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Bifurcations of holonomic systems of general Clairaut type

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Abstract. In the previous paper [20], we have classified generic holonomic systems of general Clairaut type. The notion of holonomic systems of general Clairaut type is one of the generalized notions of the classical Clairaut equations and Clairaut type. We give a generic classification of bifurcations of them as an application of the theory of complete Legendrian unfoldings and transversality theorem. In the list of normal forms, the Lagrangian equation appears. Moreover, there appears several new equations.

Key words: holonomic system, holonomic system of general Clairaut type, bifurcation, Legendrian singularity theory.

1. Introduction

In the classical theory of first order differential equations the notion of classical (or, smooth) complete solutions plays an important role (cf. [3], [6]). The Clairaut equation (Alex Claude Clairaut [4], 1734) is one of the typical example of first order differential equations which has a classical complete solution and a singular solution such that the singular solution is the envelope of the family of hyperplanes given by the complete solution (cf. [16], [17]). In this paper, we consider systems of first order partial differential equations with classical complete solution like as the Clairaut equation which is called a *system of general Clairaut type*. In particular, the system of general Clairaut type with regular property which is called a *system of Clairaut type* has been investigated in [13, 18]. Also in the previous paper [20], we have classified holonomic systems of general Clairaut type. The next problem is to consider the bifurcations of these systems. We give a generic classification of one-parameter families of holonomic systems of general Clairaut type in any dimension as an application of singularity theory (see, [1], [15], [21]).

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Since our concern is the local classification of differential equations, we can formulate as follows (cf. [13], [14], [18], [19], [20]): Let $\pi\colon PT^*\mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ be the projective cotangent bundle over $\mathbb{R}^{n+1}$. We have a local coordinate $(x, y, p) = (x_1, \dots, x_n, y, p_1, \dots, p_n)$ of $PT^*\mathbb{R}^{n+1}$ so that (x, y) gives the canonical coordinate of $\mathbb{R}^{n+1}$ and the hypersurface in $T_{(x,y)}\mathbb{R}^{n+1}$ given by $dy - \sum_{i=1}^n p_i\, dx_i = 0$. This coordinate is called *the canonical coordinate* of $PT^*\mathbb{R}^{n+1}$. *The canonical contact form* on $PT^*\mathbb{R}^{n+1}$ is defined by $\theta = dy - \sum_{i=1}^n p_i\, dx_i$ on the canonical coordinate. Using this approach, a first order differential equation is most naturally interpreted as being a closed subset of $PT^*\mathbb{R}^{n+1}$. However, here we consider that a *holonomic system of first order differential equation germ* (or, briefly, a *holonomic system*) is defined to be the image of a smooth map germ $f\colon (\mathbb{R}^{n+1}, 0) \longrightarrow PT^*\mathbb{R}^{n+1}$. We say that f is *completely integrable* if there exists a submersion germ $\mu\colon (\mathbb{R}^{n+1}, 0) \longrightarrow (\mathbb{R}, 0)$ such that $d\mu \wedge f^*\theta = 0$. The pair $(\mu, f)\colon (\mathbb{R}^{n+1}, 0) \longrightarrow \mathbb{R} \times PT^*\mathbb{R}^{n+1}$ is called *a holonomic system with complete integral.* If $\pi \circ f|_{\mu^{-1}(s)}$ is a non-singular map for each $s \in (\mathbb{R}, 0)$, then f is called a *holonomic system of general Clairaut type* (cf. [20]). Moreover, if the equation f is an immersion germ then we simply call f a *holonomic system of Clairaut type.* The term “general” means that f is not necessarily an immersion germ, that is, f might have a singularity at the origin.

One-parameter family of first order differential equation germ (or, *one-parameter family of holonomic system*) is defined to be a smooth map germ $f\colon (\mathbb{R}^{n+1} \times \mathbb{R}, 0) \longrightarrow PT^*\mathbb{R}^{n+1}$. Denote a smooth map germ $f_t\colon (\mathbb{R}^{n+1}, 0) \longrightarrow PT^*\mathbb{R}^{n+1}$ by $f_t(u_1, \dots, u_{n+1}) = f(u_1, \dots, u_{n+1}, t)$. We also say that f is a *one-parameter completely integrable* if there exist a smooth function germ $\mu\colon (\mathbb{R}^{n+1} \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ such that μ_t is a submersion germ and $d\mu_t \wedge f_t^*\theta = 0$ for each $t \in (\mathbb{R}, 0)$, where $\mu_t(u_1, \dots, u_{n+1}) = \mu(u_1, \dots, u_{n+1}, t)$. The pair $(\mu, f)\colon (\mathbb{R}^{n+1} \times \mathbb{R}, 0) \longrightarrow \mathbb{R} \times PT^*\mathbb{R}^{n+1}$ is called a *one-parameter family of holonomic system with complete integral.* If $\pi \circ f_t|_{\mu_t^{-1}(s)}$ is a non-singular map for each t, $s \in (\mathbb{R}, 0)$, then f is called a *one-parameter family of holonomic system of general Clairaut type.* Furthermore, if f_t is an immersion germ for each $t \in (\mathbb{R}, 0)$, then f is called a *one-parameter family of holonomic system of Clairaut type* which has been investigated in [18, 19].

Let (μ, g) be a pair of smooth map germs $g\colon (\mathbb{R}^{n+1} \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}^{n+1}, 0)$ and $\mu\colon (\mathbb{R}^{n+1} \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ such that μ_t is a submersion for each $t \in (\mathbb{R}, 0)$. Then the divergence diagram

$$(\mathbb{R},0) \xleftarrow{\mu} (\mathbb{R}^{n+1}\times\mathbb{R},0) \xrightarrow{g} (\mathbb{R}^{n+1},0)$$

or, briefly, (μ, g) is called a *one-parameter family of integral diagram* if there exists a one-parameter family of holonomic system $f\colon (\mathbb{R}^{n+1}\times\mathbb{R},0) \longrightarrow PT^*\mathbb{R}^{n+1}$ such that (μ, f) is a one-parameter family of holonomic system with complete integral and $\pi\circ f = g$.

Following S. Lie, the most natural equivalence relation among equation germs is given by point transformation. In order to define the one-parameter point transformation, we consider unfolding germs of f and μ. Define $F\colon (\mathbb{R}^{n+1}\times\mathbb{R},0) \longrightarrow PT^*\mathbb{R}^{n+1}\times\mathbb{R}$ by $F(u_1,\dots,u_{n+1},t) = (f(u_1,\dots,u_{n+1},t),t)$ for a one-parameter family of holonomic system f. In this case, F is called a *one-parameter unfolding of holonomic system associated to* f. We also define $\hat{\mu}\colon (\mathbb{R}^{n+1}\times\mathbb{R},0) \longrightarrow (\mathbb{R}\times\mathbb{R},0)$ by $\hat{\mu}(u_1,\dots,u_{n+1},t) = (\mu(u_1,\dots,u_{n+1},t),t)$. The pair $(\hat{\mu},F)$ or (μ,F) is called a *one-parameter unfolding of holonomic system with complete integral*.

Let F and F' be one-parameter unfoldings of holonomic system associated to f and f' respectively. We define F and F' are *equivalent* if the diagram

$$\begin{array}{ccccc}
(\mathbb{R}^{n+1}\times\mathbb{R},0) & \xrightarrow{F} & (PT^*\mathbb{R}^{n+1}\times\mathbb{R},(z,0)) & \xrightarrow{\pi\times\mathrm{id}} & (\mathbb{R}^{n+1}\times\mathbb{R},0)\\
\psi\downarrow & & \downarrow\Phi & & \downarrow\phi\\
(\mathbb{R}^{n+1}\times\mathbb{R},0) & \xrightarrow[F']{} & (PT^*\mathbb{R}^{n+1}\times\mathbb{R},(z',0)) & \xrightarrow[\pi\times\mathrm{id}]{} & (\mathbb{R}^{n+1}\times\mathbb{R},0)
\end{array}$$

commutes for some germs of diffeomorphisms ψ, Φ and ϕ of the following form

$$\begin{aligned}
&\psi(u_1,\dots,u_{n+1},t) = (\psi_1(u_1,\dots,u_{n+1},t),\varphi(t)),\\
&\Phi(x_1,\dots,x_{2n+1},t) = (\hat{\phi}_t(x_1,\dots,x_{2n+1}),\varphi(t)),\\
&\phi(x_1,\dots,x_{n+1},t) = (\phi_1(x_1,\dots,x_{n+1},t),\varphi(t)).
\end{aligned}$$

Here $\hat{\phi}_t$ is the unique contact lift of ϕ_t and $\phi_t = \phi|_{\mathbb{R}^{n+1}\times\{t\}}\colon (\mathbb{R}^{n+1}\times\{t\},0) \longrightarrow (\mathbb{R}^{n+1}\times\{\varphi(t)\},0)$.

Let $(\hat{\mu},G)$ be the pair of a smooth map germ $G\colon (\mathbb{R}^{n+1}\times\mathbb{R},0) \longrightarrow (\mathbb{R}^{n+1},0)$ and a smooth map germ $\hat{\mu}\colon (\mathbb{R}^{n+1}\times\mathbb{R},0) \longrightarrow (\mathbb{R}\times\mathbb{R},0)$ such that μ_t is a submersion for each $t\in(\mathbb{R},0)$. Then the diagram

$$(\mathbb{R}\times\mathbb{R},0) \xleftarrow{\hat{\mu}} (\mathbb{R}^{n+1}\times\mathbb{R},0) \xrightarrow{G} (\mathbb{R}^{n+1}\times\mathbb{R},0)$$

or, briefly, $(\hat{\mu}, G)$ is called a *one-parameter unfolding of integral diagram* if there exists a one-parameter family of holonomic system f such that $(\hat{\mu}, F)$ is a one-parameter unfolding of holonomic system with complete integral and $(\pi \times \mathrm{id}) \circ F = G$ where F is the one-parameter unfolding of holonomic system associated to f.

We introduce equivalence relations among one-parameter unfoldings of integral diagrams. Let $(\hat{\mu}, G)$ and $(\hat{\mu}', G')$ be one-parameter unfoldings of integral diagrams. We define $(\hat{\mu}, G)$ and $(\hat{\mu}', G')$ are *equivalent as one-parameter unfoldings of integral diagrams* (respectively, *strictly equivalent*) if the diagram

$$\begin{array}{ccccc} (\mathbb{R} \times \mathbb{R}, 0) & \xleftarrow{\hat{\mu}} & (\mathbb{R}^{n+1} \times \mathbb{R}, 0) & \xrightarrow{G} & (\mathbb{R}^{n+1} \times \mathbb{R}, 0) \\ {\scriptstyle \kappa}\downarrow & & {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \phi} \\ (\mathbb{R} \times \mathbb{R}, 0) & \xleftarrow[\hat{\mu}']{} & (\mathbb{R}^{n+1} \times \mathbb{R}, 0) & \xrightarrow[G']{} & (\mathbb{R}^{n+1} \times \mathbb{R}, 0) \end{array}$$

commutes for some germs of diffeomorphisms κ, ψ and ϕ of the form

$$\begin{aligned} &\kappa(s,t) = (\kappa_1(s,t), \varphi(t)) \quad (\text{respectively}, \kappa_1(s,t) = s), \\ &\psi(u_1, \dots, u_{n+1}, t) = (\psi_1(u_1, \dots, u_{n+1}, t), \varphi(t)), \\ &\phi(x_1, \dots, x_{n+1}, t) = (\phi_1(x_1, \dots, x_{n+1}, t), \varphi(t)). \end{aligned}$$

If $(\hat{\mu}, G)$ and $(\hat{\mu}', G')$ are strictly equivalent, then we also say that (μ, G) and (μ', G') are *strictly equivalent*.

We shall show that two one-parameter unfoldings of holonomic systems F and F' are equivalent if and only if induced one-parameter unfoldings of integral diagrams $(\hat{\mu}, (\pi \times \mathrm{id}) \circ F)$ and $(\hat{\mu}', (\pi \times \mathrm{id}) \circ F')$ are equivalent for generic (μ, F) and (μ', F') (cf. Theorem 2.4).

If (μ, f) is a one-parameter family of holonomic system of general Clairaut type, then $(\hat{\mu}, F)$ or (μ, F) is called a *one-parameter unfolding of holonomic system of general Clairaut type*.

We denote by $\mathcal{E}_{(x_1, \dots, x_{n+1}, t)}$ the ring of all smooth function germs of $\mathbb{R}^{n+1} \times \mathbb{R}$ at 0 with coordinate $(x_1, \dots, x_{n+1}, t)$ and denote by $\mathfrak{M}_{(x_1, \dots, x_{n+1}, t)}$ the unique maximal ideal of $\mathcal{E}_{(x_1, \dots, x_{n+1}, t)}$.

The main result in this paper is the following theorem which gives a generic classification of one-parameter unfoldings of holonomic systems of general Clairaut type:

Theorem 1.1 *For a generic one-parameter unfolding of holonomic system of general Clairaut type*

$$(\mu, F)\colon\ (\mathbb{R}^{n+1}\times\mathbb{R},0)\longrightarrow \mathbb{R}\times PT^*\mathbb{R}^{n+1}\times\mathbb{R},$$

the one-parameter unfolding of integral diagram $(\mu, G=(\pi\times\mathrm{id})\circ F)$ *is strictly equivalent to one of germs in the following list*:

(DA_1): $\mu = u_{n+1}$, $G=(u_1,\dots,u_{n+1},t)$.

(DA_2): $\mu = u_{n+1}-\frac{1}{2}u_1$, $G=(u_1,\dots,u_n,u_{n+1}^2,t)$.

(DA_2^0): $\mu = u_{n+1}+\frac{1}{2}(u_1^2+\cdots+u_n^2)-\frac{1}{2}t$, $G=(u_1,\dots,u_n,u_{n+1}^2,t)$.

(DA_2^k) $(1\le k\le n)$: $\mu = u_{n+1}-\frac{1}{2}(u_1^2+\cdots+u_k^2-u_{k+1}^2-\cdots-u_n^2)-\frac{1}{2}t$, $G=(u_1,\dots,u_n,u_{n+1}^2,t)$.

$(\widetilde{DA}_3^0)$: $\mu = u_{n+1}+\alpha\circ G$ for $\alpha\in\mathfrak{M}_{(x_1,\dots,x_{n+1},t)}$, $G=\big(u_1,\dots,u_n,u_{n+1}^3+(t+u_1^2+\cdots+u_n^2)u_{n+1}\big)$.

$(\widetilde{DA}_3^k)$ $(1\le k\le n)$: $\mu = u_{n+1}+\alpha\circ G$ for $\alpha\in\mathfrak{M}_{(x_1,\dots,x_{n+1},t)}$, $G=\big(u_1,\dots,u_n,u_{n+1}^3+(t-(u_1^2+\cdots+u_k^2-u_{k+1}^2-\cdots-u_n^2))u_{n+1}\big)$.

(DA_ℓ) $(3\le \ell\le n+1)$: $\mu = u_{n+1}$, $G=\left(u_1,\dots,u_n,u_{n+1}^\ell+\sum_{i=1}^{\ell-1}u_iu_{n+1}^i,t\right)$.

$(\widetilde{DA}_{n+2})$: $\mu = u_{n+1}+\alpha\circ G$ for $\alpha\in\mathfrak{M}_{(x_1,\dots,x_{n+1},t)}$, $G=\left(u_1,\dots,u_n,u_{n+1}^{n+2}+\sum_{i=1}^{n}u_iu_{n+1}^i,t\right)$.

$$(\widetilde{DA}_{n+3}):\qquad \mu = u_{n+1} + \alpha\circ G \quad \text{for } \alpha\in\mathfrak{M}_{(x_1,\dots,x_{n+1},t)},$$
$$G = \Big(u_1,\dots,u_n, u_{n+1}^{n+3} + \sum_{i=1}^{n} u_i u_{n+1}^i$$
$$+ \beta(u_1,\dots,u_n,t)u_{n+1}^{n+1}, t\Big)$$
$$\text{for } \beta\in\mathfrak{M}_{(u_1,\dots,u_n,t)} \text{ and } \frac{\partial\beta}{\partial t}(0)\neq 0.$$

We remark that the normal forms DA_ℓ $(1 \le \ell \le n+1)$, $\widetilde{DA}_{n+2}$ and $\widetilde{DA}_{n+3}$ are holonomic systems of Clairaut type, namely, the map germ f_t is an immersion germ for each $t\in(\mathbb{R},0)$ which have been already classified in [19].

Also the normal forms DA_2^k $(0 \le k \le n)$ is equivalent to $(\mu, G) = \big(u_{n+1} - (1/2)(u_1^2 + \cdots + u_k^2 - u_{k+1}^2 - \cdots - u_n^2), u_1,\dots,u_n,u_{n+1}^2\big)$. These types already have been appeared as a generic holonomic system of general Clairaut type with the generalized cross-cap singularity in [8, 20].

We call the function germ α which appears in the normal form $\widetilde{DA}_3^k$ $(0 \le k \le n)$, $\widetilde{DA}_{n+2}$ and $\widetilde{DA}_{n+3}$, *a one-parameter functional moduli* and the function germ β which appears in the normal form $\widetilde{DA}_{n+3}$, *a second functional moduli*.

In the final remark in §4, we will describe why the second functional moduli appears in the normal form $\widetilde{DA}_{n+3}$. The meaning of the genericity in the above theorem will be described in §2 (cf. Theorem 2.2).

The normal forms $\widetilde{DA}_3^k$ $(0 \le k \le n)$ are the new one which shall go into details in Example 1.3 and 1.4. We now give the typical examples of one-parameter families of holonomic systems of general Clairaut type.

Example 1.2 (One-parameter families of ordinary Clairaut differential equations) Consider the following one-parameter family of ordinary classical Clairaut equations

$$y = \frac{dy}{dx} + g_t\left(\frac{dy}{dx}\right),$$

where $g(t,p)$ is a smooth function and $p = dy/dx$. A germ of a smooth map $(\mu, f)\colon (\mathbb{R}^2\times\mathbb{R},0) \longrightarrow \mathbb{R}\times PT^*\mathbb{R}^2$ given by

$$(\mu, f)(u_1,u_2,t) = (u_2, u_1, u_1u_2 + g_t(u_2), u_2).$$

Then we can show that (μ, f) is one-parameter family of ordinary differential equation of Clairaut type. By definition, of course, (μ, f) is also one-parameter family of general Clairaut type. For the normal forms DA_1, $\widetilde{DA}_3$ where $\alpha = 0$ and $\widetilde{DA}_4$ where $\alpha = 0$, $\beta = t$ in Theorem 1.1 for $n = 1$ are examples of one-parameter families of classical Clairaut equations. In this case, the complete solution is a family of straight lines. We have been drawn the picture of the phase portrait $\{\pi \circ f_t(\mu_t^{-1}(c))\}_{c \in \mathbb{R}}$ of $\widetilde{DA}_3$ where $\alpha = 0$ and $\widetilde{DA}_4$ where $\alpha = 0$, $\beta = t$ in [18].

Example 1.3 (One-parameter families of ordinary Lagrangian equations) We consider classical equations which are more general than Clairaut differential equations. The equations of the following form is called a *Lagrangian equation* (cf. [5, page 466]):

$$y = \varphi\left(\frac{dy}{dx}\right)x + \psi\left(\frac{dy}{dx}\right),$$

where φ and ψ are smooth functions. Therefore we consider the one-parameter family of Lagrangian equations given by

$$y = \varphi_t\left(\frac{dy}{dx}\right)x + \psi_t\left(\frac{dy}{dx}\right),$$

where $\varphi(t, p)$ and $\psi(t, p)$ are smooth functions and $p = dy/dx$. We consider a germ of a smooth map $(\mu, f)\colon (\mathbb{R}^2 \times \mathbb{R}, 0) \longrightarrow \mathbb{R} \times PT^*\mathbb{R}^2$ defined by

$$(\mu, f)(u_1, u_2, t) = \big(u_2, u_1, \varphi_t(u_2)u_1 + \psi_t(u_2), \varphi_t(u_2)\big),$$

and also call it a *one-parameter family of Lagrangian equations.*
If $\varphi_t(p)$ has critical point at the origin, then f_t is not an immersion germ. In this case, (μ, f) is a one-parameter family of holonomic system of general Clairaut type, but not Clairaut type.

We put $\varphi_t(p) = p^2$ and $\psi_t(p) = p^3 + tp$, the one-parameter family of Lagrangian equation is strictly equivalent to the normal form $\widetilde{DA}_3^1$ where $\alpha = -(1/\sqrt{3})x$ in Theorem 1.1 for $n = 1$.

Then the above equation is one of the examples of one-parameter families of general Clairaut type, but not Clairaut type. We can draw the picture of the image of equations (see, Fig. 1) and the phase portraits $\{\pi \circ f_t(\mu_t^{-1})(c)\}_{c \in \mathbb{R}}$ (see, Fig. 2).

We remark that in generic classification of general (single) Clairaut type equations, the Lagrangian equations except for the immersive equa-

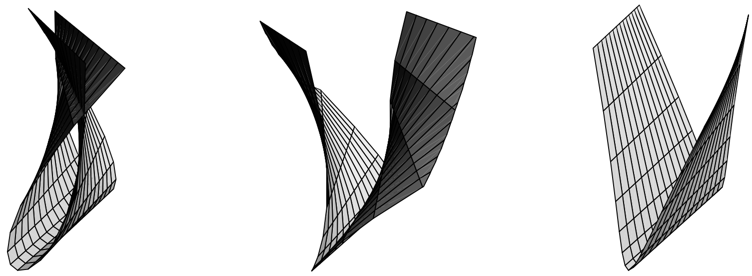


Fig. 1. Image of equations of one-parameter family of Lagrangian equations where $\varphi_t(u_2) = u_2^2$, $\psi_t(u_2) = u_2^3 + tu_2$.

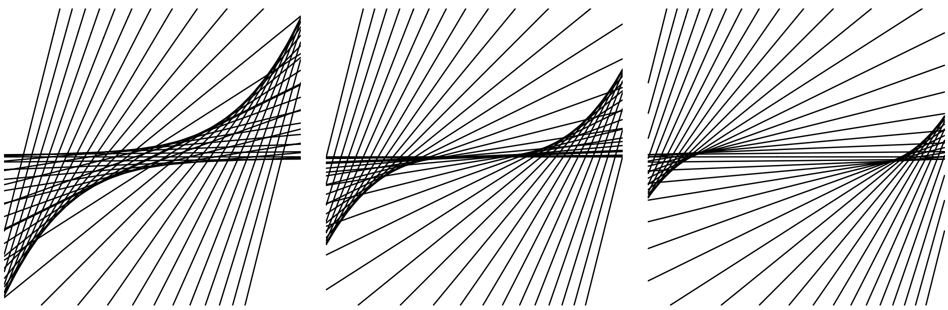


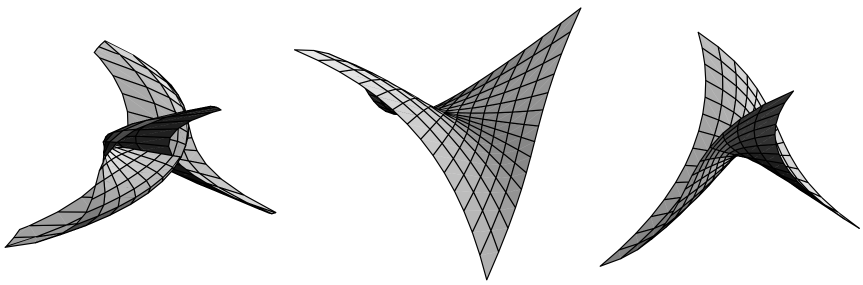
Fig. 2. Bifurcation of the phase portraits $\{\pi \circ f_t(\mu_t^{-1})(c)\}_{c\in\mathbb{R}}$ of the equation of Fig. 1.

tions does not appear (cf. [8, 20]). But our theorem assert that in generic classification of one-parameter family of general Clairaut type equations, a one-parameter family of Lagrangian equations appears.

Moreover, we can also draw the picture of the image of equations (see, Fig. 3 and Fig. 5) about $\widetilde{DA}_3^0$ and $\widetilde{DA}_3^1$ where $\alpha = 0$ in Theorem 1.1 for $n = 1$ and the corresponding to the phase portraits $\{\pi \circ f_t(\mu_t^{-1})(c)\}_{c\in\mathbb{R}}$ of them respectively (see, Fig. 4 and Fig. 6).

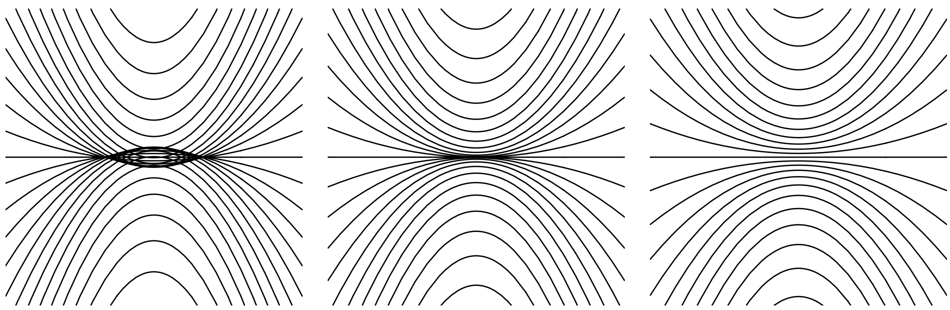
The above examples describe how two cross caps meat and how the corresponding web structures bifurcate.

Example 1.4 (cf. [20, Example 1.3]) Here we consider the one-parameter unfoldings of holonomic systems of integral diagram of $\widetilde{DA}_3^k$ where $\alpha = 0$ for $n = 2$. For the case of $\widetilde{DA}_3^0$ where $\alpha = 0$, one of component of phase



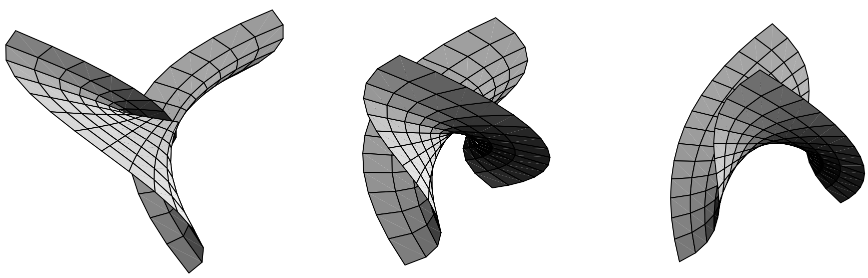
$t < 0$ $t = 0$ $t > 0$

Fig. 3. Image of equations of the normal form $\widetilde{DA}_3^0$ where $\alpha = 0$.



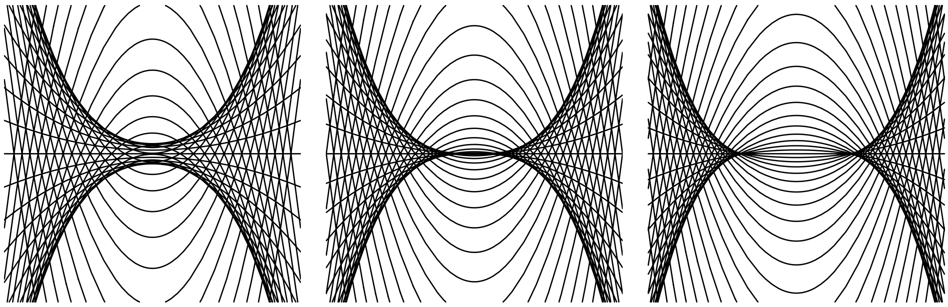
$t < 0$ $t = 0$ $t > 0$

Fig. 4. Bifurcation of the phase portraits $\{\pi \circ f_t(\mu_t^{-1})(c)\}_{c \in \mathbb{R}}$ of $\widetilde{DA}_3^0$ where $\alpha = 0$.



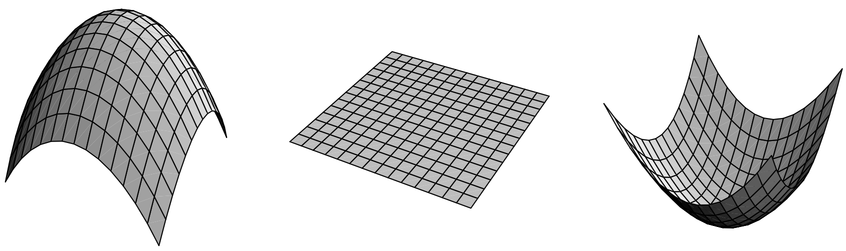
$t < 0$ $t = 0$ $t > 0$

Fig. 5. Image of equations of the normal form $\widetilde{DA}_3^1$ where $\alpha = 0$.



$t < 0$ $t = 0$ $t > 0$

Fig. 6. Bifurcation of the phase portraits $\{\pi \circ f_t(\mu_t^{-1})(c)\}_{c\in\mathbb{R}}$ of $\widetilde{DA}_3^1$ where $\alpha = 0$.



$c = -1$ $c = 0$ $c = 1$

Fig. 7.

portrait is given by

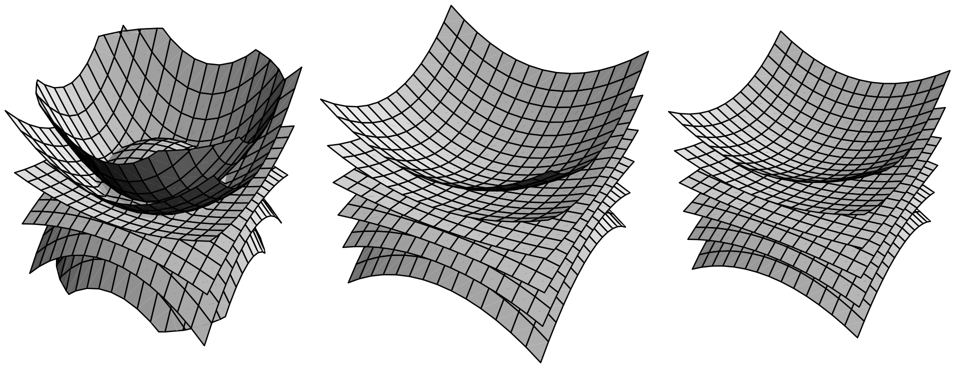
$$\pi \circ f_t(\mu_t^{-1}(c)) = (u_1, u_2, c^3 + (t + u_1^2 + u_2^2)c)$$

We can draw these pictures when $t < 0$ and $c = -1$, 0, 1, respectively in Fig. 7 and superimpose these pictures (that is, the phase portrait) $\{\pi \circ f_t(\mu_t^{-1}(c))\}_{c\in\mathbb{R}}$ of $\widetilde{DA}_3^0$ when $t < 0$, see on the left hand side of Fig. 8. In a similar way, we can draw the phase portraits of $\widetilde{DA}_3^0$ when $t = 0$ and $t > 0$, see Fig. 8.

Furthermore, for $\widetilde{DA}_3^1$ and $\widetilde{DA}_3^2$ where $\alpha = 0$, the phase portraits are given by

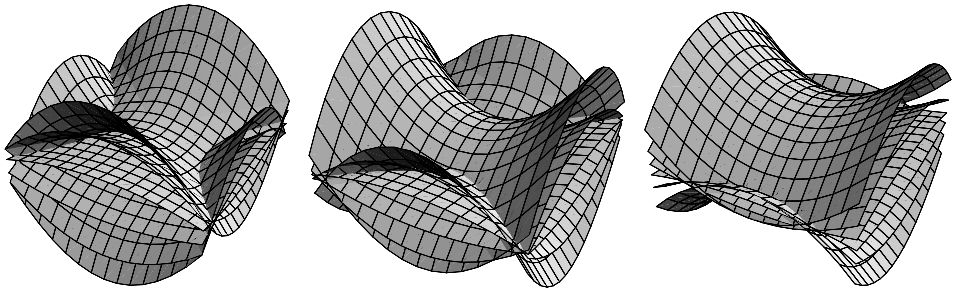
$$\pi \circ f_t(\mu_t^{-1}(c)) = (u_1, u_2, c^3 + (t + u_1^2 - u_2^2)c)$$

and



$t < 0$ $t = 0$ $t > 0$

Fig. 8. Bifurcation of the phase portraits $\{\pi \circ f_t(\mu_t^{-1}(c))\}_{c\in\mathbb{R}}$ of $\widetilde{DA}_3^0$ where $\alpha = 0$.



$t < 0$ $t = 0$ $t > 0$

Fig. 9. Bifurcation of the phase portraits $\{\pi \circ f_t(\mu_t^{-1}(c))\}_{c\in\mathbb{R}}$ of $\widetilde{DA}_3^1$ where $\alpha = 0$.

$$\pi \circ f_t(\mu_t^{-1}(c)) = (u_1, u_2, c^3 + (t - u_1^2 - u_2^2)c)$$

respectively. We draw these phase portraits $\{\pi \circ f_t(\mu_t^{-1}(c))\}_{c\in\mathbb{R}}$, see Fig. 9 and 10.

In §2, we define one-parameter complete Legendrian unfoldings corresponding to the one-parameter family of holonomic systems of general Clairaut type and describe the meaning of the genericity of properties for one-parameter unfoldings of holonomic systems of general Clairaut type. In §3, we consider the equivalence relations among the one-parameter complete Legendrian unfoldings and generating families. In order to prove The-

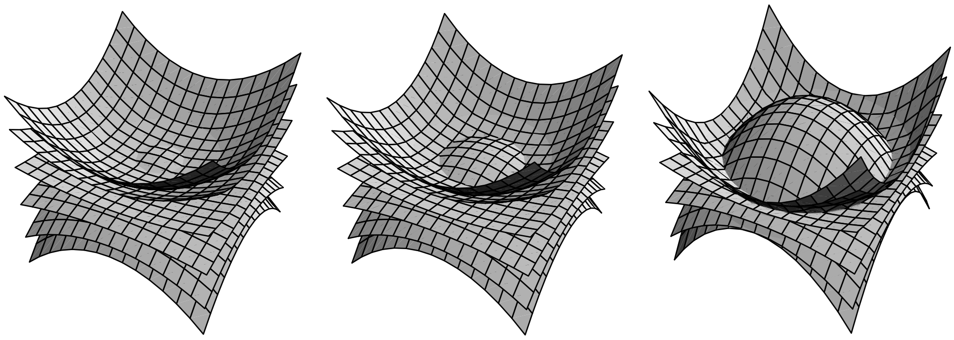


Fig. 10. Bifurcation of the phase portraits $\{\pi \circ f_t(\mu_t^{-1}(c))\}_{c\in\mathbb{R}}$ of $\widetilde{DA}_3^2$ where $\alpha = 0$.

orem 1.1, we use a kind of the versality theory (cf. [7]). In §5 Appendix, we introduce the unfolding theory of function germs which will be useful for the proof of Theorem 1.1. We shall give a proof of Theorem 1.1 in §4.

All map germs considered in this paper are of class C^∞, unless stated otherwise.

2. Genericity and one-parameter unfolding of holonomic systems of general Clairaut type

Our aim is to construct a family of Legendrian immersions depending on holonomic system of general Clairaut type (μ, f). We therefore consider the projective cotangent bundle $\Pi\colon PT^*(\mathbb{R} \times \mathbb{R}^{n+1}) \longrightarrow \mathbb{R} \times \mathbb{R}^{n+1}$. We have a local coordinate $(\sigma, x, y, \rho, p) = (\sigma, x_1, \ldots, x_n, y, \rho, p_1, \ldots, p_n)$ of $PT^*(\mathbb{R}\times\mathbb{R}^{n+1})$, such that (σ, x, y) gives the canonical coordinate of $\mathbb{R}\times\mathbb{R}^{n+1}$ and the hyperplane in $T_{(\sigma,x,y)}(\mathbb{R}\times\mathbb{R}^{n+1})$ given by $dy - \sum_{i=1}^n p_i\, dx_i - \rho\, d\sigma = 0$. This coordinate is called *the canonical coordinate* of $PT^*(\mathbb{R} \times \mathbb{R}^{n+1})$. The canonical contact form is defined by $\Theta = dy - \sum_{i=1}^n p_i\, dx_i - \rho\, d\sigma = \theta - \rho\, d\sigma$. Let (μ, f) be a one-parameter family of holonomic systems with complete integral. Then there exists a unique element $h_t \in \mathcal{E}_u$ such that $h_t \cdot d\mu_t = f_t^*\theta$, where $\mathcal{E}_u$ is the ring of function germs of $u = (u_1, \ldots, u_{n+1})$-variables. Define a map germ

$$\ell_{(\mu,f)}\colon\ (\mathbb{R}^{n+1} \times \mathbb{R}, 0) \longrightarrow PT^*(\mathbb{R} \times \mathbb{R}^{n+1}).$$

by

$$\ell_{(\mu,f)}(u,t) = \big(\mu(u,t), x \circ f(u,t), y \circ f(u,t), h(u,t), p \circ f(u,t)\big),$$

where $h(u,t) = h_t(u)$. Then we have $(\ell_{(\mu,f)}|_{\mathbb{R}^{n+1}\times\{t\}})^*\Theta = 0$ for each $t \in (\mathbb{R}, 0)$.

We say that the pair $(\mu, \ell)\colon (\mathbb{R}^{n+1} \times \mathbb{R}, 0) \longrightarrow \mathbb{R} \times PT^*(\mathbb{R} \times \mathbb{R}^{n+1})$ is a *Legendrian family* if $\ell_s = \ell|_{\mu^{-1}(s)}$ is a Legendrian immersion germ for each $s \in (\mathbb{R}, 0)$ (cf. [11]). We have the following simple but important lemma:

Lemma 2.1 ([11, Lemma 2.1]) *Let (μ, ℓ) be a Legendrian family. Then there exists a unique element $k \in \mathcal{E}_{(u,t)}$ such that $\ell^*\Theta = k \cdot d\mu$.*

If (μ, f) is a one-parameter family of holonomic systems of general Clairaut type, then $\ell_{(\mu,f)}|_{\mathbb{R}^{n+1}\times\{t\}}$ is an immersion germ for each $t \in (\mathbb{R}, 0)$. We can apply Lemma 2.1 to $(\pi_2, \ell_{(\mu,f)})$, where $\pi_2\colon (\mathbb{R}^{n+1} \times \mathbb{R}) \longrightarrow (\mathbb{R}, 0)$; $\pi_2(u,t) = t$, so that there exists a unique element $k \in \mathcal{E}_{(u,t)}$ such that $\ell^*_{(\mu,f)}\Theta = k \cdot dt$. We also consider the projective cotangent bundle $\widetilde{\Pi}\colon PT^*(\mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}) \longrightarrow \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}$. Let $(\sigma, x, t, y, \rho, p, \tau)$ be the canonical coordinate on $PT^*(\mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R})$. Here the canonical 1-form is given by $\widetilde{\Theta} = dy - \sum_{i=1}^{n} p_i\, dx_i - \rho\, d\sigma - \tau\, dt = \Theta - \tau\, dt$. We define a map germ

$$\ell_{(\mu,F)}\colon\ (\mathbb{R}^{n+1} \times \mathbb{R}, 0) \longrightarrow PT^*(\mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R})$$

by

$$\begin{aligned}&\ell_{(\mu,F)}(u,t)\\ &\quad = \big(\mu(u,t), x \circ F(u,t), t, y \circ F(u,t), h(u,t), p \circ F(u,t), k(u,t)\big).\end{aligned}$$

We can easily show that $\ell_{(\mu,F)}$ is a Legendrian immersion germ (i.e., $\ell_{(\mu,F)}$ is an immersion germ with $\ell^*_{(\mu,F)}\widetilde{\Theta} = 0$). We call $\ell_{(\mu,F)}$ a *one-parameter complete Legendrian unfolding associated to* (μ, F). If F is a one-parameter unfolding of equation associated to f, then we also call $\ell_{(\mu,F)}$ a *one-parameter complete Legendrian unfolding associated to* (μ, f).

By the aid of the notion of one-parameter complete Legendrian unfoldings, one-parameter unfoldings of holonomic systems of general Clairaut type are characterized as follows (cf. [18, Proposition 4.1]):

Proposition 2.2 *Let $(\mu, F)\colon (\mathbb{R}^{n+1}\times\mathbb{R}, 0) \longrightarrow \mathbb{R}\times PT^*\mathbb{R}^{n+1}\times\mathbb{R}$ be a one-parameter unfolding of holonomic system with complete integral. Then (μ, F) is a one-parameter unfolding of holonomic system of general Clairaut* *type if and only if* $\ell_{(\mu,F)}$ *is a Legendrian non-singular, that is,* $\widetilde{\Pi}\circ\ell_{(\mu,F)}$ *is non-singular.*

We now establish the notion of *the genericity*. Let $U\times V\subset\mathbb{R}^{n+1}\times\mathbb{R}$ be an open set. We denote by $\mathrm{Clair}(U\times V,\mathbb{R}\times PT^*\mathbb{R}^{n+1}\times\mathbb{R})$ the set of one-parameter unfoldings of holonomic systems of general Clairaut type $(\mu,F)\colon U\times V\longrightarrow\mathbb{R}\times PT^*\mathbb{R}^{n+1}\times\mathbb{R}$. We also define $L_R(U\times V,PT^*(\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}))$ to be the set of one-parameter complete Legendrian unfoldings $\ell_{(\mu,F)}\colon U\times V\longrightarrow PT^*(\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R})$ such that $\ell_{(\mu,F)}$ is Legendrian non-singular.

These sets are topological spaces equipped with the Whitney C^∞-topology. A subset of either spaces is said to be *generic* if it is an open dense subset in the space.

The genericity of a property of germs are defined as follows: Let P be a property of one-parameter unfoldings of holonomic systems of general Clairaut type $(\mu,F)\colon U\times V\longrightarrow\mathbb{R}\times PT^*\mathbb{R}^{n+1}\times\mathbb{R}$ (respectively, one-parameter complete Legendrian unfoldings $\ell_{(\mu,F)}\colon U\times V\longrightarrow PT^*(\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R})$). For an open set $U\times V\subset\mathbb{R}^{n+1}\times\mathbb{R}$, we define $\mathcal{P}(U\times V)$ to be the set of $(\mu,F)\in\mathrm{Clair}(U\times V,\mathbb{R}\times PT^*\mathbb{R}^{n+1}\times\mathbb{R})$ (respectively, $\ell_{(\mu,F)}\in L_R(U\times V,PT^*(\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}))$) such that the germ at (u,t) whose representative is given by (μ,F) (respectively, $\ell_{(\mu,F)}$) has property P for any $(u,t)\in U\times V$.

The property P is said to be *generic* if for some neighbourhood $U\times V$ of 0 in $\mathbb{R}^{n+1}\times\mathbb{R}$, the set $\mathcal{P}(U\times V)$ is a generic subset in $\mathrm{Clair}(U\times V,\mathbb{R}\times PT^*\mathbb{R}^{n+1}\times\mathbb{R})$ (respectively, $L_R(U\times V,PT^*(\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}))$).

By the construction, we have a well-defined continuous mapping

$$(\Pi_1)_*\colon\ L_R(U\times V,PT^*(\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}))\longrightarrow\mathrm{Clair}(U\times V,\mathbb{R}\times PT^*\mathbb{R}^{n+1}\times\mathbb{R})$$

defined by $(\Pi_1)_*(\ell_{(\mu,F)})=\Pi_1\circ\ell_{(\mu,F)}=(\mu,F)$, where $\Pi_1\colon PT^*(\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R})\longrightarrow\mathbb{R}\times PT^*\mathbb{R}^{n+1}\times\mathbb{R}$ is canonical projection. Then we have the following fundamental theorem.

Theorem 2.3 *The continuous map*

$$(\Pi_1)_*\colon\ L_R(U\times V,PT^*(\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}))\longrightarrow\mathrm{Clair}(U\times V,\mathbb{R}\times PT^*\mathbb{R}^{n+1}\times\mathbb{R})$$

is a homeomorphism.

The proof follows from a direct analogy of the proof for Theorem 4.4 in [12], so that we omit it.

This theorem asserts that the genericity of a property of one-parameter unfoldings of holonomic system of general Clairaut type can be interpreted by the genericity of the corresponding property of one-parameter complete Legendrian unfoldings.

We can assert the following theorem which reduces the equivalence problem for one-parameter unfoldings of holonomic systems with complete integral to that for the corresponding induced one-parameter unfolding of integral diagrams.

Theorem 2.4 ([18, Theorem 3.3]) *Let* $(\mu, F): (\mathbb{R}^{n+1} \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}\times PT^*\mathbb{R}^{n+1}\times\mathbb{R}, (0, z, 0))$ *and* $(\mu', F'): (\mathbb{R}^{n+1}\times\mathbb{R}, 0) \longrightarrow (\mathbb{R}\times PT^*\mathbb{R}^{n+1}\times\mathbb{R}, (0, z', 0))$ *be one-parameter unfoldings of holonomic systems with complete integral such that the sets of singular points of* $\pi \circ f_t$ *and* $\pi \circ f'_t$ *are closed sets without interior points except for isolated* t*. Then the following are equivalent*:

(1) F *and* F' *are equivalent*

(2) $(\hat{\mu}, (\pi\times\mathrm{id})\circ F)$ *and* $(\hat{\mu}', (\pi\times\mathrm{id})\circ F')$ *are equivalent as one-parameter unfoldings of integral diagrams.*

Remark that in the assumption of Theorem 2.4, the condition that singular points of $\pi \circ f_t$ is closed sets without interior points except for isolated t is satisfied for generic equations.

3. Equivalence of one-parameter complete Legendrian unfoldings and generating families

The main idea of the proof for Theorem 1.1 is to define an equivalence relation which can ignore one-parameter functional moduli and to do everything in terms of generating families for one-parameter complete Legendrian unfoldings analogous to those of in [12, 13, 18].

Let (μ, f) be a one-parameter family of holonomic system of general Clairaut type. Since $\ell_{(\mu,F)}$ is a Legendrian immersion germ, there exists a generating family of $\ell_{(\mu,F)}$ by the theory of Legendrian singularities (see, [1, 21]). Let $G: ((\mathbb{R}\times\mathbb{R}^n\times\mathbb{R})\times\mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ be a function germ such that $d_2G|_{0\times\mathbb{R}^n\times\{t\}\times\mathbb{R}^k}$ is non-singular, where

$$d_2G(s,x,t,q) = \left(\frac{\partial G}{\partial q_1}(s,x,t,q), \dots, \frac{\partial G}{\partial q_k}(s,x,t,q)\right).$$

Then $C(G) = d_2G^{-1}(0)$ is a smooth $(n+2)$-dimensional manifold germ and $\pi_G|_{C(G)\cap(0\times\mathbb{R}^n\times\{t\}\times\mathbb{R}^k)}\colon (C(G)\cap(0\times\mathbb{R}^n\times\{t\}\times\mathbb{R}^k),0) \longrightarrow \mathbb{R}$ is a submersion germ for each $t\in(\mathbb{R},0)$, where $\pi_G\colon (C(G),0)\longrightarrow\mathbb{R}$ is given by $\pi_G(s,x,t,q)=s$. Define map germs

$$\widetilde{\mathcal{L}_G}\colon\ (C(G),0)\longrightarrow J^1(\mathbb{R}^n,\mathbb{R})$$

by

$$\widetilde{\mathcal{L}_G}(s,x,t,q) = \left(x, G(s,x,t,q), \frac{\partial G}{\partial x}(s,x,t,q)\right),$$

and

$$\mathcal{L}_G\colon\ (C(G),0)\longrightarrow J^1(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R},\mathbb{R})$$

by

$$\begin{aligned}&\mathcal{L}_G(s,x,t,q)\\ &\quad=\left(s,x,t,G(s,x,t,q),\frac{\partial G}{\partial s}(s,x,t,q),\frac{\partial G}{\partial x}(s,x,t,q),\frac{\partial G}{\partial t}(s,x,t,q)\right).\end{aligned}$$

Since $\partial G/\partial q_i = 0$ $(i=1,\dots,k)$ on $C(G)$, we can easily show that

$$\left(\widetilde{\mathcal{L}_{Gt}}|_{\pi_G^{-1}(s)}\right)^*\theta = 0.$$

By definition, $\mathcal{L}_G$ is a one-parameter complete Legendrian unfolding associated to $(\pi_G,\widetilde{\mathcal{L}_G})$. By the same method as the theory of Legendrian singularities in [1, 21], we can also show the following proposition.

Proposition 3.1 *All one-parameter complete Legendrian unfolding germs are constructed by the above method.*

We say that G is *a generalized phase family* of the complete Legendrian unfolding $\mathcal{L}_G$.

Moreover, by Proposition 2.2, $\ell_{(\mu,F)}$ is Legendrian non-singular. Then we can choose a family of function germ

$$G\colon\ (\mathbb{R}\times\mathbb{R}^n\times\mathbb{R},0)\longrightarrow(\mathbb{R},0)$$

such that $\mathrm{Image}\, j^1G_t = \ell_{(\mu,f)}|_{\mathbb{R}^{n+1}\times\{t\}}$ and $\mathrm{Image}\, j^1G_{(s,t)} = f_t(\mu_t^{-1}(s))$ for each s, $t\in(\mathbb{R},0)$ where $G_t(s,x)=G(s,x,t)$ and $G_{(s,t)}(x)=G(s,x,t)$. We remark that a smooth germ

$$j_1^1 G_t\colon\ (\mathbb{R}\times\mathbb{R}^n, 0) \longrightarrow J^1(\mathbb{R}^n, \mathbb{R})$$

defined by $j_1^1 G_t(s,x) = j^1 G_{(s,t)}(x)$ is not necessarily an immersion germ.

In this case, we have $(C(G), 0) = (\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}, 0)$ and

$$\mathcal{L}_G = j^1 G\colon\ (\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}, 0) \longrightarrow J^1(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}, \mathbb{R}),$$

so that it is a one-parameter complete Legendrian unfolding associated to $(\pi_G, j_1^1 G)$. Thus the generalized phase family of a one-parameter complete Legendrian unfolding of holonomic system of general Clairaut type $\mathcal{L}_G$ is given by the above germ. We define $\widetilde{G}\colon (\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ by $\widetilde{G}(s,x,t,y) = G(s,x,t) - y$. We call $\widetilde{G}$ a *generating family* of a one-parameter complete Legengrian unfolding of general Clairaut type.

We now consider an equivalence relation among one-parameter unfoldings of integral diagrams which ignore one-parameter functional moduli. Let $(\hat{\mu}, G)$ and $(\hat{\mu}', G')$ be one-parameter unfolding of integral diagrams. Then $(\hat{\mu}, G)$ and $(\hat{\mu}', G')$ are *one-parameter $\mathcal{R}^+$-equivalent* if there exist a germ of diffeomorphism $\Psi\colon (\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}, 0) \longrightarrow (\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}, 0)$ of the form $\Psi(s,x,t) = (s+\alpha(x,t), \psi(x,t), \varphi(t))$ and a germ of diffeomorphism $\Phi\colon (\mathbb{R}^{n+1}\times\mathbb{R}, 0) \longrightarrow (\mathbb{R}^{n+1}\times\mathbb{R}, 0)$ of the form $\Phi(u,t) = (\phi(u,t), \varphi(t))$ such that $\Psi\circ(\mu, G) = (\mu', G')\circ\Phi$. We remark that if $(\hat{\mu}, G)$ and $(\hat{\mu}', G')$ are one-parameter $\mathcal{R}^+$-equivalent by the above diffeomorphisms, then we have $\mu(u,t) + \alpha\circ G(u,t) = \mu'\circ\Phi(u,t)$ and $(\psi,\varphi)\circ G(u,t) = G'\circ\Phi(u,t)$ for any $(u,t)\in(\mathbb{R}^{n+1}\times\mathbb{R}, 0)$. Thus the one-parameter unfolding of integral diagram $(\mu+\alpha\circ G, G)$ is strictly equivalent to (μ', G').

We now define the corresponding equivalence relation among one-parameter Legendrian unfoldings. Let $\ell_{(\mu,F)}\colon (\mathbb{R}^{n+1}\times\mathbb{R}, 0) \longrightarrow (PT^*(\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}), z)$ and $\ell_{(\mu',F')}\colon (\mathbb{R}^{n+1}\times\mathbb{R}, 0) \longrightarrow (PT^*(\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}), z')$ be one-parameter complete Legendrian unfoldings. We say that $\ell_{(\mu,F)}$ and $\ell_{(\mu',F')}$ are *one-parameter SP^+-Legendrian equivalent* (respectively, *one-parameter SP-Legendrian equivalent*) if there exist a germ of contact diffeomorphism $K\colon (PT^*(\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}), z) \longrightarrow (PT^*(\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}), z')$, a germ of diffeomorphism $\Phi\colon (\mathbb{R}^{n+1}\times\mathbb{R}, 0) \longrightarrow (\mathbb{R}^{n+1}\times\mathbb{R}, 0)$ of the form $\Phi(u,t) = (\phi(u,t), \varphi(t))$ and a germ of diffeomorphism $\Psi\colon (\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}, \widetilde{\Pi}(z)) \longrightarrow (\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}, \widetilde{\Pi}(z'))$ of the form $\Psi(s,x,t) = (s+\alpha(x,t), \psi(x,t), \varphi(t))$ (respectively, $\Psi(s,x,t) = (s, \psi(x,t), \varphi(t))$), such that $\widetilde{\Pi}\circ K = \Psi\circ\widetilde{\Pi}$ and $K\circ\ell_{(\mu,F)} =$

$\ell_{(\mu',F')}\circ\Phi$, where $\widetilde{\Pi}\colon (PT^*(\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}),z)\longrightarrow(\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R},\widetilde{\Pi}(z))$ is the canonical projection. It is clear that if $\ell_{(\mu,F)}$ and $\ell_{(\mu',F')}$ are one-parameter SP^+-Legendrian equivalent (respectively, one-parameter SP-Legendrian equivalent), then $(\mu,(\pi\times\mathrm{id})\circ F)$ and $(\mu',(\pi\times\mathrm{id})\circ F')$ are one-parameter $\mathcal{R}^+$-equivalent (respectively, strictly equivalent). By [21, Theorem 1.1], the converse is also true for generic (μ,F) and (μ',F').

We also say that $\ell_{(\mu,F)}$ and $\ell_{(\mu',F')}$ are *P-Legendrian equivalent* if there exist a germ of contact diffeomorphism $K\colon (PT^*(\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}),z)\longrightarrow(PT^*(\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}),z')$, a germ of diffeomorphism $\Phi\colon(\mathbb{R}^{n+1}\times\mathbb{R},0)\longrightarrow(\mathbb{R}^{n+1}\times\mathbb{R},0)$ of the form $\Phi(u,t)=(\phi(u,t),\varphi(t))$ and a germ of diffeomorphism $\Psi\colon(\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R},\widetilde{\Pi}(z))\longrightarrow(\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R},\widetilde{\Pi}(z'))$ of the form $\Psi(s,x,t)=(q(s,x,t),\psi(x,t),\varphi(t))$, such that $\widetilde{\Pi}\circ K=\Psi\circ\widetilde{\Pi}$ and $K\circ\ell_{(\mu,F)}=\ell_{(\mu',F')}\circ\Phi$.

The notion of *stability* of one-parameter complete Legendrian unfoldings with respect to one-parameter SP^+-Legendrian equivalence (respectively, one-parameter SP-Legendrian equivalence and P-Legendrian equivalence) is analogous to the usual notion of the stability of Legendrian immersion germs with respect to Legendrian equivalence. (cf. [1, Part *III*]).

On the other hand, we can interpret the above equivalence relation in terms of generating families. Let $\widetilde{G}$, $\widetilde{G}'\colon(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R},0)\longrightarrow(\mathbb{R},0)$ be generating families of one-parameter Legendrian unfoldings of general Clairaut type, where $\widetilde{G}(s,x,t,y)=G(s,x,t)-y$, $\widetilde{G}'(s,x,t,y)=G'(s,x,t)-y$. We say that $\widetilde{G}$ and $\widetilde{G}'$ are *one-parameter P-$\mathcal{C}^+$-equivalent* (respectively, *one-parameter P-$\mathcal{C}$-equivalent*) if there exists a germ of diffeomorphism $\Phi\colon(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R},0)\longrightarrow(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R},0)$ of the form

$$\Phi(s,x,t,y)=(s+\alpha(x,t,y),\phi_1(x,t,y),\varphi(t),\phi_2(x,t,y))$$

(respectively,

$$\Phi(s,x,t,y)=(s,\phi_1(x,t,y),\varphi(t),\phi_2(x,t,y)))$$

such that $\langle\widetilde{G}\circ\Phi\rangle_{\mathcal{E}_{(s,x,t,y)}}=\langle\widetilde{G}'\rangle_{\mathcal{E}_{(s,x,t,y)}}$ where $\langle\widetilde{G}'\rangle_{\mathcal{E}_{(s,x,t,y)}}$ is the ideal generated by G' in $\mathcal{E}_{(s,x,t,y)}$. We say that $\widetilde{G}(s,x,t,y)$ is a P-$\mathcal{C}^+$ (respectively, P-$\mathcal{C}$)-*versal deformation* of $g=\widetilde{G}|_{t=0}$ if

$$\mathcal{E}_{(s,x,y)}=\left\langle\frac{\partial g}{\partial s}\right\rangle_{\mathcal{E}_{(x,y)}}+\langle g\rangle_{\mathcal{E}_{(s,x,y)}}+\left\langle\frac{\partial g}{\partial x_1},\dots,\frac{\partial g}{\partial x_n},1\right\rangle_{\mathcal{E}_{(x,y)}}+\left\langle\left.\frac{\partial G}{\partial t}\right|_{t=0}\right\rangle_{\mathbb{R}}$$

(respectively,

$$\mathcal{E}_{(s,x,y)} = \langle g\rangle_{\mathcal{E}_{(s,x,y)}} + \left\langle \frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n}, 1\right\rangle_{\mathcal{E}_{(x,y)}} + \left\langle \left.\frac{\partial G}{\partial t}\right|_{t=0}\right\rangle_{\mathbb{R}}\Bigg).$$

We also say that $\widetilde{G}$ and $\widetilde{G}'$ are *t-P-$\mathcal{K}$-equivalent* if there exists a germ of diffeomorphism $\Phi\colon (\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R},0) \longrightarrow (\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R},0)$ of the form

$$\Phi(s,x,t,y) = \big(q(s,x,t,y), \phi_1(x,t,y), \varphi(t), \phi_2(x,t,y)\big)$$

such that $\langle \widetilde{G}\circ\Phi\rangle_{\mathcal{E}_{(s,x,t,y)}} = \langle \widetilde{G}'\rangle_{\mathcal{E}_{(s,x,t,y)}}$ and $\widetilde{G}(s,x,t,y)$ is a *P-$\mathcal{K}$-versal deformation* of $g = \widetilde{G}|_{t=0}$ if

$$\mathcal{E}_{(s,x,y)} = \left\langle \frac{\partial g}{\partial s}, g\right\rangle_{\mathcal{E}_{(s,x,y)}} + \left\langle \frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n}, 1\right\rangle_{\mathcal{E}_{(x,y)}} + \left\langle \left.\frac{\partial G}{\partial t}\right|_{t=0}\right\rangle_{\mathbb{R}}.$$

By the similar arguments like as those of [1, Theorems 20.8 and 21.4], we can show the following:

Theorem 3.2 *Let $\widetilde{G}$, $\widetilde{G}'\colon (\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R},0) \longrightarrow (\mathbb{R},0)$ be generating families of one-parameter Legendrian unfoldings of general Clairaut type $\mathcal{L}_G$, $\mathcal{L}_{G'}$ respectively. Then*

(1) $\mathcal{L}_G$ and $\mathcal{L}_{G'}$ are one-parameter SP^+ (respectively, one-parameter SP)-Legendrian equivalent if and only if $\widetilde{G}$ and $\widetilde{G}'$ are one-parameter P-$\mathcal{C}^+$ (respectively, one-parameter P-$\mathcal{C}$)-equivalent.

(2) $\mathcal{L}_G$ is a one-parameter SP^+ (respectively, one-parameter SP)-Legendrian stable if and only if $\widetilde{G}$ is a P-$\mathcal{C}^+$ (respectively, P-$\mathcal{C}$)-versal deformation of $g = \widetilde{G}|_{t=0}$.

(3) $\mathcal{L}_G$ is a P-Legendrian stable if and only if $\widetilde{G}$ is a P-$\mathcal{K}$-versal deformation of $g = \widetilde{G}|_{t=0}$.

For each function germ $g\colon (\mathbb{R}^{n+2},0) \longrightarrow (\mathbb{R},0)$, we set

$$P\text{-}\mathcal{C}^+\text{-}\mathrm{cod}(g) = \dim_{\mathbb{R}} \mathcal{E}_{(s,x,y)} \Bigg/ \left(\left\langle \frac{\partial g}{\partial s}\right\rangle_{\mathcal{E}_{(x,y)}} + \langle g\rangle_{\mathcal{E}_{(s,x,y)}} + \left\langle \frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n}, 1\right\rangle_{\mathcal{E}_{(x,y)}} \right).$$

By the definition, P-$\mathcal{C}^+$-$\mathrm{cod}(g) \leq 1$ then $\mathcal{C}^+$-$\mathrm{cod}(g_0) \leq n+2$ (cf. §5

Appendix), where $g_0 = g|_{\mathbb{R}\times 0}$. And we set

$$P\text{-}\mathcal{K}\text{-}\operatorname{cod}(g) = \dim_{\mathbb{R}} \mathcal{E}_{(s,x,y)} \Big/ \left(\left\langle \frac{\partial g}{\partial s}, g \right\rangle_{\mathcal{E}_{(s,x,y)}} + \left\langle \frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_n}, 1 \right\rangle_{\mathcal{E}_{(x,y)}} \right).$$

Since $g_0 = g|_{\mathbb{R}\times 0}$ is a function germ of one-variable, we have $\langle dg_0/ds\rangle_{\mathbb{R}} + \langle g_0\rangle_{\mathcal{E}_s} = \langle dg_0/ds, g_0\rangle_{\mathcal{E}_s}$. Hence if g satisfy the condition $P\text{-}\mathcal{K}\text{-}\operatorname{cod}(g) \leq 1$, then $\mathcal{C}^+\text{-}\operatorname{cod}(g_0) \leq n+2$.

4. Proof of Theorem 1.1

The set of P-Legendrian stable one-parameter complete unfoldings is an open and dense subset in $L_R(U \times V, PT^*(\mathbb{R}\times\mathbb{R}^{n+1}\times\mathbb{R}))$. Therefore by Theorem 2.3, it give a classification of P-Legendrian stable one-parameter complete Legendrian unfoldings under the one-parameter SP^+-Legendrian equivalence (or, SP-Legendrian equivalent). Let (μ, f) be a one-parameter family of holonomic system of general Clairaut type such that the corresponding one-parameter complete Legendrian unfolding $\ell_{(\mu,F)}$ is P-Legendrian stable.

We also consider a generic condition of generalized phase family G. Let $J^{n+3}(n+2,1)$ be the set of $(n+3)$-jets of function $h\colon (\mathbb{R}^{n+2},0) \longrightarrow (\mathbb{R},0)$. We consider the following three algebraic subset of $J^{n+3}(n+2,1)$:

$$\Sigma_1 = \left\{ j^{n+3}h(0) \;\middle|\; \frac{\partial h}{\partial s}(0) = \cdots = \frac{\partial^{n+2} h}{\partial s^{n+2}}(0) = \frac{\partial^{n+3} h}{\partial s^{n+3}}(0)\, \frac{\partial^2 h}{\partial s \partial x_1}(0) \cdots \frac{\partial^{n+1} h}{\partial s^n \partial x_n}(0)\, \frac{\partial^{n+2} h}{\partial s^{n+1}\partial t}(0) = 0 \right\}$$

$$\Sigma_2 = \left\{ j^{n+3}h(0) \;\middle|\; \frac{\partial h}{\partial s}(0) = \frac{\partial^2 h}{\partial s \partial x_1}(0) = \cdots = \frac{\partial^2 h}{\partial s \partial x_n}(0) = \frac{\partial^3 h}{\partial s^3}(0) = \frac{\partial^2 h}{\partial s^2}(0)\, \frac{\partial^3 h}{\partial s \partial x_1^2}(0) \cdots \frac{\partial^3 h}{\partial s \partial x_n^2}(0)\, \frac{\partial^2 h}{\partial s \partial t}(0) = 0 \right\}$$

$$\Sigma_3 = \left\{ j^{n+3}h(0) \;\middle|\; \frac{\partial h}{\partial s}(0) = \frac{\partial^2 h}{\partial s \partial x_1}(0) = \cdots = \frac{\partial^2 h}{\partial s \partial x_n}(0) = \frac{\partial^2 h}{\partial s^2}(0) = \frac{\partial^3 h}{\partial s^3}(0)\, \frac{\partial^3 h}{\partial s \partial x_1^2}(0) \cdots \frac{\partial^3 h}{\partial s \partial x_n^2}(0)\, \frac{\partial^2 h}{\partial s \partial t}(0) = 0 \right\}$$

We consider the union $W = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$, then it is also an algebraic subset of $J^{n+3}(n+2,1)$. We can stratify the algebraic set W by submanifolds whose codimensions are at least $n+3$ in $J^{n+3}(n+2,1)$. By Thom's jet transversality theorem, $j^{n+3}G(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R})\cap(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\times W)=\emptyset$ for a generic function G. Therefore we might assume that $\widetilde{G}$ is satisfied above the condition and the corresponding one-parameter complete Legendrian unfolding $\ell_{(\mu,F)}$ is P-Legendrian stable.

By the assumption and Theorem 3.2, the generating family $\widetilde{G}$ of $\ell_{(\mu,F)}$ is a P-$\mathcal{K}$-versal deformation of $g=\widetilde{G}|_{t=0}$, then $\widetilde{G}$ is a $\mathcal{K}$-versal deformation of $g_0=g|_{\mathbb{R}\times 0}$. We remark that $\widetilde{G}$ is a $\mathcal{K}$-versal deformation of g_0 if and only if $\widetilde{G}$ is a $\mathcal{C}^+$-versal deformation of g_0. Since Lemma 5.1 and Theorem 5.2 in §5 Appendix, $\widetilde{G}$ is P-$\mathcal{C}^+$-equivalent to one of germs in the following list:

$$(A_\ell)\ (1\le\ell\le n+1):\quad s^\ell+\sum_{i=0}^{\ell-1}u_{i+1}s^i+\sum_{j=\ell+1}^{n+2}u_j,$$

$$(A_{n+2}):\quad s^{n+2}+\sum_{i=0}^{n+1}u_{i+1}s^i,$$

$$(\widetilde{A}_{n+3}):\quad s^{n+3}+\sum_{i=0}^{n+1}u_{i+1}s^i,$$

where $(s,u_1,\dots,u_{n+2})\in(\mathbb{R}\times\mathbb{R}^{n+2},0)$. We would like to classify these germs by the one-parameter P-$\mathcal{C}^+$-equivalence. By the above normal forms, there exists a diffeomorphism germ $\phi\colon(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R},0)\longrightarrow(\mathbb{R}^{n+2},0)$ such that $\widetilde{G}$ is one-parameter P-$\mathcal{C}^+$-equivalent the following list:

$$(A'_\ell)\ (1\le\ell\le n+1):\quad s^\ell+\sum_{i=0}^{\ell-1}u_{i+1}(x,t,y)s^i+\sum_{j=\ell+1}^{n+2}u_j(x,t,y),$$

$$(A'_{n+2}):\quad s^{n+2}+\sum_{i=0}^{n+1}u_{i+1}(x,t,y)s^i,$$

$$(\widetilde{A'}_{n+3}):\quad s^{n+3}+\sum_{i=0}^{n+1}u_{i+1}(x,t,y)s^i.$$

Since $\widetilde{G}$ has the form $G-y$, we assume that $(\partial u_1/\partial y)(0)\neq 0$. Furthermore, we perform local coordinate change, so that $u_1(x,t,y)=-y$, $u_i(x,t,y)=u_i(x,t)$ $(i=2,\dots,n+2)$. Hence we classify these germs by the one-parameter P-$\mathcal{C}^+$-equivalence under the condition P-$\mathcal{K}$-$\operatorname{cod}(g)\le 1$

where $g = \widetilde{G}|_{t=0}$ and satisfy the transversality conditions.

By transversality conditions, generalized phase family G satisfies one of the following condition for generic $\ell_{(\mu,F)}$:

$$(\alpha_1):\quad \frac{\partial G}{\partial s}(0) \neq 0,$$

$$(\alpha_i)\ (2 \leq i \leq n+2):\quad \frac{\partial G}{\partial s}(0) = \cdots = \frac{\partial^{i-1} G}{\partial s^{i-1}}(0) = 0,$$
$$\frac{\partial^i G}{\partial s^i}(0) \neq 0,\ \frac{\partial^2 G}{\partial s \partial x_j}(0) \neq 0,$$

$$(\alpha_{n+3}):\quad \frac{\partial G}{\partial s}(0) = \cdots = \frac{\partial^{n+2} G}{\partial s^{n+2}}(0) = 0,$$
$$\frac{\partial^{n+3} G}{\partial s^{n+3}}(0)\,\frac{\partial^2 G}{\partial s \partial x_1}(0) \cdots \frac{\partial^{n+1} G}{\partial s^n \partial x_n}(0)\,\frac{\partial^{n+2} G}{\partial s^{n+1} \partial t}(0) \neq 0,$$

$$(\beta):\quad \frac{\partial G}{\partial s}(0) = \frac{\partial^2 G}{\partial s \partial x_k}(0) = 0,$$
$$\frac{\partial^2 h}{\partial s^2}(0)\,\frac{\partial^3 h}{\partial s \partial x_1^2}(0) \cdots \frac{\partial^3 h}{\partial s \partial x_n^2}(0)\,\frac{\partial^2 h}{\partial s \partial t}(0) \neq 0,$$

$$(\gamma):\quad \frac{\partial G}{\partial s}(0) = \frac{\partial^2 G}{\partial s^2}(0) = \frac{\partial^2 G}{\partial s \partial x_k}(0) = 0,$$
$$\frac{\partial^3 h}{\partial s^3}(0)\,\frac{\partial^3 h}{\partial s \partial x_1^2}(0) \cdots \frac{\partial^3 h}{\partial s \partial x_n^2}(0)\,\frac{\partial^2 h}{\partial s \partial t}(0) = 0,$$

where j is some integer from 1 to n and k is any integer from 1 to n.

First, we consider the condition (α_i) $(1 \leq i \leq n+3)$. In order to classify germs of (A'_ℓ) $(1 \leq \ell \leq n+2)$ and $(\widetilde{A}'_{n+3})$ with respect to one-parameter P-$\mathcal{C}^+$-equaivalence (one-parameter P-$\mathcal{C}$-equivalence) under the condition (α_i), we consider the t-P-$\mathcal{K}$-equivalence. In this case, we remark that $j^1_1 G_t$ is an immersion germ by the condition (α_i). The corresponding one-parameter family of holonomic system of general Clairaut type is Clairaut type. We can apply the analogous method of the proof for Theorem 1.1 in [18].

Since the P-$\mathcal{C}^+$-equivalence is a stronger equivalence relation than the P-$\mathcal{K}$-equivalence, $\widetilde{G}$ is P-$\mathcal{K}$-equivalent to one of germs in the above list.

We now classify these germs by the t-P-$\mathcal{K}$-equivalence under the condition that P-$\mathcal{K}$-$\operatorname{cod} \leq 1$. We use the result of [2, Theorem 1.2] such that $j^1_1 G_t$ is an immersion germ, i.e.,

$$\operatorname{rank}\left(\frac{\partial G}{\partial s}\ \frac{\partial^2 G}{\partial s\partial x_1}\cdots\frac{\partial^2 G}{\partial s\partial x_n}\right)=1,$$

then such a germ is t-P-$\mathcal{K}$-equivalent to one of the following germ:

$$(a_\ell)\ (1\le \ell\le n+1):\quad s^\ell - y + t + \sum_{i=1}^{\ell-1} x_i s^i + \sum_{j=\ell}^{n} x_j,$$

$$(a_{n+2}):\quad s^{n+2} - y + \sum_{i=1}^{n} x_i s^i + t s^{n+1},$$

$$(\widetilde{a}_{n+3}):\quad s^{n+3} - y + \sum_{i=1}^{n} x_i s^i + t s^{n+1}.$$

Now $\widetilde{G}$ is t-P-$\mathcal{K}$-equivalent to one of germs in the above list, then g is P-$\mathcal{K}$-equivalent to one of the following germ:

$$(a'_\ell)\ (1\le \ell\le n+1):\quad s^\ell - y + \sum_{i=1}^{\ell-1} x_i s^i + \sum_{j=\ell}^{n} x_j,$$

$$(a'_{n+2}):\quad s^{n+2} - y + \sum_{i=1}^{n} x_i s^n,$$

$$(\widetilde{a}'_{n+3}):\quad s^{n+3} - y + \sum_{i=1}^{n} x_i s^n.$$

For (a'_ℓ) $(1\le \ell\le n+1)$ and (a'_{n+2}), g is a $\mathcal{K}$-versal deformation of g_0, so g is a $\mathcal{C}^+$-versal deformation of g_0. We remark that g_0 and s^i $(i=1,\dots,n+2)$ are $\mathcal{K}$-equivalent, then g_0 and s^i $(i=1,\dots,n+2)$ are $\mathcal{C}$-equivalent. By Theorem 5.2, g is P-$\mathcal{C}^+$-equivalent to one of the following germ:

$$(a'_\ell)\ (1\le \ell\le n+1):\quad s^\ell - y + \sum_{i=1}^{\ell-1} x_i s^i + \sum_{j=\ell}^{n} x_j,$$

$$(a'_{n+2}):\quad s^{n+2} - y + \sum_{i=1}^{n} x_i s^i.$$

These germ satisfy the condition that P-$\mathcal{C}^+$-$\operatorname{cod}=0$, so that we have P-$\mathcal{C}^+$-$\operatorname{cod}(g)=0$. By definition, $\widetilde{G}$ is a P-$\mathcal{C}^+$-versal deformation of g. By the uniqueness result of P-$\mathcal{C}^+$-versal deformation, $\widetilde{G}$ is one-parameter

P-$\mathcal{C}^+$-equivalent to one of the following germs:

$$(a_\ell)\ (1 \le \ell \le n+1):\quad s^\ell - y + \sum_{i=1}^{\ell-1} x_i s^i + \sum_{j=\ell}^{n} x_j + t,$$

$$(\widetilde{a}_{n+2}):\qquad s^{n+2} - y + \sum_{i=1}^{n} x_i s^i + t.$$

On the other hand, we assume that g is a t-P-$\mathcal{K}$-equivalent to the germ $(\widetilde{a}_{n+3})$: $s^{n+3} - y + \sum_{i=1}^n x_i s^i$. By the previous arguments, we consider the germ of the form $\widetilde{G}(s,x,t,y) = s^{n+3} - y + \sum_{i=1}^{n+1} u_i(x,t)s^i$. We remark that the P-$\mathcal{K}$-codimension of $s^{n+3} - y + \sum_{i=1}^n x_i s^i$ is 1, so that P-$\mathcal{K}$-$\mathrm{cod}(g) = 1$. We use the transversality condition (α_{n+3}):

$$\frac{\partial^2 h}{\partial s \partial x_1}(0) \cdots \frac{\partial^{n+1} h}{\partial s^n \partial x_n}(0)\, \frac{\partial^{n+2} h}{\partial s^{n+1} \partial t}(0) \neq 0.$$

Therefore we have germ of a diffeomorphism defined by $X_i = u_i(x,t)$ $(i = 1,\dots,n)$ and $T = t$. It follows that the germ $s^{n+3} - y + \sum_{i=1}^n u_i(x,t)s^i$ is one-parameter P-$\mathcal{C}^+$-equivalent to the germ

$$(\widetilde{a}''_{n+3})\quad s^{n+3} - y + \sum_{i=1}^{n} x_i s^i + \beta(x,t)s^{n+1},$$

where $\beta \in \mathcal{E}_{(x_1,\dots,x_n,t)}$ and $(\partial\beta/\partial t)(0) \neq 0$.

Second, we consider the condition (β). It follows that $\widetilde{G}$ is a one-parameter P-$\mathcal{C}$-equivalent to $s^2 - y + (t + v(x_1,\dots,x_n) + w(x_1,\dots,x_n,t))s$, where $v(x_1,\dots,x_n)$ is a non-degenerate function and $w(x_1,\dots,x_n,t)$ is a degenerate function with respect to $(x_1,\dots,x_n)$ at $t = 0$. We put $h(x,t) = v(x_1,\dots,x_n) + w(x_1,\dots,x_n,t)$, then $h(x,0)$ is $\mathcal{R}$-equivalent to $v(x_1,\dots,x_n)$. A non-degenerate function $v(x_1,\dots,x_n)$ is $\mathcal{R}$-equivalent to $x_1^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_n^2$ for some integer k $(0 \le k \le n)$ by Morse's Lemma (cf. [15]). By the uniqueness result of $\mathcal{R}^+$-versal deformation (cf. [7, Corollary 9.9]), $h(x,t)$ is P-$\mathcal{R}^+$-equivalent to $x_1^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_n^2$. Hence $\widetilde{G}$ is one-parameter P-$\mathcal{C}$-equivalent to $s^2 - y + (t + \alpha(t) + x_1^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_n^2)s$, where $\alpha \in \mathfrak{M}_t$. Since the condition $(\partial^2 G/\partial s \partial t)(0) \neq 0$, $\widetilde{G}$ is one-parameter P-$\mathcal{C}$-equivalent to

$$(b_k):\quad s^2 - y + (t + x_1^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_n^2)s.$$

Finally, we consider the condition (γ). It follows that $\widetilde{G}$ is one-parameter

P-$\mathcal{C}^+$-equivalent to (A_3'). By the condition (γ), $\widetilde{G}$ is one-parameter P-$\mathcal{C}^+$-equivalent to $s^3 - y + (t + v(x_1, \dots, x_n) + w(x_1, \dots, x_n, t))s$, where $v(x_1, \dots, x_n)$ is a non-degenerate function and $w(x_1, \dots, x_n, t)$ is a degenerate function with respect to $(x_1, \dots, x_n)$ at $t = 0$. We also consider the same way as (β), the function germ $v(x_1, \dots, x_n) + w(x_1, \dots, x_n, t)$ is P-$\mathcal{R}^+$-equivalent to $x_1^2 + \dots + x_k^2 - x_{k+1}^2 - \dots - x_n^2$ for some integer k ($0 \le k \le n$). Therefore $\widetilde{G}$ is one-parameter P-$\mathcal{C}^+$-equivalent to

$$(c_k): \quad s^3 - y + (t + x_1^2 + \dots + x_k^2 - x_{k+1}^2 - \dots - x_n^2)s.$$

We now detect the corresponding normal forms of one-parameter integral diagrams as follows:

For the case (a_ℓ) $(1 \le \ell \le n+1)$, we can choose

$$G(s, x_1, \dots, x_n, t) = s^\ell + \sum_{i=1}^{\ell-1} x_i s^i + \sum_{j=\ell}^{n} x_j + t,$$

as a generalized phase family, so that

$$\mathcal{L}_G = \Bigg(s, x_1, \dots, x_n, t, s^\ell + \sum_{i=1}^{\ell-1} x_i s^i + \sum_{j=1}^{n} x_j + t, \\ \ell s^{\ell-1} + \sum_{i=1}^{\ell-1} i x_i s^i, s, \dots, s^{\ell-1}, 1, \dots, 1 \Bigg).$$

Then we can easily calculate that the corresponding one-parameter unfolding of integral diagram is strictly equivalent to

$$(DA_\ell)\ (1 \le \ell \le n+1): \quad \mu = u_{n+1}, \\ G = \left(u_1, \dots, u_n, u_{n+1}^\ell + \sum_{i=1}^{\ell-1} u_i u_{n+1}^i, t \right).$$

This is the normal form in the case of DA_ℓ $(1 \le \ell \le n+1)$. If $\ell = 2$, we have

$$\mu = u_{n+1}, \quad G = (u_1, \dots, u_n, u_{n+1}^2 + u_1 u_{n+1}, t).$$

We define a local coordinate transformation by $U_i = u_i$ $(i = 1, \dots, n)$,

$U_{n+1} = u_{n+1} + (1/2)u_1$, then (μ, G) is strictly equivalent to

$$\mu = u_{n+1} - \frac{1}{2}\, u_1, \quad G = \left(u_1, \ldots, u_n, u_{n+1}^2 - \frac{1}{4}\, u_1^2, t\right).$$

We also apply a local coordinate transformation which is defined by

$$X_i = x_i \quad (i = 1, \ldots, n), \qquad Y = y + \frac{1}{4}\, x_1^2,$$

then we have the normal form DA_2 in Theorem 1.1.

For the case $(\widetilde{a}_{n+2})$, we can also choose

$$G(s, x_1, \ldots, x_n, t) = s^{n+2} + \sum_{i=1}^{n} x_i s^i + t$$

as a generalized phase family, so that

$$\mathcal{L}_G = \Bigg(s, x_1, \ldots, x_n, t, s^{n+2} + \sum_{i=1}^{n} x_i s^i + t, \\ (n+2)s^{n+1} + \sum_{i=1}^{n} i x_i s^{i-1}, s, \ldots, s^n, 1\Bigg).$$

Then we can easily calculate that the corresponding one-parameter unfolding of integral diagram is one-parameter $\mathcal{R}^+$-equivalent to

$$\widetilde{DA}_{n+2}\colon \quad \mu = u_{n+1}, \quad G = \left(u_1, \ldots, u_n, u_{n+1}^{n+2} + \sum_{i=1}^{n} u_i s^i, t\right).$$

For the case $(\widetilde{a}''_{n+3})$, we can also choose

$$G(s, x_1, \ldots, x_n, t) = s^{n+3} + \sum_{i=1}^{n} x_i s^i + \beta(x_1, \ldots, x_n, t)s^{n+1}$$

as a generalized phase family and calculate one-parameter complete Legendrian unfolding like as above. Then we can easily calculate that the corresponding one-parameter unfolding of integral diagram is one-parameter $\mathcal{R}^+$-equivalent to

$$\widetilde{DA}_{n+3}\colon \quad \mu = u_{n+1}, \\ G = \left(u_1, \ldots, u_n, u_{n+1}^{n+3} + \sum_{i=1}^{n} u_i u_{n+1}^i + \beta(u_1, \ldots, u_n, t)u_{n+1}^{n+1}, t\right).$$

For the cases (b_k) and (c_k), we also apply the same arguments as the above process, we have the normal forms DA_2^k and $\widetilde{DA}_3^k$ in the list of Theorem 1.1.

Since each generalized phase family for $(\widetilde{a}_{n+2})$, $(\widetilde{a}''_{n+3})$ and (c_k) are one-parameter P-$\mathcal{C}^+$-equivalent to the normal form but there are not one-parameter P-$\mathcal{C}$-equivalent, the corresponding one-parameter integral diagram is strictly equivalent to the normal form $\widetilde{DA}_{n+2}$, $\widetilde{DA}_{n+3}$ and $\widetilde{DA}_3^k$ in Theorem 1.1.

This completes the proof of Theorem 1.1. □

Remark In the above proof, we can show that the P-$\mathcal{C}^+$-codimension of the germ: $s^{n+3} - y + \sum_{i=1}^n x_i s^i$ is infinite. This means that there appears the second functional moduli in the normal form $\widetilde{DA}_{n+3}$ in Theorem 1.1.

5. Appendix

In this section we now give a quick review of the theory of unfoldings of function germs [1, 7, 9, 12, 13].

Let $f,\ f'\colon (\mathbb{R},0) \longrightarrow (\mathbb{R},0)$ be germs of function, $F,\ F'\colon (\mathbb{R}\times\mathbb{R}^n,0) \longrightarrow (\mathbb{R},0)$ be unfoldings of $f,\ f'$ respectively and $\widetilde{F},\ \widetilde{F}'\colon (\mathbb{R}\times\mathbb{R}^n\times\mathbb{R},0) \longrightarrow (\mathbb{R},0)$ be function germs given by $\widetilde{F}(s,x,y) = F(s,x) - y$, $\widetilde{F}'(s,x,y) = F'(s,x) - y$. We say that $\widetilde{F}$ and $\widetilde{F}'$ are P-$\mathcal{C}^+$-equivalent (respectively, P-$\mathcal{C}$-equivalent) if there exists a germ of diffeomorphism $\Phi\colon (\mathbb{R}\times\mathbb{R}^n\times\mathbb{R},0) \longrightarrow (\mathbb{R}\times\mathbb{R}^n\times\mathbb{R},0)$ of the form $\Phi(s,x,y) = (s+\alpha(x,y), \phi_1(x,y), \phi_2(x,y))$ (respectively, $\Phi(s,x,y) = (s, \phi_1(x,y), \phi_2(x,y))$) such that $\langle \widetilde{F}\circ\Phi\rangle_{\mathcal{E}_{(s,x,y)}} = \langle \widetilde{F}'\rangle_{\mathcal{E}_{(s,x,y)}}$ where, $\langle \widetilde{F}'\rangle_{\mathcal{E}_{(s,x,y)}}$ is the ideal generated by $\widetilde{F}'$ in $\mathcal{E}_{(s,x,y)}$. We also say that $\widetilde{F}(s,x,y)$ is $\mathcal{C}^+$ (respectively, $\mathcal{C}$ and $\mathcal{K}$)-versal deformation of $f = F|_{\mathbb{R}\times 0}$ if

$$\mathcal{E}_s = \left\langle \frac{df}{ds} \right\rangle_{\mathbb{R}} + \langle f\rangle_{\mathcal{E}_s} + \left\langle \left.\frac{\partial F}{\partial x_1}\right|_{\mathbb{R}\times 0}, \dots, \left.\frac{\partial F}{\partial x_n}\right|_{\mathbb{R}\times 0}, 1 \right\rangle_{\mathbb{R}}$$

(respectively,

$$\mathcal{E}_s = \langle f\rangle_{\mathcal{E}_s} + \left\langle \left.\frac{\partial F}{\partial x_1}\right|_{\mathbb{R}\times 0}, \dots, \left.\frac{\partial F}{\partial x_n}\right|_{\mathbb{R}\times 0}, 1 \right\rangle_{\mathbb{R}}$$

$$\text{and } \mathcal{E}_s = \left\langle \frac{df}{ds}, f \right\rangle_{\mathcal{E}_s} + \left\langle \left.\frac{\partial F}{\partial x_1}\right|_{\mathbb{R}\times 0}, \dots, \left.\frac{\partial F}{\partial x_n}\right|_{\mathbb{R}\times 0}, 1 \right\rangle_{\mathbb{R}} \Bigg).$$

Let f and $f'\colon (\mathbb{R},0) \longrightarrow (\mathbb{R},0)$ be germs of function. We say that f and f' are *$\mathcal{C}$-equivalent* if and only if $\langle f\rangle_{\mathcal{E}_s} = \langle f'\rangle_{\mathcal{E}_s}$. Then the classification theory of function germs by the $\mathcal{C}$-equivalence is quite useful for our purpose. For each function germ $f\colon (\mathbb{R},0) \longrightarrow (\mathbb{R},0)$, we set

$$\begin{aligned}
&\mathcal{C}\text{-}\operatorname{cod}(f) = \dim_{\mathbb{R}} \mathcal{E}_s/\langle f\rangle_{\mathcal{E}_s},\\
&\mathcal{C}^{+}\text{-}\operatorname{cod}(f) = \dim_{\mathbb{R}} \mathcal{E}_s\bigg/\left(\langle f\rangle_{\mathcal{E}_s} + \left\langle \frac{df}{ds}\right\rangle_{\mathbb{R}}\right),\\
&\mathcal{K}\text{-}\operatorname{cod}(f) = \dim_{\mathbb{R}} \mathcal{E}_s\bigg/\left(\langle f\rangle_{\mathcal{E}_s} + \left\langle \frac{df}{ds}\right\rangle_{\mathcal{E}_s}\right).
\end{aligned}$$

Then we have the following well-known classification (cf. [9]).

Lemma 5.1 *Let $f\colon (\mathbb{R},0) \longrightarrow (\mathbb{R},0)$ be a function germ with $\mathcal{K}\text{-}\operatorname{cod}(f) < \infty$. Then f is $\mathcal{C}$-equivalent to the map germ $s^{\ell+1}$ for some $\ell \in \mathbb{N}$.*

By a direct calculation, we have

$$\mathcal{C}\text{-}\operatorname{cod}(s^{\ell+1}) = \ell+1, \quad \mathcal{C}^{+}\text{-}\operatorname{cod}(s^{\ell+1}) = \ell.$$

Thus we can easily determine $\mathcal{C}$ (respectively, $\mathcal{C}^{+}$)-versal deformations of the above germs by using the usual method:

$$\begin{aligned}
\mathcal{C}\text{-versal deformation:}\quad & s^{\ell+1} + \sum_{i=0}^{\ell} u_{i+1}s^{i},\\
\mathcal{C}^{+}\text{-versal deformation:}\quad & s^{\ell+1} + \sum_{i=0}^{\ell-1} u_{i+1}s^{i}.
\end{aligned}$$

The following theorem is useful and important for our purpose (cf. [7]).

Theorem 5.2 *Let $\widetilde{F}$ and $\widetilde{F}'\colon (\mathbb{R}\times\mathbb{R}^n\times\mathbb{R},0) \longrightarrow (\mathbb{R},0)$ be function germs such that $\widetilde{F}$ and $\widetilde{F}'$ are $\mathcal{C}^{+}$ (respectively, $\mathcal{C}$)-versal deformations of $f = F|_{\mathbb{R}\times 0}$ and $f' = F'|_{\mathbb{R}\times 0}$. Then $\widetilde{F}$ and $\widetilde{F}'$ are P-$\mathcal{C}^{+}$-equivalent (respectively, P-$\mathcal{C}$-equivalent) if and only if f and f' are $\mathcal{C}$-equivalent.*

Let function germ $\widetilde{F}(s,x,y)$ be a $\mathcal{C}^{+}$-versal deformation of $f = F|_{\mathbb{R}\times 0}$. By Lemma 5.1 and Theorem 5.2, $\widetilde{F}(s,x,y)$ is P-$\mathcal{C}^{+}$-equivalent to one of germs in the following list:

$$A_\ell \ (1 \le \ell \le n+1): \quad s^\ell + \sum_{i=1}^{\ell-1} x_i s^i + \sum_{j=\ell}^{n} x_j - y,$$

$$\widetilde{A}_{n+2}: \quad s^{n+2} + \sum_{i=1}^{n} x_i s^i - y.$$

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