L^2(\Omega))$ to problem* (1.1)-(1.4) *satisfying*

$$E(t) = E(0), \quad t \ge 0. \tag{1.6}$$

Our main result reads as follows.

Theorem 1.1 *Let $N \ge 3$ and assume* (A-1), (A-2) *and* (A-3). *If the initial data $[u_0,\, u_1] \in H_0^1(\Omega) \times L^2(\Omega)$ further satisfy*

$$\int_\Omega (|x|^2|u_1(x)|^2 + |x||\nabla u_0(x)|^2)dx < +\infty, \tag{1.7}$$

then the unique solution $u(t,x)$ *to problem* (1.1)-(1.4) *has the uniform local energy decay property: for each* $R > r_0$ *it holds that*

$$E_R(t) \leq \frac{C}{t - (R/a_m)}$$

for any $t > R/a_m$*, where* $C > 0$ *is a constant depending only on some quantities* a_0*,* r_0*,* N*, and the initial data.*

Remark 1.1 Since we are dealing with the weakest weight condition on the initial data as far as previously known in the non-constant variable coefficient case, as the first step it is quite natural that we do not care about the optimal decay order of the local energy as in [16], [23] and so on. Conversely speaking, it is open to obtain optimal decay rates of the local energy. In the present case this problem will be much difficult as compared with the compactly supported solution cases because even in the 3-dimensional case one can not use so called the Huygens principle as in [16].

Remark 1.2 It seems difficult to derive the similar result to Theorem 1.1 for the equation (1.5), and it is still open, too. This is because the useful identity (3.10) below breaks.

Remark 1.3 Uniform local energy decay estimate can be also derived without assuming the compactness of the support on the initial data if one assumes further regularity on the initial data, and for this we refer the reader to (for example) Secchi-Shibata [20]. Therefore, in this paper it is also essential that we are dealing with weak solutions to problem (1.1)-(1.3).

Next we shall introduce the $N = 1, 2$ dimensional cases. For this we shall define a weight function $d(x)$ as follows (see Dan-Shibata [5]).

$$d(x) = |x| \log(B|x|), \quad N = 2, \tag{1.8}$$

where $B > 0$ is a constant such that $\inf_{x \in \Omega} |x| \geq 2/B > 0$. Then our results read as follows.

Theorem 1.2 *Let* $N = 2$ *and assume* (A-1), (A-2) *and* (A-3). *If the initial data* $[u_0, u_1] \in H_0^1(\Omega) \times L^2(\Omega)$ *further satisfy*

$$\int_\Omega (d(x)^2 |u_1(x)|^2 + |x||\nabla u_0(x)|^2) dx < +\infty,$$

then one has the same conclusion as in Theorem 1.1.

Theorem 1.3 *Let* $N = 1$ *and assume* (A-2) *and* (A-3). *If the initial data* $[u_0, u_1] \in H_0^1(0, \infty) \times L^2(0, \infty)$ *further satisfy*

$$\int_0^{+\infty} x(|u_1(x)|^2 + |\partial_x u_0(x)|^2)dx < +\infty,$$

then one has the same conclusion as in Theorem 1.1.

Furthermore, one can reconsider the assumption (A-2) in a more general framework. We shall state only a result for the case of $N \geq 3$ (for $N = 1$ and 2 cases it is left to the reader's exercise).

Theorem 1.4 *Let* $N \geq 3$ *and assume* (A-1), (A-3) *and* (A-4). *Under the same assumptions of* Theorem 1.1 *on the initial data, it holds that*

$$E_R(t) \leq \frac{C}{(t - (R/a_m))^{1-\gamma_0}},$$

for large $t \gg 1$.

2. Preliminaries

In this section we shall prepare an identity, which is used later in the proof of Theorems. The following identity is a modification of that introduced in [15] in the case when (at least) $a(x) \equiv 1$. Note that since we are dealing with a weak solution to problem (1.1)-(1.4) we may assume that in the following calculations the corresponding solution is sufficiently smooth and vanishes as $|x| \to +\infty$.

Proposition 2.1 (cf. [25, page 511, (4.2)]) *Let* $N \geq 1$. *Under the assumptions of* Proposition 1.1 *it holds that*

$$\begin{aligned} tE(t) = &\frac{N-1}{2}\left(\frac{1}{a^2}u_1, u_0\right) + \left(\frac{1}{a^2}u_1, x \cdot \nabla u_0\right) \\ &-\frac{N-1}{2}\left(\frac{1}{a^2}u_t(t, \cdot), u(t, \cdot)\right) - \left(\frac{1}{a^2}u_t(t, \cdot), x \cdot \nabla u(t, \cdot)\right) \\ &+\frac{1}{2}\int_0^t \int_{\partial\Omega} \left(\frac{\partial u(s, \sigma)}{\partial n}\right)^2 \sigma \cdot n(\sigma) dS_\sigma ds \\ &+\int_0^t \int_\Omega \frac{1}{a(x)^3}(x \cdot \nabla a(x)) u_t(s, x)^2 dx ds, \end{aligned}$$

where $n = n(\sigma)$ *is the unit outward normal vector relative to* Ω *at* $\sigma \in \partial\Omega$.

In the following paragraph we shall give some elementary lemmas without proof, which are used in the proof of Proposition 2.1.

Lemma 2.1 *Under the assumptions of* Proposition 2.1 *it holds that*

$$\begin{aligned}&\int_\Omega \Delta u(t,\,x)(x\cdot\nabla u(t,\,x))dx\\&=\frac{1}{2}\int_{\partial\Omega}\left(\frac{\partial u(s,\,\sigma)}{\partial n}\right)^2\sigma\cdot n(\sigma)dS_\sigma+\frac{N-2}{2}\int_\Omega|\nabla u(t,\,x)|^2dx.\end{aligned}$$

Lemma 2.2 *Under the assumptions of* Proposition 2.1 *it holds that*

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{a^2}u_t(t,\,\cdot),\,x\cdot\nabla u(t,\,\cdot)\right)+\frac{N}{2}\|\frac{1}{a}u_t(t,\,\cdot)\|^2\\&=\left(\frac{1}{a^2}u_{tt}(t,\,\cdot),\,x\cdot\nabla u(t,\,\cdot)\right)+\int_\Omega\frac{1}{a(x)^3}(x\cdot\nabla a)u_t(t,\,x)^2dx.\end{aligned}$$

Lemma 2.3 *Under the assumptions of* Proposition 2.1 *it holds that*

$$\frac{d}{dt}(tE(t))-E(t)=0.$$

Lemma 2.4 *Under the assumptions of* Proposition 2.1 *it holds that*

$$\begin{aligned}&\frac{N-1}{2}\frac{d}{dt}\left(\frac{1}{a^2}u_t(t,\,\cdot),\,u(t,\,\cdot)\right)\\&\qquad-\frac{N-1}{2}\left\|\frac{1}{a}u_t(t,\,\cdot)\right\|^2+\frac{N-1}{2}\|\nabla u\|^2=0.\end{aligned}$$

Sum up the four identities integrated over $[0,\,t]$ in lemmas above. Then the desired identity in Proposition 2.1 can be derived with several cancellations.

3. Proof of theorems

In this section let us prove our results by relying on the identity in Proposition 2.1. To begin with, the following basic inequality is useful for the proof (see Dan-Shibata [5, Theorem 2.3]).

Lemma 3.1 (Hardy-Sobolev) *Let $N \geq 3$. Then for $u \in H_0^1(\Omega)$, it holds that*

$$\left\| \frac{u}{|x|} \right\| \leq C\|\nabla u\|$$

with some constant $C > 0$.

The following lemma is a generalization of a result, which was derived by a "device" due to Ikehata-Matsuyama [10] (which is an essential modification of the Morawetz [15] method) to problem (1.1)-(1.3) with $a(x) \equiv 1$. The advantage is that we do not assume any compactness of the support on the initial data as in previous results like [15], [21] and [25] (especially, compare the result below with [25, Theorem 3]).

Lemma 3.2 *Let $N \geq 3$, and $[u_0, u_1] \in H_0^1(\Omega) \times L^2(\Omega)$ satisfy $\||x|u_1\| < +\infty$. Then the unique solution $u(t, x)$ to problem* (1.1)-(1.4) *as in* Proposition 1.1 *satisfies*

$$\left\| \frac{1}{a} u(t, \cdot) \right\| \leq C\left(\left\| \frac{1}{a} u_0 \right\| + \left\| \frac{1}{a^2} |x| u_1 \right\| \right).$$

Remark 3.1 Because of (1.4) under the assumptions on the initial data as in Lemma 3.2 it holds that

$$\left\| \frac{1}{a} u_0 \right\| + \left\| \frac{1}{a^2} |x| u_1 \right\| < +\infty.$$

Proof of Lemma 3.2. As in [10] we first set

$$w(t, x) = \int_0^t u(s, x) ds.$$

Since the equation (1.1) is linear,

$$w(t, x) \in C^1([0, +\infty); H_0^1(\Omega)) \cap C^2([0, +\infty); L^2(\Omega))$$

satisfies the transformed mixed problem:

$$\frac{1}{a(x)^2} w_{tt} - \Delta w = \frac{1}{a(x)^2} u_1, \quad (t, x) \in (0, \infty) \times \Omega, \tag{3.1}$$

$$w(0, x) = 0, \quad w_t(0, x) = u_0(x), \quad x \in \Omega, \tag{3.2}$$

$$w|_{\partial\Omega} = 0, \quad t \in (0, \infty). \tag{3.3}$$

Multiplying the equation (3.1) by w_t and integrating it, one has

$$\begin{aligned}
&\frac{1}{2}\left\|\frac{1}{a}w_t(t,\,\cdot\,)\right\|^2+\frac{1}{2}\|\nabla w(t,\,\cdot\,)\|^2\\
&=\frac{1}{2}\left\|\frac{1}{a}u_0\right\|^2+\int_0^t\frac{d}{ds}\Big(\frac{1}{a^2}u_1,\,w(s,\,\cdot\,)\Big)ds\\
&\qquad\qquad=\frac{1}{2}\left\|\frac{1}{a}u_0\right\|^2+\Big(\frac{1}{a^2}u_1,\,w(t,\,\cdot\,)\Big). \qquad (3.4)
\end{aligned}$$

Here, we see from the Schwarz inequality and Lemma 3.1 that

$$\begin{aligned}
\Big(\frac{1}{a^2}u_1,\,w(t,\,\cdot\,)\Big)&\le\left\|\frac{1}{a^2}|x|u_1\right\|\,\left\|\frac{w(t,\,\cdot\,)}{|x|}\right\|\\
&\le C\left\|\frac{1}{a^2}|x|u_1\right\|\,\|\nabla w(t,\,\cdot\,)\|. \qquad (3.5)
\end{aligned}$$

Hence it follows from (3.4) and (3.5) that

$$\begin{aligned}
&\frac{1}{2}\left\{\left\|\frac{1}{a}w_t(t,\,\cdot\,)\right\|^2+\|\nabla w(t,\,\cdot\,)\|^2\right\}\\
&\le\frac{1}{2}\left\|\frac{1}{a}u_0\right\|^2+C\left\|\frac{1}{a^2}|x|u_1\right\|\,\|\nabla w(t,\,\cdot\,)\|\\
&\qquad\le\frac{1}{2}\left\|\frac{1}{a}u_0\right\|^2+\frac{C}{2\varepsilon}\left\|\frac{1}{a^2}|x|u_1\right\|^2+\frac{C\varepsilon}{2}\|\nabla w(t,\,\cdot\,)\|^2
\end{aligned}$$

for any $\varepsilon>0$, which implies

$$\frac{1}{2}\left\|\frac{1}{a}w_t(t,\,\cdot\,)\right\|^2+\frac{1-C\varepsilon}{2}\|\nabla w(t,\,\cdot\,)\|^2\le\frac{1}{2}\left\|\frac{1}{a}u_0\right\|^2+\frac{C}{2\varepsilon}\left\|\frac{1}{a^2}|x|u_1\right\|^2.$$

Therefore, taking $\varepsilon\in(0,\,1/C)$ so small and noting $w_t=u$, one has the desired estimate. □

The next weighted energy estimate is important in deriving the local energy decay. That idea originally comes from the new weighted energy method due to Todorova-Yordanov [22]. Their method was originally applied to the damped wave equations. It is essential how we use the estimates below, and is not so important how we derive it. For this purpose one defines the weight function $\psi(t,\,x)$ $((t,\,x) \in [0,\,+\infty) \times \mathbf{R}^N)$ as follows.

$$\psi(t,\,x) = \begin{cases} (1 + |x| - a_0 t), & |x| \geq a_0 t, \\ (1 + a_0 t - |x|)^{-1}, & |x| < a_0 t. \end{cases}$$

It is easily checked that the function $\psi \in C^1([0,\,+\infty) \times \bar{\Omega})$ satisfies

$$\frac{\partial \psi}{\partial t}(t,\,x) < 0, \quad (t,\,x) \in [0,\,+\infty) \times \bar{\Omega}, \tag{3.6}$$

$$a_0^2 |\nabla \psi(t,\,x)|^2 - \psi_t(t,\,x)^2 = 0, \quad (t,\,x) \in [0,\,+\infty) \times \bar{\Omega}, \tag{3.7}$$

$$\psi(t,\,x) > 0. \tag{3.8}$$

(3.7) is so called the Eikonal equation. Then one has the following important lemma.

Lemma 3.3 *Let $N \geq 2$, and assume* (A-3). *If the initial data $[u_0,\,u_1] \in H_0^1(\Omega) \times L^2(\Omega)$ further satisfy* (1.7), *then the unique solution $u(t,\,x)$ to problem* (1.1)-(1.4) *as in* Proposition 1.1 *satisfies*

$$\int_{|x| \geq R} \psi(t,\,x) \left(\frac{1}{a(x)^2} |u_t(t,\,x)|^2 + |\nabla u(t,\,x)|^2 \right) dx$$
$$\leq C_{r_0} \int_{\Omega} (1 + |x|) \left(\frac{1}{a(x)^2} |u_1(x)|^2 + |\nabla u_0(x)|^2 \right) dx = C_{r_0} I_0$$

for each $R > r_0$, where $C_{r_0} > 0$ is a constant depending on $r_0 > 0$.

In order to prove Lemma 3.3 above one must prepare the auxiliary weight function $\phi \in C^1([0,\,+\infty))$ as follows. This part is essential in our idea.

$$\phi(t) = \begin{cases} (1 + r_0 - a_0 t), & r_0 \geq a_0 t, \\ (1 + a_0 t - r_0)^{-1}, & r_0 < a_0 t. \end{cases}$$

Here it is also true that

$$\phi_t(t) < 0. \tag{3.9}$$

It follows from the similar derivation to Todorova-Yordanov [22] that the following two identities can be obtained. Note that the two identities below are slightly modified as compared with that of [22] in order to obtain appropriate informations deriving the algebraic weight condition on the initial data (for the derivation of these identities see the Appendix).

$$0 = (\psi u_t)\left(\frac{1}{a(x)^2}u_{tt} - \Delta u\right) = \frac{d}{dt}(\psi E(t,\,x)) - \operatorname{div}(\psi u_t \nabla u)$$
$$-\frac{1}{2\psi_t}|\psi_t \nabla u - u_t \nabla \psi|^2 + \frac{u_t^2}{2a(x)^2\psi_t}(a(x)^2|\nabla\psi|^2 - \psi_t^2), \tag{3.10}$$

and

$$0 = (\phi u_t)\left(\frac{1}{a(x)^2}u_{tt} - \Delta u\right) = \frac{d}{dt}(\phi E(t,\,x)) - \phi(t)\operatorname{div}(u_t \nabla u)$$
$$-\frac{\phi_t(t)}{2a(x)^2}(a(x)^2|\nabla u|^2 + u_t^2), \tag{3.11}$$

where we set

$$E(t,\,x) = \frac{1}{2}\left(\frac{1}{a(x)^2}|u_t(t,\,x)|^2 + |\nabla u(t,\,x)|^2\right).$$

Proof of Lemma 3.3. Since we are dealing with a weak solution, by density argument we may assume that the initial data and the corresponding solution are sufficiently smooth and vanish as $|x| \to +\infty$.

To begin with, it follows from (3.6) and (3.10) that

$$0 \geq \frac{d}{dt}(\psi E(t,\,x)) - \operatorname{div}(\psi u_t \nabla u) + \frac{u_t^2}{2a(x)^2\psi_t}(a(x)^2|\nabla\psi|^2 - \psi_t^2).$$

Integrate the above inequality over $[0,\,t] \times \{|x| \geq r_0\}$. Because of (A-3) we see that

$$\int_{|x|\geq r_0} \psi(0,\,x)E(0,\,x)dx + \int_0^t \int_{|x|\geq r_0} \operatorname{div}(\psi u_t \nabla u)dxds$$
$$\geq \int_{|x|\geq r_0} \psi(t,\,x)E(t,\,x)dx + \int_0^t \int_{|x|\geq r_0} \frac{u_t^2}{2a_0^2\psi_t}(a_0^2|\nabla\psi|^2 - \psi_t^2)dxds.$$

By applying (3.7) one obtains

$$\int_{|x|\geq r_0} \psi(0,x)E(0,x)dx + \int_0^t \int_{|x|\geq r_0} \operatorname{div}(\psi u_t \nabla u)dxds$$
$$\geq \int_{|x|\geq r_0} \psi(t,\,x)E(t,\,x)dx. \tag{3.12}$$

On the other hand, because of (3.9) by integrating both sides of (3.11) over $[0,\, t] \times \Omega(r_0)$ one has

$$\int_{\Omega(r_0)} \phi(0)E(0,x)dx + \int_0^t \int_{\Omega(r_0)} \phi(s)\,\mathrm{div}(u_t \nabla u)dxds$$
$$\geq \int_{\Omega(r_0)} \phi(t)E(t,x)dx. \tag{3.13}$$

Since $\phi(t) = \psi(t,\, x)$ on $|x| = r_0$, it follows from the divergence formula that

$$\int_0^t \int_{\Omega(r_0)} \phi(s)\,\mathrm{div}(u_t \nabla u)dxds + \int_0^t \int_{|x|\geq r_0} \mathrm{div}(\psi u_t \nabla u)dxds = 0.$$

In order to erase the second term of the left hand side of (3.12), let us sum up both inequalities (3.12) and (3.13). Then because of the positivity of $\phi(t)$ one has arrived at

$$\int_{|x|\geq r_0} \psi(t,\, x)E(t,\, x)dx$$
$$\leq C_{r_0} \int_{\Omega(r_0)} E(0,\, x)dx + \int_{|x|\geq r_0} (1+|x|)E(0,\, x)dx.$$

Since $R > r_0$, the desired estimate follows from (3.8). $\square$

Lemma 3.4 *Let $R > r_0$ and $a_0 t > R$. Under the assumptions of* Theorem 1.1 *it holds that*

$$\left|\left(x \cdot \nabla u(t,\,\cdot),\, \frac{1}{a(\,\cdot\,)^2} u_t(t,\,\cdot)\right)\right|$$
$$\leq \frac{R}{a_m} E_R(t) + \frac{CI_0}{2a_0} + t\int_{|x|\geq R} E(t,\, x)dx.$$

Proof. Let $R > r_0$ and $t > R/a_0$. We have

$$\left|\left(x \cdot \nabla u(t,\,\cdot),\, \frac{1}{a(\,\cdot\,)^2} u_t(t,\,\cdot)\right)\right|$$
$$\leq \left(\int_{\Omega(R)} + \int_{|x|\geq R}\right) \frac{1}{a(x)^2} |x||\nabla u||u_t|dx$$
$$\leq \frac{R}{a_m} \int_{\Omega(R)} E(t,\, x)dx + \int_{|x|\geq R} \frac{1}{a(x)^2} |x||\nabla u||u_t|dx$$

$$
\begin{aligned}
&\le \frac{R}{a_m}E_R(t) + \int_{\{|x|\ge a_0 t\}} \frac{1}{a(x)^2}|x||\nabla u||u_t|dx \\
&\quad + \int_{\{a_0 t\ge |x|\ge R\}} \frac{1}{a(x)^2}|x||\nabla u||u_t|dx \\
&\le \frac{R}{a_m}E_R(t) + \int_{\{|x|\ge a_0 t\}} \frac{1}{a(x)^2}(|x|-a_0 t)|\nabla u||u_t|dx \\
&\quad + a_0 t\int_{\{|x|\ge a_0 t\}} \frac{1}{a(x)^2}|\nabla u||u_t|dx \\
&\quad + a_0 t\int_{\{a_0 t\ge |x|\ge R\}} \frac{1}{a(x)^2}|\nabla u||u_t|dx \\
&\le \frac{R}{a_m}E_R(t) + \frac{1}{a_0}\int_{\{|x|\ge a_0 t\}} (1+|x|-a_0 t)E(t,\,x)dx \\
&\quad + t\int_{|x|\ge R} E(t,\,x)dx
\end{aligned}
$$

(because of (3.8) and (A-3))

$$
\le \frac{R}{a_m}E_R(t) + \frac{1}{a_0}\int_{|x|\ge R} \psi(t,\,x)E(t,\,x)dx + t\int_{|x|\ge R} E(t,\,x)dx.
$$

By using Lemma 3.3 one has the desired statement. □

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. First it follows from Proposition 2.1, and the assumptions (A-1) and (A-2) that

$$
\begin{aligned}
tE(t) &\le \left(x\cdot\nabla u_0,\, \frac{1}{a^2}u_1\right) + \frac{N-1}{2}\left(u_0,\, \frac{1}{a^2}u_1\right) \\
&\quad - \left(x\cdot\nabla u(t,\,\cdot),\, \frac{1}{a^2}u_t(t,\,\cdot)\right) - \frac{N-1}{2}\left(u(t,\,\cdot),\, \frac{1}{a^2}u_t(t,\,\cdot)\right),
\end{aligned} \tag{3.14}
$$

where we have used the assumption (A-1), that is,

$$
\int_0^t\int_{\partial\Omega}\{\sigma\cdot n(\sigma)\}\left|\frac{\partial u(s,\,\sigma)}{\partial n}\right|^2 dS_\sigma ds \le 0.
$$

Thus we have only to estimate the last two terms in the right hand side of (3.14). Indeed, since one obtains

$$
tE(t) = tE_R(t) + t\int_{|x|\ge R} E(t,\,x)dx, \tag{3.15}
$$

from (3.14), (3.15) and Lemma 3.4 it follows that

$$\Big(t-\frac{R}{a_m}\Big)E_R(t)\le\Big(x\cdot\nabla u_0,\,\frac{1}{a^2}u_1\Big)+\frac{N-1}{2}\Big(u_0,\,\frac{1}{a^2}u_1\Big)\\+\frac{CI_0}{2a_0}+\frac{N-1}{2}\Big|\Big(u(t,\,\cdot\,),\,\frac{1}{a^2}u_t(t,\,\cdot\,)\Big)\Big|. \quad (3.16)$$

Now, let us apply Lemma 3.2 in order to control the last term in the right hand side of (3.16). This part is also crucial in deriving the local energy decay. In fact, because of (1.6) and Lemma 3.2 one can evaluate as

$$\Big|\Big(u(t,\,\cdot\,),\,\frac{1}{a^2}u_t(t,\,\cdot\,)\Big)\Big|\le\frac{1}{2}\Big\|\frac{1}{a}u_t(t,\,\cdot\,)\Big\|^2+\frac{1}{2}\Big\|\frac{1}{a}u(t,\,\cdot\,)\Big\|^2\\\le E(0)+C\Big\{\Big\|\frac{1}{a}u_0\Big\|^2+\Big\|\frac{1}{a^2}|x|u_1\Big\|^2\Big\}\equiv I_1. \quad (3.17)$$

Thus, one obtains

$$\Big(t-\frac{R}{a_m}\Big)E_R(t)\le\Big(x\cdot\nabla u_0,\,\frac{1}{a^2}u_1\Big)+\frac{N-1}{2}\Big(u_0,\,\frac{1}{a^2}u_1\Big)\\+\frac{CI_0}{2a_0}+\frac{N-1}{2}I_1\\\equiv C,$$

which completes the proof of Theorem 1.1. □

Remark 3.2 Note that the quantity $|(x\cdot\nabla u_0,\,(1/a^2)u_1)|$ above has a finite value. In fact, by assumptions (1.4) and (1.7) it follows that

$$\Big|\Big(x\cdot\nabla u_0,\,\frac{1}{a^2}u_1\Big)\Big|\le\frac{1}{2a_m^2}\int_\Omega|x|(|u_1|^2+|\nabla u_0|^2)dx<+\infty.$$

Let us prove Theorem 1.2. Instead of Lemma 3.1 the following Hardy-Sobolev type inequality is needed (see Dan-Shibata [5]).

Lemma 3.5 (Hardy-Sobolev) *Let $N=2$. Then for each $u\in H_0^1(\Omega)$, it holds that*

$$\Big\|\frac{u}{d(\,\cdot\,)}\Big\|\le C\|\nabla u\|$$

with some constant $C>0$, where $d(x)$ is a function defined in (1.8).

In the case when $N = 2$, by using Lemma 3.5 in place of Lemma 3.1 one can give the proof of Theorem 1.2 quite similarly to that of Theorem 1.1, and for a result corresponding to Lemma 3.2 we can state as follows.

Lemma 3.6 *Let $N = 2$. If the initial data $[u_0, u_1] \in H_0^1(\Omega) \times L^2(\Omega)$ further satisfy*

$$\|d(\,\cdot\,)u_1\| < +\infty,$$

then the unique solution $u(t, x)$ to problem (1.1)-(1.4) *as in* Proposition 1.1 *satisfies*

$$\|u(t, \cdot\,)\| \le C(\|u_0\| + \|d(\,\cdot\,)u_1\|).$$

Outline of proof of Lemma 3.6. It is sufficient to check the following part because the other part of proof is done quite similarly to that in Lemma 3.2. Indeed, we see from Lemma 3.5 and the Schwarz inequality that

$$\begin{aligned}\left|\left(\frac{1}{a^2}u_1,\, w(t,\,\cdot\,)\right)\right| &\le \left\|\frac{d(\,\cdot\,)}{a^2}u_1\right\| \left\|\frac{w(t,\,\cdot\,)}{d(\,\cdot\,)}\right\| \\ &\le C\left\|\frac{d(\,\cdot\,)}{a^2}u_1\right\| \|\nabla w(t,\,\cdot\,)\| \\ &\le C\frac{1}{a_m^2}\|d(\,\cdot\,)u_1\| \, \|\nabla w(t,\,\cdot\,)\|.\end{aligned}$$

This completes the proof of Lemma 3.6. □

In the case when $N = 1$, we can proceed the proof of Theorem 1.3 in a quite similar fashion to Theorem 1.1 with a slight modification. We shall omit the detail. Note that in this case we use Proposition 2.1 replaced by the following identity:

$$\begin{aligned}tE(t) =& \left(\frac{1}{a^2}u_1,\, x\cdot\nabla u_0\right) - \left(\frac{1}{a^2}u_t,\, x\cdot\nabla u\right) \\ &+ \int_0^t\int_0^\infty \frac{1}{a(x)^3}(x\cdot\nabla a(x))u_t(s,\,x)^2dxds.\end{aligned}$$

So, the weght condition on the initial data in Theorem 1.3 only comes from (modified) Lemma 3.3.

Proof of Theorem 1.4. First it follows from (A-1), (A-3), (A-4), Lemma 3.4, Proposition 2.1 and (3.17) that

$$\begin{aligned} tE_R(t) + t\int_{|x|\geq R} E(t,\,x)dx &\leq A_0 + \frac{N-1}{2}I_1 + \frac{R}{a_m}E_R(t) \\ &+ \frac{CI_0}{2a_0} + t\int_{|x|\geq R} E(t,\,x)dx + \gamma_0\int_0^t\int_{\Omega(r_0)}\left(\frac{u_t(s,\,x)^2}{2a(x)^2}\right)dxds, \end{aligned}$$

where

$$A_0 = \frac{N-1}{2}\left(u_0,\,\frac{u_1}{a^2}\right) + \left(x\cdot\nabla u_0,\,\frac{u_1}{a^2}\right),$$

so that one obtains

$$\left(t - \frac{R}{a_m}\right)E_R(t) \leq K_0 + \gamma_0\int_0^t E_R(s)ds, \tag{3.18}$$

for an arbitrarily fixed $R > r_0$ and large $t \gg 1$, where

$$K_0 = A_0 + \frac{N-1}{2}I_1 + \frac{CI_0}{2a_0}.$$

Set

$$\xi(t) = \left(t - \frac{R}{a_m}\right)^{-\gamma_0}\int_0^t E_R(s)ds.$$

Then by (3.18) one can calculate as follows:

$$\begin{aligned} \xi'(t) &= \left(t - \frac{R}{a_m}\right)^{-1-\gamma_0}\left\{\left(t - \frac{R}{a_m}\right)E_R(t) - \gamma_0\int_0^t E_R(s)ds\right\} \\ &\leq K_0\left(t - \frac{R}{a_m}\right)^{-1-\gamma_0} \end{aligned}$$

for large $t \gg 1$. By integrating the inequality just above one has

$$\begin{aligned} \xi(t) &\leq \xi(t_0) + K_0\int_{t_0}^t\left(s - \frac{R}{a_m}\right)^{-1-\gamma_0}ds \\ &\leq \xi(t_0) + K_0\frac{1}{\gamma_0}\left(t_0 - \frac{R}{a_m}\right)^{-\gamma_0} \equiv L_0. \end{aligned}$$

By combining this inequality together with (3.18) one obtains

$$\left(t - \frac{R}{a_m}\right)E_R(t) \leq K_0 + \gamma_0\int_0^t E_R(s)ds \leq K_0 + \gamma_0 L_0\left(t - \frac{R}{a_m}\right)^{\gamma_0}$$

for large $t \gg 1$, which implies the desired estimate. $\square$

4. Appendix

In this appendix let us make sure the identity (3.10) ((3.11) is a special case of (3.10)). In fact,

$$\begin{aligned}
&-\psi(t,x)u_t\Delta u \\
&= -\operatorname{div}(\psi u_t \nabla u) - \frac{1}{2\psi_t}|\psi_t\nabla u - u_t\nabla\psi|^2 \\
&\quad + \frac{u_t^2}{2\psi_t}|\nabla\psi|^2 + \frac{\psi}{2}\frac{\partial}{\partial t}|\nabla u|^2 + \frac{\psi_t^2}{2\psi_t}|\nabla u|^2.
\end{aligned}$$

On the other hand, one has

$$\begin{aligned}
\psi u_t \frac{1}{a^2}u_{tt} = {}& \frac{\partial}{\partial t}\{\psi(t,x)E(t,x)\} \\
& - \frac{\psi_t}{2}\frac{1}{a^2}|u_t|^2 - \frac{\psi}{2}\frac{\partial}{\partial t}|\nabla u|^2 - \frac{\psi_t}{2}|\nabla u|^2.
\end{aligned}$$

Sum up the above two equalities. By several cancellations one has the desired identity. $\square$

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