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Differential superordination defined by Ruscheweyh derivative

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Abstract. By using the Ruscheweyh operator $D^n f(z)$, $z \in U$, we obtain sharp superordinations results related to some normalized holomorphic functions in the unit disk U.

Key words: differential subordination, differential superordination, univalent function.

1. Introduction

Let Ω be any set in the complex plane $\mathbb{C}$, let p be analytic in the unit disk U and let $\psi(r, s, t; z)\colon \mathbb{C}^3 \times U \to \mathbb{C}$. In a series of articles the authors and many others [1] have determined properties of functions p that satisfy the differential subordination

$$\{\psi(p(z), zp'(z), z^2p''(z); z) \mid z \in U\} \subset \Omega.$$

In this article we consider the dual problem of determining properties of functions p that satisfy the differential superordination

$$\Omega \subset \{\psi(p(z),\ zp'(z),\ z^2p''(z); z) \mid z \in U\}.$$

This problem was introduced in [2].

We let $\mathcal{H}(U)$ denote the class of holomorphic functions in the unit disk $U = \{z \in \mathbb{C}\colon |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$ we let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U),\ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots,\ z \in U\}$$

and

$$A = \{f \in \mathcal{H}(U),\ f(z) = z + a_2 z^2 + \cdots,\ z \in U\}.$$

For $0 < r < 1$, we let $U_r = \{z,\ |z| < r\}$.

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Definition 1 ([2]) Let $\varphi\colon \mathbb{C}^2\times U \to \mathbb{C}$ and let h be analytic in U. If p and $\varphi(p(z),\, zp'(z); z)$ are univalent in U and satisfy the (first-order) differential superordination

$$h(z) \prec \varphi(p(z),\, zp'(z); z) \tag{1}$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if $q \prec p$ for all p satisfying (1). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1) is said to be the best subordinant. Note that the best subordinant is unique up to a rotation of U.

For Ω a set in $\mathbb{C}$, with φ and p as given in Definition 1, suppose (1) is replaced by

$$\Omega \subset \{\varphi(p(z),\, zp'(z); z) \mid z \in U\}. \tag{$1'$}$$

Although this more general situation is a "differential containment", the condition in ($1'$) will also be referred to as a differential superordination, and the definitions of solution, subordinant and best dominant as given above can be extended to this generalization.

Definition 2 ([2]) We denote by Q the set of functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \{\zeta \in \partial U\colon \lim_{z\to\zeta} f(z) = \infty\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.

In order to prove the new results we shall use the following lemma:

Lemma A ([2]) *Let h be convex in U, with $h(0) = a$, $\gamma \neq 0$ with $\operatorname{Re}\gamma \geq 0$, and $p \in \mathcal{H}[a,\, 1] \cap Q$. If $p(z) + zp'(z)/\gamma$ is univalent in U,*

$$h(z) \prec p(z) + \frac{zp'(z)}{\gamma}$$

then

$$q(z) \prec p(z),$$

where

$$q(z) = \frac{\gamma}{z^{\gamma}} \int_0^z h(t)t^{\gamma-1}dt, \quad z \in U.$$

The function q is convex and is the best subordinant.

Lemma B ([2]) *Let q be convex in U and let h be defined by*

$$h(z) = q(z) + \frac{zq'(z)}{\gamma}, \quad z \in U,$$

with $\mathrm{Re}\,\gamma \geq 0$*. If $p \in \mathcal{H}[a, 1] \cap Q$, $p(z) + zp'(z)/\gamma$ is univalent in U, and*

$$q(z) + \frac{zq'(z)}{\gamma} \prec p(z) + \frac{zp'(z)}{\gamma}, \quad z \in U$$

then

$$q(z) \prec p(z),$$

where

$$q(z) = \frac{\gamma}{z^{\gamma}} \int_0^z h(t)t^{\gamma-1}dt.$$

The function q is the best subordinant.

Definition 3 ([3]) For $f \in A$ and $n \geq 0$, $n \in \mathbb{N}$, the operator $D^n f$ is defined by

$$D^n f(z) = f(z) * \frac{z}{(1-z)^{n+1}} = \frac{z}{n!}[z^{n-1}f(z)]^{(n)}, \quad z \in U,$$

where $*$ stands for convolution.

Remark 1 We have

$$\begin{aligned} &D^0 f(z) = f(z) \\ &D^1 f(z) = zf'(z) \\ &\qquad \vdots \\ &(n+1)D^{n+1}f(z) = z[D^n f(z)]' + nD^n f(z),\ z \in U. \end{aligned}$$

2. Main results

Theorem 1 *Let*

$$h(z) = \frac{1+(2\alpha-1)z}{1+z}$$

be convex in U*, with* $h(0) = 1$.

Let $f \in A$*, and suppose that* $[D^{n+1}f(z)]'$ *is univalent and* $[D^n f(z)]' \in \mathcal{H}[1, 1] \cap Q$.

If

$$h(z) \prec [D^{n+1}f(z)]', \quad z \in U, \tag{2}$$

then

$$q(z) \prec [D^n f(z)]', \quad z \in U,$$

where

$$q(z) = \frac{n+1}{z^{n+1}} \int_0^z \frac{1+(2\alpha-1)t}{1+t} t^n dt, \quad z \in U. \tag{3}$$

The function q *is convex and is the best subordinant.*

Proof. Let $f \in A$. By using the properties of the operator $D^n f(z)$ we have

$$(n+1)D^{n+1}f(z) = z[D^n f(z)]' + nD^n f(z), \quad z \in U. \tag{4}$$

Differentiating (4), we obtain

$$\begin{aligned}(n+1)[D^{n+1}f(z)]' &= [D^n f(z)]' + z[D^n f(z)]'' + n[D^n f(z)]' \\ &= (n+1)[D^n f(z)]' + z[D^n f(z)]'', \quad z \in U.\end{aligned} \tag{5}$$

If we let $p(z) = [D^n f(z)]'$ then (5) becomes

$$[D^{n+1}f(z)]' = p(z) + \frac{1}{n+1} zp'(z), \quad z \in U.$$

Then (2) becomes

$$h(z) \prec p(z) + \frac{1}{n+1} zp'(z), \quad z \in U.$$

By using Lemma A, we have

$$q(z) \prec p(z) = [D^n f(z)]', \quad z \in U,$$

where q is given by (3).

The function q is the best subordinant. $\square$

Theorem 2 *Let*

$$h(z) = \frac{1+(2\alpha-1)z}{1+z}$$

be convex in U*, with* $h(0) = 1$*. Let* $f \in A$ *and suppose that* $[D^n f(z)]'$ *is univalent and* $D^n f(z)/z \in \mathcal{H}[1, 1] \cap Q$*.*

If

$$h(z) \prec [D^n f(z)]', \quad z \in U, \tag{6}$$

then

$$q(z) \prec \frac{D^n f(z)}{z}, \quad z \in U,$$

where

$$q(z) = \frac{1}{z}\int_0^z \frac{1+(2\alpha-1)t}{1+t}dt = 2\alpha - 1 + (2-2\alpha)\frac{\ln(1+z)}{z}.$$

The function q *is convex and is the best subordinant.*

Proof. We let

$$p(z) = \frac{D^n f(z)}{z}, \quad z \in U,$$

and we obtain

$$D^n f(z) = zp(z), \quad z \in U. \tag{7}$$

By differentiating (7) we obtain

$$[D^n f(z)]' = p(z) + zp'(z), \quad z \in U.$$

Then (6) becomes

$$h(z) \prec p(z) + zp'(z), \quad z \in U.$$

By using Lemma A we have

$$q(z) \prec p(z) = \frac{D^n f(z)}{z}, \quad z \in U,$$

where

$$q(z)=\frac{1}{z}\int_0^z \frac{1+(2\alpha-1)t}{1+t}dt=\frac{1}{z}\int_0^z \left[(2\alpha-1)+\frac{2-2\alpha}{1+t}\right) dt$$
$$=\frac{1}{z}\left[(2\alpha-1)t\Big|_0^z+(2-2\alpha)\int_0^z \frac{1}{1+t}dt\right]$$
$$=2\alpha-1+(2-2\alpha)\frac{\ln(1+z)}{z}.$$

The function q is convex and is the best subordinant. □

Theorem 3 *Let q be convex in U and let h be defined by*

$$h(z)=q(z)+\frac{1}{n+1}zq'(z), \quad z\in U.$$

Let $f\in A$ and suppose that $[D^{n+1}f(z)]'$ is univalent in U, $[D^n f(z)]'\in \mathcal{H}[1,\,1]\cap Q$ and

$$h(z)=q(z)+\frac{1}{n+1}zq'(z)\prec [D^{n+1}f(z)]', \quad z\in U, \tag{8}$$

then

$$q(z)\prec [D^n f(z)]', \quad z\in U$$

where

$$q(z)=\frac{n+1}{z^{n+1}}\int_0^z h(t)t^n dt.$$

The function q is the best subordinant.

Proof. Let $f\in A$. By using the properties of the operator $D^n f(z)$, we have

$$(n+1)[D^{n+1}f(z)]'=(n+1)[D^n f(z)]'+z[D^n f(z)]''. \tag{9}$$

If we let $p(z)=[D^n f(z)]'$ then (9) becomes

$$[D^{n+1}f(z)]'=p(z)+\frac{1}{n+1}zp'(z), \quad z\in U.$$

Then (8) becomes

$$q(z)+\frac{1}{n+1}zq'(z)\prec p(z)+\frac{1}{n+1}zp'(z), \quad z\in U.$$

By using Lemma B, we have

$$q(z) \prec p(z) = [D^n f(z)]', \quad z \in U,$$

where

$$q(z) = \frac{n+1}{z^{n+1}} \int_0^z h(t)t^n dt.$$

□

Theorem 4 *Let q be convex in U and let h be defined by*

$$h(z) = q(z) + zq'(z), \quad z \in U.$$

Let $f \in A$ and suppose that $[D^n f(z)]'$ is univalent in U, $D^n f(z)/z \in \mathcal{H}[1, 1] \cap Q$ and

$$h(z) = q(z)+zq'(z) \prec [D^n f(z)]', \quad z \in U \tag{10}$$

then

$$q(z) \prec \frac{D^n f(z)}{z}, \quad z \in U,$$

where

$$q(z) = \frac{1}{z} \int_0^z h(t)dt.$$

The function q is the best subordinant.

Proof. We let

$$p(z) = \frac{D^n f(z)}{z}, \quad z \in U$$

and we obtain

$$D^n f(z) = zp(z), \quad z \in U.$$

By differentiating, we obtain

$$[D^n f(z)]' = p(z) + zp'(z), \quad z \in U.$$

Then (10) becomes

$$q(z) + zq'(z) \prec p(z) + zp'(z), \quad z \in U.$$

By using Lemma B we have

$$q(z) \prec p(z) = \frac{D^n f(z)}{z}, \quad z \in U,$$

where

$$q(z) = \frac{1}{z}\int_0^z h(t)dt, \quad z \in U.$$

The function q is the best subordinant. □

References

[1] Miller S.S. and Mocanu P.T., *Differential Subordinations. Theory and Applications.* Marcel Dekker Inc., New York, Basel, 2000.

[2] Miller S.S. and Mocanu P.T., *Subordinants of Differential Superordinations.* Complex Variables (10) **48** (2003), 815–826.

[3] Ruscheweyh St., *New criteria for univalent functions.* Proc. Amer. Math. Soc. **49** (1975), 109–115.

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