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A maximal inequality associated to Schrödinger type equation

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Abstract. In this note, we consider a maximal operator $\sup_{t\in\mathbb{R}}|u(x,t)| = \sup_{t\in\mathbb{R}}|e^{it\Omega(D)}f(x)|$, where u is the solution to the initial value problem $u_t = i\Omega(D)u$, $u(0) = f$ for a C^2 function Ω with some growth rate at infinity. We prove that the operator $\sup_{t\in\mathbb{R}}|u(x,t)|$ has a mapping property from a fractional Sobolev space $H^{\frac{1}{4}}$ with additional angular regularity in which the data lives to $L^2((1+|x|)^{-b}dx)$ $(b>1)$. This mapping property implies the almost everywhere convergence of $u(x,t)$ to f as $t\to 0$, if the data f has an angular regularity as well as $H^{1/4}$ regularity.

Key words: Schrödinger type equation, maximal operator, angular regularity.

1. Introduction

We consider the following free Schrödinger type equation:

$$\frac{\partial}{\partial t}u(x,t) = i\Omega(D)u(x,t) \quad \text{in} \quad \mathbb{R}^{n+1}\ (n\ge 2), \quad u(x,0) = f(x),$$

where $\Omega(D)$ is a generalized differential operator defined by a C^2 function Ω and $D = (-\Delta)^{1/2}$. For smooth initial data f, the solution $u(x,t) = e^{it\Omega(D)}f$ can be written as

$$u(x,t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x\cdot\xi + t\Omega(\xi))}\widehat{f}(\xi)\,d\xi, \qquad f\in\mathcal{S}(\mathbb{R}^n),$$

where $\widehat{f}(\xi) = \int e^{-ix\cdot\xi}f(x)\,dx$. In this note, we assume that the initial data f has H^s regularity for some $s>0$ as well as some regularity in the angular direction. For α, $\beta\ge 0$, we define an initial data space $H_r^\alpha H_\omega^\beta$ by

$$H_r^\alpha H_\omega^\beta = \Big\{f : \|f\|_{H_r^\alpha H_\omega^\beta} := \|(1-\Delta)^{\alpha/2}f\|_{L_r^2 H_\omega^\beta} < \infty\Big\},$$

where $\|g\|_{L_r^2}^2 = \int_0^\infty |g(r)|^2 r^{n-1}\,dr$, $\|g\|_{L_r^2 H_\omega^\beta} = \big\|\|(1-\Delta_\omega)^{\beta/2}f(r\omega)\|_{L_\omega^2}\big\|_{L_r^2}$ (here, $(r,\omega)\in\mathbb{R}_+\times S^{n-1}$ is the spherical coordinates), and Δ_ω is the Laplace-Beltrami operator on S^{n-1}. Since Δ_ω commutes with Δ, one can

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readily check that $\|g\|_{H_r^\alpha H_\omega^\beta} \sim \|(1-\Delta_\omega)^{\beta/2} g\|_{H^\alpha}$ (for instance, see [9]). There is no embedding from or into a usual Sobolev space because not every function in $H_r^\alpha H_\omega^\beta$ has radial regularity higher than α. In particular, it should be noted that $H_r^\alpha H_\omega^\beta \not\subseteq H^{\alpha+\gamma}$ $(0 < \gamma < \beta)$ and $H_r^\alpha H_\omega^\beta \not\supseteq H^{\alpha+\gamma}$ $(\gamma \geq \beta)$.

We also assume that $\Omega \in C^2(\mathbb{R}^n)$ is radially symmetric and satisfies

$$c_1|\rho|^{a-k} \leq |\Omega^{(k)}(\rho)| \leq c_2|\rho|^{a-k} \quad (k = 0,\, 1,\, 2), \qquad \text{if} \quad |\rho| \geq N$$

for some c_1, c_2, $a > 0$ with $a \neq 1$ and a large $N > 0$. With the above assumptions, let us define a maximal function $u^*(x)$ by $u^*(x) = \sup_{t\in\mathbb{R}} |u(x,t)|$.

Our main result is the following.

Theorem 1.1 *For any $\varepsilon > 0$ and $b > 1$, if $f \in H_r^{1/4} H_\omega^{(n-1)/2-1/4+\varepsilon}$, then there exists a constant C, depending only on a, c_1, c_2, N, n, ε, b, such that*

$$\|u^*\|_{L^2((1+|x|)^{-b}dx)} \leq C\|f\|_{H_r^{1/4} H_\omega^{(n-1)/2-1/4+\varepsilon}}.$$

Now let us define a linear operator T and a maximal operator T^* for a fixed $s > 0$ by

$$Tf(x,t) = w(|x|) \int e^{i(x\cdot\xi + t\Omega(\xi))} \widehat{f}(\xi)\, \frac{d\xi}{(1+|\xi|^2)^{s/2}},$$

where $w(r) = (1+r)^{-b/2}$, $b > 0$ and

$$T^*f(x) = \sup_{t\in\mathbb{R}} |Tf(x,t)|.$$

Then Theorem 1.1 follows immediately from

Theorem 1.2 *For any $\varepsilon > 0$ and $b > 1$, if $f \in L_r^2 H_\omega^{(n-1)/2-s+\varepsilon}$ for some $s \in \big[1/4, 1/2\big)$, there exists a constant C, depending only on a, c_1, c_2, N, n, s, ε, b, such that*

$$\|T^*f\|_{L^2} \leq C\|f\|_{L_r^2 H_\omega^{(n-1)/2-s+\varepsilon}}.$$

The maximal function u^* and operator T^* have been studied by many authors ([1, 2, 3, 4, 5, 7, 8, 10, 11, 12, 13, 14, 18, 19, 21]). P. Sjölin [14] and L. Vega [19] showed that for a ball B_R of radius R

$$\|u^*\|_{L^2(B_R)} \leq C\|f\|_{H^s}, \tag{1.1}$$

only if $s \geq 1/4$. It has been known that (1.1) is true, when $n = 1$ ([5, 8]) or the initial data is radial ([4, 12]), or $s > 1/2$ and $n \geq 2$ ([11, 19]). Recently, T. Tao [18] obtained (1.1) with $\Omega = |\xi|^2$ for $s > 2/5$ and $n = 2$. However, the sufficiency remains open.

Theorem 1.1 shows that it is true for $s = 1/4$ if we assume the additional angular regularity. When the initial data is a finite linear combination of radial functions and spherical harmonics such that $f = \sum_{k \leq L} f_k Y_k$, it was proved by the authors of [4] that $\|u^*\|_{L^{4n/(2n-1)}} \leq C_L \|f\|_{H^{1/4}}$, where

$$C_L \leq CL^{1/2+\varepsilon}(n+2L)^{(n+2L)/2} \max_{1 \leq k \leq L} \frac{\|Y_k\|_{L^{4n/(2n-1)}}}{\|Y_k\|_{L^2}} \quad (0 < \varepsilon \ll 1).$$

The factor $(n+2L)^{(n+2L)/2}$ is due to the asymptotic behavior of Bessel function ($J_\nu(t) = b_+ t^{-1/2} e^{it} + b_- t^{-1/2} e^{-it} + O((n+2\nu)^{(n+2\nu)/2}) t^{-3/2}$ for $t > 1$). The tail $t^{-3/2}$ was used crucially for the non-weighted global $L^{4n/(2n-1)}$ ($4n/(2n-1) > 2$) estimate. It seems that one cannot avoid a big cost of C_L for this global estimate. In view of this point, Theorem 1.1 improves the dependency on the order of spherical harmonic up to $L^{3/4+\varepsilon}$ (see (2.2) below). This improvement results from an estimate for the tail of Bessel function Ct^{-1} for $t > 2\nu$, which enables us to use the L^2 method. The weighted L^2 estimate as in Theorem 1.1 is necessary for a global estimate because the non-weighted global L^2 estimate [11] and any local estimate in L^p $(p > 2)$ [22] are impossible for the data $f \in H^{1/4}$.

In case that $\Omega(D) = -\Delta$, recently G. Gigante and F. Soria [6] showed a local L^2 estimate that $\|u^*\|_{L^2(B_R)} \leq CL^{1/2+\varepsilon}\|f\|_{H^{\frac{1}{4}}}$. They used a finer asymptotic behavior of Bessel function $J_\nu(t)$ for $\nu + \nu^{1/3} \leq t \leq 2\nu$ but their method does not seem to be applied directly to the general phase Ω like ours.

Obvious examples of our Ω are $\Omega(\xi) = |\xi|^a$, $a > 0$, $\Omega(\xi) = \sum_{i=1}^{l} m_i |\xi|^{a_i}$ for any number $a_l > a_{l-1} > \cdots > a_1 > 0$, $a_l \neq 1$ and $m_i \in \mathbb{R}$. For more general phase Ω, we refer the readers to [3] in which a weighted L^2 estimate is discussed with the phase Ω which allows $\nabla\Omega$ to have zeros or singularities. Another use of angular regularity can be found in [9] where the endpoint Strichartz estimates of 3-d wave and Klein-Gordon equations are considered.

If not specified, throughout this paper, C denotes a generic constant that depends on a, c_1, c_2, N, n, s, b, ε. We use the notation $A \lesssim B$ and $A \sim B$ to denote $|A| \leq CB$ and $C^{-1}B \leq |A| \leq CB$ respectively.

2. Proof of Theorem 1.2

We begin with reviewing some properties of the spherical harmonic expansion. If $f(r\omega) = g(r)Y_k(\omega)$ for a radial function g and a spherical harmonic Y_k of order k, then we have

$$\widehat{f}(\rho\theta) = G(\rho)Y_k(\theta), \quad \|g\|_{L^2_r} = \|G\|_{L^2_r},$$

where

$$G(\rho) = c_{n,k}\int_0^\infty g(r)r^{n-1}(r\rho)^{-(n-2)/2}J_\nu(r\rho)\,dr$$

with $|c_{n,k}| \le C$ and $\nu = (2k+n-2)/2$ (see e.g. [16] or [22]). Since $-\Delta_\omega Y_k = k(k+n-2)Y_k$, we also have $\|f\|_{L^2_rH^\beta_\omega} \sim (1+k^2)^{\beta/2}\|g\|_{L^2_r}\|Y_k\|_{L^2_\omega}$. Furthermore, if $h \in L^2_rH^\beta_\omega$, then there exist radial functions $\{h^l_k\}$ and spherical harmonics $\{Y^l_k\}$ such that

$$h(r\omega) = \sum_{k\ge 0}\sum_{1\le l\le d(k)} h^l_k(r)Y^l_k(\omega) \quad \text{in} \quad L^2_rH^\beta_\omega,$$

where $d(k)$ is the dimension of the space of spherical harmonics of degree k, and

$$\|h\|^2_{L^2_rH^\beta_\omega} \sim \sum_{k\ge 0}\sum_{1\le l\le d(k)}(1+k^2)^\beta\|h^l_k\|^2_{L^2_r}\|Y^l_k\|^2_{L^2_\omega}. \tag{2.1}$$

Thus for the proof of theorem, we have only to consider the case $f(r\omega) = g(r)Y_k(\omega)$ and to show that for large k

$$\|T^*f\|_{L^2} \lesssim k^{1/2-s}\|g\|_{L^2_r}\|Y_k\|_{L^2_\omega}, \tag{2.2}$$

since for the function $h(r\omega) = \sum_{k\ge 0}\sum_{1\le l\le d(k)} h^l_k(r)Y^l_k(\omega)$ in $L^2_rH^\beta_\omega$, using (2.2), we have

$$\begin{aligned}\|T^*h\|_{L^2} &\lesssim \sum_k\sum_{1\le l\le d(k)} k^{1/2-s}\|h^l_k\|_{L^2_r}\|Y^l_k\|_{L^2_\omega}\\ &\lesssim \sum_k k^{1/2-s}d(k)^{1/2}\Bigg(\sum_{1\le l\le d(k)}\|h^l_k\|^2_{L^2_r}\|Y^l_k\|^2_{L^2_\omega}\Bigg)^{1/2}\\ &\lesssim \sum_k k^{(n-1)/2-s}\Bigg(\sum_{1\le l\le d(k)}\|h^l_k\|^2_{L^2_r}\|Y^l_k\|^2_{L^2_\omega}\Bigg)^{1/2}\end{aligned}$$

$$\lesssim \left(\sum_k \sum_{1\le l\le d(k)} k^{n-1-2s+\varepsilon} \|h_k^l\|_{L_r^2}^2 \|Y_k^l\|_{L_\omega^2}^2 \right)^{1/2}.$$

Here we used the estimate

$$d(k) = \frac{n+2k-2}{k}\binom{n+k-3}{k-1} \lesssim k^{n-2}$$

for the third inequality (see [16]).

Now if $\widehat{f}(\rho\omega) = G(\rho)Y_k(\omega)$, from the definition of T,

$$\begin{aligned} &Tf(r\omega,t) \\ &= w(r)\int_{S^{n-1}}\int_0^\infty e^{i(r\omega\cdot\rho\theta+t\Omega(\rho))}G(\rho)Y_k(\theta)\rho^{n-1}\frac{d\rho}{(1+\rho^2)^{s/2}}\,d\theta \\ &= c_{n,k}w(r)\int_0^\infty e^{it\Omega(\rho)}(r\rho)^{-(n-2)/2}J_\nu(r\rho)\rho^{n-1}G(\rho)\,\frac{d\rho}{(1+\rho^2)^{s/2}}Y_k(-\omega). \end{aligned}$$

We define an operator S by

$$\begin{aligned} SG(r,t) &= c_{n,k}r^{(n-1)/2}w(r) \\ &\qquad\times\int_0^\infty e^{it\Omega(\rho)}(r\rho)^{-(n-2)/2}J_\nu(r\rho)\rho^{(n-1)/2}G(\rho)\frac{d\rho}{(1+\rho^2)^{s/2}}. \end{aligned}$$

Let us denote by $\|F\|_{L^pL^q}$ the mixed norm $\|(\|F(r,t)\|_{L^q(dt)})\|_{L^p(dr)}$. Here we use the notation $\|F\|_{L^p(dr)}^p$ for $\int |F(r)|^p\,dr$ to avoid the confusion with $\|F\|_{L_r^p}$. To prove (2.2) it suffices to show that

$$\|S\tilde{G}\|_{L^2L^\infty} \lesssim k^{1/2-s}\|\tilde{G}\|_{L^2(dr)}, \tag{2.3}$$

where $\tilde{G}(\rho) = \rho^{(n-1)/2}G(\rho)$. Now the dual operator S^d of S is given by

$$S^dF(\rho) = \frac{c_{n,k}}{(1+\rho^2)^{s/2}}\int_{\mathbb{R}}\int_0^\infty e^{-it\Omega(\rho)}(r\rho)^{1/2}J_\nu(r\rho)w(r)F(r,t)\,dr\,dt$$

for $F \in C_0^\infty(\mathbb{R}_+\times\mathbb{R})$. Then, by duality (2.3) follows from

$$\|S^dF\|_{L^2(dr)} \le Ck^{1/2-s}\|F\|_{L^2L^1}. \tag{2.4}$$

Choose smooth cut-off functions ϕ_0, ϕ_1 and ϕ_3 so that $\phi_0 = 1$ on $\{|s| < 1/4\}$, $\phi_0 = 0$ on $\{|s| > 1/2\}$, $\phi_1 = 1$ on $\{|s| \sim 1\}$, $\phi_1 = 0$ otherwise, $\phi_2 = 0$ on $\{|s| < 2\}$, $\phi_2 = 1$ on $\{|s| > 3\}$, and $\phi_0+\phi_1+\phi_2 = 1$.

Then we decompose S^d as

$$S^d F(\rho) = S_0 F + S_1 F + S_2 F,$$

where for $i = 0, 1, 2$,

$$S_i F(\rho) = \frac{c_{n,k}}{(1+\rho^2)^{s/2}} \times \int_{\mathbb{R}} \int_0^\infty e^{-it\Omega(\rho)} (r\rho)^{1/2} J_\nu(r\rho) \phi_i\Big(\frac{r\rho}{\nu}\Big) w(r) F(r,t)\, dr\, dt.$$

Now we need to show each S_i satisfies (2.4) in the place of S^d. Each estimate is to be shown using the following asymptotic behavior of Bessel functions:

$$|J_\nu(t)| \le C \exp(-C\nu), \qquad \text{if} \quad t \le \frac{\nu}{2}, \tag{2.5}$$

$$\frac{1}{r}\int_0^r |J_\nu(t)|^2 t\, dt \le C \qquad \text{for all} \quad r > 0, \tag{2.6}$$

$$J_\nu(t)\phi_2\Big(\frac{t}{\nu}\Big) = t^{-1/2}(b_+ e^{it} + b_- e^{-it})\phi_2\Big(\frac{t}{\nu}\Big) + \Phi_\nu(t)\phi_2\Big(\frac{t}{\nu}\Big), \tag{2.7}$$

where $|\Phi_\nu(t)| \le C/t$, $|b_\pm| \le C$ and the constant C is independent of ν. For the proof of (2.5), see [17]. The mean value estimate (2.6) can be found in Section 4.10 of [20]. Invoking the Schläfli's integral representation (see p. 176 in [23]):

$$J_\nu(t) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(t\sin\theta - \nu\theta)}\, d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-\nu\tau - t\sinh\tau}\, d\tau,$$

(2.7) follow from the easy estimate

$$\left|\frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-\nu\tau - t\sinh\tau}\, d\tau\right| \le \frac{C}{\nu + t}$$

and the method of stationary phase, which gives

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i(t\sin\theta - \nu\theta)}\, d\theta = (b_+ e^{it} + b_- e^{-it}) t^{-1/2} + O\big(t^{-3/2}\big) \qquad \text{for } t > 2\nu.$$

Using (2.5), we now see

$$|S_0 F(\rho)| \lesssim \nu^{1/2} e^{-C\nu} (1+\rho^2)^{-s/2} \int_0^{\nu/\rho} w(r) \|F(r,\cdot)\|_{L^1}\, dr$$

$$
\begin{aligned}
&= \nu^{1/2} e^{-C\nu}(1+\rho^2)^{-s/2}\Big(\int_0^{\min(\nu/\rho,2)} \|F(r,\,\cdot\,)\|_{L^1}\,dr \\
&\qquad\qquad + \int_0^{\nu/\rho} \chi_{[2,\infty)}(r) w(r)\|F(r,\,\cdot\,)\|_{L^1}\,dr\Big) \\
&\lesssim \nu^{1/2} e^{-C\nu}(1+\rho^2)^{-s/2} \\
&\quad \times \bigg(\Big(\min\Big(\frac{\nu}{\rho},\,2\Big)\Big)^{1/2} + \chi_{[0,\nu/2](\rho)}\bigg)\|F\|_{L^2L^1}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\|S_0F\|_{L^2(dr)} \\
&\lesssim \nu^{1/2}e^{-C\nu}\bigg(\int_0^\infty (1+\rho^2)^{-s}\Big(\min\Big(\frac{\nu}{\rho},2\Big) + \chi_{[0,\nu/2]}(\rho)\Big)\,d\rho\bigg)^{1/2}\|F\|_{L^2L^1} \\
&\lesssim \nu^{1-s}e^{-C\nu}\|F\|_{L^2L^1}. \qquad (2.8)
\end{aligned}
$$

For S_1,

$$
\begin{aligned}
|S_1F(\rho)| &\lesssim (1+\rho^2)^{-s/2}\bigg(\int_0^\infty J_\nu^2(r\rho)r\rho\phi_1^2\Big(\frac{r\rho}{\nu}\Big)w(r)^2\,dr\bigg)^{1/2}\|F\|_{L^2L^1} \\
&\lesssim (1+\rho^2)^{-s/2}\bigg(\int_0^2 + \int_2^\infty\bigg)^{1/2}\|F\|_{L^2L^1}.
\end{aligned}
$$

Changing variables $r \mapsto r/\rho$, the first part in the middle parenthesis is bounded by $\chi_{[\nu/4,\,\infty)}(\rho)(1/\rho)\int_0^{2\rho} J_\nu^2(r)r\phi_1^2(r/\nu)\,dr$. By (2.6), it follows that

$$
\int_0^2 \lesssim \nu\rho^{-1}\chi_{[\nu/4,\,\infty)}(\rho).
$$

For the second part, by the change of variable $r \mapsto r/\rho$ and (2.6)

$$
\int_2^\infty \lesssim \rho^{b-1}\int_{\max(2\rho,\,\nu/2)}^{3\nu} J_\nu^2(r)r^{1-b}\,dr \lesssim \nu\rho^{b-1}\Big(\max\Big(2\rho,\,\frac{\nu}{2}\Big)\Big)^{-b}.
$$

We thus obtain

$$
\begin{aligned}
&\|S_1F\|_{L^2(dr)} \\
&\lesssim \bigg(\int_0^\infty (1+\rho^2)^{-s}\Big(\nu\rho^{-1}\chi_{[\nu/4,\infty)}(\rho) + \nu\rho^{b-1}\Big(\max\Big(2\rho,\,\frac{\nu}{2}\Big)\Big)^{-b}\Big)\,d\rho\bigg)^{1/2} \\
&\quad \times \|F\|_{L^2L^1} \\
&\lesssim \nu^{1/2-s}\|F\|_{L^2L^1}. \qquad (2.9)
\end{aligned}
$$

Now we estimate S_2F. Let us set $S_2F = S_+F + S_-F + S_3F$, where

$$S_\pm F(\rho) = \frac{c_{n,k}b_\pm}{(1+\rho^2)^{s/2}} \int_{\mathbb{R}} \int_0^\infty e^{i(\pm r\rho - t\Omega(\rho))} \phi_2\Big(\frac{r\rho}{\nu}\Big) w(r)F(r,t)\,dr\,dt,$$

$$S_3F(\rho) = \frac{c_{n,k}}{(1+\rho^2)^{s/2}} \int_{\mathbb{R}} \int_0^\infty e^{-it\Omega(\rho)}(r\rho)^{1/2} \times \Phi_\nu(r\rho)\phi_2\Big(\frac{r\rho}{\nu}\Big) w(r)F(r,t)dr\,dt.$$

For the estimate $S_\pm F$, it suffices to consider S_+F. We decompose it into two parts as follows:

$$S_+F(\rho) = I + II$$

where

$$I = \frac{c_{n,k}b_+}{(1+\rho^2)^{s/2}} \int_{\mathbb{R}} \int_0^\infty e^{i(r\rho - t\Omega(\rho))} w(r)F(r,t)\,dr\,dt,$$

$$II = \frac{c_{n,k}b_+}{(1+\rho^2)^{s/2}} \int_{\mathbb{R}} \int_0^\infty e^{i(r\rho - t\Omega(\rho))} \Big(\phi_2\Big(\frac{r\rho}{\nu}\Big) - 1\Big) w(r)F(r,t)\,dr\,dt.$$

For II, we have

$$|II(\rho)| \lesssim (1+\rho^2)^{-s/2} \int_0^{3\nu/\rho} w(r)\|F(r,\cdot)\|_{L^1}\,dr \lesssim (1+\rho^2)^{-s/2} \Big(\int_0^{3\nu/\rho} w(r)^2\,dr\Big)^{1/2} \|F\|_{L^2L^1}$$

and hence by the same computation as in (2.8)

$$\|II\|_{L^2(dr)} \lesssim \nu^{1/2-s}\|F\|_{L^2L^1}. \tag{2.10}$$

Now we estimate I. Since F is in $C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$, obviously we may assume

$$I = \frac{c_{n,k}b_+}{(1+\rho^2)^{s/2}} \int_{\mathbb{R}^2} e^{i(r\rho - t\Omega(\rho))} w(|r|)F(r,t)\,dr\,dt.$$

Squaring and integrating I over $\{|\rho| \le N\}$ (here N is the number in the condition of Ω), we have

$$\int_{|\rho|<N} |I|^2\,d\rho \le C\|F\|^2_{L^2L^1}. \tag{2.11}$$

It is easy to see

$$\int_{|\rho|>N} |I|^2 \, d\rho$$
$$\le C \iiiint |K(r-r', t-t')w(|r|)|F(r,t)|w(|r'|)|F(r',t')| \, dr \, dr' dt \, dt',$$

where

$$K(r,t) = \int_{|\rho|>N} e^{i(r\rho - t\Omega(\rho))} \frac{d\rho}{|\rho|^{2s}}.$$

To estimate K, we use a lemma which gives a uniform bound for kernel K in t.

Lemma 2.1 (see Lemma 2.3 in [4]) *For any real number A, B $(A \neq 0)$ and $s \in \big[1/2, 1\big)$, there exists a constant C, independent of A and B, such that*

$$\left| \int_{|\rho|>N} e^{i(A\Omega(\rho)+B\rho)} \frac{d\rho}{|\rho|^s} \right| \le C|B|^{-(1-s)}.$$

Applying Lemma 2.1 with $2s$ $(1/4 \le s < 1/2)$ and $B = r - r'$, from fractional integration and Hölder inequality it follows

$$\int_{|\rho|>N} |I|^2 d\rho$$
$$\lesssim \iint |r-r'|^{-(1-2s)} w(|r|) \|F(r,\cdot)\|_{L^1(dt)} w(|r'|) \|F(r',\cdot)\|_{L^1(dt)} \, dr \, dr'$$
$$\lesssim \big\| \mathcal{I}_{2s}(w\|F\|_{L^1(dt)}) \big\|_{L^p(dr)} \big\| w\|F\|_{L^1(dt)} \big\|_{L^{p'}(dr)} \qquad \left(\frac{1}{p} = \frac{1}{p'} - 2s \right)$$
$$\lesssim \|wF\|^2_{L^{2/(1+2s)}L^1} \lesssim \|w\|^2_{L^{1/s}} \|F\|^2_{L^2L^1} \qquad \left(\frac{b}{2} \cdot \frac{1}{s} > 1 \right)$$
$$\lesssim \|F\|^2_{L^2L^1}, \tag{2.12}$$

where $\mathcal{I}_{2s}$ is the Riesz potential of order $2s$.

Finally, we estimate S_3F. From the uniform bound of Φ_ν on ν, for small $\varepsilon > 0$, we have

$$|S_3F(\rho)| \lesssim \frac{1}{(1+\rho^2)^{s/2}} \int (r\rho)^{-1/2} \phi_2\Big(\frac{r\rho}{\nu}\Big) w(r) \|F(r,\cdot)\|_{L^1} \, dr$$

$$
\begin{aligned}
&\lesssim \rho^{-s-1/2}\chi_{[\nu,\infty)}(\rho)\int_{2\nu/\rho}^{2} r^{-1/2}\|F(r,\cdot)\|_{L^1}\,dr \\
&\quad + \rho^{-s-1/2}\int_{\max(2,2\nu/\rho)} r^{-1/2-b/2}\|F(r,\cdot)\|_{L^1}\,dr \\
&\lesssim \nu^{-\delta}\rho^{-s-1/2+\delta}\chi_{[\nu,\infty)}(\rho)\int_{2\nu/\rho}^{2} r^{-1/2+\delta}\|F(r,\cdot)\|_{L^1}\,dr \\
&\quad + \rho^{-s-1/2}\Big(\max\Big(2,\frac{2\nu}{\rho}\Big)\Big)^{-b/2}\|F\|_{L^2L^1} \\
&\lesssim \Big(\nu^{-\delta}\rho^{-s-1/2+\delta}\chi_{[\nu,\infty)}(\rho)+\rho^{-s-1/2}\Big(\max\Big(2,\frac{2\nu}{\rho}\Big)\Big)^{-b/2}\Big) \\
&\qquad\qquad \times\|F\|_{L^2L^1}.
\end{aligned}
$$

Choosing δ as $1/8$, we obtain

$$\|S_3F\|_{L^2(dr)} \lesssim \nu^{-s}\|F\|_{L^2L^1}. \tag{2.13}$$

Combining all the estimates from (2.8) to (2.13) and recalling $\nu = (2k+n-2)/2$, we get (2.4) and hence Theorem 1.2.

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References

[1] Bourgain J., *A remark on Schrödinger operators*. Israel J. Math. **77** (1992), 1–16.

[2] Carleson L., *Some analytical problems related to statistical mechanics*. Euclidean Harmonic Analysis, Lecture Notes in Math. **779** (1979), 5–45.

[3] Cho Y. and Shim Y., *Weighted L^2 estimates for maximal operators associated to dispersive equation*. Illinois J. Math. **48** (2004), 1081–1092.

[4] Cho Y. and Shim Y., *Global estimates of maximal operators generated by dispersive equations*. Hokkaido Univ. Preprint Series in Mathematics ♯704, (http://eprints.math.sci.hokudai.ac.jp/archive/00000883/).

[5] Dahlberg B.E.J. and Kenig C.E., *A note on almost everywhere behavior of solutions to the Schrödinger equation*. Harmonic Analysis, Lecture Notes in Math. **908** (1982), 205–209.

[6] Gigante G. and Soria F., *On the boundedness in $H^{\frac{1}{4}}$ of the maximal square function associated with the Schrödinger equation*. in preprint.

[7] Heinig H.P. and Wang S., *Maximal function estimates of solutions to general dispersive partial differential equations*. Trans. Amer. Math. Soc. **351**(1) (1999), 79–108.

[8] Kenig C.E. and Ruiz A., *A strong type* $(2,2)$ *estimate for a maximal oprerator associated to the Schrödinger equation*. Trans. Amer. Math. Soc. **280** (1983), 239–246.

[9] Machihara S., Nakamura M., Nakanishi K. and Ozawa T., *Endpoint Strichartz estimates and global solutions for the noninear Dirac equation*. J. Func. Anal. **219** (2005), 1–20.

[10] Moyua A., Vargas A. and Vega L., *Restriction theorems and Maximal operators related to oscillatory integrals in* $\mathbb{R}^3$. Duke Math. J. **96**(3) (1999), 547–574.

[11] Sjölin P., *Global maximal estimates for solutions to the Schrödinger equation*. Studia. Math. **110**(2) (1994), 105–114.

[12] Sjölin P., *Radial functions and maximal estimates for solutions to the Schrödinger equation*. J. Austral. Math. Soc. (Series A) **59** (1995), 134–142.

[13] Sjölin P., L^p *Maximal estimates for solutions to the Schrödinger equation*. Math. Scand. **81** (1997), 35–68.

[14] Sjölin P., *A Counter-example Concerning Maximal Estimates for Solutions to Equations of Schrödinger Type*. Indiana Univ. Math. J. **47**(2) (1998), 593–599.

[15] Stein E.M., Harmonic Analysis: Real-Varialble Methods, Orthogonality, and Oscillatory Integrals, Princeton Univ. Press, Princeton, N.J., 1993.

[16] Stein E.M. and Weiss G., Introduction to Fourier Analysis on Euclidean Spaces, Princeton Univ. Press, 1971.

[17] Stempak K., *A weighted uniform* L^p*-estimate of Bessel functions: a note on a paper of Guo*. Proc. Amer. Math. Soc. **128** (2000), 2943–2945.

[18] Tao T., *A sharp bilinear restriction estimate for paraboloids*. Geom. Funct. Anal. **13** (2003), 1359–1384.

[19] Vega L., *Schrödinger equations: pointwise convergence to the initial data*. Proc. Amer. Math. Soc. **102** (1988), 874–878.

[20] Walther B.G., *Homogeneous estimates for oscillatory integrals*. Acta Math. Univ. Comenianae **9** (2000), 151–171.

[21] Walther B.G., *Higher integrability for maximal oscillatory Fourier integrals*. Annales Academiæ Scientiarum Fennicæ Mathematica **26** (2001), 189–204.

[22] Wang S., *On the maximal operator associated with the free Schrödinger equation*. Studia Math. **122** (1997), 167–182.

[23] Watson G., A treatise on the theory of Bessel functions, Reprint of the second (1944) edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1995.

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