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Scattering theory for the Zakharov system

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Abstract. We study the theory of scattering for the Zakharov system in space dimension 3. We prove in particular the existence of wave operators for that system with no size restriction on the data in larger spaces and for more general asymptotic states than were previously considered, and we determine convergence rates in time of solutions in the range of the wave operators to the solutions of the underlying linear system. We also consider the same system in space dimension 2, where we prove the existence of wave operators for small Schrödinger data in the special case of vanishing asymptotic data for the wave field.

Key words: scattering theory, Zakharov system.

1. Introduction

This paper is devoted to the theory of scattering and more precisely to the construction of wave operators for the Zakharov system $(\mathrm{Z})_n$ in space dimension $n = 3$ and 2 (in that order), namely

$$\begin{cases} i\partial_t u = -\dfrac{1}{2}\Delta u + Au \\ \square A = \Delta |u|^2 \end{cases} \tag{1.1}$$

where u and A are respectively a complex valued and a real valued function defined in space time $\mathbb{R}^{n+1}$, Δ is the Laplacian in $\mathbb{R}^n$ and $\square = \partial_t^2 - \Delta$ is the d'Alembertian in $\mathbb{R}^{n+1}$. The $(Z)_3$ system is used in plasma physics to describe the Langmuir turbulence. The function u is the slowly varying complex amplitude of the rapidly oscillating electric field and A is the deviation of the ion density from its average value [27]. In this paper we use the notation (u, A) for those variables instead of the more common (E, n) in order to allow for a better contact with the existing literature on related nonlinear systems based on the Schrödinger equation, in particular with the Wave-Schrödinger system $(\mathrm{WS})_3$ and the Maxwell-Schrödinger sys-

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tem $(\mathrm{MS})_3$ in $\mathbb{R}^{3+1}$, and with the Klein-Gordon-Schrödinger system $(\mathrm{KGS})_2$ in $\mathbb{R}^{2+1}$ (see [13] for a review). The Zakharov system is Lagrangian and admits a number of formally conserved quantities, among which the L^2 norm of u and the energy

$$E(u, A) = \int dx \left\{ \frac{1}{2} \left(|\nabla u|^2 + |\omega^{-1} \partial_t A|^2 + |A|^2 \right) + A|u|^2 \right\} \qquad (1.2)$$

where $\omega = (-\Delta)^{1/2}$. The Cauchy problem for the $(\mathrm{Z})_n$ system has been extensively studied [1] [2] [3] [6] [15] [25] and is known to be globally well posed for $n = 2$, 3 in the energy space $X_e = H^1 \oplus L^2 \oplus \dot{H}^{-1}$ for $(u, A, \partial_t A)$.

A large amount of work has been devoted to the theory of scattering for non linear equations and systems related to the Schrödinger equation, in particular for non linear Schrödinger $(\mathrm{NLS})_n$ and Hartree $(\mathrm{R3})_n$ equations in $\mathbb{R}^{n+1}$ and for the above mentioned $(\mathrm{WS})_3$, $(\mathrm{MS})_3$, $(\mathrm{KGS})_2$ and $(\mathrm{Z})_3$ systems. As in the case of the linear Schrödinger equation, one must distinguish the short range case from the long range case. In the former case, ordinary wave operators are expected and in a number of cases proved to exist, describing solutions where the Schrödinger function behaves asymptotically like a solution of the free Schrödinger equation. In the latter case, ordinary wave operators do not exist and have to be replaced by modified wave operators including a suitable phase in their definition. In that respect, the $(\mathrm{WS})_3$ and $(\mathrm{MS})_3$ systems belong to the borderline (Coulomb) long range case, as does the $(\mathrm{R3})_n$ equation with $|x|^{-1}$ potential, the $(\mathrm{Z})_3$ system is short range, and the $(\mathrm{KGS})_2$ and $(\mathrm{Z})_2$ systems, although not really long range, exhibit some difficulties typical of the long range case.

The construction of (possibly modified) wave operators for the previous equations and systems in the long range cases has been tackled by two methods. The first one was initiated on the example of $(\mathrm{NLS})_1$ [14] and subsequently applied to the $(\mathrm{NLS})_n$ equation for $n = 2$, 3 and to the $(\mathrm{R3})_n$ equation for $n \geq 2$ [5], to the $(\mathrm{KGS})_2$ system [17] [18] [19] [20], to the $(\mathrm{WS})_3$ system [11] [21], to the $(\mathrm{MS})_3$ system [12] [22] [26] and to the $(\mathrm{Z})_3$ system [16] [23]. See [13] for a review. That method is rather direct, starting from the original equation or system. It will be sketched below on the example of the $(\mathrm{Z})_n$ system. In long range cases, it is restricted to the limiting Coulomb case and requires a smallness condition on the asymptotic state of the Schrödinger function. Early applications of the method required in addition a support condition on the Fourier transform of the Schrödinger asymptotic state and a smallness condition of the Klein-Gordon or Maxwell field in the case of the $(\mathrm{KGS})_2$ or $(\mathrm{MS})_3$ system respectively [17] [26]. A support condition was also required in the case of the $(\mathrm{Z})_3$ system when both the Schrödinger and the wave field are large, which is allowed by the fact that the $(\mathrm{Z})_3$ system is short range [16]. The support condition was subsequently removed for the $(\mathrm{KGS})_2$, $(\mathrm{MS})_3$ and $(\mathrm{Z})_3$ systems and the method was applied to the $(\mathrm{WS})_3$ system without a support condition, at the expense of adding a correction term to the Schrödinger asymptotic function [18] [21] [22] [23]. The smallness condition of the KG field was then removed for the $(\mathrm{KGS})_2$ system, first with and then without a support condition [19] [20]. All the previous papers on $(\mathrm{KGS})_2$, $(\mathrm{WS})_3$, $(\mathrm{MS})_3$ and $(\mathrm{Z})_3$ use spaces of fairly regular solutions, with at least H^2 regularity for the Schrödinger function. Finally the smallness condition of the wave or Maxwell field was removed for the $(\mathrm{WS})_3$ and $(\mathrm{MS})_3$ systems [11] [12]. Furthermore larger function spaces than previously considered are used in [11] [12], thereby allowing for more general asymptotic states.

In the present paper, we reconsider the same problem for the $(\mathrm{Z})_3$ and $(\mathrm{Z})_2$ systems in the framework of the previous method. We treat again the $(\mathrm{Z})_3$ system with no smallness condition on either field and no support condition. In the same spirit as in [11] [12], we use function spaces that are as large as possible, namely with regularity as low as possible, and with convergence in time as slow as possible. In particular we treat the problem with only a weak convergence in time of the solutions to their asymptotic form, namely $t^{-\lambda}$ with $\lambda > 1/4$. Under such a weak condition, neither a support condition nor a correction term for the asymptotic Schrödinger function is needed as long as $\lambda \leq 1/2$ and much weaker assumptions on the asymptotic state than previously considered can be accomodated. We also consider the case of more regular data but still more general than previously considered, where the use of a correction term yields a stronger convergence in time, namely $\lambda = 3/2$. We finally apply the method to the $(\mathrm{Z})_2$ system. Again no support condition is needed, but we need a smallness condition of the Schrödinger function and we can only treat the case where the asymptotic state of the wave field is zero.

For completeness and although we shall not make use of that fact in the present paper, we mention that the same problem for the Hartree equation and for the $(\mathrm{WS})_3$ and $(\mathrm{MS})_3$ system can also be treated by a more complex method where one first applies a phase-amplitude separation to the Schrödinger function. The main interest of that method is to remove the smallness condition on the Schrödinger function, and to go beyond the Coulomb limiting case for the Hartree equation. That method has been applied in particular to the $(\mathrm{WS})_3$ system and to the $(\mathrm{MS})_3$ system in a special case [7] [8] [9].

We now sketch briefly the method of construction of the modified wave operators initiated in [14]. That construction basically consists in solving the Cauchy problem for the system (1.1) with infinite initial time, namely in constructing solutions (u, A) with prescribed asymptotic behaviour at infinity in time. We restrict our attention to time going to $+\infty$. That asymptotic behaviour is imposed in the form of suitable approximate solutions (u_a, A_a) of the system (1.1). The approximate solutions are parametrized by data $(u_+, A_+, \dot{A}_+)$ which in the simplest cases are initial data at time zero for a simpler evolution. One then looks for exact solutions (u, A) of the system (1.1), the difference of which with the given asymptotic ones tends to zero at infinity in time in a suitable sense, more precisely, in suitable norms. The wave operator is then defined traditionally as the map $\Omega_+ \colon (u_+, A_+, \dot{A}_+) \to (u, A, \partial_t A)(0)$. However what really matters is the solution (u, A) in the neighborhood of infinity in time, namely in some interval $[T, \infty)$, and we shall restrict our attention to the construction of such solutions. Continuing such solutions down to $t = 0$ is a somewhat different question, connected with the global Cauchy problem at finite times, which we shall not touch here. That problem is well controlled for the $(\mathrm{Z})_n$ system for $n = 2$, 3.

The construction of solutions (u, A) with prescribed asymptotic behaviour (u_a, A_a) is performed in two steps.

Step 1. One looks for (u, A) in the form $(u, A) = (u_a + v, A_a + B)$. The system satisfied by (v, B) is

$$\begin{cases} i\partial_t v = -\dfrac{1}{2}\Delta v + Av + Bu_a - R_1 \\ \square B = \Delta(|v|^2 + 2\,\mathrm{Re}\,\overline{u}_a v) - R_2 \end{cases} \tag{1.3}$$

where the remainders R_1, R_2 are defined by

$$\begin{cases} R_1 = i\partial_t u_a + \dfrac{1}{2}\Delta u_a - A_a u_a \\ R_2 = \square A_a - \Delta|u_a|^2. \end{cases} \tag{1.4}$$

It is technically useful to consider also the partly linearized system for functions (v', B')

$$\begin{cases} i\partial_t v' = -\dfrac{1}{2}\Delta v' + Av' + Bu_a - R_1 \\ \square B' = \Delta(|v|^2 + 2\,\mathrm{Re}\,\overline{u}_a v) - R_2. \end{cases} \tag{1.5}$$

The first step of the method consists in solving the system (1.3) for (v, B), with (v, B) tending to zero at infinity in time in suitable norms, under assumptions on (u_a, A_a) of a general nature, the most important of which being decay assumptions on the remainders R_1 and R_2. That can be done as follows. One first solves the linearized system (1.5) for (v', B') with given (v, B) and initial data $(v', B')(t_0) = 0$ for some large finite t_0. One then takes the limit $t_0 \to \infty$ of that solution, thereby obtaining a solution (v', B') of (1.5) which tends to zero at infinity in time. That construction defines a map $\phi\colon (v, B) \to (v', B')$. One then shows by a contraction method that the map ϕ has a fixed point.

Step 2. The second step of the method consists in constructing approximate asymptotic solutions (u_a, A_a) satisfying the general estimates needed to perform Step 1. With the weak time decay allowed by our treatment of Step 1, and taking advantage of the fact that the $(\mathrm{Z})_3$ system is short range, one can take for (u_a, A_a) solutions of the free Schrödinger and wave equations in that case. One can also improve u_a by a correction term as in [23], thereby obtaining faster convergence rates for more regular asymptotic states. In the case of the $(\mathrm{Z})_2$ system, one can again take for u_a a solution of the free Schrödinger equation, but one is forced to take $A_a = 0$.

In order to state our results we introduce some notation. We denote by F the Fourier transform and by $\|\cdot\|_r$ the norm in $L^r \equiv L^r(\mathbb{R}^n)$, $1 \le r \le \infty$. For any nonnegative integer k and for $1 \le r \le \infty$, we denote by W_r^k the Sobolev spaces

$$W_r^k = \left\{ u : \|u; W_r^k\| = \sum_{\alpha\colon 0 \le |\alpha| \le k} \|\partial_x^\alpha u\|_r < \infty \right\}$$

where α is a multiindex, so that $H^k = W_2^k$. We shall need the weighted Sobolev spaces $H^{k,s}$ defined for k, $s \in \mathbb{R}$ by

$$H^{k,s} = \left\{ u : \|u; H^{k,s}\| = \|(1 + x^2)^{s/2}(1 - \Delta)^{k/2} u\|_2 < \infty \right\}$$

so that $H^k = H^{k,0}$. For any interval I, for any Banach space X and for any q, $1 \le q \le \infty$, we denote by $L^q(I,X)$ (resp. $L^q_{loc}(I,X)$) the space of L^q integrable (resp. locally L^q integrable) functions from I to X if $q < \infty$ and the space of measurable essentially bounded (resp. locally essentially bounded) functions from I to X if $q = \infty$. We shall occasionally use the notation

$$\|f; L^q(I, L^r)\| = \big\| \|f\|_r \big\|_q$$

when there is no ambiguity in the choice of the interval I. For any $h \in \mathcal{C}([1,\infty), \mathbb{R}^+)$, non increasing and tending to zero at infinity and for any interval $I \subset [1,\infty)$, we define the space

$$\begin{aligned} X(I) = \Big\{ (v,B) : (v,B) \in \mathcal{C}(I, H^2 \oplus H^1) \cap \mathcal{C}^1(I, L^2 \oplus L^2),\\ \|(v,B); X(I)\| \equiv \mathop{\mathrm{Sup}}_{t\in I} h(t)^{-1} \big(\|v(t); H^2\| + \|\partial_t v(t)\|_2 \\ + \|v; L^{8/n}(J, W^2_4)\| + \|\partial_t v; L^{8/n}(J, L^4)\| \\ + \|B(t); H^1\| + \|\partial_t B(t)\|_2 \big) < \infty \Big\} \end{aligned} \qquad (1.6)$$

where $J = [t,\infty) \cap I$, for $n = 2, 3$. Finally we denote by

$$u_0(t) = U(t)u_+ = \exp\left(i \left(\frac{t}{2} \right) \Delta \right) u_+, \qquad (1.7)$$

$$A_0(t) = \cos \omega t \, A_+ + \omega^{-1} \sin \omega t \, \dot{A}_+ \qquad (1.8)$$

the solutions of the free Schrödinger and wave equations with initial data u_+ and $(A_+, \dot{A}_+)$ at time zero.

We can now state our results. We first state the result obtained for the $(\mathrm{Z})_3$ system by using only the simplest asymptotics (1.7) (1.8).

Proposition 1.1 *Let $n = 3$. Let $h(t) = t^{-1/2}$ and let $X(\,\cdot\,)$ be defined by* (1.6). *Let $u_+ \in H^2 \cap W^2_1$, let $A_+, \omega^{-1}\dot{A}_+ \in H^1$ and $\nabla^2 A_+, \nabla \dot{A}_+ \in W^1_1$. Let (u_0, A_0) be defined by* (1.7) (1.8). *Then there exists T, $1 \le T < \infty$, and there exists a unique solution (u, A) of the $(Z)_3$ system* (1.1) *such that $(v,B) \equiv (u - u_0, A - A_0) \in X([T,\infty))$. If in addition $u_+ \in H^{0,2}$, then B satisfies the estimate*

$$\|B(t); H^1\| \vee \|\omega^{-1}\partial_t B(t); H^1\| \le Ct^{-3/4} \qquad (1.9)$$

for some constant C and for all $t \ge T$.

We next state the result obtained for the $(\mathrm{Z})_3$ system by using an improved asymptotic u_a, for more regular asymptotic states $(u_+, A_+, \dot{A}_+)$ and with stronger asymptotic convergence in time.

Proposition 1.2 *Let $n = 3$. Let $h(t) = t^{-3/2}$ and let $X(\cdot)$ be defined by* (1.6). *Let $u_+ \in H^2 \cap H^{0,2} \cap W_1^2$ with $xu_+ \in W_1^2$. Let $(A_+, \dot{A}_+)$ satisfy*

$$\begin{aligned} &A_+,\, \omega^{-1}\dot{A}_+ \in \dot{H}^{-2} \cap H^1, \quad \nabla^2 A_+,\, \nabla\dot{A}_+ \in W_1^1, \\ &x \cdot \nabla A_+,\, \omega^{-1} x \cdot \nabla \dot{A}_+ \in \dot{H}^{-2} \cap \dot{H}^{-1}. \end{aligned} \tag{1.10}$$

Let (u_0, A_0) be defined by (1.7) (1.8) *and let $u_a = (1 + f)u_0$ with $f = 2\Delta^{-1}A_0$. Then:*

(1) *There exists T, $1 \le T < \infty$ and there exists a unique solution (u, A) of the $(Z)_3$ system* (1.1) *such that $(v, B) \equiv (u - u_a, A - A_0) \in X([T, \infty))$.*

(2) *Assume in addition that $\omega^{-1}A_+$, $\omega^{-2}\dot{A}_+ \in W_{4/3}^1$. Then there exists T, $1 \le T < \infty$ and there exists a unique solution (u, A) of the $(Z)_3$ system* (1.1) *such that $(u - u_0, A - A_0) \in X([T, \infty))$. One can take the same T and the solution (u, A) is the same as in Part* (1).

We finally state the result for the $(\mathrm{Z})_2$ system. As already mentioned, that result requires small Schrödinger data, namely small u_+, and requires $A_+ = \dot{A}_+ = 0$.

Proposition 1.3 *Let $n = 2$. Let $h(t) = t^{-1}$ and let $X(\cdot)$ be defined by* (1.6). *Let $u_+ \in H^2 \cap H^{0,2} \cap W_1^2$ with $\|u_+; W_1^2\|$ sufficiently small and let $u_0(t) = U(t)u_+$. Then there exists T, $1 \le T < \infty$, and there exists a unique solution (u, A) of the $(Z)_2$ system* (1.1) *such that $(u - u_0, A) \in X([T, \infty))$.*

Remark 1.1 We could have included the norm $\|\omega^{-1}\partial_t B\|_2$, which is part of the energy, in the definition of $X(\cdot)$. That norm is never used in the proofs to perform the estimates and comes out at the end as a by product thereof. We have omitted it for simplicity.

The results of this paper have been announced in [13].

2. The Zakharov system $(\mathrm{Z})_3$ in space dimension $n = 3$

In this section we treat the $(\mathrm{Z})_3$ system and eventually prove Propositions 1.1 and 1.2. We follow the sketch given in the introduction and begin with the first step of the method. The treatment of that step follows exactly the same pattern as for the $(\mathrm{WS})_3$ system treated in [11]. We shall therefore be rather sketchy as regards the general arguments of the proofs, for which we refer to [11] for more details, and we shall mostly concentrate on the parts which are specific to the $(\mathrm{Z})_3$ system, namely the estimates. We shall make extensive use of the Strichartz inequalities for the Schrödinger equation [4] which we recall for completeness, in space dimension $n \geq 2$. A pair of exponents q, r with $2 \leq q, r \leq \infty$ is called admissible if

$$\begin{aligned} 0 \leq \frac{2}{q} = \frac{n}{2} - \frac{n}{r} &\leq 1 \quad \text{for } n \geq 3 \\ &< 1 \quad \text{for } n = 2. \end{aligned} \tag{2.1}$$

Lemma 2.1 *Let (q_i, r_i), $i = 1, 2$, be two admissible pairs. Let v satisfy the equation*

$$i\partial_t v = -\frac{1}{2}\Delta v + f$$

in some interval I with $v(t_0) = v_0$ for some $t_0 \in I$. Then the following estimates hold:

$$\|v; L^{q_1}(I, L^{r_1})\| \leq C\big(\|v_0\|_2 + \|f; L^{\overline{q}_2}(I, L^{\overline{r}_2})\|\big) \tag{2.2}$$

where C is a constant independent of I, and with $1/p + 1/\overline{p} = 1$.

Note that the pair $(8/n, 4)$ which appears in the definition (1.6) of $X(\,\cdot\,)$ is an admissible pair.

We shall also need some information on the Cauchy problem at finite times for the Schrödinger equation with time dependent real potential and time dependent inhomogeneity.

$$i\partial_t v = -\frac{1}{2}\Delta v + Vv + f. \tag{2.3}$$

We refer to Proposition 3.2 in [10] for sufficient conditions on V, f under which that problem is (globally) well posed with solutions in $\mathcal{C}(\cdot, H^2) \cap \mathcal{C}^1(\cdot, L^2)$.

We denote by h a function in $\mathcal{C}([1, \infty), \mathbb{R}^+)$ such that for some $\lambda > 0$, the function $\overline{h}(t) = t^\lambda h(t)$ is nonincreasing and tends to zero at infinity. We shall make repeated use of the following lemma, which is proved in [11].

Lemma 2.2 *Let* $1 \le T < t_0 \le \infty$, *let* $I = [T, t_0)$, *and for* $t \in I$, *let* $J = [t, \infty) \cap I$. *Let* $1 \le q$, $q_k \le \infty$ $(1 \le k \le n)$ *be such that*

$$\mu \equiv \frac{1}{q} - \sum_k \frac{1}{q_k} \ge 0.$$

Let $f_k \in L^{q_k}(I)$ *satisfy*

$$\|f_k; L^{q_k}(J)\| \le N_k h(t) \tag{2.4}$$

for $1 \le k \le n$, *for some constants* N_k *and for all* $t \in I$.

Let $\rho \ge 0$ *be such that* $n\lambda + \rho > \mu$. *Then the following inequality holds for all* $t \in I$

$$\left\| \left(\prod_k f_k \right) t^{-\rho}; L^q(J) \right\| \le C \left(\prod_k N_k \right) h(t)^n t^{\mu-\rho} \tag{2.5}$$

where

$$C = \left(1 - 2^{-q(n\lambda+\rho-\mu)}\right)^{-1/q}.$$

We now turn to Step 1 of the method, namely to the construction of solutions of the system (1.3) under general assumptions on (u_a, A_a). The main result on Step 1 can be stated as follows.

Proposition 2.1 *Let* h *be defined as above with* $\lambda = 1/4$, *and let* $X(\cdot)$ *be defined by* (1.6). *Let* u_a, A_a, R_1 *and* R_2 *be sufficiently regular* (*for the following estimates to make sense*) *and satisfy the estimates*

$$\|u_a(t)\|_\infty \vee \|\nabla u_a(t)\|_\infty \vee \|\Delta u_a(t)\|_\infty \vee \|\partial_t u_a(t)\|_\infty \le ct^{-3/2} \tag{2.6}$$

$$\|\partial_t^j A_a(t)\|_\infty \le at^{-1} \qquad \text{for } j = 0,\ 1, \tag{2.7}$$

$$\|\partial_t^j R_1; L^1([t,\infty), L^2)\| \le r_1 h(t) \qquad \text{for } j = 0,\ 1, \tag{2.8}$$

$$\|R_1; L^{8/3}([t,\infty), L^4)\| \le r_1 t^{-\eta} h(t) \qquad \text{for some } \eta \ge 0, \tag{2.9}$$

$$\|\omega^{-1} R_2; L^1([t,\infty), H^1)\| \le r_2 h(t), \tag{2.10}$$

for some constants c, a, r_1 *and* r_2 *and for all* $t \ge 1$. *Then there exists* T, $1 \le T < \infty$, *and there exists a unique solution* (v, B) *of the system* (1.3) *in* $X([T,\infty))$. *If in addition*

$$\|\omega^{-1} R_2; L^1([t,\infty), L^2)\| \le r_2 t^{-1/2} h(t) \tag{2.11}$$

for all $t \geq T$*, then* B *satisfies the estimate*

$$\|B(t);H^1\| \vee \|\omega^{-1}\partial_t B(t);H^1)\| \leq C\big(t^{-1/2} + t^{1/4}h(t)\big)h(t) \qquad (2.12)$$

for some constant C *and for all* $t \geq T$*.*

Proof. We follow the sketch given in the introduction, and more precisely the proof of Proposition 2.2 in [11]. Let $1 \leq T < \infty$ and let $(v,B) \in X([T,\infty))$. In particular (v,B) satisfies the estimates

$$\|v(t)\|_2 \leq N_0 h(t) \qquad (2.13)$$

$$\|v;L^{8/3}(J,L^4)\| \leq N_1 h(t) \qquad (2.14)$$

$$\|B(t);H^1\| \vee \|\partial_t B(t)\|_2 \leq N_2 h(t) \qquad (2.15)$$

$$\|\partial_t v(t)\|_2 \leq N_3 h(t) \qquad (2.16)$$

$$\|\partial_t v;L^{8/3}(J,L^4)\| \leq N_4 h(t) \qquad (2.17)$$

$$\|\Delta v(t)\|_2 \leq N_5 h(t) \qquad (2.18)$$

$$\|\Delta v;L^{8/3}(J,L^4)\| \leq N_6 h(t) \qquad (2.19)$$

for some constants N_i, $0 \leq i \leq 6$ and for all $t \geq T$, with $J = [t,\infty)$. We first construct a solution (v',B') of the system (1.5) in $X([T,\infty))$. For that purpose, we take t_0, $T < t_0 < \infty$ and we solve the system (1.5) in $X(I)$ where $I = [T,t_0]$ with initial condition $(v',B')(t_0) = 0$. Let (v'_{t_0}, B'_{t_0}) be the solution thereby obtained. The existence of v'_{t_0} follows from Proposition 3.2 in [10] with (A,V,f) replaced by $(0,A,-R_1)$. We want to take the limit of (v'_{t_0}, B'_{t_0}) as $t_0 \to \infty$ and for that purpose we need estimates of (v'_{t_0}, B'_{t_0}) in $X(I)$ that are uniform in t_0. Omitting the subscript t_0 for brevity, we define

$$N'_0 = \operatorname*{Sup}_{t\in I} h(t)^{-1}\|v'(t)\|_2 \qquad (2.20)$$

$$N'_1 = \operatorname*{Sup}_{t\in I} h(t)^{-1}\|v';L^{8/3}(J,L^4)\| \qquad (2.21)$$

$$N'_2 = \operatorname*{Sup}_{t\in I} h(t)^{-1}(\|B'(t);H^1\| \vee \|\partial_t B'(t)\|_2) \qquad (2.22)$$

$$N'_3 = \operatorname*{Sup}_{t\in I} h(t)^{-1}\|\partial_t v'(t)\|_2 \qquad (2.23)$$

$$N'_4 = \operatorname*{Sup}_{t\in I} h(t)^{-1}\|\partial_t v';L^{8/3}(J,L^4)\| \qquad (2.24)$$

$$N_5' = \operatorname*{Sup}_{t\in I} h(t)^{-1}\|\Delta v'(t)\|_2 \tag{2.25}$$

$$N_6' = \operatorname*{Sup}_{t\in I} h(t)^{-1}\|\Delta v'; L^{8/3}(J, L^4)\| \tag{2.26}$$

where $J = [t, \infty) \cap I$ and we set out to estimate the various N_i'. We first estimate N_0'. From (1.5) we obtain

$$\begin{aligned} \|v'(t)\|_2 &\le \|Bu_a - R_1; L^1(J, L^2)\| \\ &\le \big\| \|B\|_2 \|u_a\|_\infty + \|R_1\|_2 \big\|_1 \\ &\le (2cN_2 t^{-1/2} + r_1) h(t) \end{aligned} \tag{2.27}$$

so that

$$N_0' \le 2cN_2 T^{-1/2} + r_1. \tag{2.28}$$

We next estimate N_1'. By Lemma 2.1

$$\begin{aligned} \|v'; L^{8/3}(J, L^4)\| \le C\big(&\|A_a v'; L^1(J, L^2)\| + \|Bv'; L^{8/5}(J, L^{4/3})\| \\ &+ \|Bu_a - R_1; L^1(J, L^2)\|\big). \end{aligned} \tag{2.29}$$

The last norm has already been estimated by (2.27), while

$$\begin{aligned} &\|A_a v'; L^1(J, L^2)\| \le \big\| \|A_a\|_\infty \|v'\|_2 \big\|_1 \le 4aN_0' h(t), \\ &\|Bv'; L^{8/5}(J, L^{4/3})\| \le \big\| \|B\|_2 \|v'\|_4 \big\|_{8/5} \le CN_2 N_1' \overline{h}(t) h(t) \end{aligned}$$

by Lemma 2.2. Substituting those estimates into (2.29) yields

$$N_1' \le C\big(aN_0' + cN_2 T^{-1/2} + r_1 + N_2 N_1' \overline{h}(T)\big)$$

and therefore

$$N_1' \le C_1(aN_0' + cN_2 T^{-1/2} + r_1) \tag{2.30}$$

for T sufficiently large satisfying a condition of the type $N_2 \overline{h}(T) \le C$. We next estimate N_3'. From the time derivative of the equation for v', we obtain

$$\begin{aligned} \|\partial_t v'(t)\|_2^2 \le \|\partial_t v'(t_0)\|_2^2 + 2\big\| &\|\partial_t v'\|_2 \big(\|\partial_t A_a\|_\infty \|v'\|_2 \\ &+ \|\partial_t B\|_2 \|u_a\|_\infty + \|B\|_2 \|\partial_t u_a\|_\infty + \|\partial_t R_1\|_2\big) \\ &+ \|\partial_t v'\|_4 \|\partial_t B\|_2 \|v'\|_4 \big\|_1 \end{aligned} \tag{2.31}$$

We estimate the initial condition by

$$\begin{aligned}\|\partial_t v'(t_0)\|_2 &\le \|B(t_0)\|_2 \|u_a(t_0)\|_\infty + \|R_1(t_0)\|_2 \\ &\le \left(cN_2 t_0^{-3/2} + r_1\right) h(t_0)\end{aligned}$$

where we have used the pointwise estimate

$$\|R_1(t)\|_2 \le \|\partial_t R_1; L^1([t,\infty), L^2)\| \le r_1 h(t).$$

Using Lemma 2.2, we then obtain

$$\begin{aligned}N_3'^2 \le &\left(cN_2 T^{-3/2} + r_1\right)^2 \\ &+ C\left(N_3'(aN_0' + cN_2 T^{-1/2} + r_1) + N_2 N_4' N_1' \overline{h}(T)\right)\end{aligned}$$

and therefore

$$N_3' \le C_3\left(aN_0' + cN_2 T^{-1/2} + r_1 + (N_2 N_4' N_1' \overline{h}(T))^{1/2}\right). \tag{2.32}$$

We next estimate N_4'. By Lemma 2.1

$$\begin{aligned}\|\partial_t v'; L^{8/3}(J, L^4)\| \le C\Big(&\big\| \|A_a\|_\infty \|\partial_t v'\|_2 + \|\partial_t A_a\|_\infty \|v'\|_2 \\ &+ \|\partial_t B\|_2 \|u_a\|_\infty + \|B\|_2 \|\partial_t u_a\|_\infty + \|\partial_t R_1\|_2 \big\|_1 \\ &+ \big\| \|\partial_t B\|_2 \|v'\|_4 + \|B\|_2 \|\partial_t v'\|_4 \big\|_{8/5}\Big)\end{aligned} \tag{2.33}$$

and therefore by Lemma 2.2

$$N_4' \le C_4\left(a(N_3' + N_0') + cN_2 T^{-1/2} + r_1 + N_2(N_1' + N_4')\overline{h}(T)\right). \tag{2.34}$$

We next estimate N_5'. From (1.5) we obtain directly for $2 \le r \le 4$

$$\begin{aligned}\|\Delta v'\|_r \le 2\Big(&\|\partial_t v'\|_r + \|A_a\|_\infty \|v'\|_r + \|B\|_r \|u_a\|_\infty \\ &+ \|R_1\|_r + C\|B\|_r \|v'\|_r^{1-3/2r} \|\Delta v'\|_r^{3/2r}\Big)\end{aligned} \tag{2.35}$$

by a Sobolev inequality, and therefore for $r = 2$

$$N_5' \le 4\left(N_3' + aN_0' T^{-1} + cN_2 T^{-3/2} + r_1 + CN_0'(N_2 h(T))^4\right). \tag{2.36}$$

Similarly, taking the norm in $L^{8/3}(J)$ of (2.35) with $r = 4$, we obtain

$$N_6' \le 4\left(N_4' + aN_1' T^{-1} + CcN_2 T^{-9/8} + r_1 T^{-\eta} + CN_1'(N_2 h(T))^{8/5}\right) \tag{2.37}$$

where we have used the estimate

$$\|B\|_4 \le C\|B\|_2^{1/4} \|\nabla B\|_2^{3/4} \le CN_2 h(t).$$

We next turn to the estimate of N'_2. The equation for B' takes the form $\Box B' = \Delta F$ where

$$F = |v|^2 + 2\,\mathrm{Re}\,\overline{u}_a v - \Delta^{-1} R_2$$

and by standard energy estimates

$$\begin{cases} \|\nabla B'(t)\|_2 \vee \|\partial_t B'(t)\|_2 \le \|\Delta F; L^1(J, L^2)\| \\ \|B'(t)\|_2 \vee \|\omega^{-1}\partial_t B'(t)\|_2 \le \|\nabla F; L^1(J, L^2)\|. \end{cases} \tag{2.38}$$

We estimate

$$\begin{aligned} \|\nabla F; L^1(J, L^2)\| \le \big\| & 2\|v\|_4 \|\nabla v\|_4 + \|v\|_2 \|\nabla u_a\|_\infty \\ & + \|\nabla v\|_2 \|u_a\|_\infty + \|\omega^{-1} R_2\|_2 \big\|_1 \end{aligned} \tag{2.39}$$

$$\begin{aligned} \|\Delta F; L^1(J, L^2)\| \le & \big\| 2(\|v\|_4 \|\Delta v\|_4 + \|\nabla v\|_4^2) + \|v\|_2 \|\Delta u_a\|_\infty \\ & + 2\|\nabla v\|_2 \|\nabla u_a\|_\infty + \|\Delta v\|_2 \|u_a\|_\infty + \|R_2\|_2 \big\|_1. \end{aligned} \tag{2.40}$$

Using Lemma 2.2 and the definitions, we obtain from (2.38)–(2.40)

$$N'_2 \le C_2\big(c(N_0 + N_5)T^{-1/2} + N_1(N_1 + N_6)\overline{h}(T) + r_2\big). \tag{2.41}$$

It follows immediately from (2.28) (2.30) (2.32) (2.34) (2.36) (2.37) (2.41) that (v', B') is bounded in $X(I)$ uniformly in t_0 for T sufficiently large, more precisely for $N_2 \overline{h}(T) \le C$ for a suitable absolute constant C.

From now on the proof is very similar to that of Proposition 2.2 in [11]. We next take the limit $t_0 \to \infty$ of (v'_{t_0}, B'_{t_0}), restoring the subscript t_0 for that part of the argument. Let $T < t_0 < t_1 < \infty$ and let (v'_{t_0}, B'_{t_0}) and (v'_{t_1}, B'_{t_1}) be the corresponding solutions of (1.5). From the L^2 norm conservation of the difference $v'_{t_0} - v'_{t_1}$ and from (2.28), it follows that for all $t \in [T, t_0]$

$$\|v'_{t_0}(t) - v'_{t_1}(t)\|_2 = \|v'_{t_1}(t_0)\|_2 \le K_0 h(t_0) \tag{2.42}$$

where K_0 is the RHS of (2.28), while from (1.5) (2.38)–(2.41) and the initial conditions, it follows that

$$\begin{aligned} & \|B'_{t_0} - B'_{t_1}; L^\infty([T, t_0], H^1)\| \vee \|\partial_t(B'_{t_0} - B'_{t_1}); L^\infty([T, t_0], L^2)\| \\ & \quad \le K_2 h(t_0) \end{aligned} \tag{2.43}$$

where K_2 is the RHS of (2.41).

It follows from (2.42) (2.43) that there exists $(v', B') \in L^\infty_{loc}([T, \infty), L^2 \oplus H^1)$ with $\partial_t B' \in L^\infty_{loc}([T, \infty), L^2)$ such that (v'_{t_0}, B'_{t_0}) converges to (v', B') in that space when $t_0 \to \infty$. From the uniformity in t_0 of the estimates (2.28) (2.41), it follows that (v', B') satisfies the same estimates in $[T, \infty)$, namely that (2.28) (2.41) hold with N'_i defined by (2.20) (2.22) with $I = [T, \infty)$. Furthermore it follows by a standard compactness argument that $(v', B') \in X([T, \infty))$ and that v' satisfies the remaining estimates, namely (2.30) (2.32) (2.34) (2.36) (2.37) with the remaining N'_i again defined by (2.21) (2.23)–(2.26) with $I = [T, \infty)$. Clearly (v', B') satisfies the system (1.5).

From now on, I denotes the interval $[T, \infty)$. The previous construction defines a map ϕ: $(v, B) \to (v', B')$ from $X(I)$ to itself. The next step consists in proving that the map ϕ is a contraction on a suitable closed bounded set $\mathcal{R}$ of $X(I)$. We define $\mathcal{R}$ by the conditions (2.13)–(2.19) for some constants N_i and for all $t \in I$. We first show that for a suitable choice of N_i and for sufficiently large T, the map ϕ maps $\mathcal{R}$ into $\mathcal{R}$. By (2.28) (2.30) (2.32) (2.34) (2.36) (2.37) (2.41), it suffices for that purpose that

$$\begin{cases} (N'_0 \leq)\ r_1 + 2cN_2T^{-1/2} \leq N_0 \\ (N'_1 \leq)\ C_1\big(r_1 + aN'_0 + cN_2T^{-1/2}\big) \leq N_1 \\ (N'_2 \leq)\ C_2\big(r_2 + c(N_0 + N_5)T^{-1/2} + N_1(N_1 + N_6)\overline{h}(T)\big) \leq N_2 \\ (N'_3 \leq)\ C_3\big(r_1 + aN'_0 + cN_2T^{-1/2} + (N_2N'_4N'_1\overline{h}(T))^{1/2}\big) \leq N_3 \\ (N'_4 \leq)\ C_4\big(r_1 + a(N'_3 + N'_0) + cN_2T^{-1/2} + N_2(N'_1 + N'_4)\overline{h}(T)\big) \leq N_4 \\ (N'_5 \leq)\ 4\big(r_1 + N'_3 + aN'_0T^{-1} + cN_2T^{-3/2} + CN'_0(N_2h(T))^4\big) \leq N_5 \\ (N'_6 \leq)\ 4\big(r_1 + N'_4 + aN'_1T^{-1} + CcN_2T^{-9/8} + CN'_1(N_2h(T))^{8/5}\big) \leq N_6. \end{cases} \tag{2.44}$$

We ensure those conditions as follows. We ensure the first two conditions by taking

$$\begin{cases} N_0 = r_1 + 1 \\ N_1 = C_1(r_1 + aN_0 + 1) \end{cases} \tag{2.45}$$

and by taking T sufficiently large for the $o(1)$ terms in those conditions not to exceed 1. It is then easy to see that the conditions on N_3, N_4 are satisfied by taking

$$\begin{cases} N_3 = C_3(r_1 + aN_0 + 1) \\ N_4 = C_4(r_1 + a(N_3 + N_0) + 1) \end{cases} \tag{2.46}$$

and by taking T sufficiently large for the $o(1)$ terms in those conditions with the N'_i replaced by N_i not to exceed 1. We finally take

$$\begin{cases} N_5 = 4(r_1 + N_3 + 1) \\ N_6 = 4(r_1 + N_4 + 1) \\ N_2 = C_2(r_2 + 1) \end{cases} \tag{2.47}$$

and we take in addition T sufficiently large to ensure that the $o(1)$ terms in the corresponding conditions do not exceed 1. This completes the proof of the stability of $\mathcal{R}$.

We next show that the map ϕ is a contraction on $\mathcal{R}$. Let $(v_i, B_i) \in \mathcal{R}$, $i = 1, 2$, and let $(v'_i, B'_i) = \phi((v_i, B_i))$. For any pair of functions (f_1, f_2) we define $f_\pm = (1/2)(f_1 \pm f_2)$ so that $(fg)_\pm = f_+ g_\pm + f_- g_\mp$. In particular $u_+ = u_a + v_+$, $u_- = v_-$, $A_+ = A_a + B_+$ and $A_- = B_-$. Corresponding to (1.5), (v'_-, B'_-) satisfies the system

$$\begin{cases} i\partial_t v'_- = -\dfrac{1}{2}\Delta v'_- + A_+ v'_- + B_- u_a + B_- v'_+ \\ \square B'_- = 2\Delta \operatorname{Re}(\overline{u}_a + \overline{v}_+) v_-. \end{cases} \tag{2.48}$$

Since $\mathcal{R}$ is convex and stable under ϕ, (v_+, B_+) and (v'_+, B'_+) belong to $\mathcal{R}$, namely satisfy (2.13)–(2.19). Let N_{i-} and N'_{i-} be the seminorms of (v_-, B_-) and (v'_-, B'_-) corresponding to (2.20)–(2.26), namely the constants obtained by replacing (v', B', N'_i) by (v_-, B_-, N_{i-}) and (v'_-, B'_-, N'_{i-}) in (2.20)–(2.26). We have to estimate the N'_{i-} in terms of the N_{i-}. The estimates are essentially the same as those of N'_i in terms of N_i with minor differences: the contribution of the remainders disappear, the linear terms are the same, and the quadratic terms are in general obtained by polarization. The only exceptions to that rule are the $B_- v'_+$ term in the estimate of N'_{0-} and the $B_- \partial_t v'_+$ term in the estimate of N'_{3-} because the corresponding terms in the estimate of one single function disappear for algebraic reasons. Thus we estimate

$$\begin{aligned} \|v'_-(t)\|_2^2 &\le 2\|\langle v'_-, B_-(u_a + v'_+)\rangle\|_1 \\ &\le 2\Big\| \|v'_-\|_2 \|B_-\|_2 \|u_a\|_\infty + \|v'_-\|_4 \|B_-\|_2 \|v'_+\|_4 \Big\|_1 \end{aligned} \tag{2.49}$$

and therefore by Lemma 2.2

$$N'^2_{0-} \le 2cN'_{0-} N_{2-} T^{-1/2} + CN'_{1-} N_{2-} N_1 \overline{h}(T)$$

so that

$$N'_{0-} \le 2cN_{2-}T^{-1/2} + C\big(N'_{1-}N_{2-}N_1\overline{h}(T)\big)^{1/2}. \tag{2.50}$$

Similarly

$$\begin{aligned}\|\partial_t v'_-(t)\|_2^2 \le 2\Big\| &\|\partial_t v'_-\|_2\big(\|\partial_t A_a\|_\infty \|v'_-\|_2 + \|\partial_t B_-\|_2\|u_a\|_\infty \\ &+ \|B_-\|_2\|\partial_t u_a\|_\infty\big) + \|\partial_t v'_-\|_4\big(\|\partial_t B_+\|_2\|v'_-\|_4 + \|\partial_t B_-\|_2\|v'_+\|_4 \\ &+ \|B_-\|_2\|\partial_t v'_+\|_4\big)\Big\|_1\end{aligned} \tag{2.51}$$

and therefore by Lemma 2.2

$$\begin{aligned}N'^2_{3-} &\le C\big(N'_{3-}(aN'_{0-} + cN_{2-}T^{-1/2}) \\ &\quad + N'_{4-}(N_2N'_{1-} + N_{2-}(N_1+N_4))\overline{h}(T)\big)\end{aligned} \tag{2.52}$$

$$\begin{aligned}N'_{3-} &\le C_3\big(aN'_{0-} + cN_{2-}T^{-1/2} \\ &\quad + \big(N'_{4-}(N_2N'_{1-} + N_{2-}(N_1+N_4))\overline{h}(T)\big)^{1/2}\big).\end{aligned} \tag{2.53}$$

The estimates of the other N'_{i-} follow the general rule and are thus given by

$$N'_{1-} \le C_1\big(aN'_{0-} + cN_{2-}T^{-1/2} + N_{2-}N_1\overline{h}(T)\big) \tag{2.54}$$

$$\begin{aligned}N'_{4-} &\le C_4\big(a(N'_{3-} + N'_{0-}) + cN_{2-}T^{-1/2} \\ &\quad + (N_2N'_{1-} + N_{2-}(N_1+N_4))\overline{h}(T)\big)\end{aligned} \tag{2.55}$$

$$\begin{aligned}N'_{5-} &\le 4\big(N'_{3-} + aN'_{0-}T^{-1} + cN_{2-}T^{-3/2} + CN'_{0-}(N_2h(T))^4 \\ &\quad + CN_{2-}N_0^{1/4}N_5^{3/4}h(t)\big)\end{aligned} \tag{2.56}$$

$$\begin{aligned}N'_{6-} &\le 4\big(N'_{4-} + aN'_{1-}T^{-1} + CcN_{2-}T^{-9/8} + CN'_{1-}(N_2h(T))^{8/5} \\ &\quad + CN_{2-}N_1^{5/8}N_6^{3/8}h(t)\big)\end{aligned} \tag{2.57}$$

where in the last term we have estimated

$$\|B_-v'_+\|_4 \le C\|B_-\|_4\|v'_+\|_4^{5/8}\|\Delta v'_+\|_4^{3/8}.$$

Finally,

$$N'_{2-} \le C_2\big(c(N_{0-} + N_{5-})T^{-1/2} + (N_{1-}(N_1+N_6) + N_{6-}N_1)\overline{h}(T)\big). \tag{2.58}$$

We have kept the same constants C_i in (2.54) (2.53) (2.55) (2.58) as in (2.30) (2.32) (2.34) (2.41). In fact those constants are determined by the linear terms in the estimates, which are the same in both cases. There may occur additional different constants coming from the quadratic terms. They have been omitted in (2.53)–(2.58).

From the fact that most of the terms in the RHS of (2.50) and of (2.53)–(2.58) are $o(1)$ when $T \to \infty$ and that this system of inequalities is strictly triangular in the $O(1)$ terms, it follows easily as in [11] [12] that the map ϕ is a contraction in the set of semi norms N_i, $0 \le i \le 6$, for T sufficiently large. It follows therefrom that the system (1.3) has a unique solution in $\mathcal{R}$. Uniqueness in $X(\,\cdot\,)$ follows from the same estimates.

The last statement of the Proposition follows from the estimates of B' leading to (2.41) (see especially (2.38)–(2.40)) by using the stronger estimate (2.11) of R_2. □

We now turn to the second step of the method, namely to the choice of (u_a, A_a) and the derivation of the conditions (2.6)–(2.11). We shall need the standard factorisation of the free Schrödinger group

$$U(t) \equiv \exp\left(i\left(\frac{t}{2}\right)\Delta\right) = MDFM \tag{2.59}$$

where

$$M \equiv M(t) = \exp\left(\frac{ix^2}{2t}\right) \tag{2.60}$$

$$D(t) = (it)^{-n/2} D_0(t), \quad (D_0(t)f)(x) = f\left(\frac{x}{t}\right). \tag{2.61}$$

Using that decomposition, one can easily derive the following lemma, which we state for $n = 2$, 3.

Lemma 2.3 *Let $n = 2$ or 3. Let $u_+ \in H^{0,2}$ $(\subset L^1)$ and let $u_0 = U(t)u_+$. Then the following estimates hold*:

$$\|\nabla |u_0|^2\|_2 \le 2(2\pi t)^{-n/2} t^{-1} \|u_+\|_1 \|xu_+\|_2, \tag{2.62}$$

$$\|\Delta |u_0|^2\|_2 \le 4(2\pi t)^{-n/2} t^{-2} \|u_+\|_1 \|x^2 u_+\|_2. \tag{2.63}$$

Proof. From the representation (2.59), we obtain

$$\begin{aligned}\|\nabla |u_0|^2\|_2 &\le 2t^{-n-1} \|D_0(t)(\overline{FMu_+} FMxu_+)\|_2 \\ &\le 2t^{-n/2-1} \|FMu_+\|_\infty \|FMxu_+\|_2 \\ &\le 2(2\pi t)^{-n/2} t^{-1} \|u_+\|_1 \|xu_+\|_2\end{aligned}$$

by the Hausdorff-Young inequality.

Similarly

$$\begin{aligned}\|\Delta|u_0|^2\|_2 &\le 2t^{-n-2}\big(\|D_0(t)(\overline{FMu_+}FMx^2u_+)\|_2 \\ &\quad + \|D_0(t)|FMxu_+|^2\|_2\big) \\ &\le 2t^{-n/2-2}\big(\|FMu_+\|_\infty\|FMx^2u_+\|_2 + \|FMxu_+\|_4^2\big) \\ &\le 4(2\pi t)^{-n/2}t^{-2}\|u_+\|_1\|x^2u_+\|_2\end{aligned}$$

by the Hausdorff-Young and Hölder inequalities. □

We shall also need some estimates of solutions of the free wave equation, which we collect in the following lemma. A proof can be found in [24].

Lemma 2.4 *Let A_0 be defined by* (1.8)*. Let $k \ge 0$ be an integer. Let A_+ and $\dot{A}_+$ satisfy the conditions*

$$A_+,\, \omega^{-1}\dot{A}_+ \in H^k, \quad \nabla^2 A_+,\, \nabla\dot{A}_+ \in W_1^k. \tag{2.64}$$

Then A_0 satisfies estimates

$$\begin{cases}\|A_0(t); W_r^k\| \le a t^{-1+2/r}, \\ \|\partial_t A_0(t); W_r^{k-1}\| \le a t^{-1+2/r} \quad \text{for } k \ge 1\end{cases} \tag{2.65}$$

for $2 \le r \le \infty$ and for all $t \in \mathbb{R}$, where a depends on A_+, $\dot{A}_+$ through the norms associated with (2.64)*.*

We are now in a position to derive the final result with simple asymptotics (1.7) (1.8), namely Proposition 1.1.

Proof of Proposition 1.1

The result will follow from Proposition 2.1 once we have proved that (u_0, A_0) satisfies the assumptions of that proposition for (u_a, A_a). From the standard L^1-L^∞ estimates of $U(t)$, we obtain

$$\|u_0(t)\|_\infty \le (2\pi t)^{-3/2}\|u_+\|_1, \tag{2.66}$$

$$2\|\partial_t u_0\|_\infty = \|\Delta u_0\|_\infty \le (2\pi t)^{-3/2}\|\Delta u_+\|_1, \tag{2.67}$$

which proves (2.6). The assumption (2.7) on A_0 follows from Lemma 2.4. We next consider the remainders $R_1 = -A_0u_0$ and $R_2 = -\Delta|u_0|^2$. We estimate

$$\|R_1\|_2 \le \|A_0\|_2\|u_0\|_\infty \le Ct^{-3/2}$$

$$\|\partial_t R_1\|_2 \le \|A_0\|_2 \|\partial_t u_0\|_\infty + \|\partial_t A_0\|_2 \|u_0\|_\infty \le Ct^{-3/2}$$

by (2.66) (2.67) and Lemma 2.4. This proves (2.8) with $h(t) = t^{-1/2}$. On the other hand

$$\|R_1\|_4 \le \|A_0\|_4 \|u_0\|_\infty \le Ct^{-2}$$

by (2.66) and Lemma 2.4, which yields (2.9) with $\eta = 9/8$. Finally

$$\|R_2\|_2 = \|\Delta |u_0|^2\|_2 \le 2\big(\|u_0\|_\infty \|\Delta u_0\|_2 + \|\nabla u_0\|_4^2\big) \le Ct^{-3/2},$$
$$\|\omega^{-1} R_2\|_2 = \|\nabla |u_0|^2\|_2 \le 2\|u_0\|_\infty \|\nabla u_0\|_2 \le Ct^{-3/2},$$

which proves (2.10).

The last statement of the proposition follows from the corresponding statement of Proposition 2.1 and from Lemma 2.3 with $n = 3$, which yields actually

$$\|\omega^{-1} R_2; H^1\| \le Ct^{-5/2}$$

and therefore

$$\|\omega^{-1} R_2; L^1([t,\infty), H^1)\| \le Ct^{-1} h(t)$$

which is stronger than (2.11) by a factor $t^{-1/2}$. □

We next turn to the case where one uses the more accurate asymptotic form proposed in [23], thereby obtaining a stronger asymptotic convergence in time of the solution on a smaller subspace of asymptotic states. Thus we choose

$$(u_a, A_a) = ((1+f)u_0, A_0) \tag{2.68}$$

where (u_0, A_0) are defined by (1.7) (1.8) and

$$f = 2\Delta^{-1} A_0. \tag{2.69}$$

Using the operators

$$J = x + it\nabla, \quad P = t\partial_t + x\cdot\nabla, \tag{2.70}$$

we can rewrite the remainders R_1 and R_2 as

$$\begin{aligned} R_1 &= \Big(i\partial_t + \frac{1}{2}\Delta - A_0\Big)(1+f)u_0 \\ &= -fA_0u_0 - it^{-1}(\nabla f)\cdot Ju_0 + it^{-1}(Pf)u_0 \end{aligned} \tag{2.71}$$

$$R_2 = -\Delta(1+f)^2|u_0|^2. \tag{2.72}$$

We first reduce the estimates required for R_1 and R_2 to general estimates of u_+, A_0 and f. We first estimate R_1.

Lemma 2.5 *Let $u_+ \in W_1^2$, $xu_+ \in W_1^2$, and let A_0 and f satisfy*

$$\|\partial_t^j \nabla^k A_0\|_\infty \le at^{-1} \tag{2.73}$$

$$\|\partial_t^j \nabla^k f; H^1\| \vee \|\partial_t^j \nabla^k Pf\|_2 \le C \tag{2.74}$$

for $0 \le j+k \le 1$ and for all $t \ge 1$. Then the following estimates hold:

$$\|\partial_t^j \nabla^k R_1\|_2 \le Ct^{-5/2} \tag{2.75}$$

for some constant C, for $0 \le j+k \le 1$ and for all $t \ge 1$.

Proof. By the L^1-L^∞ estimate of $U(t)$ and the commutation rule $JU(t) = U(t)x$, we obtain

$$\|\partial_t^j \nabla^k u_0\|_\infty \vee \|\partial_t^j \nabla^k Ju_0\|_\infty \le Ct^{-3/2}$$

for $0 \le j+k \le 1$. We then estimate

$$\|R_1\|_2 \le \|f\|_2 \|A_0\|_\infty \|u_0\|_\infty + t^{-1}\|\nabla f\|_2 \|Ju_0\|_\infty + t^{-1}\|Pf\|_2\|u_0\|_\infty \le Ct^{-5/2}$$

which proves (2.75) for $j = k = 0$. The other cases are obtained similarly by distributing ∂_t or ∇ among the various factors. □

We next estimate R_2.

Lemma 2.6 *Let $u_+ \in W_1^2 \cap H^{0,2}$ and let f satisfy*

$$\|\nabla f(t)\|_2 \vee \|\Delta f(t)\|_2 \vee \|f(t)\|_\infty \le C \tag{2.76}$$

for all $t \ge 1$. Then the following estimates hold:

$$\|\omega^{-1} R_2\|_2 \le Ct^{-5/2}, \tag{2.77}$$

$$\|R_2\|_2 \le Ct^{-3} \tag{2.78}$$

for some constant C and for all $t \ge 1$.

Proof. For $u_+ \in W_1^2$, we know that $\|\nabla^j u_0\|_\infty \le Ct^{-3/2}$ and therefore $\|\nabla^j |u_0|^2\|_\infty \le Ct^{-3}$ for $j = 0, 1, 2$. For $u_+ \in H^{0,2}$, we know that $\|\nabla |u_0|^2\|_2 \le$

$Ct^{-5/2}$ and $\|\Delta|u_0|^2\|_2 \leq Ct^{-7/2}$ by Lemma 2.3. We then estimate R_2 as follows

$$\begin{aligned}\|\omega^{-1}R_2\|_2 &= \|\nabla(1+f)^2|u_0|^2\|_2 \\ &\leq (1+\|f\|_\infty)^2\|\nabla|u_0|^2\|_2 + 2(1+\|f\|_\infty)\|\nabla f\|_2\|u_0\|_\infty^2 \\ &\leq Ct^{-5/2}, \\ \|R_2\|_2 &= \|\Delta(1+f)^2|u_0|^2\|_2 \\ &\leq (1+\|f\|_\infty)^2\|\Delta|u_0|^2\|_2 + 4(1+\|f\|_\infty)\|\nabla f\|_2\|\nabla|u_0|^2\|_\infty \\ &+ 2\big((1+\|f\|_\infty)\|\Delta f\|_2 + \|\nabla f\|_4^2\big)\|u_0\|_\infty^2 \leq Ct^{-3}. \end{aligned}$$ □

Remark 2.1 In practice the bound on $\|f\|_\infty$ in (2.76) will follow from the Sobolev inequality

$$\|f\|_\infty^2 \leq C\|\nabla f\|_2\|\Delta f\|_2 \tag{2.79}$$

for f tending to zero at infinity in some weak sense.

We are now in a position to derive the final result with improved asymptotics (2.68) (2.69), namely Proposition 1.2.

Proof of Proposition 1.2
Part 1. The result follows from Proposition 2.1 and from the fact that (u_a, A_0) satisfies the assumptions of that proposition for (u_a, A_a). The condition (2.6) for u_a follows from the same condition for u_0, which follows from (2.66) (2.67), and from L^∞ estimates for f. For $\partial = 1$, ∂_t, ∇, we estimate

$$\|\partial f\|_\infty^2 \leq C\|\nabla\partial f\|_2^{1/2}\|\Delta\partial f\|_2^{1/2} \leq C \tag{2.80}$$

by a Sobolev inequality, the definition (2.69) of f and Lemma 2.4, while

$$\|\Delta f\|_\infty = C\|A_0\|_\infty \leq Ct^{-1} \tag{2.81}$$

as a special case of (2.7), which also follows from (1.10) and from Lemma 2.4 as before. The conditions (2.8) (2.9) with $h(t) = t^{-3/2}$ follow from Lemma 2.5, especially (2.75), under the assumptions made on u_+ and the conditions (2.73) (2.74). The latter follow from (1.10), from Lemma 2.4, from the definition (2.69) of f and from the fact that Pf is a solution of the free wave equation with initial data $(2x\cdot\nabla\Delta^{-1}A_+, 2(1+x\cdot\nabla)\Delta^{-1}\dot{A}_+)$. Finally the condition (2.10) follows from Lemma 2.6, from (2.69), from (1.10) and from Lemma 2.4.

Part 2. The result follows from the fact that $(fu_0, 0) \in X(I)$, namely that fu_0 satisfies the conditions on v that appear in the definition of $X(I)$, as we now show. For $u_+ \in W_1^2$, we estimate

$$\begin{aligned}\|\partial^\alpha(fu_0)\|_r &\leq \sum_{\beta\leq\alpha} \|\partial^\beta f\|_r \|\partial^{\alpha-\beta} u_0\|_\infty \\ &\leq Ct^{-3/2} \sum_{\beta\leq\alpha} \|\partial^\beta f\|_r \end{aligned} \tag{2.82}$$

for $2 \leq r \leq \infty$ and α a multiindex with $|\alpha| \leq 2$, and

$$\begin{aligned}\|\partial_t(fu_0)\|_r &\leq \|f\|_r \|\partial_t u_0\|_\infty + \|\partial_t f\|_r \|u_0\|_\infty \\ &\leq Ct^{-3/2}(\|f\|_r + \|\partial_t f\|_r). \end{aligned} \tag{2.83}$$

For $r = 2$, it follows from (2.82) (2.83), from (1.10), from Lemma 2.4 and from the definition (2.69) of f that

$$\|\partial_t(fu_0)\|_2 \vee \|fu_0; H^2\| \leq Ct^{-3/2}. \tag{2.84}$$

For $r = 4$, it follows from the standard L^p-L^q estimates for the wave equation [24] and from the definition of f that

$$\|f\|_4 \leq Ct^{-1/2}(\|\omega^{-1}A_+\|_{4/3} + \|\omega^{-2}\dot{A}_+\|_{4/3}) \tag{2.85}$$

$$\|\nabla f\|_4 \vee \|\partial_t f\|_4 \leq Ct^{-1/2}(\|A_+\|_{4/3} + \|\omega^{-1}\dot{A}_+\|_{4/3}) \tag{2.86}$$

while for β a multiindex with $\beta = 2$

$$\|\partial^\beta f\|_4 \leq C\|\Delta f\|_4 = 2C\|A_0\|_4 \leq Ct^{-1/2} \tag{2.87}$$

by the Mikhlin theorem, by (1.10) and Lemma 2.4. From (2.82) (2.83) (2.85)–(2.87) it follows that

$$\|fu_0; L^{8/3}([t,\infty), W_4^2)\| \vee \|\partial_t(fu_0); L^{8/3}([t,\infty), L^4)\| \leq Ct^{-13/8} \tag{2.88}$$

which together with (2.84) proves that $(fu_0, 0) \in X([1,\infty))$. □

3. The Zakharov system $(\mathrm{Z})_2$ in space dimension $n = 2$

In this section, we treat the $(\mathrm{Z})_2$ system and eventually prove Proposition 1.3. As mentioned in the introduction, the situation is much less satisfactory than in space dimension $n = 3$. The free part A_0 of the asymptotic field is estimated at best as

$$\|A_0(t)\|_\infty \le Ct^{-1/2} \tag{3.1}$$

and we are unable to handle such a slow decay in Step 1, so that the final result will eventually be restricted to the special case of zero asymptotic state $(A_+, \dot{A}_+)$ for A. On the other hand, in a suitable limit, the Zakharov system formally yields the cubic NLS equation, which is short range for $n = 2$, and one might naively expect a similar situation for the $(\mathrm{Z})_2$ system, allowing for a treatment of that system without a smallness condition on u. This turns out not to be the case, and the $(\mathrm{Z})_2$ system does actually require such a smallness condition at the level of Step 1. The treatment of that step is very similar to the case of $(\mathrm{Z})_3$. The relevant space $X(\,\cdot\,)$ is again given by (1.6), now with $n = 2$, and the main result can be stated as follows.

Proposition 3.1 *Let h be defined as in* Section 2 *with $\lambda = 1/2$ and let $X(\,\cdot\,)$ be defined by* (1.6). *Let u_a, A_a, R_1 and R_2 be sufficiently regular and satisfy the estimates*

$$\|u_a(t)\|_\infty \vee \|\nabla u_a(t)\|_\infty \vee \|\Delta u_a(t)\|_\infty \vee \|\partial_t u_a(t)\|_\infty \le ct^{-1}, \tag{3.2}$$

$$\|\partial_t^j A_a(t)\|_\infty \le at^{-1-j\theta} \qquad \textit{for some } \theta > 0 \textit{ and for } j = 0,\, 1, \tag{3.3}$$

$$\|\partial_t^j R_1; L^1([t,\infty), L^2)\| \le r_1 h(t) \qquad \textit{for } j = 0,\, 1, \tag{3.4}$$

$$\|R_1; L^4([t,\infty), L^4)\| \le r_1 t^{-\eta} h(t) \qquad \textit{for some } \eta \ge 0, \tag{3.5}$$

$$\|\omega^{-1} R_2; L^1([t,\infty), H^1)\| \le r_2 h(t) \tag{3.6}$$

for some constants c, a, r_1 and r_2 with c sufficiently small and for all $t \ge 1$. Then there exists T, $1 \le T < \infty$, and there exists a unique solution (v, B) of the system (1.3) *in $X([T,\infty))$.*

Sketch of proof. The proof is essentially the same as that of Proposition 2.1 with minor differences in the estimates, and we concentrate on the latter. We take again $(v, B) \in X([T,\infty))$ for some T, $1 \le T < \infty$, so that (v, B) satisfies

$$\|v(t)\|_2 \le N_0 h(t) \tag{3.7}$$

$$\|v; L^4(J, L^4)\| \le N_1 h(t) \tag{3.8}$$

$$\|B(t); H^1\| \vee \|\partial_t B(t)\|_2 \le N_2 h(t) \tag{3.9}$$

$$\|\partial_t v(t)\|_2 \le N_3 h(t) \tag{3.10}$$

$$\|\partial_t v; L^4(J, L^4)\| \le N_4 h(t) \tag{3.11}$$

$$\|\Delta v(t)\|_2 \le N_5 h(t) \tag{3.12}$$

$$\|\Delta v; L^4(J, L^4)\| \le N_6 h(t) \tag{3.13}$$

for some constants N_i, $0 \le i \le 6$ and for all $t \ge T$, with $J = [t, \infty)$. We construct a solution (v', B') of the system (1.5) in $X(I)$ first for $I = [T, t_0]$ and then for $I = [T, \infty)$. For that purpose we define again

$$N'_0 = \operatorname*{Sup}_{t \in I} h(t)^{-1} \|v'(t)\|_2 \tag{3.14}$$

$$N'_1 = \operatorname*{Sup}_{t \in I} h(t)^{-1} \|v'; L^4(J, L^4)\| \tag{3.15}$$

$$N'_2 = \operatorname*{Sup}_{t \in I} h(t)^{-1} (\|B'(t); H^1\| \vee \|\partial_t B'(t)\|_2) \tag{3.16}$$

$$N'_3 = \operatorname*{Sup}_{t \in I} h(t)^{-1} \|\partial_t v'(t)\|_2 \tag{3.17}$$

$$N'_4 = \operatorname*{Sup}_{t \in I} h(t)^{-1} \|\partial_t v'; L^4(J, L^4)\| \tag{3.18}$$

$$N'_5 = \operatorname*{Sup}_{t \in I} h(t)^{-1} \|\Delta v'(t)\|_2 \tag{3.19}$$

$$N'_6 = \operatorname*{Sup}_{t \in I} h(t)^{-1} \|\Delta v'; L^4(J, L^4)\| \tag{3.20}$$

where $J = [t, \infty) \cap I$. The crux of the proof is to estimate the N'_i in terms of the N_i. By exactly the same method as in the proof of Proposition 2.1, we obtain

$$N'_0 \le 2cN_2 + r_1 \tag{3.21}$$

$$N'_1 \le C_1(aN'_0 + cN_2 + r_1) \tag{3.22}$$

for $N_2 \overline{h}(T) \le C$,

$$N'_3 \le C_3\big(aN'_0 T^{-\theta} + cN_2 + r_1 + (N_2 N'_4 N'_1 \overline{h}(T))^{1/2}\big) \tag{3.23}$$

$$N'_4 \le C_4\big(a(N'_3 + N'_0 T^{-\theta}) + cN_2 + r_1 + N_2(N'_1 + N'_4)\overline{h}(T)\big) \tag{3.24}$$

$$N'_5 \le 4\big(N'_3 + aN'_0 T^{-1} + cN_2 T^{-1} + r_1 + CN'_0(N_2 h(T))^2\big) \tag{3.25}$$

$$N'_6 \le 4\big(N'_4 + aN'_1 T^{-1} + CcN_2 T^{-3/4} + r_1 T^{-\eta} + CN'_1(N_2 h(T))^{4/3}\big) \tag{3.26}$$

$$N'_2 \le C_2\big(c(N_0 + N_5) + N_1(N_1 + N_6)\overline{h}(T) + r_2\big). \tag{3.27}$$

The estimates (3.21)–(3.27) are very similar to the corresponding estimates (2.28) (2.30) (2.32) (2.34) (2.36) (2.37) (2.41) of the case $n = 3$. Aside from unimportant changes in the exponents in (3.25) (3.26), the main differences are (i) the occurrence of the factor $T^{-\theta}$ in (3.23) and (3.24), coming from the assumption (3.3) with $j = 1$ on $\|\partial_t A_a\|_\infty$, and (ii) the replacement of $cT^{-1/2}$ by c everywhere, coming from the assumption (3.2) as compared with (2.6). The latter difference is responsible for the need of the smallness condition on c. In fact, with the estimates (3.21)–(3.27) available, the proof proceeds as that of Proposition 2.1. The main step is to prove that the set $\mathcal{R}$ defined by (3.7)–(3.13) is stable under the map ϕ: $(v, B) \to (v', B')$. This is ensured by taking

$$\begin{cases} N_0 = 2cN_2 + r_1 \\ N_1 = C_1(aN_0 + cN_2 + r_1) \\ N_3 = C_3(cN_2 + r_1 + 1) \\ N_4 = C_4(aN_3 + cN_2 + r_1 + 1) \\ N_5 = 4(N_3 + r_1 + 1) \\ N_6 = 4(N_4 + r_1 + 1) \\ N_2 = C_2(c(N_0 + N_5) + r_2 + 1) \end{cases} \tag{3.28}$$

and by taking T sufficiently large so that the remaining $o(1)$ terms in the RHS of (3.21)–(3.27) do not exceed 1. In order to solve the system (3.28), we remark that the constants N_1 and N_6 associated with the Strichartz norms do not occur in the RHS and can therefore be determined at the very end. Eliminating N_3 and N_4 (to be determined at the end as functions of N_2) one is left with the reduced system

$$\begin{cases} N_0 = 2cN_2 + r_1 \\ N_5 = 4C_3cN_2 + 4(C_3 + 1)(r_1 + 1) \\ N_2 = C_2(c(N_0 + N_5) + r_2 + 1) \end{cases} \tag{3.29}$$

which can obviously be solved for N_0, N_5 and N_2 positive for c sufficiently small.

The remaining part of the proof proceeds as that of Proposition 2.1 with appropriate changes in the contraction argument and will be omitted. $\square$

Remark 3.1 The assumption (3.3) on A_a is rather arbitrary. It is too strong to accomodate a non zero A_0 satisfying only (3.1). On the other hand it is weaker by one power of t than the condition that would be satisfied by an A_1 devised to ensure that $R_2 = 0$. It has been chosen so as to ensure that the proof of the proposition proceeds smoothly.

We are now in a position to derive the final result, namely Proposition 1.3. As already mentioned, the assumption (3.3) forces us to take $A_0 = 0$.

Proof of Proposition 1.3

The result will follow from Proposition 3.1 once we have proved that $(u_0, 0)$ satisfies the assumptions of that proposition for (u_a, A_a). From the standard L^1-L^∞ estimates of $U(t)$, we obtain

$$\begin{aligned} &\|u_0(t)\|_\infty \le (2\pi t)^{-1}\|u_+\|_1 \\ &2\|\partial_t u_0\|_\infty = \|\Delta u_0\|_\infty \le (2\pi t)^{-1}\|\Delta u_+\|_1 \end{aligned}$$

which proves (3.2). Since $A_a = 0$ and $R_1 = 0$, (3.3)–(3.5) are obvious. Finally $R_2 = -\Delta|u_0|^2$, so that (3.6) with $h(t) = t^{-1}$ follows from Lemma 2.3 with $n = 2$. □

References

[1] Added H. and Added S., *Existence globale de solutions fortes pour les équations de la turbulence de Langmuir en dimension* 2. C. R. Acad. Sci. Paris **299** (1984), 551–554.

[2] Added H. and Added S., *Equations of Langmuir turbulence and nonlinear Schrödinger equation: Smoothness and approximation*. J. Funct. Anal. **79** (1988), 183–210.

[3] Bourgain J. and Colliander J., *On wellposedness of the Zakharov system*. Int. Math. Res. Not. **11** (1996), 515–546.

[4] Cazenave T., *Semilinear Schrödinger Equations*, Courant Inst. Lect. Notes **10**, Am. Math. Soc., Providence 2003.

[5] Ginibre J. and Ozawa T., *Long range scattering for nonlinear Schrödinger and Hartree equations in space dimension* $n \ge 2$. Commun. Math. Phys. **151** (1993), 619–645.

[6] Ginibre J., Tsutsumi Y. and Velo G., *On the Cauchy problem for the Zakharov system*. J. Funct. Anal. **151** (1997), 384–436.

[7] Ginibre J. and Velo G., *Long range scattering and modified wave operators for the Wave-Schrödinger system*. Ann. H.P. **3** (2002), 537–612.

[8] Ginibre J. and Velo G., *Long range scattering and modified wave operators for the Wave-Schrödinger system II*. Ann. H.P. **4** (2003), 973–999.

[9] Ginibre J. and Velo G., *Long range scattering and modified wave operators for the Maxwell-Schrödinger system I. The case of vanishing asymptotic magnetic field.* Commun. Math. Phys. **236** (2003), 395–448.

[10] Ginibre J. and Velo G., *Scattering theory for the Schrödinger equation in some external time dependent magnetic fields.* J. Diff. Eq. **215** (2005), 108–177.

[11] Ginibre J. and Velo G., *Long range scattering for the Wave-Schrödinger system with large wave data and small Schrödinger data.* Hokkaido Math. J., **35** (2006), 261–287.

[12] Ginibre J. and Velo G., *Long range scattering for the Maxwell-Schrödinger system with large magnetic field data and small Schrödinger data.* Publ. RIMS, Kyoto Univ., **42** (2006), 421–459.

[13] Ginibre J. and Velo G., *Long range scattering for some Schrödinger related non linear systems.* preprint, math.AP/0412430, to appear in "Nonlinear Dispersive Equations" (T. Ozawa and Y. Tsutsumi Eds), GAKUTO International Series, Mathematical Sciences and Applications.

[14] Ozawa T., *Long range scattering for nonlinear Schrödinger equations in one space dimension.* Commun. Math. Phys. **139** (1991), 479–493.

[15] Ozawa T. and Tsutsumi Y., *Existence and Smoothing effect of solutions for the Zakharov equations.* Publ. RIMS, Kyoto Univ. **28** (1992), 329–361.

[16] Ozawa T. and Tsutsumi Y., *Global existence and asymptotic behaviour of solutions for the Zakharov equations in three space dimensions.* Adv. Math. Sci. Appl. **3** (1993), 301–334.

[17] Ozawa T. and Tsutsumi Y., *Asymptotic behaviour of solutions for the coupled Klein-Gordon-Schrödinger equations.* in Spectral and Scattering Theory and Applications, Adv. Stud. in Pure Math., Jap. Math. Soc. **23** (1994), 295–305.

[18] Shimomura A., *Wave operators for the coupled Klein-Gordon-Schrödinger equations in two space dimensions.* Funkcial. Ekvac. **47** (2004), 63–82.

[19] Shimomura A., *Scattering theory for the coupled Klein-Gordon-Schrödinger equations in two space dimensions.* J. Math. Sci. Univ. Tokyo **10** (2003), 661–685.

[20] Shimomura A., *Scattering theory for the coupled Klein-Gordon-Schrödinger equations in two space dimensions II.* Hokkaido Math. J. **34** (2005), 405–433.

[21] Shimomura A., *Modified wave operators for the coupled Wave-Schrödinger equations in three space dimensions.* Disc. Cont. Dyn. Syst. **9** (2003), 1571–1586.

[22] Shimomura A., *Modified wave operators for Maxwell-Schrödinger equations in three space dimensions.* Ann. H.P. **4** (2003), 661–683.

[23] Shimomura A., *Scattering theory for Zakharov equations in three space dimensions with large data.* Commun. Contemp. Math **6** (2004), 881–899.

[24] Strauss W., *Non linear Wave Equations.* CMBS Lecture notes **73**, Am. Math. Soc., Providence, 1989.

[25] Sulem C. and Sulem P.L., *Quelques résultats de régularité pour les équations de la turbulence de Langmuir.* C. R. Acad. Sci. Paris **289** (1979), 173–176.

[26] Tsutsumi Y., *Global existence and asymptotic behaviour of solutions for the Maxwell-Schrödinger system in three space dimensions.* Commun. Math. Phys. **151** (1993), 543–576.

[27] Zakharov V.E., *Collapse of Langmuir waves.* Sov. Phys. JETP **35** (1972), 908–914.

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