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Rank of a Hermitian symmetric space of noncompact type and Kähler magnetic fields on a product manifold

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Abstract. In this paper we study asymptotic behavior of trajectories for a Kähler magnetic fields on a product of complex hyperbolic spaces and investigate the rank of a Hermitian symmetric space of noncompact type in terms of horocycle trajectories for Kähler magnetic fields.

Key words: Kähler magnetic field, horocycle, products of Kähler manifolds, Hermitian symmetric space, Euclidean factor, ideal boundary.

1. Introduction

On a complete Kähler manifold $(M, J, \langle\ ,\ \rangle)$ with complex structure J, we have natural uniform magnetic fields which are constant multiples of the Kähler form $\mathbb{B}_J$. We call them *Kähler magnetic fields*. Given a Kähler magnetic field $\mathbb{B}_\kappa = \kappa\mathbb{B}_J$ of constant κ we say a smooth curve γ parameterized by its arclength to be a *trajectory* for $\mathbb{B}_\kappa$ if it satisfies the differential equation $\nabla_{\dot\gamma}\dot\gamma = \kappa J\dot\gamma$. Trajectories represent motions of unit charged particles with unit speed under the action of magnetic fields. Without the force of magnetic fields, which is the case $\kappa = 0$, trajectories turn to geodesics. The author hence consider trajectories for Kähler magnetic fields are natural extension of geodesics when we study Kähler manifolds from real Riemannian geometric point of view. In this paper we study asymptotic behaviors of trajectories for Kähler magnetic fields on a Hermitian symmetric space of noncompact type.

A Hermitian symmetric space of noncompact type is a typical example of a Hadamard manifold, which is a simply connected complete Riemannian manifold of nonpositive curvature. On a Hadamard manifold M we call two geodesic rays σ_1, σ_2 of unit speed *asymptotic* if the distance function $t \mapsto d(\sigma_1(t), \sigma_2(t))$ is bounded on $[0, \infty)$. The ideal boundary ∂M of M is

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defined as the set of all asymptotic classes of geodesic rays on M. Since the restriction of the exponential map $\exp_x$ of the unit tangent space U_xM gives a bijection onto ∂M at each point $x \in M$, we get a natural completion $\overline{M} = M \cup \partial M$ of M with a canonical topology which is called the cone topology ([5, 3]). The notion of the ideal boundary is quite useful in the study of Hadamard manifolds. As some geometric properties of a Hadamard manifold M are inherited on its ideal boundary, the geometry of the ideal boundary gives some information on the geometry of M. For example, Gromov [3] introduced Tits metric on ∂M, which is closely related to asymptotical flatness of M, and Ohtsuka [7, 8] studied Hadamard manifolds with compact ideal boundaries.

In view of classification of Kähler magnetic flows for a complex Euclidean space $\mathbb{C}^n$ and a complex hyperbolic space $\mathbb{C}H^n$ given in [1, 4], horocycle flows for $\mathbb{C}H^n$ have a similar conjugacy property as for geodesic flow for $\mathbb{C}^n$. As Ohtsuka and the author [2] studied the dimension of the Euclidean factor of a Hadamard manifold by the Tits geometry of its ideal boundary, the author is interested in the relationship between horocycle trajectories for Kähler magnetic fields and the geometry of a Kähler Hadamard manifold. In this paper we study the relationship between rank of Hermitian symmetric spaces of noncompact type and horocycle trajectories. Unfortunately our proof is a bit tiresome, but we carefully study asymptotic behaviors of trajectories on products of complex hyperbolic spaces.

2. Rank of a Hermitian symmetric space

We call a smooth curve γ parameterized by its arclength on a Hadamard manifold M *unbounded in both directions* if both of the sets $\gamma([0,\infty))$ and $\gamma(-\infty,0])$ are unbounded sets. For such a curve γ we define its limit points at infinity by

$$\gamma(\infty) = \lim_{t\to\infty}\gamma(t), \quad \gamma(-\infty) = \lim_{t\to-\infty}\gamma(t) \in \partial M$$

if they exist in the ideal boundary ∂M of M. We shall call γ a *horocycle* if the following conditions hold:

i) it has a single limit point; $\gamma(\infty) = \gamma(-\infty)$,
ii) if γ crosses a geodesic σ with $\sigma(\infty) = \gamma(\infty)$, then they cross orthogonally at their crossing point.

We here put our main results in this paper, which will be proved in the following sections.

Theorem 1 *Let M be a Hermitian symmetric space of noncompact type.*
(1) *The minimum of the sectional curvature of M coincides with*

$$-\max\{\kappa^2 \,|\, \mathbb{B}_\kappa \textit{ has a horocycle trajectory}\}.$$

(2) *The number*

$$\frac{\max\{\kappa^2 \,|\, \mathbb{B}_\kappa \textit{ has a horocycle trajectory}\}}{\min\{\kappa^2 \,|\, \mathbb{B}_\kappa \textit{ has a horocycle trajectory}\}}$$

is an integer and coincides with the rank of M.
(3) *If each of Kähler magnetic fields $\mathbb{B}_{\kappa_1}$, $\mathbb{B}_{\kappa_2}$ ($|\kappa_1| \le |\kappa_2|$) has a horocycle trajectory on M, then the rank of M is not smaller than $(\kappa_2/\kappa_1)^2$.*
(4) *Suppose the minimum of the sectional curvature of M is $-\delta$. If there is a Kähler magnetic field $\mathbb{B}_\kappa$ on M which has a horocycle trajectory, then the rank of M is not smaller than δ/κ^2.*

The similar results hold for a product of complex hyperbolic spaces and a complex Euclidean space.

Theorem 2 *Let M be a product of complex space forms of nonpositive curvature.*
(1) *The minimum of the sectional curvature of M coincides with*

$$-\max\{\kappa^2 \,|\, \mathbb{B}_\kappa \textit{ has a horocycle trajectory}\}.$$

(2) *The manifold M has a complex Euclidean factor if and only if*

$$\inf\{\kappa^2 \,|\, \mathbb{B}_\kappa \textit{ has a horocycle trajectory}\} = 0.$$

When we investigate these theorems Ikawa's basic study on trajectories for Kähler magnetic fields on Hermitian symmetric spaces plays an important role. It should be noted that he call our Kähler magnetic fields "electromagnetic fields" without distinguishing other magnetic fields because he also treat such magnetic fields on Sasakian manifolds. In his speech for [6], he pointed out that every trajectory for a Kähler magnetic field on a Hermitian symmetric space M of noncompact type and of rank r with minimum sectional curvature $-\delta$ lies on some totally geodesic r product $N = \mathbb{C}H^1(-\delta) \times \cdots \times \mathbb{C}H^1(-\delta)$ of real hyperbolic planes of curvature $-\delta$. This holds because for each unit tangent vector $v \in TM$ there is a totally geodesic complex r product N of complex hyperbolic lines with $v \in TN$. Taking account of this result, we study trajectories on products of complex hyperbolic spaces in the following sections.

3. Trajectories on a complex hyperbolic space

In order to show our results we begin by recalling some features of trajectories on a complex hyperbolic space $\mathbb{C}H^n$. Though the author gave in [1] their explicit expressions, we need to study them a bit different point of view to investigate asymptotic behavior of trajectories on a product of complex hyperbolic spaces.

Let $\varpi\colon H_1^{2n+1} \to \mathbb{C}H^n$ be a Hopf fibration of an anti-de Sitter space $H_1^{2n+1} = \{\hat{x} \in \mathbb{C}^{n+1} \mid \langle\!\langle \hat{x}, \hat{x} \rangle\!\rangle = -1\}$, where a Hermitian form $\langle\!\langle\ ,\ \rangle\!\rangle$ on $\mathbb{C}^{n+1}$ is defined by

$$\langle\!\langle \hat{x}, \hat{y} \rangle\!\rangle = -\hat{x}_0\overline{\hat{y}_0} + \sum_{j=1}^{n} \hat{x}_j\overline{\hat{y}_j}$$

for $\hat{x} = (\hat{x}_0, \dots, \hat{x}_n)$, $\hat{y} = (\hat{y}_0, \dots, \hat{y}_n) \in \mathbb{C}^{n+1}$. The tangent space $T_x\mathbb{C}H^n$ at x is identified with the horizontal subspace $\mathcal{H}_{\hat{x}} = \{(\hat{x}, \hat{u}) \mid \hat{u} \in \mathbb{C}^{n+1}, \langle\!\langle \hat{u}, \hat{x} \rangle\!\rangle = 0\}$ at $\hat{x} \in H_1^{2n+1}$ with $\varpi(\hat{x}) = x$. Since a geodesic σ with initial unit vector $\dot{\sigma}(0) = u \in T_x\mathbb{C}H^n$ on a complex hyperbolic space $\mathbb{C}H^n(-4)$ of constant holomorphic sectional curvature -4 is expressed as $\sigma(t) = \varpi(\hat{x}\cosh t + \hat{u}\sinh t)$, where $\varpi(\hat{x}) = x$ and $(\hat{x}, \hat{u}) \in \mathcal{H}_{\hat{x}}$ with $d\varpi(\hat{x}, \hat{u}) = u$, we see the following:

Lemma 1 *On a complex hyperbolic space $\mathbb{C}H^n(-\beta)$ of constant holomorphic sectional curvature $-\beta$, the distance $d(x, y)$ between two points $x = \varpi(\hat{x})$ and $y = \varpi(\hat{y})$ satisfies the equality $\sinh^2((2/\sqrt{\beta})d(x, y)) = |\langle\!\langle \hat{x}, \hat{y} \rangle\!\rangle|^2 - 1$.*

The following was shown in [1] by use of the expression of trajectories on the ball model of $\mathbb{C}H^n(-4)$.

Proposition 1 *We consider a Kähler magnetic field $\mathbb{B}_\kappa$ on $\mathbb{C}H^n(-\beta)$.*

(1) *If $|\kappa| < \sqrt{\beta}$, every trajectory for $\mathbb{B}_\kappa$ has two distinct points at infinity.*
(2) *Every trajectory for $\mathbb{B}_{\pm\sqrt{\beta}}$ is a horocycle.*
(3) *If $|\kappa| < \sqrt{\beta}$, for arbitrary distinct points z, $w \in \partial\mathbb{C}H^n(-\beta)$, we have a unique trajectory γ for $\mathbb{B}_\kappa$ with $\gamma(-\infty) = z$ and $\gamma(\infty) = w$ up to the translation of parameter.*

(4) *If* $|\kappa| \le \sqrt{\beta}$*, for arbitrary points* $x \in \mathbb{C}H^n(-\beta)$ *and* $z \in \partial\mathbb{C}H^n(-\beta)$ *we have a unique trajectory* γ *for* $\mathbb{B}_\kappa$ *with* $\gamma(0) = x$ *and* $\gamma(\infty) = z$.

We here study this result a bit more precisely by comparing asymptotic behaviors of trajectories and geodesics. For a Kähler magnetic field $\mathbb{B}_\kappa$ on a Kähler manifold M, we define the Kähler magnetic flow $\mathbb{B}_\kappa\varphi_t$ on the unit tangent bundle UM of M by $\mathbb{B}_\kappa\varphi_t(v) = \dot\gamma_v(t)$, where γ_v denotes the trajectory for $\mathbb{B}_\kappa$ with $\dot\gamma_v(0) = v$. It was also shown in [1] that for $\mathbb{C}H^n(-\beta)$ a Kähler magnetic flow $\mathbb{B}_\kappa\varphi_t$ with $|\kappa| < \sqrt{\beta}$ and the geodesic flow φ_t are smoothly conjugate in the strong sense: There exists a diffeomorphism $g_\kappa = g_{\kappa,\beta}$ of $U\mathbb{C}H^n$ with $\mathbb{B}_\kappa\varphi_t \circ g_\kappa = g_\kappa \circ \varphi_{\sqrt{1-\kappa^2/\beta}\,t}$. When $\beta = 4$, this diffeomorphism is given by

$$g_{\kappa,4}\big(d\varpi(\hat x, \hat v)\big) = d\varpi\big(a_\kappa^{-1}(\kappa\epsilon_\kappa^{-1}\hat x - \sqrt{-1}\,\epsilon_\kappa\hat v),\, a_\kappa^{-1}(\sqrt{-1}\,\epsilon_\kappa\hat x + \kappa\epsilon_\kappa^{-1}\hat v)\big)$$

with $a_\kappa = 2^{-1/2}(4-\kappa^2)^{-1/4}$ and $\epsilon_\kappa = (2 - \sqrt{4-\kappa^2})^{-1/2}$. For general β the diffeomorphism $g_{\kappa,\beta}$ is derived from the diffeomorphism $g_{2\kappa/\sqrt{\beta},4}$. Let $p\colon U\mathbb{C}H^n \to \mathbb{C}H^n$ be the canonical projection and $f_\kappa = f_{\kappa,\beta}$ be the inverse map of $g_{\kappa,\beta}$. One can easily find by use of Lemma 1 that the distance between $p(v)$ and $p(g_\kappa(v))$ does not depend on $v \in U\mathbb{C}H^n$. For a unit tangent vector v, we denote by σ_v the geodesic with initial vector $\dot\sigma_v(0) = v$.

Lemma 2 *For each trajectory* γ_v *for* $\mathbb{B}_\kappa$ *with* $|\kappa| < \sqrt{\beta}$ *on* $\mathbb{C}H^n(-\beta)$*, the function* $t \mapsto d\big(\gamma_v(t), \sigma_{f_\kappa(v)}\big(\sqrt{1-(\kappa^2/\beta)}\,t\big)\big)$ *is constant on* $\mathbb{R}$.

Proof. As we have

$$\dot\sigma_{f_\kappa(v)}(\sqrt{1-\kappa^2/\beta}\,t) = \varphi_{\sqrt{1-\kappa^2/\beta}\,t} \circ f_\kappa(v) = f_\kappa \circ \mathbb{B}_\kappa\varphi_t(v) = f_\kappa(\dot\gamma_v(t)),$$

the distance

$$d\big(\gamma_v(t), \sigma_{f_\kappa(v)}(\sqrt{1-(\kappa^2/\beta)}\,t)\big) = d\Big(p\big(\dot\gamma_v(t)\big), p\big(f_\kappa(\dot\gamma_v(t))\big)\Big)$$

does not depend on t. □

This Lemma shows that the geodesic σ_v and the trajectory $\gamma_{g_\kappa(v)}$ for $\mathbb{B}_\kappa$ satisfy $\sigma_v(\infty) = \gamma_{g_\kappa(v)}(\infty)$, $\sigma_v(-\infty) = \gamma_{g_\kappa(v)}(-\infty)$ for every unit tangent vector $v \in U\mathbb{C}H^n$. Since $\mathbb{C}H^n(-\beta)$ is of strictly negative curvature, it is known that arbitrary distinct points in the ideal boundary can be joined by a geodesic, hence we can reprove the third assertion of Proposition 1.

On a Hadamard manifold M the exponential map restricted on the unit tangent space at an arbitrary point gives a bijection to the ideal boundary. So it is clear that there exists a unique geodesic joining arbitrary $x \in M$ and $z \in M(\infty)$. The fourth assertion of Proposition 1 says that the same property holds for unbounded trajectories. In order to reprove this property, we define for κ with $|\kappa| \le \sqrt{\beta}$ an endomorphism $\Psi_\kappa \colon T\mathbb{C}H^n(-\beta) \to T\mathbb{C}H^n(-\beta)$ of the tangent bundle by

$$\Psi_\kappa(v) = \sqrt{1 - \frac{\kappa^2}{\beta}}\, v + \frac{\kappa}{\sqrt{\beta}}\, Jv.$$

Lemma 3 *For each trajectory γ_v for $\mathbb{B}_\kappa$ with $|\kappa| < \sqrt{\beta}$ on $\mathbb{C}H^n(-\beta)$, the distance function $t \mapsto d\big(\gamma_v(t), \sigma_{\Psi_\kappa(v)}\big(\sqrt{1-(\kappa^2/\beta)}\,t\big)\big)$ is bounded for $t \ge 0$. Hence $\gamma_v(\infty) = \sigma_{\Psi_\kappa(v)}(\infty)$.*

Proof. We are enough to show the assertion in the case $\beta = 4$. If $p(v) = \varpi(\hat{x})$ and $v = d\varpi(\hat{x}, \hat{v})$ with $(\hat{x}, \hat{v}) \in \mathcal{H}_{\hat{x}}$, it was shown in [1] that

$$\gamma_v(t) = \varpi\bigg(\cosh \frac{1}{2}\sqrt{4-\kappa^2}\, t \cdot \hat{x} + \frac{1}{\sqrt{4-\kappa^2}} \sinh \frac{1}{2}\sqrt{4-\kappa^2}\, t \cdot (2\hat{v} - \sqrt{-1}\,\kappa \hat{x})\bigg).$$

Therefore by Lemma 1 we have

$$\begin{aligned} &\sinh d\big(\gamma_v(t), \sigma_{\Psi_\kappa(v)}\big(\sqrt{1-(\kappa^2/4)}\,t\big)\big) \\ &\quad = \frac{\kappa}{\sqrt{4-\kappa^2}} \sinh \frac{1}{2}\sqrt{4-\kappa^2}\; t \left| \cosh \frac{1}{2}\sqrt{4-\kappa^2}\, t - \sinh \frac{1}{2}\sqrt{4-\kappa^2}\, t \right| \\ &\quad < \kappa(4-\kappa^2)^{-1/2}/2 \end{aligned}$$

for $t \ge 0$, and get the assertion. □

Next we study the case $\kappa = \pm\sqrt{\beta}$. We should note that magnetic flows $\mathbb{B}_{\pm\sqrt{\beta}}\varphi_t$ on $\mathbb{C}H^n(-\beta)$ are horocycle flows and are not conjugate to the geodesic flow. We here note that $\Psi_{\pm\sqrt{\beta}}(v) = \pm Jv$, hence $\Psi_{\pm\sqrt{\beta}} = -\Psi_{\mp\sqrt{\beta}}$. If γ is a trajectory for $\mathbb{B}_\kappa$, then the curve $\rho(t) = \gamma(-t)$ being reversed the parameter is a trajectory for $\mathbb{B}_{-\kappa}$. Therefore we only need to treat the case $\kappa = \sqrt{\beta}$.

Lemma 4 *For a trajectory γ_v for a Kähler magnetic field $\mathbb{B}_{\sqrt{\beta}}$ on $\mathbb{C}H^n(-\beta)$ we have the following for $s > 0$:*

$$d\Big(\gamma_v\big(\pm(2/\sqrt{\beta})\sinh\sqrt{\beta}\, s/2\big), \gamma_v(0)\Big) = s, \tag{3.1}$$

$$s < d\Big(\gamma_v\big(\pm(2/\sqrt{\beta})\sinh\sqrt{\beta}\, s/2\big), \sigma_{\Psi_{\sqrt{\beta}}(v)}(s)\Big) < s + 2/\sqrt{\beta}, \tag{3.2}$$

and

$$\gamma_v\big(\pm(2/\sqrt{\beta})\sinh\sqrt{\beta}\, s/2\big) = \sigma_{v_\pm(s)}(s), \tag{3.3}$$

with $v_\pm(s) = (\pm 1 + \sqrt{-1}\sinh s)\cosh^{-1} s \cdot v$. *Hence we see* $\gamma_v(\infty) = \gamma_v(-\infty) = \sigma_{\Psi_{\sqrt{\beta}}(v)}(\infty)$.

Proof. We also study only the case $\beta = 4$, and hence $k = 2$. If $p(v) = \varpi(\hat{x})$ and $v = d\varpi(\hat{x}, \hat{v})$ with $(\hat{x}, \hat{v}) \in \mathcal{H}_{\hat{x}}$, it was shown in [1] that $\gamma_v(t) = \varpi\big((1 - \sqrt{-1}\, t)\hat{x} + t\hat{v}\big)$. By Lemma 1 we have $\sinh^2 d(\gamma_v(t), \gamma_v(0)) = -1 + \big|1 - \sqrt{-1}\, t\big|^2 = t^2$, hence get the equality (3.1). Similarly as we have

$$\begin{aligned}
&\sinh^2 d\big(\gamma_v(\pm\sinh s), \sigma_{\Psi_2(v)}(s)\big) \\
&\quad = -1 + \big|(-1 \pm \sqrt{-1}\sinh s)\cosh s \mp \sqrt{-1}\sinh s \sinh s\big|^2 \\
&\quad = (1 + e^{-2s})\sinh^2 s
\end{aligned}$$

and $\sqrt{2}\sinh s < \sinh(s+1)$ for $s \geq 0$, we get the estimate (3.2). By the expression of γ_v one can easily see (3.3). Thus we obtain the last assertion on asymptotic behaviour of γ_v in view of the definition of cone topology on $\overline{\mathbb{C}H^n}$. We should note that (3.1) and (3.2) also show the asymptotic behavior of γ_v because $\mathbb{C}H^n$ is of strictly negative curvature. □

Since the ideal boundary is naturally identified with the unit tangent space by the exponential map, we can reprove the fourth assertion of Proposition 1 by Lemmas 3 and 4.

4. Asymptotic behavior of trajectories on a product manifold

We now study asymptotic behavior of trajectories on a product of complex hyperbolic spaces. Suppose a smooth curve γ which is not necessarily parameterized by its arclength on a Kähler manifold (M, J) satisfies the equation $\nabla_{\dot{\gamma}}\dot{\gamma} = \kappa J\dot{\gamma}$ with some constant κ. Then it has constant speed because

$$\frac{d}{dt}\|\dot{\gamma}\|^2 = 2\langle \nabla_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma}\rangle = 2\kappa\langle J\dot{\gamma}, \dot{\gamma}\rangle = 0.$$

When γ is not a point curve, we put $\lambda = \|\dot{\gamma}\|\ (\neq 0)$, and define a smooth curve $\tilde{\gamma}$ which is parameterized by its arclength by $\tilde{\gamma}(t) = \gamma(t/\lambda)$. Since it satisfies

$$\nabla_{\dot{\tilde{\gamma}}}\dot{\tilde{\gamma}} = \frac{1}{\lambda^2}\nabla_{\dot{\gamma}}\dot{\gamma} = \frac{\kappa}{\lambda^2}J\dot{\gamma} = \frac{\kappa}{\lambda}J\dot{\tilde{\gamma}},$$

it is a trajectory for a Kähler magnetic field $\mathbb{B}_{\kappa/\lambda}$. This trivial property makes our study rich. If $\gamma = (\gamma_1, \ldots, \gamma_q)$ is a trajectory for a Kähler magnetic field $\mathbb{B}_\kappa$ on a product $M = M_1 \times \cdots \times M_q$ of Kähler manifolds (M_i, J), $i = 1, \ldots, q$, where γ_i is a smooth curve on M_i, then we see $\nabla_{\dot{\gamma}_i}\dot{\gamma}_i = \kappa J\dot{\gamma}_i$, $i = 1, \ldots, q$. Thus γ is of the form $\gamma(t) = \big(\tilde{\gamma}_1(\lambda_1 t), \ldots, \tilde{\gamma}_q(\lambda_q t)\big)$, where $\tilde{\gamma}_i$ is either a trajectory for a Kähler magnetic field $\mathbb{B}_{\kappa/\lambda_i}$ on M_i when $\lambda_i \neq 0$, or a point curve $\tilde{\gamma}_i(t) = \gamma_i(0)$ when $\lambda_i = 0$.

We now restrict ourselves on a product of complex hyperbolic spaces. We shall show the following which corresponds to Proposition 1 in the previous section.

Theorem 3 *Let $\gamma = (\gamma_1, \ldots, \gamma_q)$ be a trajectory for a Kähler magnetic field $\mathbb{B}_\kappa$ on $M = \mathbb{C}H^{n_1}(-\beta_1) \times \cdots \times \mathbb{C}H^{n_q}(-\beta_q)$.*
(1) *It is unbounded in both directions if and only if $\|\dot{\gamma}_{i_0}\| \geq |\kappa|/\sqrt{\beta_{i_0}}$ for some i_0.*
(2) *If $\|\dot{\gamma}_{i_0}\| > |\kappa|/\sqrt{\beta_{i_0}}$ for some i_0, then γ has distinct points at infinity.*
(3) *If $\|\dot{\gamma}_{i_0}\| = |\kappa|/\sqrt{\beta_{i_0}}$ for some i_0 and $\|\dot{\gamma}_i\| \leq |\kappa|/\sqrt{\beta_i}$ for every i, then γ is a horocycle.*

By the first and the second assertion of Proposition 1, the first assertion of Theorem 3 is trivial. Also one can easily guess the second and the third. To see these precisely we study the asymptotic behavior of trajectories. Let $\mathbb{B}_\kappa$ be a Kähler magnetic field on $M = \mathbb{C}H^{n_1}(-\beta_1) \times \cdots \times \mathbb{C}H^{n_q}(-\beta_q)$. For a unit tangent vector $v = (v_1, \ldots, v_q) \in UM$ satisfying
i) $\|v_{i_j}\| = |\kappa|/\sqrt{\beta_{i_j}}$ for $j = 1, \ldots, a$ $(a \geq 1)$,
ii) $\|v_i\| < |\kappa|/\sqrt{\beta_i}$ for $i \neq i_j$,
we set $\eta = \big\{\sum_{j=1}^a \|v_{i_j}\|^2\big\}^{1/2}$ and $\varPsi_\kappa(v) = (w_1, \ldots, w_q)$ with

$$w_i = \begin{cases} \eta^{-1}\operatorname{sgn}(\kappa)Jv_{i_j}, & \text{when } i = i_j \text{ for } 1 \leq j \leq a, \\ 0, & \text{otherwise.} \end{cases}$$

We should note that $\kappa^2\sum_{j=1}^a 1/\beta_{i_j} = \sum_{j=1}^a \|v_{i_j}\|^2$ and $\varPsi_\kappa(v)$ is orthogonal to v.

Lemma 5 *Under the above situation there are constants c_1, c_2 (which do not depend on s) with*

$$s \le d\Big(\gamma_v\big(\pm(2/|\kappa|)\sinh|\kappa|s/(2\eta)\big), \gamma_v(0)\Big) \le s + c_1,$$
$$s < d\Big(\gamma_v\big(\pm(2/|\kappa|)\sinh|\kappa|s/(2\eta)\big), \sigma_{\Psi_\kappa(v)}(s)\Big) < s + c_2$$

for $s > 0$. Hence $\gamma_v(\infty) = \gamma_v(-\infty) = \sigma_{\Psi_\kappa(v)}(\infty)$ and γ_v is a horocycle.

Proof. Let $\tilde{\gamma}_{i_j}$ be a trajectory for $\mathbb{B}_{\mathrm{sgn}(\kappa)\sqrt{\beta_{i_j}}}$ on $\mathbb{C}H^{n_{i_j}}(-\beta_{i_j})$. We find by Lemma 4 that the trajectory $\gamma_v = (\gamma_1, \ldots, \gamma_q)$ satisfies

$$\begin{aligned} &d\Big(\gamma_{i_j}\big(\pm(2/|\kappa|)\sinh|\kappa|s/(2\eta)\big), \gamma_{i_j}(0)\Big) \\ &\quad = d\Big(\tilde{\gamma}_{i_j}\big(\pm(2/\sqrt{\beta_{i_j}})\sinh|\kappa|s/(2\eta)\big), \tilde{\gamma}_{i_j}(0)\Big) = \|v_{i_j}\|s/\eta. \end{aligned}$$

Since γ_i with $i \ne i_j$ is bounded, we get the first estimate. Similarly, we obtain

$$\begin{aligned} \|v_{i_j}\|s/\eta &< d\big(\gamma_{i_j}(\pm(2/|\kappa|)\sinh|\kappa|s/(2\eta)), \sigma_{w_{i_j}/\|w_{i_j}\|}(\|w_{i_j}\|s)\big) \\ &< \|v_{i_j}\|s/\eta + 2/\sqrt{\beta_{i_j}}, \end{aligned}$$

hence get the second estimate. Since M is a product of Hadamard manifolds of strictly negative curvature, we see these estimates show the asymptotic behavior of γ_v. □

The third assertion of Theorem 3 is a direct consequence of Lemma 5. Next we study the case for the second assertion of Theorem 3. For a unit tangent vector $v = (v_1, \ldots, v_q) \in UM$ satisfying

i) $\|v_{i_\ell}\| > |\kappa|/\sqrt{\beta_{i_\ell}}$ for $\ell = 1, \ldots, b$ $(b \ge 1)$

ii) $\|v_i\| < |\kappa|/\sqrt{\beta_i}$ for $i \ne i_\ell$,

we set $\xi = \big\{\sum_{\ell=1}^{b}\{\|v_{i_\ell}\|^2 - (\kappa^2/\beta_{i_\ell})\}\big\}^{1/2}$ and define $F_\kappa(v) = (\hat{w}_1, \ldots, \hat{w}_q)$ as

$$\hat{w}_i = \begin{cases} \xi^{-1}\{\|v_{i_\ell}\|^2 - (\kappa^2/\beta_{i_\ell})\}^{1/2} f^{(i_\ell)}_{\kappa/\|v_{i_\ell}\|}\Big(\frac{v_{i_\ell}}{\|v_{i_\ell}\|}\Big), & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{when } i = i_\ell \text{ for } 1 \le \ell \le b, \\ 0, & \text{otherwise,} \end{cases}$$

where $f^{(i)}_\nu$ denotes the diffeomorphism of $U\mathbb{C}H^{n_i}$ defined in Section 3.

Clearly $F_\kappa(v) = F_\kappa(v')$ if and only if

$$\begin{cases} v_{i_\ell} = v'_{i_\ell}, & \text{for } 1 \le \ell \le b, \\ \|v_i\| \le |\kappa|/\sqrt{\beta},\ \ \|v'_i\| \le |\kappa|/\sqrt{\beta}, & \text{for } i \ne i_\ell \end{cases}$$

hold.

Lemma 6 *For a unit tangent vector $v \in UM$ satisfying the above conditions* i) *and* ii), *the distance function $t \mapsto d(\gamma_v(t), \sigma_{F_\kappa(v)}(\xi t))$ is bounded on $\mathbb{R}$. Hence*

$$\gamma_v(\infty) = \sigma_{F_\kappa(v)}(\infty), \quad \gamma_v(-\infty) = \sigma_{F_\kappa(v)}(-\infty)$$

and γ_v has distinct points at infinity.

Proof. Let $\tilde\gamma_{i_\ell}$ be a trajectory for $\mathbb{B}_{\kappa/\|v_{i_\ell}\|}$ on $\mathbb{C}H^{n_{i_\ell}}(-\beta_{i_\ell})$. As $\gamma_{i_\ell}(t) = \tilde\gamma_{i_\ell}(\|v_{i_\ell}\|t)$, we find by Lemma 2 that $d\big(\gamma_{i_\ell}(t), \sigma_{u_{i_\ell}}\big(\sqrt{\|v_{i_\ell}\|^2 - (\kappa^2/\beta_{i_\ell})}\,t\big)\big)$ is a constant function, where $u_{i_\ell} = f^{(i_\ell)}_{\kappa/\|v_{i_\ell}\|}(v_{i_\ell}/\|v_{i_\ell}\|)$. Since γ_i with $i \ne i_\ell$ is bounded, and the geodesic $\sigma_{F_\kappa(v)} = (\sigma_1, \dots, \sigma_q)$ is of the form

$$\sigma_i(t) = \begin{cases} \sigma_{u_{i_\ell}}\big(\xi^{-1}\sqrt{\|v_{i_\ell}\|^2 - (\kappa^2/\beta_{i_\ell})}\,t\big), & \text{when } i = i_\ell, \\ \sigma_i(0), & \text{otherwise,} \end{cases}$$

we obtain the assertion. □

Remark 1 Corresponding to Lemmas 3 and 5, we define $\Psi_\kappa(v) = (w_1, \dots, w_q)$ for $v \in UM$ satisfying the above conditions i) and ii) by

$$w_i = \begin{cases} \xi^{-1}\{\|v_{i_\ell}\|^2 - (\kappa^2/\beta_{i_\ell})\}^{1/2}\Psi^{(i_\ell)}_{\kappa/\|v_{i_\ell}\|}\Big(\frac{v_{i_\ell}}{\|v_{i_\ell}\|}\Big), & \\ & \text{when } i = i_\ell \text{ for } 1 \le \ell \le b, \\ 0, & \text{otherwise,} \end{cases}$$

where $\Psi^{(i)}_\nu$ denotes the endomorphism on $T\mathbb{C}H^{n_i}$ defined in Section 3. Then the distance function $t \mapsto d(\gamma_v(t), \sigma_{\Psi_\kappa(v)}(\xi t))$ is also bounded on $\mathbb{R}$.

Finally we study a unit tangent vector $v = (v_1, \dots, v_q) \in UM$ satisfying
i) $\|v_{i_j}\| = |\kappa|/\sqrt{\beta_{i_j}}$ for $j = 1, \dots, a$ $(a \ge 1)$,
ii) $\|v_{i_\ell}\| > |\kappa|/\sqrt{\beta_{i_\ell}}$ for $\ell = a+1, \dots, a+b$ $(b \ge 1)$
iii) $\|v_i\| < |\kappa|/\sqrt{\beta_i}$ for $i \ne i_j$, i_ℓ.
For such a unit tangent vector v we define $\Psi_\kappa(v)$ just the same way as for a unit tangent vector in Remark 1: We set $\xi = \big\{\sum_{\ell=a+1}^{a+b}\{\|v_{i_\ell}\|^2 - (\kappa^2/\beta_{i_\ell})\}\big\}^{1/2}$

and define $\Psi_\kappa(v) = (w_1, \dots, w_q)$ as

$$w_i = \begin{cases} \xi^{-1}\{\|v_{i_\ell}\|^2 - (\kappa^2/\beta_{i_\ell})\}^{1/2}\, \Psi^{(i_\ell)}_{\kappa/\|v_{i_\ell}\|}\Big(\frac{v_{i_\ell}}{\|v_{i_\ell}\|}\Big), & \\ & \text{when } i = i_\ell \text{ for } a+1 \le \ell \le a+b, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 7 *For a unit tangent vector $v \in UM$ satisfying the above conditions* i), ii) *and* iii), *there is a constant c (which does not depend on t) satisfying $d(\gamma_v(t), \gamma_v(0)) > \xi t - c$ and*

$$\sum_{j=1}^{a} 4\beta_{i_j}^{-1}\Big\{\log\big(|\kappa|t + \sqrt{\kappa^2 t^2 + 4}\big)/2\Big\}^2 \le d^2\big(\gamma_v(t), \sigma_{\Psi_\kappa(v)}(\xi t)\big)$$
$$\le \sum_{j=1}^{a} 4\beta_{i_j}^{-1}\Big\{\log\big(|\kappa|t + \sqrt{\kappa^2 t^2 + 4}\big)/2\Big\}^2 + c.$$

Hence these trajectory γ_v and geodesic $\sigma_{\Psi_\kappa(v)}$ satisfy

$$\gamma_v(\infty) = \sigma_{\Psi_\kappa(v)}(\infty), \quad \gamma_v(-\infty) = \sigma_{\Psi_\kappa(v)}(-\infty),$$

and γ_v has distinct points at infinity.

Proof. Let $\tilde\gamma_{i_j}$ be a trajectory for $\mathbb{B}_{\mathrm{sgn}(\kappa)\sqrt{\beta_{i_j}}}$ on $\mathbb{C}H^{n_{i_j}}(-\beta_{i_j})$ for $j = 1, \dots, a$, and $\tilde\gamma_{i_\ell}$ be a trajectory for $\mathbb{B}_{\kappa/\|v_{i_\ell}\|}$ on $\mathbb{C}H^{n_{i_\ell}}(-\beta_{i_\ell})$ for $\ell = a+1, \dots, a+b$. Along the same lines as in the proof of Lemma 6 we see the function $d\big(\gamma_{i_\ell}(t), \sigma_{u_{i_\ell}}\big(\sqrt{\|v_{i_\ell}\|^2 - (\kappa^2/\beta_{i_\ell})}\, t\big)\big)$ is constant for $a+1 \le \ell \le a+b$, where $u_{i_\ell} = \Psi^{(i_\ell)}_{\kappa/\|v_{i_\ell}\|}(v_{i_\ell}/\|v_{i_\ell}\|)$. On the other hand, we have

$$\begin{aligned} d\big(\gamma_{i_j}(t), \gamma_{i_j}(0)\big) &= d\big(\tilde\gamma_{i_j}(\|v_{i_j}\|t), \tilde\gamma_{i_j}(0)\big) \\ &= 2\beta_{i_j}^{-1/2}\log\big(|\kappa|t + \sqrt{\kappa^2 t^2 + 4}\big)/2 \end{aligned}$$

for $1 \le j \le a$ by Lemma 4, hence we obtain our assertion. □

We obtain the second assertion of Theorem 3 by Lemmas 6 and 7, and complete the proof of Theorem 3.

As a direct consequence of Theorem 3 we obtain the following:

Corollary 1 *For a product $M = \mathbb{C}H^{n_1}(-\beta_1) \times \cdots \times \mathbb{C}H^{n_q}(-\beta_q)$ of complex hyperbolic spaces we put $\beta = 1\big/\big(\sum_{i=1}^{q} 1/\beta_i\big)$.*
(1) *If $|\kappa| > \max_{1\le i\le q}\sqrt{\beta_i}$, every trajectory for $\mathbb{B}_\kappa$ on M is bounded.*

(2) *If* $|\kappa| \le \sqrt{\beta}$*, every trajectory for* $\mathbb{B}_\kappa$ *on* M *is unbounded in both directions.*

We now focus our attention on horocycle trajectories. By Theorem 3 we see a trajectory γ_v for $\mathbb{B}_\kappa$ on $M = \mathbb{C}H^{n_1}(-\beta_1) \times \cdots \times \mathbb{C}H^{n_q}(-\beta_q)$ with a unit vector $v = (v_1, \ldots, v_q)$ is horocycle if and only if the vector v satisfies
 i) $\|v_{i_j}\| = |\kappa|/\sqrt{\beta_{i_j}}$ for $j = 1, \ldots, a$ with $a \ge 1$,
 ii) $\|v_i\| < |\kappa|/\sqrt{\beta_i}$ for $i \ne i_j$.
We hence obtain the following:

Corollary 2 *There is a horocycle trajectory for* $\mathbb{B}_\kappa$ *on* $M = \mathbb{C}H^{n_1}(-\beta_1) \times \cdots \times \mathbb{C}H^{n_q}(-\beta_q)$ *if and only if* $\beta \le \kappa^2 \le \max_{1 \le i \le q} \beta_i$*, where* $\beta = 1\big/\big(\sum_{i=1}^{q} 1/\beta_i\big)$.

5. Hyperbolic and Euclidean factors

Our study on horocycle trajectories on a product of complex hyperbolic spaces gives some information on hyperbolic and Euclidean factors for some Kähler manifolds. We are now in the position to prove Theorems 1 and 2 in Section 2.

Proof of Theorem 1. Let M be a Hermitian symmetric space of noncompact type and of rank r with minimum sectional curvature $-\delta$. As we mentioned in Section 2, every trajectory for a Kähler magnetic field on M lies on some totally geodesic r product $\mathbb{C}H^1(-\delta) \times \cdots \times \mathbb{C}H^1(-\delta)$ of real hyperbolic planes of curvature $-\delta$. Therefore, by Corollary 2, we see that there is a horocycle trajectory for $\mathbb{B}_\kappa$ if and only if $\delta/r \le \kappa^2 \le \delta$. Thus we find the first and the second assertions in Theorem 1 hold. Other two assertions of Theorem 1 follow these assertions. □

Here we also study a Kähler Hadamard manifold with complex Euclidean factor. On a complex Euclidean space $\mathbb{C}^r$, every trajectory for $\mathbb{B}_\kappa$ ($\kappa \ne 0$) is a circle of radius $1/|\kappa|$ in the sense of Euclidean geometry, hence it is bounded. For a Kähler Hadamard manifold with complex Euclidean factor we have the following:

Theorem 4 *Let* N *be a Kähler Hadamard manifold. If there is a horocycle trajectory for a Kähler magnetic field* $\mathbb{B}_{\kappa_0}$ *on* N *with some positive* κ_0*, then for every* κ *with* $0 < \kappa \le \kappa_0$ *there exists a horocycle trajectory for a Kähler magnetic field* $\mathbb{B}_\kappa$ *on a product* $\mathbb{C}^r \times N$.

Proof. Let γ_0 be a horocycle trajectory for $\mathbb{B}_{\kappa_0}$ on N. For κ with $0 < \kappa \le \kappa_0$, we find that every trajectory for $\mathbb{B}_\kappa$ on $\mathbb{C}^r \times N$ with initial vector $\big(u, (\kappa/\kappa_0)\dot{\gamma}_0(0)\big)$ is a horocycle, where $u \in T\mathbb{C}^r$ satisfies $\|u\| = \sqrt{1-(\kappa/\kappa_0)^2}$. $\square$

In particular, the same results as for Corollaries 1 and 2 hold for a Kähler Hadamard manifold with complex Euclidean factor.

Corollary 3 *For a product $\mathbb{C}^r \times \mathbb{C}H^{n_1}(-\beta_1) \times \cdots \times \mathbb{C}H^{n_q}(-\beta_q)$ we have the following*:
(1) *If $|\kappa| > \max_{1\le i\le q}\sqrt{\beta_i}$, every trajectory for $\mathbb{B}_\kappa$ is bounded.*
(2) *There exists a horocycle trajectories for $\mathbb{B}_\kappa$ if and only if $0 < \kappa^2 \le \max_{1\le i\le q}\beta_i$.*

One can easily see that this Corollary guarantees Theorem 2.

By the definition of cone topology we find Theorem 3 and the proof of Theorem 4 show the following on a horocycle trajectories on a Kähler manifold with complex hyperbolic factors.

Theorem 5 *Let N be a Kähler Hadamard manifold. If there is a horocycle trajectory for a Kähler magnetic field $\mathbb{B}_{\kappa_0}$ on N with some positive κ_0, then for every κ with $\kappa_0\sqrt{\beta/(\kappa_0^2+\beta)} \le \kappa \le \kappa_0$ there exists a horocycle trajectory for a Kähler magnetic field $\mathbb{B}_\kappa$ on a product $N \times \mathbb{C}H^{n_1}(-\beta_1) \times \cdots \times \mathbb{C}H^{n_q}(-\beta_q)$, where $\beta = 1\big/\big(\sum_{i=1}^q 1/\beta_i\big)$.*

The author is interested in a relationship between the existence of horocycle trajectories and hyperbolic and Euclidean factors of a Hadamard Kähler manifold under a curvature condition from below.

6. Trajectories and the ideal boundary

We devote this section to study trajectories joining an interior point and a point at infinity, and trajectories joining two points at infinity. As we see in Proposition 1, on a complex hyperbolic space $\mathbb{C}H^n(-\beta)$ for each Kähler magnetic field $\mathbb{B}_\kappa$ with $|\kappa| \le \sqrt{\beta}$, there exists a unique trajectory γ for $\mathbb{B}_\kappa$ with $\gamma(0) = x$ and $\gamma(\infty) = z$ for arbitrary $x \in \mathbb{C}H^n$, $z \in \partial\mathbb{C}H^n$. We first study this property on a product manifold.

Let $M = \mathbb{C}H^{n_1}(-\beta_1) \times \cdots \times \mathbb{C}H^{n_q}(-\beta_q)$ be a product of complex hyperbolic spaces. We consider a dense open subset S of the unit tangent bundle UM given by

$$\begin{aligned} S &= S(\mathbb{C}H^{n_1}, \dots, \mathbb{C}H^{n_q}) \\ &= \big\{ v = (v_1, \dots, v_q) \in UM \,\big|\, v_i \in T\mathbb{C}H^{n_i},\ v_i \neq 0,\ 1 \le i \le q \big\}, \end{aligned}$$

and put $S(\infty)$ the set $\{\sigma_v(\infty) \,|\, v \in S\}$ of all limit points at infinity of geodesics with initial vector contained in S.

Proposition 2 *Let $\mathbb{B}_\kappa$ be a Kähler magnetic field with $|\kappa| \le \sqrt{\beta}$ on a product $M = \mathbb{C}H^{n_1}(-\beta_1) \times \cdots \times \mathbb{C}H^{n_q}(-\beta_q)$, where $\beta = 1\big/\big(\sum_{i=1}^{q} 1/\beta_i\big)$.*

(1) *For arbitrary points $x \in M$ and $z \in \partial M$, there is a trajectory γ for $\mathbb{B}_\kappa$ on M with $\gamma(0) = x$ and $\gamma(\infty) = z$.*

(2) *If $z \in S(\infty)$, then such a trajectory is unique.*

Proof. First we show the existence of trajectories. We define a smooth map $\lambda = (\lambda_1, \dots, \lambda_q) \colon S \to \mathbb{R}^q$ by

$$\lambda_i(u) = \sqrt{\{1 - (\kappa^2/\beta)\}\|u_i\|^2 + (\kappa^2/\beta_i)}, \qquad i = 1, \dots, q.$$

We should note that the Euclidean vector $\lambda(u)$ satisfies $\|\lambda(u)\| = 1$ and

$$|\kappa|/\sqrt{\beta_i} \le \lambda_i(u) < \sqrt{1 + (\kappa^2/\beta_i) - (\kappa^2/\beta)}\ \ (< 1).$$

We also define an endomorphism $\Phi_\nu^{(i)}$ of $T\mathbb{C}H^{n_i}$ for each ν with $|\nu| \le \beta_i$ by

$$\Phi_\nu^{(i)}(u_i) = \sqrt{1 - \frac{\nu^2}{\beta_i}}\, u_i - \frac{\nu}{\sqrt{\beta_i}}\, J u_i.$$

This is the inverse of the endomorphism $\Psi_\nu^{(i)}$ of $T\mathbb{C}H^{n_i}$ defined in Section 3, and preserves the norm (i.e. $\|\Phi_\nu^{(i)}(u_i)\| = \|u_i\|$). By using these we define for each κ with $|\kappa| \le \sqrt{\beta}$ a map $\Phi_\kappa \colon S \to S$ which preserve each fiber by

$$\Phi_\kappa(u) = \left(\lambda_1(u)\Phi^{(1)}_{\kappa/\lambda_1(u)}\left(\frac{u_1}{\|u_1\|}\right), \dots, \lambda_q(u)\Phi^{(q)}_{\kappa/\lambda_q(u)}\left(\frac{u_q}{\|u_q\|}\right)\right),$$

where $u = (u_1, \dots, u_q)$ with $u_i \in T\mathbb{C}H^{n_i}$. For a unit tangent vector $u \in S$ we put $v = (v_1, \dots, v_q) = \Phi_\kappa(u)$, and denote by $\sigma_u = (\sigma_1, \dots, \sigma_q)$ the geodesic with $\dot\sigma_u(0) = u$ and by $\gamma_v = (\gamma_1, \dots, \gamma_q)$ the trajectory for $\mathbb{B}_\kappa$ with $\dot\gamma_v(0) = v$. Let $\tilde\sigma_i$ be the geodesic with initial vector $u_i/\|u_i\|$, and $\tilde\gamma_i$ be a smooth curve defined by $\tilde\gamma_i(t) = \gamma_i(t/\lambda_i(u))$, which is a trajectory for $\mathbb{B}_{\kappa/\lambda_i(u)}$ on $\mathbb{C}H^{n_i}$ with $\dot{\tilde\gamma}_i(0) = v_i/\|v_i\|$.

When $|\kappa| < \sqrt{\beta}$, as $|\kappa|/\lambda_i(u) < \sqrt{\beta_i}$, we see by Lemma 3 that the distance function

$$t \mapsto d\Big(\tilde{\sigma}_i\big(\sqrt{1-\{\kappa^2/(\lambda_i(u)^2\beta_i)\}}\,t\big), \tilde{\gamma}_i(t)\Big)$$

is bounded for $t \geq 0$. Since

$$\begin{aligned} & d\Big(\sigma_i\big(\sqrt{1-(\kappa^2/\beta)}\,t\big), \gamma_i(t)\Big) \\ &\quad = d\Big(\tilde{\sigma}_i\big(\|u_i\|\sqrt{1-(\kappa^2/\beta)}\,t\big), \tilde{\gamma}_i(\lambda_i(u)t)\Big) \\ &\quad = d\Big(\tilde{\sigma}_i\big(\sqrt{1-\{\kappa^2/(\lambda_i(u)^2\beta_i)\}}\,\lambda_i(u)t\big), \tilde{\gamma}_i(\lambda_i(u)t)\Big), \end{aligned}$$

we see the distance function $t \mapsto d\big(\sigma_u\big(\sqrt{1-(\kappa^2/\beta)}\,t\big), \gamma_v(t)\big)$ is bounded for $t \geq 0$, hence $\sigma_u(\infty) = \gamma_v(\infty)$. When $\kappa = \pm\sqrt{\beta}$, as $\|v_i\| = \lambda_i(u) = \sqrt{\beta/\beta_i}$, we find by Lemma 5 that $\sigma_u(\infty) = \gamma_v(\infty)$. In both cases, as we have

$$\begin{aligned} UM = \Big(\bigcup_i U\mathbb{C}H^{n_i}\Big) \cup \Big(\bigcup_{i_1<i_2} S(\mathbb{C}H^{n_{i_1}}, \mathbb{C}H^{n_{i_2}})\Big) \\ \cup \cdots \cup S(\mathbb{C}H^{n_1}, \ldots, \mathbb{C}H^{n_q}), \end{aligned}$$

we obtain the first assertion.

Next we study the uniqueness. If a trajectory $\gamma = (\gamma_1, \ldots, \gamma_q)$ for a Kähler magnetic field $\mathbb{B}_\kappa$ ($|\kappa| \leq \sqrt{\beta}$) on M satisfies $\gamma(\infty) \in S(\infty)$, then each γ_i should be unbounded, hence it satisfies $\|\dot{\gamma}_i\| \geq |\kappa|/\sqrt{\beta_i}$. Since $\|\dot{\gamma}\| = 1$, we find for each i that

$$|\kappa|/\sqrt{\beta_i} \leq \|\dot{\gamma}_i\| \leq \sqrt{1+(\kappa^2/\beta_i)-(\kappa^2/\beta)}.$$

When $\kappa = \pm\sqrt{\beta}$, we see $\|\dot{\gamma}_i\| = \sqrt{\beta/\beta_i}$, hence $\tilde{\gamma}_i$ is a horocycle for all i. Thus the uniqueness follows from the uniqueness of horocycle trajectories joining an interior point and a point at infinity on complex hyperbolic spaces (Proposition 1 (4)). When $|\kappa| < \sqrt{\beta}$, we see $\|\dot{\gamma}_{i_0}\| > \kappa/\sqrt{\beta_{i_0}}$ for some i_0, hence $\tilde{\gamma}_{i_0}$ is not a horocycle. Therefore, as $\gamma(\infty) \in S(\infty)$, we get by Lemma 7 that $\tilde{\gamma}_i$ is not a horocycle for all i, hence $\|\dot{\gamma}_i\| > \kappa/\sqrt{\beta_i}$ for all i. Thus we find the initial vector of γ is contained in the image of Φ_κ. Therefore we obtain the second assertion because Φ_κ is injective. □

Remark 2 Under the situation in Proposition 2, there exist infinitely many trajectories for $\mathbb{B}_\kappa$ joining x and z when $z \notin S(\infty)$. For example, we consider the case $M = \mathbb{C}H^{n_1}(-\beta_1) \times \mathbb{C}H^{n_2}(-\beta_2)$. For a unit tangent vector

$v = (v_1, 0) \in U_xM$ at a point $x = (x_1, x_2)$, we have $\gamma_v(\infty) \in \partial\mathbb{C}H^{n_1} \subset \partial M$. Given a tangent vector $u_2 \in T_{x_2}\mathbb{C}H^{n_2}$ with $0 < \|u_2\| < |\kappa|/\sqrt{\beta_2}$ we put $u = \bigl(\sqrt{1-\|u_2\|^2}\, v_1, u_2\bigr) \in U_xM$. We find that the distance between $\gamma_v\bigl(\sqrt{1-\|u_2\|^2}\, t\bigr)$ and $\gamma_u(t)$ is bounded, hence $\gamma_v(\infty) = \gamma_u(\infty)$.

In the last stage we study the property joining two points in the ideal boundary of M. Although two distinct points in the ideal boundary of a Hadamard manifold of strictly negative curvature can be joined by some geodesic, such property does not necessarily hold for a Hadamard manifold in general. We shall call two points z and w in ∂M are *antipodal* if there exists a geodesic σ with $\sigma(\infty) = z$ and $\sigma(-\infty) = w$. On a product $M_1 \times M_2$ of Hadamard manifolds, it is clear that if two points z, w are antipodal and $z \in S(M_1, M_2)(\infty)$ then w is also contained in $S(M_1, M_2)(\infty)$. Corresponding to the third assertion of Proposition 1, we have the following for a product of complex hyperbolic spaces.

Proposition 3 *We consider a Kähler magnetic field $\mathbb{B}_\kappa$ with $|\kappa| < \sqrt{\beta}$ on a product $M = \mathbb{C}H^{n_1}(-\beta_1) \times \cdots \times \mathbb{C}H^{n_q}(-\beta_q)$, where $\beta = 1/\bigl(\sum_{i=1}^{q} 1/\beta_i\bigr)$.*

(1) *Two points $z, w \in \partial M$ are antipodal if and only if there is a trajectory γ for $\mathbb{B}_\kappa$ with $\gamma(\infty) = z$ and $\gamma(-\infty) = w$.*

(2) *If two trajectories $\gamma = (\gamma_1, \ldots, \gamma_q)$ and $\rho = (\rho_1, \ldots, \rho_q)$ satisfy $\gamma(\infty) = \rho(\infty)$, $\gamma(-\infty) = \rho(-\infty)$ and these points at infinity are contained in $S(\infty)$, then there exists a vector $(t_1, \ldots, t_q)$ with $\rho_i(t) = \gamma_i(t + t_i)$, $i = 1, \ldots, q$.*

Proof. We are enough to show the first assertion in the case z, $w \in S(\infty)$, because

$$UM = \Bigl(\bigcup_i U\mathbb{C}H^{n_i}\Bigr) \cup \Bigl(\bigcup_{i_1 < i_2} S(\mathbb{C}H^{n_{i_1}}, \mathbb{C}H^{n_{i_2}})\Bigr) \cup \cdots \cup S(\mathbb{C}H^{n_1}, \ldots, \mathbb{C}H^{n_q}).$$

We define an injective map $G_\kappa \colon S \to S$ by

$$G_\kappa(u) = \left(\lambda_1(u) g^{(1)}_{\kappa/\lambda_1(u)}\left(\frac{u_1}{\|u_1\|}\right), \ldots, \lambda_q(u) g^{(q)}_{\kappa/\lambda_q(u)}\left(\frac{u_q}{\|u_q\|}\right)\right),$$

where $g^{(i)}_\nu$ denotes the diffeomorphism of $U\mathbb{C}H^{n_i}$ defined in Section 3 and $\lambda = (\lambda_1, \ldots, \lambda_q) \colon S \to (0,1) \times \cdots \times (0,1)$ is given by

$$\lambda_i(u) = \sqrt{\{1-(\kappa^2/\beta)\}\|u_i\|^2 + (\kappa^2/\beta_i)}, \qquad i = 1, \dots, q.$$

As $\lambda_i(u) > |\kappa|/\sqrt{\beta_i}$, we find that $F_\kappa \circ G_\kappa$ is the identity map with the map F_κ defined in Section 4. As a matter of fact on $G_\kappa(S)$ we see F_κ is given as

$$F_\kappa(v) = \left(\mu_1(v) f^{(1)}_{\kappa/\|v_1\|}\left(\frac{v_1}{\|v_1\|}\right), \dots, \mu_q(v) f^{(q)}_{\kappa/\|v_q\|}\left(\frac{v_q}{\|v_q\|}\right)\right)$$

with $\mu = (\mu_1, \dots, \mu_q) \colon G_\kappa(S) \to (0,1) \times \cdots \times (0,1)$ defined by

$$\mu_j(v) = \sqrt{\{\|v_j\|^2 - (\kappa^2/\beta_j)\}/\{1-(\kappa^2/\beta)\}}, \qquad j = 1, \dots, q,$$

and the diffeomorphism $f^{(j)}_\nu = (g^{(j)}_\nu)^{-1}$. Therefore Lemma 6 guarantees the assertion.

The second assertion follows directly from the third assertion of Proposition 1. $\square$

Remark 3 When z, $w \notin S(\infty)$, we have infinitely many trajectories joining them.

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