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Sharing three values with small weights

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Abstract. We prove a uniqueness theorem for meromorphic functions sharing three values with small weights which improves some known results. We also exhibit some applications of the main result.

Key words: weighted sharing, uniqueness, meromorphic functions.

1. Introduction, Definitions and Results

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. For $b \in \mathbb{C} \cup \{\infty\}$ we say that f and g share the value b CM (counting multiplicities) if f and g have the same b-points with the same multiplicities. If we do not take multiplicities into account, we say that f and g share the value b IM (ignoring multiplicities). For standard definitions and notations of the value distribution theory we refer [1].

H. Ueda [9] proved the following result

Theorem A ([9]) *Let f and g be two distinct nonconstant entire functions sharing 0, 1 CM and let a $(\neq 0, 1)$ be a finite complex number. If a is lacunary for f then $1-a$ is lacunary for g and $(f-a)(g+a-1) \equiv a(1-a)$.*

Improving *Theorem A* H.X. Yi [11] proved the following theorem.

Theorem B ([11]) *Let f and g be two distinct nonconstant entire functions sharing 0, 1 CM and let a $(\neq 0, 1)$ be a finite complex number. If $\delta(a; f) > 1/3$ then a and $1-a$ are Picard exceptional values of f and g respectively and $(f-a)(g+a-1) \equiv a(1-a)$.*

Extending *Theorem B* to meromorphic functions S.Z. Ye [10] proved the following results.

Theorem C ([10]) *Let f and g be two distinct nonconstant meromorphic functions such that f and g share 0, 1, ∞ CM. Let a $(\neq 0, 1)$ be a finite complex number. If $\delta(a; f) + \delta(\infty; f) > 4/3$ then a and $1-a$ are Picard*

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exceptional values of f and g respectively and also ∞ is so and $(f-a)(g+a-1) \equiv a(1-a)$.

Theorem D ([10]) *Let f and g be two distinct nonconstant meromorphic functions sharing $0,\ 1,\ \infty$ CM. Let $a_1,\ a_2,\ \ldots,\ a_p$ be $p\ (\geq 1)$ distinct finite complex numbers and $a_j \neq 0,\ 1$ for $j = 1,\ 2,\ 3, \ldots,\ p$. If $\sum_{j=1}^{p} \delta(a_j; f) + \delta(\infty; f) > 2(p+1)/(p+2)$ then there exist one and only one a_k in $a_1,\ a_2,\ \ldots,\ a_p$ such that a_k and $1-a_k$ are Picard exceptional values of f and g respectively and also ∞ is so and $(f-a_k)(g+a_k-1) \equiv a_k(1-a_k)$.*

Improving above results H.X. Yi [12] proved the following theorem.

Theorem E ([12]) *Let f and g be two distinct nonconstant meromorphic functions such that f and g share $0,\ 1,\ \infty$ CM. Let $a\ (\neq 0,\ 1)$ be a finite complex number. If $N(r,\ a; f) \neq T(r,\ f) + S(r,\ f)$ and $N(r,\ f) \neq T(r,\ f) + S(r,\ f)$ then a and $1-a$ are Picard exceptional values of f and g respectively and also ∞ is so and $(f-a)(g+a-1) \equiv a(1-a)$.*

Definition 1 Let p be a positive integer and $b \in \mathbb{C} \cup \{\infty\}$. Then by $N(r,\ b; f \mid\leq p)$ we denote the counting function of those b-points of f (counted with proper multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r,\ b; f \mid\leq p)$ we denote the corresponding reduced counting function.

In an analogous manner we define $N(r,\ b; f \mid\geq p)$ and $\overline{N}(r,\ b; f \mid\geq p)$.

Also we put

$$\delta_{p)}(a; f) = 1 - \limsup_{r\to\infty} \frac{N(r,\ a; f \mid\leq p)}{T(r,\ f)}.$$

Hua and Fang [2] proved that if two nonconstant distinct meromorphic functions f and g share $0,\ 1,\ \infty$ CM then $N(r,\ a; f \mid\geq 3) = S(r,\ f)$ for any complex number $a\ (\neq 0,\ 1,\ \infty)$.

Also Yi [12] proved that if two nonconstant distinct meromorphic functions f and g share $0,\ 1,\ \infty$ CM then $N(r,\ \infty; f \mid\geq 2) = S(r,\ f)$.

Therefore *Theorem E* of Yi can easily be improved to the following result.

Theorem F ([5]) *Let f and g be distinct nonconstant meromorphic functions sharing $0,\ 1,\ \infty$ CM. If $a\ (\neq 0,\ 1)$ is a finite complex number such that $N(r,\ a; f \mid\leq 2) \neq T(r,\ f) + S(r,\ f)$ and $N(r,\ \infty; f \mid\leq 1) \neq T(r,\ f) + S(r,\ f)$ then a and $1-a$ are Picard exceptional values of f and g respectively and* *also* ∞ *is so and* $(f-a)(g+a-1) \equiv a(1-a)$.

Following examples show that *Theorem F* is sharp.

Example 1 ([5]) Let $f=(e^z-1)/(e^z+1)$, $g=(1-e^z)/(1+e^z)$, $a_1=-1$ and $a_2=2$. Then f, g share $0, 1, \infty$ **CM**. Also $N(r,\infty;f\mid\leq 1)=T(r,f)+S(r,f)$, $N(r,a_1;f\mid\leq 2)\neq T(r,f)+S(r,f)$ and $N(r,a_2;f\mid\leq 2)=T(r,f)+S(r,f)$. Clearly $(f-a_i)(g+a_i-1)\not\equiv a_i(1-a_i)$ for $i=1,2$.

Example 2 ([5]) Let $f=e^z$, $g=e^{-z}$ and $a=2$. Then f , g share $0, 1, \infty$ **CM**. Also $N(r,\infty;f\mid\leq 1)\neq T(r,f)+S(r,f)$, $N(r,a;f\mid\leq 2)=T(r,f)+S(r,f)$. Clearly $(f-a)(g+a-1)\not\equiv a(1-a)$.

It is shown in [5] by the following example that the condition $N(r,a;f\mid\leq 2)\neq T(r,f)+S(r,f)$ of *Theorem F* cannot be replaced by any one of $N(r,a;f\mid\leq 1)\neq T(r,f)+S(r,f)$ and $\overline{N}(r,a;f\mid\leq 2)\neq T(r,f)+S(r,f)$.

Example 3 ([5]) Let $f=e^z(1-e^z)$, $g=e^{-z}(1-e^{-z})$ and $a=1/4$. Then f , g share $0, 1, \infty$ CM. Also $N(r,\infty;f\mid\leq 1)\neq T(r,f)+S(r,f)$. Since $f-a=-(e^z-2a)^2$, we see the following
(i) $N(r,a;f\mid\leq 1)\equiv 0$,
(ii) $\overline{N}(r,a;f\mid\leq 2)=N(r,2a;e^z)=(1/2)T(r,f)+S(r,f)$ and
(iii) $N(r,a;f\mid\leq 2)=2N(r,2a;e^z)=T(r,f)+S(r,f)$.
Also clearly $(f-a)(g+a-1)\not\equiv a(1-a)$.

Following two examples show that in the above theorems the sharing of 0 and 1 can not be relaxed from CM to IM.

Example 4 ([5]) Let $f=e^z-1$, $g=(e^z-1)^2$ and $a=-1$. Then f, g share 0 IM and $1, \infty$ CM. Also $N(r,\infty;f)\equiv 0$ and $N(r,a;f)\equiv 0$ but $(f-a)(g+a-1)\not\equiv a(1-a)$.

Example 5 ([5]) Let $f=2-e^z$, $g=e^z(2-e^z)$ and $a=2$. Then f, g share 1 IM and $0, \infty$ CM. Also $N(r,\infty;f)\equiv 0$ and $N(r,a;f)\equiv 0$ but $(f-a)(g+a-1)\not\equiv a(1-a)$.

In [5] following question is asked: Is it really impossible to relax in any way the nature of sharing of any one of 0 and 1 in the above theorems?

The notion of weighted sharing of values is used in [5] to deal this problem. We now explain the notion in the following definition which measures how close a shared value is to being shared CM or to being shared IM.

Definition 2 ([3, 4]) Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \le k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_o is a zero of $f - a$ with multiplicity m $(\le k)$ if and only if it is a zero of $g - a$ with multiplicity m $(\le k)$ and z_o is a zero of $f - a$ with multiplicity m $(> k)$ if and only if it is a zero of $g - a$ with multiplicity n $(> k)$ where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integer p, $0 \le p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Improving *Theorem C* in [5] following result is proved.

Theorem G ([5]) *Let f and g be two distinct meromorphic functions sharing* $(0, 1)$*,* $(1, \infty)$ *and* (∞, ∞)*. If* a $(\ne 0, 1)$ *is a finite complex number such that* $3\delta_{2)}(a; f) + 2\delta_{1)}(\infty; f) > 3$ *then* a *and* $1-a$ *are Picard exceptional values of* f *and* g *and also* ∞ *is so and* $(f - a)(g + a - 1) \equiv a(1 - a)$.

In [5] we were unable to relax the nature of sharing of values in *Theorem F.* We now take up this problem and prove the following result which improve *Theorem F* and so all previous results.

Theorem 1 *Let* f *and* g *be two distinct meromorphic functions sharing* $(0, 1)$*,* $(1, m)$ *and* (∞, k)*, where* $(m-1)(mk-1) > (1+m)^2$*. If* $a (\ne 0, 1)$ *is a finite complex number such that* $N(r, a; f \mid\le 2) \ne T(r, f) + S(r, f)$ *and* $N(r, \infty; f \mid\le 1) \ne T(r, f) + S(r, f)$ *then* a *and* $1-a$ *are Picard exceptional values of* f *and* g *respectively and also* ∞ *is so and* $(f - a)(g + a - 1) \equiv a(1 - a)$.

We note that the condition $(m-1)(mk-1) > (1+m)^2$ is equivalent to $(m-1)(k-1) > 4$ and so is symmetric in m and k. We also note that *Theorem 1* holds for the following pairs of least values of m and k: (i) $m = 3$, $k = 4$; (ii) $m = 4$, $k = 3$; (iii) $m = 2$, $k = 6$; (iv) $m = 6$, $k = 2$.

Definition 3 Let f and g share a value a IM. Let z be an a-point of f and g with multiplicities $p_f(z)$ and $p_g(z)$ respectively.

We put

$$\overline{\nu}_f(z) = 1 \quad \text{if } p_f(z) > p_g(z)$$
$$= 0 \quad \text{if } p_f(z) \leq p_g(z)$$

and

$$\overline{\mu}_f(z) = 1 \quad \text{if } p_f(z) < p_g(z)$$
$$= 0 \quad \text{if } p_f(z) \geq p_g(z).$$

Let $\overline{n}(r, a; f > g) = \sum_{|z| \leq r} \overline{\nu}_f(z)$ and $\overline{n}(r, a; f < g) = \sum_{|z| \leq r} \overline{\mu}_f(z)$. We now denote by $\overline{N}(r, a; f > g)$ and $\overline{N}(r, a; f < g)$ the integrated counting functions obtained from $\overline{n}(r, a; f > g)$ and $\overline{n}(r, a; f < g)$ respectively.

Finally we put $\overline{N}_*(r, a; f, g) = \overline{N}(r, a; f > g) + \overline{N}(r, a; f < g)$.

Definition 4 Let f and g share a value a IM. Let z be an a-point of f and g with multiplicities $p_f(z)$ and $p_g(z)$ respectively.

We put

$$\nu_f(z) = p_f(z) \quad \text{if } p_f(z) > p_g(z)$$
$$= 0 \quad \text{if } p_f(z) \leq p_g(z)$$

and

$$\mu_f(z) = p_f(z) \quad \text{if } p_f(z) < p_g(z)$$
$$= 0 \quad \text{if } p_f(z) \geq p_g(z).$$

Let $n(r, a; f > g) = \sum_{|z| \leq r} \nu_f(z)$ and $n(r, a; f < g) = \sum_{|z| \leq r} \mu_f(z)$. We now denote by $N(r, a; f > g)$ and $N(r, a; f < g)$ the integrated counting functions obtained from $n(r, a; f > g)$ and $n(r, a; f < g)$ respectively.

Throughout the paper we denote by f and g two nonconstant meromorphic functions defined in $\mathbb{C}$.

2. Lemmas

In this section we present some lemmas which are needed in the sequel.

Lemma 1 ([3]) *If* f, g *share* $(0, 0)$, $(1, 0)$, $(\infty, 0)$ *then* (i) $T(r, f) \leq 3T(r, g) + S(r, f)$, (ii) $T(r, g) \leq 3T(r, f) + S(r, g)$.

This shows that $S(r, f) = S(r, g)$ and we denote them by $S(r)$.

Lemma 2 ([6]) *Let* f, g *share* $(0, 1)$, $(1, m)$, (∞, k) *and* $f \not\equiv g$, *where* $(m-1)(mk-1) > (1+m)^2$. *Then* $\overline{N}(r, a; f \mid\geq 2) = S(r)$ *and* $\overline{N}(r, a; g \mid\geq$

2) $= S(r)$ *for* $a = 0,\ 1,\ \infty$.

Following lemma can be proved in the line of statements (iii) and (iv) of Lemma 2.3 of [7].

Lemma 3 *Let* $f,\ g$ *share* $(0,\ 0),\ (1,\ 0),\ (\infty,\ 0)$ *and* $f \not\equiv g$. *If* $\alpha = (f-1)/(g-1)$ *and* $h = g/f$ *then*
(i) $\overline{N}(r,\ 0;\alpha) = \overline{N}(r,\ \infty; f<g) + \overline{N}(r,\ 1; f>g)$,
(ii) $\overline{N}(r,\ \infty;\alpha) = \overline{N}(r,\ \infty; f>g) + \overline{N}(r,\ 1; f<g)$,
(iii) $\overline{N}(r,\ 0;h) = \overline{N}(r,\ 0; f<g) + \overline{N}(r,\ \infty; f>g)$,
(iv) $\overline{N}(r,\ \infty;h) = \overline{N}(r,\ 0; f>g) + \overline{N}(r,\ \infty; f<g)$.

Lemma 4 *Let* $f,\ g$ *share* $(0,\ 1),\ (1,\ m),\ (\infty,\ k)$ *and* $f \not\equiv g$, *where* $(m-1)(mk-1) > (1+m)^2$. *If* α *and* h *are defined as in* Lemma 3 *then* $\overline{N}(r,\ a;\alpha) = S(r)$ *and* $\overline{N}(r,\ a;h) = S(r)$ *for* $a = 0,\ \infty$.

Proof. The lemma follows from Lemmas 2 and 3 because $\overline{N}_*(r,\ a; f, g) \le \overline{N}(r,\ a; f \mid\ge 2)$ for $a = 0,\ 1,\ \infty$. □

Lemma 5 ([8]) *Let* f *and* g *share* $(0,\ 0),\ (1,\ 0),\ (\infty,\ 0)$. *If* f *is a bilinear transformation of* g *then* f *and* g *satisfy exactly one of the following:* (i) $f \equiv g$, (ii) $f + g \equiv 1$, (iii) $(f-1)(g-1) \equiv 1$, (iv) $fg \equiv 1$, (v) $f \equiv Ag + 1 - A$, (vi) $f \equiv Ag$, (vii) $f(g + A - 1) \equiv Ag$, *where* $A\ (\ne 0,\ 1)$ *is a constant.*

Following lemma is of independent interest.

Lemma 6 *Let* $f,\ g$ *share* $(0,\ 1),\ (1,\ m),\ (\infty,\ k)$ *and* $f \not\equiv g$, *where* $(m-1)(mk-1) > (1+m)^2$. *If* f *is not a bilinear transformation of* g *then each of the following holds*:
(i) $T(r,\ f) + T(r,\ g)$
$= N(r,\ 0; g \mid\le 1) + N(r,\ 1; g \mid\le 1) + N(r,\ \infty; g \mid\le 1) + N_0(r) + S(r)$,
(ii) $T(r,\ f) + T(r,\ g)$
$= N(r,\ 0; f \mid\le 1) + N(r,\ 1; f \mid\le 1) + N(r,\ \infty; f \mid\le 1) + N_0(r) + S(r)$,
(iii) $T(r,\ f) = N(r,\ 0; g' \mid\le 1) + N_0(r) + S(r)$,
(iv) $T(r,\ g) = N(r,\ 0; f' \mid\le 1) + N_0(r) + S(r)$,
(v) $N_1(r) = S(r)$,
(vi) $N_0(r,\ 0; g' \mid\ge 2) = S(r)$,
(vii) $N_0(r,\ 0; f' \mid\ge 2) = S(r)$,
(viii) $\overline{N}(r,\ 0; g' \mid\ge 2) = S(r)$,
(ix) $\overline{N}(r,\ 0; f' \mid\ge 2) = S(r)$,

where $N_0(r)(N_1(r))$ *denotes the counting function of those simple (multiple) zeros of* $f-g$ *which are not the zeros of* $g(g-1)$, $1/g$ *and so are not the zeros of* $f(f-1)$, $1/f$, *also* $N_0(r,\, 0; g' \mid\geq 2)$ $(N_0(r,\, 0; f' \mid\geq 2))$ *is the counting function of those multiple zeros of* $g'(f')$ *which are not the zeros of* $g(g-1)$ *and so not of* $f(f-1)$.

Proof. We see that $f = (1-\alpha)/(1-\alpha h)$ and $g = (1-\alpha)h/(1-\alpha h)$, where α and h are defined as in Lemma 3. Since f is not a bilinear transformation of g, α, h and αh are nonconstant. Let $b = \alpha' h/(\alpha h' + \alpha' h)$. Then

$$f - b = \frac{(1-\alpha) - b(1-\alpha h)}{(1-\alpha h)}.$$

Let $F = (f-b)(1-\alpha h) = (1-\alpha) - b(1-\alpha h)$. Also $(f-g)(1-\alpha h) = (1-\alpha)(1-h)$ and $(g-1)(1-\alpha h) = h-1$ so that $f-g = (g-1)(\alpha-1)$. Again

$$\frac{g'}{g} = \frac{h'(1-\alpha h) + (h-1)(\alpha' h + \alpha h')}{h(1-\alpha)(1-\alpha h)}.$$

Therefore

$$\frac{g'(g-f)}{g(g-1)} = \frac{h'(1-\alpha h) + (h-1)(\alpha' h + \alpha h')}{h(1-\alpha h)} \tag{1}$$
$$= \frac{(1-\alpha)(\alpha h' + \alpha' h) - \alpha' h(1-\alpha h)}{\alpha h(1-\alpha h)}.$$

Again

$$(f-b)(1-\alpha h) = (1-\alpha) - b(1-\alpha h)$$
$$= \frac{(1-\alpha)(\alpha h' + \alpha' h) - \alpha' h(1-\alpha h)}{\alpha h' + \alpha' h}$$

and so

$$(f-b)\frac{\alpha h' + \alpha' h}{\alpha h} = \frac{(1-\alpha)(\alpha h' + \alpha' h) - \alpha' h(1-\alpha h)}{\alpha h(1-\alpha h)}. \tag{2}$$

From (1) and (2) we get

$$\frac{g'(g-f)}{g(g-1)} = (f-b)\left(\frac{h'}{h} + \frac{\alpha'}{\alpha}\right). \tag{3}$$

Since $F' = -\alpha' - b'(1-\alpha h) + b(\alpha' h + \alpha h') = -\alpha' - b'(1-\alpha h) + \alpha' h$, we get

$$\frac{F'}{F} - \frac{\alpha'}{\alpha} = \frac{-\alpha' - b'(1-\alpha h) + \alpha' h - (\alpha'/\alpha)F}{F}$$

$$
\begin{aligned}
&= \frac{(1-\alpha h)(-\alpha' - b'\alpha + b\alpha')}{\alpha(f-b)(1-\alpha h)} \\
&= \frac{1}{f-b}\left[\frac{\alpha'}{\alpha}(b-1) - b'\right]
\end{aligned}
$$

and so

$$
\frac{1}{f-b} = \frac{F'/F - \alpha'/\alpha}{(\alpha'/\alpha)(b-1) - b'}. \tag{4}
$$

Since $T(r,\, \alpha) \le T(r,\, f) + T(r,\, g) + O(1)$ and $T(r,\, h) \le T(r,\, f) + T(r,\, g) + O(1)$, in view of Lemmas 1 and 4 we obtain

$$
\begin{aligned}
T\Big(r,\, \frac{\alpha'}{\alpha}\Big) &= m\Big(r,\, \frac{\alpha'}{\alpha}\Big) + N\Big(r,\, \frac{\alpha'}{\alpha}\Big) \\
&\le \overline{N}(r,\, 0; \alpha) + \overline{N}(r,\, \infty; \alpha) + S(r,\, \alpha) = S(r)
\end{aligned}
$$

and

$$
\begin{aligned}
T\Big(r,\, \frac{h'}{h}\Big) &= m\Big(r,\, \frac{h'}{h}\Big) + N\Big(r,\, \frac{h'}{h}\Big) \\
&\le \overline{N}(r,\, 0; h) + \overline{N}(r,\, \infty; h) + S(r,\, h) = S(r).
\end{aligned}
$$

Since $1/b = 1 + \alpha h'/\alpha' h$, we get

$$
\begin{aligned}
T(r,\, b) = T\Big(r,\, \frac{1}{b}\Big) + O(1) &\le T\Big(r,\, \frac{\alpha}{\alpha'}\Big) + T\Big(r,\, \frac{h'}{h}\Big) + O(1) \\
&= T\Big(r,\, \frac{\alpha'}{\alpha}\Big) + S(r) = S(r).
\end{aligned}
$$

From (4) we now obtain

$$
m\Big(r,\, \frac{1}{f-b}\Big) \le m\Big(r,\, \frac{F'}{F}\Big) + S(r) = S(r,\, f) + S(r) = S(r). \tag{5}
$$

Since F'/F and α'/α have no multiple pole and $T(r,\, b') \le 2T(r,\, b) + S(r,\, b)$, it follows from the above and (4) that

$$
\begin{aligned}
N(r,\, 0; f-b \mid\ge 2) &\le 2N\Big(r,\, 0; \frac{\alpha'}{\alpha}(b-1) - b'\Big) + S(r) \\
&\le 2T\Big(r,\, \frac{\alpha'}{\alpha}(b-1) - b'\Big) + S(r) \\
&\le 2T\Big(r,\, \frac{\alpha'}{\alpha}\Big) + 2T(r,\, b-1) + 2T(r,\, b') + S(r) \\
&\le 2T\Big(r,\, \frac{\alpha'}{\alpha}\Big) + 6T(r,\, b) + S(r) = S(r). \qquad (6)
\end{aligned}
$$

Since f, g share (0, 1), (1, m), (∞, k) and $b = f - (f - b)$, we see that if z_0 is a zero, pole or 1-point of g which is also a simple zero of $f - b$, then z_0 is a zero, pole or 1-point of b and so the counting function of such simple zeros of $f - b$ is $S(r)$. So we get from (3) and (6)

$$\begin{aligned} N(r,\,0; f-b) &= N(r,\,0; f-b \mid\le 1) + N(r,\,0; f-b \mid\ge 2)\\ &= N(r,\,0; f-b \mid\le 1) + S(r)\\ &= N(r,\,0; g' \mid\le 1) + N_0(r) + S(r). \end{aligned} \tag{7}$$

From (5) and (7) we obtain

$$\begin{aligned} T(r,\,f) &= T(r,\,f-b) + S(r)\\ &= m\Big(r,\,\frac{1}{f-b}\Big) + N\Big(r,\,\frac{1}{f-b}\Big) + S(r)\\ &= N(r,\,0; g' \mid\le 1) + N_0(r) + S(r), \end{aligned} \tag{8}$$

which is (iii).

Similarly we get

$$T(r,\,g) = N(r,\,0; f' \mid\le 1) + N_0(r) + S(r), \tag{9}$$

which is (iv).

Again from (3) and (6) we obtain $N_1(r) \le N(r,\,0; f-b \mid\ge 2) + S(r) = S(r)$ and $N_0(r,\,0; g' \mid\ge 2) \le N(r,\,0; f-b \mid\ge 2) + S(r) = S(r)$, which are respectively (v) and (vi). Similarly we can prove (vii).

Since $\overline{N}(r,\,0; g' \mid\ge 2) \le N_0(r,\,0; g' \mid\ge 2) + \overline{N}(r,\,0; g \mid\ge 2) + \overline{N}(r,\,1; g \mid\ge 2)$ and $\overline{N}(r,\,0; f' \mid\ge 2) \le N_0(r,\,0; f' \mid\ge 2) + \overline{N}(r,\,0; f \mid\ge 2) + \overline{N}(r,\,1; f \mid\ge 2)$, (viii) and (ix) follow from (vi), (vii) and Lemma 2.

By the second fundamental theorem, Lemma 2 and (8) we get

$$\begin{aligned} &T(r,\,f) + T(r,\,g)\\ &\quad\le T(r,\,f) + N(r,\,0; g \mid\le 1) + N(r,\,1; g \mid\le 1)\\ &\qquad + N(r,\,\infty; g \mid\le 1) - \overline{N}_0(r,\,0; g') + S(r)\\ &\quad= N(r,\,0; g \mid\le 1) + N(r,\,1; g \mid\le 1) + N(r,\,\infty; g \mid\le 1)\\ &\qquad + N(r,\,0; g' \mid\le 1) + N_0(r) - \overline{N}_0(r,\,0; g') + S(r), \end{aligned} \tag{10}$$

where $\overline{N}_0(r,\,0; g')$ denotes the reduced counting function of those zeros of g' which are not the zeros of $g(g-1)$.

By Lemma 2 we see that

$$N(r,\,0; g' \mid\le 1) = N_0(r,\,0; g' \mid\le 1) + S(r), \tag{11}$$

where $N_0(r, 0; g' \mid\leq 1)$ is the counting function of those simple zeros of g' which are not the zeros of $g(g-1)$.

Similarly

$$N(r, 0; f' \mid\leq 1) = N_0(r, 0; f' \mid\leq 1) + S(r). \tag{12}$$

From (10) and (11) we get

$$\begin{aligned}
T(r, f) + T(r, g) & \\
\leq & N(r, 0; g \mid\leq 1) + N(r, 1; g \mid\leq 1) + N(r, \infty; g \mid\leq 1) \\
& + N_0(r, 0; g' \mid\leq 1) + N_0(r) - \overline{N}_0(r, 0; g') + S(r) \\
\leq & N(r, 0; g \mid\leq 1) + N(r, 1; g \mid\leq 1) + N(r, \infty; g \mid\leq 1) \\
& + N_0(r) + S(r) \\
\leq & N(r, 0; f - g) + N(r, \infty; g \mid\leq 1) + S(r) \\
\leq & T(r, f - g) + N(r, \infty; g \mid\leq 1) + S(r) \\
\leq & m(r, f) + m(r, g) \\
& \qquad + N(r, f - g) + N(r, \infty; g \mid\leq 1) + S(r) \\
\leq & m(r, f) + m(f, g) + N(r, f) \\
& \qquad + N(r, \infty; g > f) + N(r, \infty; g \mid\leq 1) + S(r) \\
\leq & m(r, f) + N(r, f) + m(r, g) + N(r, g) + S(r) \\
\leq & T(r, f) + T(r, g) + S(r),
\end{aligned}$$

from which (i) follows.

Now (ii) follows from (i) because $N(r, a; f \mid\leq 1) = N(r, a; g \mid\leq 1)$ for $a = 0, 1, \infty$. This proves the lemma. □

Lemma 7 ([6]) *Let f, g share $(0, 1)$, $(1, m)$, (∞, k) and $f \not\equiv g$, where $(m-1)(mk-1) > (1+m)^2$. Then for any complex number a $(\neq 0, 1, \infty)$, $\overline{N}(r, a; f \mid\geq 3) = S(r)$ and $\overline{N}(r, a; g \mid\geq 3) = S(r)$.*

3. Proof of the main result

Proof of Theorem 1. If possible, let f be not a bilinear transformation of g. Then by Lemma 6(vii), Lemma 2, Lemma 7 and the second fundamental theorem we get

$$\begin{aligned}
2T(r, f) \leq & N(r, 0; f \mid\leq 1) + N(r, 1; f \mid\leq 1) + N(r, \infty; f \mid\leq 1) \\
& + \overline{N}(r, a; f \mid\leq 2) - N_1(r, 0; f' \mid\leq 1) + S(r),
\end{aligned} \tag{13}$$

where $N_1(r, 0; f' \mid\leq 1)$ is the counting function of those simple zeros of f' which are not the zeros of $f(f-1)(f-a)$.

Since a double a-point of f is a simple zero of f', it follows that

$$\begin{aligned}&\overline{N}(r, a; f \mid\leq 2) - N_1(r, 0; f' \mid\leq 1)\\&\qquad\qquad = N(r, a; f \mid\leq 2) - N_0(r, 0; f' \mid\leq 1).\end{aligned}$$

So from (13) we get by (12) and Lemma 6 (ii) and (iv)

$$\begin{aligned}2T(r, f) &\leq T(r, f) + T(r, g) - N_0(r)\\&\quad + N(r, a; f \mid\leq 2) - N_0(r, 0; f' \mid\leq 1) + S(r)\\&= T(r, f) + N(r, a; f \mid\leq 2) + S(r, f)\\&\leq 2T(r, f) + S(r, f),\end{aligned}$$

which is a contradiction.

Hence f is a bilinear transformation of g. So any one of the possibilities of (ii)-(vii) of Lemma 5 will occur. We now examine each of these possibilities one by one.

Let $f + g \equiv 1$ Since f, g share $(0, 1)$, $(1, m)$, it follows that 0 and 1 are Picard exceptional values (evP) of f and so by the second fundamental theorem and Lemma 7 we get

$$\begin{aligned}T(r, f) &\leq \overline{N}(r, a; f \mid\leq 2) + S(r, f)\\&\leq N(r, a; f \mid\leq 2) + S(r, f)\\&\leq T(r, f) + S(r, f),\end{aligned}$$

a contradiction.

Let $(f-1)(g-1) \equiv 1$. Since f, g share $(1, m)$, (∞, k), it follows that 1 and ∞ are evP of f and so as above we get $N(r, a; f \mid\leq 2) = T(r, f) + S(r, f)$, a contradiction.

If $fg \equiv 1$. Since f, g share $(0, 1)$, (∞, k), it follows that 0 and ∞ are evP of f and so $N(r, a; f \mid\leq 2) = T(r, f) + S(r, f)$, a contradiction.

Let $f \equiv Ag + 1 - A$, where A $(\neq 0, 1)$ is a constant. Since f, g share $(0, 1)$, it follows that 0, $1 - A$ are evP of f and so by the second fundamental theorem and Lemma 2 we get $T(r, f) \leq N(r, \infty; f \mid\leq 1) + S(r, f) \leq T(r, f) + S(r, f)$, a contradiction.

Let $f \equiv Ag$, where A $(\neq 0, 1)$ is a constant. Since f, g share $(1, m)$, it follows that 1, A are evP of f and so $N(r, \infty; f \mid\leq 1) = T(r, f) + S(r, f)$, a contradiction.

Let $f(g+A-1) \equiv Ag$, where A ($\neq 0, 1$) is a constant. Since f, g share (∞, k), it follows that ∞ is an evP of f and so of g.

If $A \neq a$, by the second fundamental theorem and Lemma 7 we get

$$\begin{aligned} T(r, f) &\leq \overline{N}(r, a; f \mid\leq 2) + \overline{N}(r, a; f) + S(r, f) \\ &\leq N(r, a; f \mid\leq 2) + \overline{N}(r, \infty; g) + S(r, f) \\ &= N(r, a; f \mid\leq 2) + S(r, f) \\ &\leq T(r, f) + S(r, f), \end{aligned}$$

a contradiction.

Therefore $A = a$ and so $(f-a)(g+a-1) \equiv a(1-a)$. This proves the theorem. □

Remark 1 If in Theorem 1 we remove the condition $N(r, \infty; f \mid\leq 1) \neq T(r, f) + S(r, f)$, in a like manner we can prove that one of the following possibilities occurs, which improves Theorem 4 [12]:

(i) $(f-a)(g+a-1) \equiv a(1-a)$. This occurs only when ∞ is an evP of f. In this case a, $1-a$ are evP of f and g respectively and ∞ is an evP of g.

(ii) $f + (a-1)g \equiv a$. This occurs only when 0 is an evP of f. In this case a is an evP of f and 0, $a/(a-1)$ are evP of g.

(iii) $f \equiv ag$. This occurs only when 1 is an evP of f. In this case a is an evP of f and 1, $1/a$ are evP of g.

4. Applications

In this section we discuss two applications of Theorem 1.

Definition 5 ([3]) For $S \subset \mathbb{C} \cup \{\infty\}$ we define $E_f(S, k)$ as $E_f(S, k) = \cup_{a \in S} E_k(a; f)$, where k is a nonnegative integer or infinity.

H.X. Yi [12] proved the following result.

Theorem H ([12]) *Let $S_1 = \{a_1, a_2\}$ and $S_2 = \{b_1, b_2\}$ be two pairs of distinct elements with $a_1 + a_2 = b_1 + b_2$ but $a_1 a_2 \neq b_1 b_2$ and let $S_3 = \{\infty\}$. If $E_f(S_i, \infty) = E_g(S_i, \infty)$ for $i = 1, 2, 3$ and $\delta(c/2; f) > 0$ for $c = a_1 + a_2$ then one of the following holds*: (i) $f \equiv g$, (ii) $f + g \equiv a_1 + a_2$, (iii) $(f - c/2)(g - c/2) \equiv \pm(a_1 - a_2)^2/4$, *which occurs only for* $(a_1 - a_2)^2 + (b_1 - b_2)^2 = 0$.

H.X. Yi [12] considered the following example to establish the necessity of the condition $\delta(c/2; f) > 0$ for *Theorem H.*

Example 6 ([12]) Let $f = 1 - 4e^z$, $g = 1 - e^{-z}$, $a_1 = -1$, $a_2 = 1$, $b_1 = -i\sqrt{3}$, $b_2 = i\sqrt{3}$, $S_1 = \{a_1, a_2\}$, $S_2 = \{b_1, b_2\}$ and $S_3 = \{\infty\}$. Then clearly $(f - a_1)(f - a_2) = -8e^{2z}(g - a_1)(g - a_2)$ and $(f - b_1)(f - b_2) = 4e^z(g - b_1)(g - b_2)$ so that $E_f(S_i, \infty) = E_g(S_i, \infty)$ for $i = 1, 2, 3$. Also we see that $c = a_1 + a_2 = 0$, $\delta(c/2; f) = 0$ and $f \not\equiv g$, $f + g \not\equiv a_1 + a_2$, $(f - c/2)(g - c/2) \not\equiv \pm(a_1 - a_2)^2/4$.

In the following theorem we improve *Theorem H* and show that the condition $\delta(c/2; f) > 0$ can further be relaxed.

Theorem 2 *Let $S_1 = \{a_1, a_2\}$ and $S_2 = \{b_1, b_2\}$ be two pairs of distinct elements with $a_1 + a_2 = b_1 + b_2$ but $a_1a_2 \neq b_1b_2$ and let $S_3 = \{\infty\}$. Suppose that $E_f(S_1, 1) = E_g(S_1, 1)$, $E_f(S_2, m) = E_g(S_2, m)$, $E_f(S_3, k) = E_g(S_3, k)$ and $\delta_{1)}(c/2; f) > 0$, where $(m - 1)(mk - 1) > (1 + m)^2$ and $c = a_1 + a_2$. Then the conclusion of* Theorem H *holds.*

Proof. Let $A = (b_1 - b_2)^2/4 - (a_1 - a_2)^2/4$ and

$$F = \frac{1}{A}\left[\left(f - \frac{c}{2}\right)^2 - \frac{(a_1 - a_2)^2}{4}\right], \; G = \frac{1}{A}\left[\left(g - \frac{c}{2}\right)^2 - \frac{(a_1 - a_2)^2}{4}\right].$$

If $F \equiv G$ then clearly either $f \equiv g$ or $f + g \equiv a_1 + a_2$. So we suppose that $F \not\equiv G$. Also let $(a_1 - a_2)^2 + (b_1 - b_2)^2 = 0$ and $a = 1/2$. Then we see that $A(F - a) = (f - c/2)^2$ and so $N(r, \infty; F \mid\leq 1) \equiv 0$ and $N(r, a; F \mid\leq 2) = 2N(r, c/2; f \mid\leq 1) \neq 2T(r, f) + S(r, f) = T(r, F) + S(r, F)$. Since F, G share $(0, 1)$, $(1, m)$, (∞, k), by Theorem 1 we get $(F - a)(G + a - 1) \equiv a(1 - a)$ and so $(f - c/2)(g - c/2) \equiv \pm(a_1 - a_2)^2/4$. This proves the theorem. $\square$

Remark 2 Example 6 shows that the condition $\delta_{1)}(c/2; f) > 0$ is essential.

In [5] following result is proved.

Theorem I ([5]) *Let a and b $(\neq 0, 1)$ be two finite complex numbers and $S_1 = \{a + \alpha \colon \alpha^n + b = 0\}$, $S_2 = \{a + \beta \colon \beta^n + b = 1\}$, $S_3 = \{\infty\}$ where n (≥ 3) be a positive integer. If $E_f(S_1, 1) = E_g(S_1, 1)$, $E_f(S_2, \infty) = E_g(S_2, \infty)$, $E_f(S_3, \infty) = E_g(S_3, \infty)$ then one of the following holds:* (i) $f - a \equiv t(g - a)$, *where $t^n = 1$ and* (ii) $(f - a)(g - a) \equiv s$, *where $4s^n = 1$.*

In the next theorem we improve *Theorem I.*

Theorem 3 Theorem I *holds if* $E_f(S_1, 1) = E_g(S_1, 1)$, $E_f(S_2, m) = E_g(S_2, m)$ *and* $E_f(S_3, k) = E_g(S_3, k)$, *where* $(m-1)(mk-1) > (1+m)^2$.

We omit the proof as it can be done in the line of Theorem I using Theorem 1.

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