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# A lower bound for the class number of $P^n(C)$ and $P^n(H)$

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**Abstract.** We obtain new lower bounds on the codimension of local isometric imbeddings of complex and quaternion projective spaces. We show that any open set of the complex projective space $P^n(\boldsymbol{C})$ (resp. quaternion projective space $P^n(\boldsymbol{H})$) cannot be locally isometrically imbedded into the euclidean space of dimension $4n-3$ (resp. $8n-4$). These estimates improve the previously known results obtained in [2] and [7].

*Key words*: curvature invariant, isometric imbedding, complex projective space, quaternion projective space, root space decomposition.

## 1. Introduction

Let $M$ be a Riemannian manifold. As is known, $M$ can be locally or globally isometrically imbedded into a euclidean space of sufficiently large dimension (see Janet [19], Cartan [14], Nash [24], Greene-Jacobowitz [16], Gromov-Rokhlin [17]). It is a natural and interesting question to ask the least dimension of euclidean spaces into which $M$ can be locally or globally isometrically imbedded. In this paper we will investigate the problem of local isometric imbeddings of the projective spaces $P^n(\boldsymbol{C})$ and $P^n(\boldsymbol{H})$ and give a new estimate on the least dimension of the ambient euclidean spaces.

Let $x \in M$. Assume that there is a neighborhood $U$ of $x$ in $M$ such that $U$ is isometrically imbedded into a euclidean space $\boldsymbol{R}^D$. If any neighborhood of $x$ cannot be isometrically imbedded into $\boldsymbol{R}^{D-1}$, then the codimension $D-\dim M$ is called the *class number* of $M$ at $x$ and is denoted by $\mathrm{class}(M)_x$.

Let $G/K$ be a Riemannian symmetric space. By homogeneity, the class number of $G/K$ is constant everywhere on $G/K$, which is denoted by $\mathrm{class}(G/K)$. In Agaoka-Kaneda [4], [5], [7], [8], [9] and [10] we have tried to estimate $\mathrm{class}(G/K)$ from below. In doing this we mainly used the following inequality

$$\mathrm{class}(G/K) \geq \dim G/K - p(G/K),$$

where $p(G/K)$ is the pseudo-nullity of $G/K$ (see §2 below or [4]). For

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the following Riemannian symmetric spaces $G/K$ our estimates just hit class$(G/K)$, i.e., class$(G/K) = \dim G/K - p(G/K)$:

*a*) The sphere $S^n$ $(n \geq 2)$;

*b*) $CI$: $Sp(n)/U(n)$ $(n \geq 1)$ (see [4]);

*c*) The symplectic group $Sp(n)$ $(n \geq 1)$ (see [5]).

As for the class numbers of the projective spaces such as the complex projective space $P^n(\boldsymbol{C})$, the quaternion projective space $P^n(\boldsymbol{H})$ and the Cayley projective plane $P^2(\boldsymbol{Cay})$, the following are known:

(1) $\mathrm{class}(P^n(\boldsymbol{C})) \geq \max\left\{n+1, \left[\frac{6}{5}n\right]\right\}$ $(n \geq 2)$ (see [2] and [7]);

(2) $\mathrm{class}(P^n(\boldsymbol{H})) \geq \min\{4n-3, 3n+1\}$ $(n \geq 3)$ (see [7]);

(3) $\mathrm{class}(P^n(\boldsymbol{C})) \leq n^2$ $(n \geq 2)$; $\mathrm{class}(P^n(\boldsymbol{H})) \leq 2n^2 - n$ $(n \geq 2)$ (see [22]);

(4) $\mathrm{class}(P^2(\boldsymbol{H})) = 6$; $\mathrm{class}(P^2(\boldsymbol{Cay})) = 10$ (see [8] and [22]).

It should be noted that any local isometric imbedding of $P^2(\boldsymbol{H})$ (resp. $P^2(\boldsymbol{Cay})$) into the euclidean space $\boldsymbol{R}^{14}$ (resp. $\boldsymbol{R}^{26}$) is rigid in the strongest sense (see [9] and [10]).

In this paper we will propose a new type of estimate and by applying it we will prove

**Theorem 1** *Let $G/K$ denote the complex projective space $P^n(\boldsymbol{C})$ $(n \geq 3)$ or the quaternion projective space $P^n(\boldsymbol{H})$ $(n \geq 3)$. Define an integer $q(G/K)$ by*

$$q(G/K) = \begin{cases} 4n-2, & \text{if } G/K = P^n(\boldsymbol{C}) \ (n \geq 3); \\ 8n-3, & \text{if } G/K = P^n(\boldsymbol{H}) \ (n \geq 3). \end{cases}$$

*Then, any open set of $G/K$ cannot be isometrically imbedded into the euclidean space $\boldsymbol{R}^D$ with $D \leq q(G/K) - 1$. In other words,*

$$\mathrm{class}(P^n(\boldsymbol{C})) \geq 2n-2 \quad (n \geq 3);$$
$$\mathrm{class}(P^n(\boldsymbol{H})) \geq 4n-3 \quad (n \geq 3).$$

It is clearly seen that Theorem 1 improves the estimates (1) and (2) stated above. However, we have to recognize a large gap between our estimate and the upper bound stated in (3), which cannot be filled at present.

Throughout this paper we will assume the differentiability of class $C^\infty$. For the notations of Lie algebras and Riemannian symmetric spaces, see Helgason [18].

## 2. The Gauss equation

Let $M$ be a Riemannian manifold and $g$ be the Riemannian metric of $M$. We denote by $R$ the Riemannian curvature tensor of type $(1,3)$ with respect to $g$.

For each $x \in M$ we denote by $T_x(M)$ (resp. $T_x^*(M)$) the tangent (resp. cotangent) vector space of $M$ at $x \in M$. Let $r$ be a non-negative integer. We define a quadratic equation with respect to an unknown $\boldsymbol{\Psi} \in S^2T_x^*(M) \otimes \boldsymbol{R}^r$ by

$$-g(R(X,Y)Z,W) = \big\langle \boldsymbol{\Psi}(X,Z), \boldsymbol{\Psi}(Y,W)\big\rangle - \big\langle \boldsymbol{\Psi}(X,W), \boldsymbol{\Psi}(Y,Z)\big\rangle, \tag{2.1}$$

where $X$, $Y$, $Z$, $W \in T_x(M)$ and $\langle\ ,\ \rangle$ is the standard inner product of $\boldsymbol{R}^r$. We call (2.1) the *Gauss equation in codimension* $r$ at $x$. It is well-known that for a sufficiently large $r$ the Gauss equation (2.1) in codimension $r$ admits a solution (see Berger [12], Berger-Bryant-Griffiths [13]). On the other hand, in general, for a small $r$ (2.1) does not admit any solution. By $\mathrm{Crank}(M)_x$ we denote the least value of $r$ with which (2.1) admits a solution and call it the *curvature rank* of $M$ at $x$. It should be noted that $\mathrm{Crank}(M)_x$ is a curvature invariant, i.e., it can be determined only by the curvature $R$ of $M$ at $x$.

As is well-known, if there is an isometric immersion $\boldsymbol{f}$ of $M$ into $\boldsymbol{R}^D$, then the second fundamental form of $\boldsymbol{f}$ at $x$ satisfies the Gauss equation in codimension $r = D - \dim M$. Therefore, we have

**Lemma 2** $\mathrm{class}(M)_x \geq \mathrm{Crank}(M)_x$ *holds for any* $x \in M$.

In the following, we assume that $\boldsymbol{\Psi} \in S^2T_x^*(M) \otimes \boldsymbol{R}^r$ is a solution of the Gauss equation in codimension $r$. Let $X \in T_x(M)$. We define a linear mapping $\boldsymbol{\Psi}_X \colon T_x(M) \longrightarrow \boldsymbol{R}^r$ by $\boldsymbol{\Psi}_X(Y) = \boldsymbol{\Psi}(X,Y)$ $(Y \in T_x(M))$. The kernel of this map $\boldsymbol{\Psi}_X$ is denoted by $\mathbf{Ker}(\boldsymbol{\Psi}_X)$. Then we can easily show the following

**Lemma 3** *Let* $X \in T_x(M)$. *Then* $R(\mathbf{Ker}(\boldsymbol{\Psi}_X), \mathbf{Ker}(\boldsymbol{\Psi}_X))X = 0$.

For the proof, see [4]. By this lemma we can get the following estimate for $\mathrm{Crank}(M)_x$: Let $X \in T_x(M)$. By $d(X)$ we denote the maximum value of the dimensions of those subspaces $V \subset T_x(M)$ such that $R(V,V)X = 0$. Then by Lemma 3 it is easily seen that $d(X) \geq \dim \mathbf{Ker}(\boldsymbol{\Psi}_X) \geq \dim M - r$. Set $p_M(x) = \min\{d(X) \mid X \in T_x(M)\}$. Then $p_M(x) \geq \dim M - r$, i.e.,

$r \geq \dim M - p_M(x)$. The number $p_M(x)$ thus defined is also a curvature invariant, which is called the *pseudo-nullity* of $M$ at $x$. By the above discussion we have

**Lemma 4** $\mathrm{Crank}(M)_x \geq \dim M - p_M(x)$.

In the case where $M$ is a Riemannian homogeneous space $G/K$, the class number, the curvature rank and the pseudo-nullity of $G/K$ are constant everywhere on $G/K$, which are denoted by $\mathrm{class}(G/K)$, $\mathrm{Crank}(G/K)$ and $p(G/K)$, respectively. Combining Lemma 4 with Lemma 2, we obtain

**Proposition 5** *Let $G/K$ be a Riemannian homogeneous space. Then*:

$$\mathrm{class}(G/K) \geq \dim G/K - p(G/K).$$

This is a fundamental tool in our works [5] and [7] to estimate the class numbers of Riemannian symmetric spaces from below.

Now, we show a new type of estimate:

**Theorem 6** *Let $\boldsymbol{\Psi} \in S^2T_x^*(M) \otimes \boldsymbol{R}^r$ be a solution of the Gauss equation in codimension $r$. Assume that there are tangent vectors $X, Y \in T_x(M)$ and a subspace $\boldsymbol{U}$ of $T_x(M)$ satisfying*
(1) $\boldsymbol{\Psi}(X, Y) = 0$;
(2) $\boldsymbol{U} \supset \mathbf{Ker}(\boldsymbol{\Psi}_X)$ *and* $R(\boldsymbol{U}, Y)X = 0$.
*Then the following inequality holds*:

$$r \geq \dim M + \dim \boldsymbol{U} - \dim \mathbf{Ker}(\boldsymbol{\Psi}_X) - \dim \mathbf{Ker}(\boldsymbol{\Psi}_Y). \tag{2.2}$$

*Proof.* Let $Z$ be an arbitrary element of $T_x(M)$. Then by the Gauss equation (2.1) it follows that

$$\begin{aligned} 0 &= -g(R(\boldsymbol{U}, Y)X, Z) \\ &= \big\langle \boldsymbol{\Psi}(\boldsymbol{U}, X), \boldsymbol{\Psi}(Y, Z) \big\rangle - \big\langle \boldsymbol{\Psi}(\boldsymbol{U}, Z), \boldsymbol{\Psi}(Y, X) \big\rangle \\ &= \big\langle \boldsymbol{\Psi}_X(\boldsymbol{U}), \boldsymbol{\Psi}_Y(Z) \big\rangle - 0. \end{aligned}$$

Hence, we have $\big\langle \boldsymbol{\Psi}_X(\boldsymbol{U}), \boldsymbol{\Psi}_Y(Z) \big\rangle = 0$. This implies that the image of $T_x(M)$ via the map $\boldsymbol{\Psi}_Y$ is included in the orthogonal complement of $\boldsymbol{\Psi}_X(\boldsymbol{U})$. Since $\dim \boldsymbol{\Psi}_X(\boldsymbol{U}) = \dim \boldsymbol{U} - \dim \mathbf{Ker}(\boldsymbol{\Psi}_X)$, we have $\dim \boldsymbol{\Psi}_Y(T_x(M)) \leq r - \dim \boldsymbol{U} + \dim \mathbf{Ker}(\boldsymbol{\Psi}_X)$. Moreover, since $\dim \boldsymbol{\Psi}_Y(T_x(M)) = \dim M - \dim \mathbf{Ker}(\boldsymbol{\Psi}_Y)$, we immediately obtain the inequality (2.2). □

As is easily seen, the right side of the inequality (2.2) heavily depends on tangent vectors $X$, $Y$ and $\boldsymbol{\Psi}$. Accordingly, only by (2.2) we cannot obtain an estimate for $\mathrm{Crank}(M)_x$. In the following sections, by applying Theorem 6 to the complex and quaternion projective spaces we will show Theorem 1.

## 3. Projective spaces $P^n(C)$ and $P^n(H)$

In this section we make several preparations that are needed in the succeeding sections. Hereafter, $G/K$ denotes one of the following projective spaces:

(1) The complex projective spaces $P^n(\boldsymbol{C}) = SU(n+1)/S(U(n) \times U(1))$ $(n \geq 2)$.

(2) The quaternion projective spaces $P^n(\boldsymbol{H}) = Sp(n+1)/Sp(n) \times Sp(1)$ $(n \geq 2)$.

Let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of $G$ (resp. $K$) and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition of $\mathfrak{g}$ associated with the Riemannian symmetric pair $(G, K)$. Let ( , ) be the inner product of $\mathfrak{g}$ given by the $(-1)$-multiple of the Killing form of $\mathfrak{g}$. We define a $G$-invariant Riemannian metric $g$ of $G/K$ by $g(X,Y) = (X,Y)$ $(X, Y \in \mathfrak{m})$, where we identify $\mathfrak{m}$ with the tangent space $T_o(G/K)$ at the origin $o = \{K\} \in G/K$. Since the curvature at $o$ is given by $R(X,Y)Z = -\big[[X,Y],Z\big]$ $(X, Y, Z \in \mathfrak{m})$ (see Helgason [18]), the Gauss equation (2.1) in codimension $r$ at $o$ can be written as follows:

$$\big(\big[[X,Y],Z\big],W\big) = \big\langle \boldsymbol{\Psi}(X,Z), \boldsymbol{\Psi}(Y,W)\big\rangle - \big\langle \boldsymbol{\Psi}(X,W), \boldsymbol{\Psi}(Y,Z)\big\rangle, \tag{3.1}$$

where $\boldsymbol{\Psi} \in S^2\mathfrak{m}^* \otimes \boldsymbol{R}^r$, $X$, $Y$, $Z$ and $W \in \mathfrak{m}$.

Let us take and fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{m}$. Then, since $\mathrm{rank}(G/K) = 1$, we have $\dim \mathfrak{a} = 1$. We call an element $\lambda \in \mathfrak{a}$ a *restricted root* when the subspaces $\mathfrak{k}(\lambda)$ $(\subset \mathfrak{k})$ and $\mathfrak{m}(\lambda)$ $(\subset \mathfrak{m})$ defined below are not non-trivial:

$$\mathfrak{k}(\lambda) = \big\{X \in \mathfrak{k} \,\big|\, \big[H, [H,X]\big] = -(\lambda, H)^2 X, \quad \forall H \in \mathfrak{a}\big\},$$
$$\mathfrak{m}(\lambda) = \big\{Y \in \mathfrak{m} \,\big|\, \big[H, [H,Y]\big] = -(\lambda, H)^2 Y, \quad \forall H \in \mathfrak{a}\big\}.$$

As is known, by use of a non-zero restricted root $\mu$ the set of non-zero restricted roots $\Sigma$ can be written as $\Sigma = \{\pm\mu, \pm 2\mu\}$. Further, we have the following orthogonal decompositions:

$$\mathfrak{k} = \mathfrak{k}(0) + \mathfrak{k}(\mu) + \mathfrak{k}(2\mu) \quad \text{(orthogonal direct sum)},$$

$$\mathfrak{m} = \mathfrak{m}(0) + \mathfrak{m}(\mu) + \mathfrak{m}(2\mu) \quad \text{(orthogonal direct sum)},$$

where $\mathfrak{m}(0) = \mathfrak{a} = \boldsymbol{R}\mu$ (see §5 of [7]).

For convenience, in the following we set $\mathfrak{k}_i = \mathfrak{k}(|i|\mu)$, $\mathfrak{m}_i = \mathfrak{m}(|i|\mu)$ $(|i| \leq 2)$ and $\mathfrak{k}_i = \mathfrak{m}_i = 0$ $(|i| > 2)$ for any integer $i$. Then for $i$, $j = 0, 1, 2$ we have a formula:

$$[\mathfrak{k}_i, \mathfrak{k}_j] \subset \mathfrak{k}_{i+j} + \mathfrak{k}_{i-j}, \quad [\mathfrak{m}_i, \mathfrak{m}_j] \subset \mathfrak{k}_{i+j} + \mathfrak{k}_{i-j}, \quad [\mathfrak{k}_i, \mathfrak{m}_j] \subset \mathfrak{m}_{i+j} + \mathfrak{m}_{i-j}.$$

We summarize in the following table the basic data for the spaces $P^n(\boldsymbol{C})$ and $P^n(\boldsymbol{H})$ (see [18], [7]):

| $G/K$ | $\dim \mathfrak{m}_1$ $(= \dim \mathfrak{k}_1)$ | $\dim \mathfrak{m}_2 (= \dim \mathfrak{k}_2)$ |
|---|---|---|
| $P^n(\boldsymbol{C})$ $(n \geq 2)$ | $2(n-1)$ | 1 |
| $P^n(\boldsymbol{H})$ $(n \geq 2)$ | $4(n-1)$ | 3 |

As is known, each non-zero element of $\mathfrak{m}$ is conjugate to a scalar multiple of $\mu$ under the action of the isotropy group $\mathrm{Ad}(K)$, because $\mathrm{rank}(P^n(\boldsymbol{C})) = \mathrm{rank}(P^n(\boldsymbol{H})) = 1$. More precisely we can show the following

**Proposition 7** *Let $Y_i \in \mathfrak{m}_i$ $(i = 0, 1, 2)$. Assume that $Y_i \neq 0$. Then there is an element $k_i \in K$ such that $\mathrm{Ad}(k_i^{\pm 1})\mu \in \boldsymbol{R}Y_i$.*

*Proof.* In the case $i = 0$ we have only to set $k_0 = e$, where $e$ is the identity element of $K$.

Now assume $i = 1$ or $2$. Set $X_i = [\mu, Y_i]$. Then we have $X_i \in \mathfrak{k}_i$. Further, we have $[X_i, [X_i, \mu]] \in \mathfrak{a}$, because $[X_i, [X_i, \mu]] \in \mathfrak{m}$ and $[\mu, [X_i, [X_i, \mu]]] = [[\mu, X_i], [X_i, \mu]] + [X_i, [\mu, [X_i, \mu]]] = 0$. Since

$$\begin{aligned}(\mu, [X_i, [X_i, \mu]]) &= ([\mu, X_i], [X_i, \mu]) = ([\mu, [\mu, X_i]], X_i) \\ &= -i^2(\mu, \mu)^2(X_i, X_i),\end{aligned}$$

we have $[X_i, [X_i, \mu]] = -i^2(\mu, \mu)(X_i, X_i)\mu$. By this equality and the fact $[X_i, \mu] = [[\mu, Y_i], \mu] = i^2(\mu, \mu)^2 Y_i$ we have

$$\begin{aligned}\mathrm{Ad}(\exp(tX_i))\mu &= \cos(i|\mu||X_i|t)\mu \\ &\quad + \frac{1}{i|\mu||X_i|}\sin(i|\mu||X_i|t)[X_i, \mu], \quad \forall t \in \boldsymbol{R}.\end{aligned}$$

Define $t_i \in \boldsymbol{R}$ by $i|\mu||X_i|t_i = \pi/2$. Then, by setting $k_i = \exp(t_i X_i) \in K$, we easily get $\mathrm{Ad}(k_i^{\pm 1})\mu \in \boldsymbol{R}Y_i$. □

## 4. Pseudo-abelian subspaces

Let $G/K = P^n(\boldsymbol{C})$ or $P^n(\boldsymbol{H})$. We say that a subspace $V$ of $\mathfrak{m}$ is *pseudo-abelian* if $[V, V] \subset \mathfrak{k}_0$. It is easily seen that a subspace $V$ of $\mathfrak{m}$ is pseudo-abelian if and only if $[[V, V], \mu] = 0$, because $\mathrm{rank}(G/K) = 1$. We note that the pseudo-nullity $p(G/K)$ coincides with the maximum dimension of pseudo-abelian subspaces in $\mathfrak{m}$ (see [4]). In [7] we have determined the pseudo-nullities for $P^n(\boldsymbol{C})$ and $P^n(\boldsymbol{H})$: $p(P^n(\boldsymbol{C})) = \max\{n-1, 2\}$ $(n \geq 2)$; $p(P^n(\boldsymbol{H})) = \max\{n-1, 3\}$ $(n \geq 2)$ (see Theorem 5.1 of [7]).

For later use, we here study more detailed facts about pseudo-abelian subspaces in $\mathfrak{m}$ for $P^n(\boldsymbol{C})$ and $P^n(\boldsymbol{H})$. We first prove

**Lemma 8** *Let $V \subset \mathfrak{m}$ be a pseudo-abelian subspace of $\mathfrak{m}$. If $V \cap \mathfrak{m}_i \neq 0$ for some $\mathfrak{m}_i$ $(i = 0,\ 1,\ 2)$, then $V \subset \mathfrak{m}_i$.*

*Proof.* Assume that $V \cap \mathfrak{m}_1 \neq 0$. Take a non-zero element $Y_1^0 \in V \cap \mathfrak{m}_1$. Let $Y = Y_0 + Y_1$ be an arbitrary element of $V$, where $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$; $Y_1 \in \mathfrak{m}_1$. Then we have $[Y_1^0, Y_0 + Y_1] = [Y_1^0, Y_0] + [Y_1^0, Y_1] \in \mathfrak{k}_0$. However, since $[Y_1^0, Y_0] \in \mathfrak{k}_1$ and $[Y_1^0, Y_1] \in \mathfrak{k}_0 + \mathfrak{k}_2$, we have $[Y_1^0, Y_0] = 0$. Therefore we have $Y_0 = 0$, because $\mathrm{rank}(G/K) = 1$. This proves $V \subset \mathfrak{m}_1$. The other cases $V \cap \mathfrak{a} \neq 0$ and $V \cap \mathfrak{m}_2 \neq 0$ are similarly dealt with. □

We say that a pseudo-abelian subspace $V$ is *categorical* if it can be decomposed into a direct sum $V = V \cap \mathfrak{a} + V \cap \mathfrak{m}_1 + V \cap \mathfrak{m}_2$. By Lemma 8 we immediately have

**Proposition 9** *Let $V \subset \mathfrak{m}$ be a pseudo-abelian subspace of $\mathfrak{m}$. If $V$ is categorical and $V \neq 0$, then $V$ is contained in one of $\mathfrak{a}$, $\mathfrak{m}_1$ and $\mathfrak{m}_2$.*

By this proposition, we can easily estimate $\dim V$ for a categorical pseudo-abelian subspace $V$ in $\mathfrak{m}$: $\dim V \leq 1$ if $V \subset \mathfrak{a}$; $\dim V \leq \dim \mathfrak{m}_2$ if $V \subset \mathfrak{m}_2$. In the case $V \subset \mathfrak{m}_1$ we proved in [7] $\dim V \leq n-1$ (see Theorem 3.2 of [7]). For completeness, we review this proof and show an additional property of $V \subset \mathfrak{m}_1$.

Let $E(\mathfrak{m}_1)$ denote the space of all linear endomorphisms of $\mathfrak{m}_1$. Let $X \in \mathfrak{k}_2$. We define an element $X^\dagger \in E(\mathfrak{m}_1)$ by

$$X^\dagger(Y) = [X, Y], \quad Y \in \mathfrak{m}_1.$$

(Note that $[\mathfrak{k}_2, \mathfrak{m}_1] \subset \mathfrak{m}_1$.) It is easy to see that $X^\dagger$ is skew-symmetric with respect to the inner product $(\ ,\ )$. We denote by $\mathfrak{k}_2^\dagger$ the subspace of $E(\mathfrak{m}_1)$

consisting of all $X^\dagger$ ($X \in \mathfrak{k}_2$). Set $\mathfrak{F}^\dagger = \boldsymbol{R}\mathbf{1}_{\mathfrak{m}_1} + \mathfrak{k}_2^\dagger$ ($\subset E(\mathfrak{m}_1)$), where $\mathbf{1}_{\mathfrak{m}_1}$ denotes the identity mapping of $\mathfrak{m}_1$. We have proved in [7] (Theorem 3.5) the following

**Proposition 10** *Let $G/K = P^n(\boldsymbol{C})$ or $P^n(\boldsymbol{H})$. Then, $\mathfrak{F}^\dagger$ forms a subalgebra of $E(\mathfrak{m}_1)$, i.e., $\mathfrak{F}^\dagger$ is closed under addition and multiplication of $E(\mathfrak{m}_1)$. Further, in the case $G/K = P^n(\boldsymbol{C})$ ($n \geq 2$), $\mathfrak{F}^\dagger$ is isomorphic to the field $\boldsymbol{C}$ of complex numbers and in the case $G/K = P^n(\boldsymbol{H})$ ($n \geq 2$), $\mathfrak{F}^\dagger$ is isomorphic to the field $\boldsymbol{H}$ of quaternion numbers.*

We now set $f = \dim_{\boldsymbol{R}} \mathfrak{F}^\dagger$, i.e., $f = 2$ if $G/K = P^n(\boldsymbol{C})$; $f = 4$ if $G/K = P^n(\boldsymbol{H})$. By the definition we have $\dim \mathfrak{m}_2 = f - 1$, $\dim \mathfrak{m}_1 = (n-1)f$ and $\dim G/K = \dim \mathfrak{m} = nf$. As seen in Proposition 10, $\mathfrak{m}_1$ can be regarded as a vector space over the field $\mathfrak{F}^\dagger$. For an element $Y_1 \in \mathfrak{m}_1$ we denote by $\mathfrak{F}^\dagger(Y_1)$ the subspace of $\mathfrak{m}_1$ spanned by $Y_1$ over $\mathfrak{F}^\dagger$. Then we easily have $\mathfrak{F}^\dagger(\mathfrak{F}^\dagger(Y_1)) = \mathfrak{F}^\dagger(Y_1)$ and $\dim_{\boldsymbol{R}} \mathfrak{F}^\dagger(Y_1) = f$ if $Y_1 \neq 0$.

**Lemma 11** *Let $Y_1$, $Y_1' \in \mathfrak{m}_1$. Then $\big[Y_1, Y_1'\big] \in \mathfrak{k}_0$ if and only if $(\mathfrak{k}_2^\dagger(Y_1), Y_1') = 0$. Accordingly, a subspace $V \subset \mathfrak{m}_1$ is pseudo-abelian if and only if $(\mathfrak{k}_2^\dagger(V), V) = 0$.*

*Proof.* Since $\big[Y_1, Y_1'\big] \in \mathfrak{k}_0 + \mathfrak{k}_2$, $\big[Y_1, Y_1'\big] \in \mathfrak{k}_0$ holds if and only if $\big(\big[Y_1, Y_1'\big], \mathfrak{k}_2\big) = 0$. Clearly, the last equality is equivalent to $(\mathfrak{k}_2^\dagger(Y_1), Y_1') = 0$. □

Utilizing the above lemma, we can show the following

**Proposition 12** *Let $V$ be a pseudo-abelian subspace of $\mathfrak{m}$. Assume that $V \subset \mathfrak{m}_1$. Then*:
(1) $\dim \mathfrak{F}^\dagger(V) = f \dim V$. *Accordingly,* $\dim V \leq n-1$.
(2) *Let $\xi \in V$ ($\xi \neq 0$). Then there is a subspace $U$ of $\mathfrak{m}_1$ satisfying $U \supset V$, $\big[\xi, U\big] \subset \mathfrak{k}_0$ and $\dim U = (n-2)f + 1$.*

*Proof.* Let $\{Y_1^1, \ldots, Y_1^s\}$ ($s = \dim V$) be an orthonormal basis of $V$. Let $i$, $j$ be integers such that $1 \leq i \neq j \leq s$. Then, since $(\mathfrak{k}_2^\dagger(Y_1^i), Y_1^j) = (Y_1^i, \mathfrak{k}_2^\dagger(Y_1^j)) = 0$ (see Lemma 11) and since $(\mathfrak{k}_2^\dagger)^2 \subset \mathfrak{F}^\dagger$, we have

$$\begin{aligned}\big(\mathfrak{F}^\dagger(Y_1^i), \mathfrak{F}^\dagger(Y_1^j)\big) &= \big(\boldsymbol{R}Y_1^i + \mathfrak{k}_2^\dagger(Y_1^i), \boldsymbol{R}Y_1^j + \mathfrak{k}_2^\dagger(Y_1^j)\big) \\ &\subset \big(Y_1^i, (\mathfrak{k}_2^\dagger)^2(Y_1^j)\big) = 0.\end{aligned}$$

This proves $\mathfrak{F}^\dagger(V) = \sum_{1 \leq i \leq s} \mathfrak{F}^\dagger(Y_1^i)$ (orthogonal direct sum) and hence

$\dim_{\boldsymbol{R}} \mathfrak{F}^{\dagger}(V) = sf$. Therefore we have $s \leq n-1$, because $\dim \mathfrak{m}_1 = (n-1)f$.

Next we prove (2). Since $V$ is pseudo-abelian and $\xi \in V$, we have $(\mathfrak{k}_2^{\dagger}(\xi), V) = 0$. Let $U$ be the orthogonal complement of $\mathfrak{k}_2^{\dagger}(\xi)$ in $\mathfrak{m}_1$. Then $U$ satisfies $U \supset V$ and $[\xi, U] \subset \mathfrak{k}_0$ (see Lemma 11). Moreover, since $\dim \mathfrak{k}_2^{\dagger}(\xi) = f - 1$ and $\dim \mathfrak{m}_1 = (n-1)f$, we immediately obtain the equality $\dim U = (n-2)f + 1$. □

Finally, we refer to non-categorical pseudo-abelian subspaces. Let $V$ be a pseudo-abelian subspace of $\mathfrak{m}$. Assume that $V$ is not categorical, i.e., $V$ cannot be represented by a direct sum of subspaces $V \cap \mathfrak{a}$, $V \cap \mathfrak{m}_1$ and $V \cap \mathfrak{m}_2$. Then it is clear that $V \not\subset \mathfrak{a}$, $V \not\subset \mathfrak{m}_1$ and $V \not\subset \mathfrak{m}_2$. In view of Lemma 8, we know that $V \cap \mathfrak{a} = V \cap \mathfrak{m}_1 = V \cap \mathfrak{m}_2 = 0$. Apparently, this condition is sufficient for a pseudo-abelian subspace $V$ to be non-categorical. Hence we have

**Proposition 13** *Let $V$ be a pseudo-abelian subspace of $\mathfrak{m}$ such that $V \neq 0$.*
(1) *$V$ is non-categorical if and only if $V \cap \mathfrak{a} = V \cap \mathfrak{m}_1 = V \cap \mathfrak{m}_2 = 0$.*
(2) *If $V$ is non-categorical, then $\dim V \leq 2$.*

For the proof of (2), see Proposition 5.2 (1) of [7].

## 5. Proof of Theorem 1

Let $G/K = P^n(\boldsymbol{C})$ $(n \geq 2)$ or $P^n(\boldsymbol{H})$ $(n \geq 2)$. In the following we assume that the Gauss equation in codimension $r$ admits a solution $\boldsymbol{\Psi} \in S^2\mathfrak{m}^* \otimes \boldsymbol{R}^r$. We first prove

**Lemma 14** *Let $X \in \mathfrak{m}$ $(X \neq 0)$ and let $k$ be an element of $K$ satisfying $\mathrm{Ad}(k)\mu \in \boldsymbol{R}X$. Then $\mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_X)$ is a pseudo-abelian subspace of $\mathfrak{m}$.*

*Proof.* By Lemma 3 we have $\big[\big[\mathbf{Ker}(\boldsymbol{\Psi}_X), \mathbf{Ker}(\boldsymbol{\Psi}_X)\big], X\big] = 0$. Applying $\mathrm{Ad}(k^{-1})$ to this equality, we have $\big[\big[\mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_X), \mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_X)\big], \mu\big] = 0$. This proves that $\mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_X)$ is a pseudo-abelian subspace of $\mathfrak{m}$. □

Let $X \in \mathfrak{m}$ $(X \neq 0)$. If $\mathbf{Ker}(\boldsymbol{\Psi}_X) = 0$, then we say $X$ is *of type $P_{inj}$*. Now assume $\mathbf{Ker}(\boldsymbol{\Psi}_X) \neq 0$. Let $k \in K$ be an element satisfying $\mathrm{Ad}(k)\mu \in \boldsymbol{R}X$. As is shown in Lemma 14, $\mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_X)$ is a pseudo-abelian subspace of $\mathfrak{m}$. If $\mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_X)$ is categorical and is contained in $\mathfrak{m}_i$ $(i = 0, 1, 2)$, then we say $X$ is *of type $P_i$* $(i = 0, 1, 2)$. We also say $X$ is *of* *type* $P_{non}$ if $\mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_X)$ is non-categorical, i.e., $\mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_X)\cap \mathfrak{m}_i = 0$ $(i = 0, 1, 2)$.

The following lemma asserts that the type of $X$ does not depend on the choice of $k \in K$ satisfying $\mathrm{Ad}(k)\mu \in \boldsymbol{R}X$.

**Lemma 15** *Let* $X \in \mathfrak{m}$ $(X \neq 0)$. *Let* $i = 0, 1$ *or* $2$ *and let* $k_j$ $(j = 1, 2)$ *be elements of* $K$ *satisfying* $\mathrm{Ad}(k_j)\mu \in \boldsymbol{R}X$. *Then*:
(1) $\mathrm{Ad}(k_1^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_X) \subset \mathfrak{m}_i$ *if and only if* $\mathrm{Ad}(k_2^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_X) \subset \mathfrak{m}_i$.
(2) $\mathrm{Ad}(k_1^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_X)\cap\mathfrak{m}_i = 0$ *if and only if* $\mathrm{Ad}(k_2^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_X)\cap\mathfrak{m}_i = 0$.

*Proof.* Set $k' = k_1^{-1}k_2 \in K$. By the assumption we have $\mathrm{Ad}(k')\mu = \pm\mu$. Therefore it is easily seen that $\mathrm{Ad}(k')\mathfrak{m}_i = \mathfrak{m}_i$ for any $i = 0, 1, 2$. Since $\mathrm{Ad}(k')\,\mathrm{Ad}(k_2^{-1}) = \mathrm{Ad}(k_1^{-1})$, the lemma follows immediately. □

Let us denote by $\mathfrak{p}_i$ $(i = 0, 1, 2, non, inj)$ the subset of $\mathfrak{m}$ consisting of all elements of type $P_i$. Then it is clear that

$$\mathfrak{m}\setminus\{0\} = \mathfrak{p}_0 \cup \mathfrak{p}_1 \cup \mathfrak{p}_2 \cup \mathfrak{p}_{non} \cup \mathfrak{p}_{inj} \quad \text{(disjoint union)}. \tag{5.1}$$

**Proposition 16** *Let* $X, Y \in \mathfrak{m}$ $(X \neq 0, Y \neq 0)$. *Assume that* $\boldsymbol{\Psi}(X, Y) = 0$. *Then* $X \in \mathfrak{p}_i$ *if and only if* $Y \in \mathfrak{p}_i$ $(i = 0, 1, 2, non)$.

*Proof.* We note that under the assumption $\boldsymbol{\Psi}(X, Y) = 0$ we have $X \notin \mathfrak{p}_{inj}$ and $Y \notin \mathfrak{p}_{inj}$, because $Y \in \mathbf{Ker}(\boldsymbol{\Psi}_X)$ and $X \in \mathbf{Ker}(\boldsymbol{\Psi}_Y)$.

First consider the case $X \in \mathfrak{p}_i$ $(i = 0, 1, 2)$. Let $k \in K$ be an element such that $\mathrm{Ad}(k)\mu \in \boldsymbol{R}X$. Then we have $\mathrm{Ad}(k^{-1})Y \in \mathfrak{m}_i$, because $\mathrm{Ad}(k^{-1})Y \in \mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_X) \subset \mathfrak{m}_i$. Take an element $k' \in K$ satisfying $\mathrm{Ad}(k'^{\pm 1})\mu \in \boldsymbol{R}\,\mathrm{Ad}(k^{-1})Y$ and set $k'' = kk'$ (see Proposition 7). Then we have $\mathrm{Ad}(k'')\mu = \mathrm{Ad}(k)\,\mathrm{Ad}(k')\mu \in \mathrm{Ad}(k)\boldsymbol{R}\,\mathrm{Ad}(k^{-1})Y = \boldsymbol{R}Y$ and $\mathrm{Ad}(k''^{-1})X = \mathrm{Ad}(k'^{-1})\,\mathrm{Ad}(k^{-1})X \in \boldsymbol{R}\,\mathrm{Ad}(k'^{-1})\mu = \boldsymbol{R}\,\mathrm{Ad}(k^{-1})Y \subset \mathfrak{m}_i$. Since $X \in \mathbf{Ker}(\boldsymbol{\Psi}_Y)$, it follows that $\mathrm{Ad}(k''^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_Y) \cap \mathfrak{m}_i \neq 0$. Hence $\mathrm{Ad}(k''^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_Y)$ is categorical (see Proposition 13) and $\mathrm{Ad}(k''^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_Y) \subset \mathfrak{m}_i$ (see Proposition 9). This means $Y \in \mathfrak{p}_i$. The converse can be proved in the same manner.

By these arguments we know that $X \in \mathfrak{p}_{non}$ if and only if $Y \in \mathfrak{p}_{non}$. □

**Lemma 17** *Let* $G/K = P^n(\boldsymbol{C})$ $(n \geq 2)$ *or* $P^n(\boldsymbol{H})$ $(n \geq 2)$. *Then*:
(1) $\mathfrak{p}_0 = \emptyset$.
(2) *Let* $X \in \mathfrak{m}$ $(X \neq 0)$. *Then*:

$$\dim \mathbf{Ker}(\boldsymbol{\Psi}_X) \leq \begin{cases} n-1, & \textit{if } X \in \mathfrak{p}_1; \\ f-1, & \textit{if } X \in \mathfrak{p}_2; \\ 2, & \textit{if } X \in \mathfrak{p}_{non}. \end{cases} \tag{5.2}$$

*Proof.* Suppose that $\mathfrak{p}_0 \neq \emptyset$. Let $X \in \mathfrak{p}_0$ and let $k \in K$ be an element such that $\mathrm{Ad}(k)\mu \in \boldsymbol{R}X$. Then we have $\mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_X) \subset \mathfrak{a} = \boldsymbol{R}\mu$. Hence we have $\mathbf{Ker}(\boldsymbol{\Psi}_X) = \boldsymbol{R}\,\mathrm{Ad}(k)\mu = \boldsymbol{R}X$, i.e., $\boldsymbol{\Psi}(X,X) = 0$. Let $Y \in \mathfrak{m}$ such that $Y \notin \boldsymbol{R}X$. By (3.1) we have

$$\begin{aligned}\big(\big[[X,Y],X\big],Y\big) &= \big\langle \boldsymbol{\Psi}(X,X), \boldsymbol{\Psi}(Y,Y)\big\rangle - \big\langle \boldsymbol{\Psi}(X,Y), \boldsymbol{\Psi}(Y,X)\big\rangle \\ &= -\big\langle \boldsymbol{\Psi}_X(Y), \boldsymbol{\Psi}_X(Y)\big\rangle.\end{aligned}$$

Since $G/K$ is of positive curvature, the left side of the above equality is $\geq 0$. Therefore we have $\boldsymbol{\Psi}_X(Y) = 0$, which contradicts $Y \notin \boldsymbol{R}X$. Thus we have, $\mathfrak{p}_0 = \emptyset$.

The assertion (2) follows from Propositions 12, Proposition 13, $\dim \mathfrak{m}_2 = f-1$ and the discussions in the previous section. □

**Proposition 18** *Let $G/K = P^n(\boldsymbol{C})$ $(n \geq 2)$ or $P^n(\boldsymbol{H})$ $(n \geq 2)$. Then:*
(1) $\mathfrak{p}_{inj} = \emptyset$ *if* $r \leq nf-1$;
(2) $\mathfrak{p}_1 = \emptyset$ *if* $r \leq 2(n-1)(f-1)$;
(3) $\mathfrak{p}_2 = \emptyset$ *if* $r \leq (n-1)f$;
(4) $\mathfrak{p}_{non} = \emptyset$ *if* $r \leq nf-3$.

*Proof.* We first note that $\dim \mathbf{Ker}(\boldsymbol{\Psi}_X) \geq \dim G/K - r = nf - r$ holds for any $X \in \mathfrak{m}$. By this fact we can easily prove (1), (3) and (4). In fact, if $r \leq nf-1$, then it is clear that $\mathbf{Ker}(\boldsymbol{\Psi}_X) \neq 0$ for any $X \in \mathfrak{m}$. Hence $X \notin \mathfrak{p}_{inj}$, which implies $\mathfrak{p}_{inj} = \emptyset$. Similarly, if $r \leq (n-1)f$ (resp. $r \leq nf-3$), then $\dim \mathbf{Ker}(\boldsymbol{\Psi}_X) \geq f$ (resp. $\dim \mathbf{Ker}(\boldsymbol{\Psi}_X) \geq 3$) holds for any $X \in \mathfrak{m}$ and hence $\mathfrak{p}_2 = \emptyset$ (resp. $\mathfrak{p}_{non} = \emptyset$) (see Lemma 17).

Next we prove (2). Suppose that $\mathfrak{p}_1 \neq \emptyset$. Let $X \in \mathfrak{p}_1$. Take $k \in K$ such that $\mathrm{Ad}(k)\mu \in \boldsymbol{R}X$ and set $V = \mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_X)$. Then $V$ is a pseudo-abelian subspace such that $V \subset \mathfrak{m}_1$. Consequently, by Lemma 17 we have $\dim V \leq n-1$.

Now let us take a non-zero element $\xi \in V$ and a subspace $U \subset \mathfrak{m}_1$ satisfying $U \supset V$, $\big[\xi, U\big] \subset \mathfrak{k}_0$ and $\dim U = (n-2)f+1$ (see Proposition 12 (2)). Put $Y = \mathrm{Ad}(k)\xi$ $(\in \mathbf{Ker}(\boldsymbol{\Psi}_X))$ and $\boldsymbol{U} = \mathrm{Ad}(k)U$ $(\subset \mathfrak{m})$. Then we have $\boldsymbol{\Psi}(X,Y) = 0$ and $\boldsymbol{U} \supset \mathbf{Ker}(\boldsymbol{\Psi}_X)$. Moreover, we have $\big[[\boldsymbol{U},Y],X\big] = 0$, because $\big[[\boldsymbol{U},Y],X\big] = \mathrm{Ad}(k)\big[[U,\xi],\mu\big] = 0$. Therefore, by Theorem 6 we have the following inequality:

$$r \geq nf + (n-2)f + 1 - \dim \mathbf{Ker}(\mathbf{\Psi}_X) - \dim \mathbf{Ker}(\mathbf{\Psi}_Y).$$

Since $X$ and $Y \in \mathfrak{p}_1$ (see Proposition 16), it follows that $\dim \mathbf{Ker}(\mathbf{\Psi}_X) \leq n-1$ and $\dim \mathbf{Ker}(\mathbf{\Psi}_Y) \leq n-1$ (see Lemma 17). Consequently, we have $r \geq 2(n-1)(f-1)+1$, which proves (2). □

We are now in a position to prove Theorem 1. If there is a solution $\mathbf{\Psi}$ of the Gauss equation in codimension $r$, then at least one of the sets $\mathfrak{p}_{inj}$, $\mathfrak{p}_0$, $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{p}_{non}$ is not empty (see (5.1)). Therefore, in view of Lemma 17 (1) and Proposition 18, we have $r \geq 1+\min\{nf-1, 2(n-1)(f-1), (n-1)f, nf-3\}$. Accordingly, we have $r \geq 2n-2$ if $G/K = P^n(\boldsymbol{C})$ and $r \geq 4n-3$ if $G/K = P^n(\boldsymbol{H})$. Hence, $\mathrm{Crank}(P^n(\boldsymbol{C})) \geq 2n-2$ and $\mathrm{Crank}(P^n(\boldsymbol{H})) \geq 4n-3$. This, together with Lemma 2, shows Theorem 1. □

**Remark 1** The proof of Theorem 1 stated above is effective in the case $n=2$. We thereby have $\mathrm{Crank}(P^2(\boldsymbol{C})) \geq 2$ and $\mathrm{Crank}(P^2(\boldsymbol{H})) \geq 5$. However, for the spaces $P^2(\boldsymbol{C})$ and $P^2(\boldsymbol{H})$, we have already known the best results: $\mathrm{Crank}(P^2(\boldsymbol{C})) = 3$ (see [1]) and $\mathrm{class}(P^2(\boldsymbol{H})) = \mathrm{Crank}(P^2(\boldsymbol{H})) = 6$ (see [8]).

As for the class number of $P^2(\boldsymbol{C})$ we have $\mathrm{class}(P^2(\boldsymbol{C})) = 3$ or $4$ (see Lemma 2 and Introduction). It is still an open question whether $\mathrm{class}(P^2(\boldsymbol{C})) = 3$ or not (cf. [20]).

**Remark 2** Consider the following two cases:
(1) $G/K = P^n(\boldsymbol{C})$ $(n \geq 3)$ and $r = 2n-2$;
(2) $G/K = P^n(\boldsymbol{H})$ $(n \geq 3)$ and $r = 4n-3$.
If there is a solution $\mathbf{\Psi}$ of the Gauss equation in codimension $r$, then it is shown by Lemma 17 (1) and Proposition 18 that $\mathbf{\Psi}$ must satisfy the following condition:
Case (1) $\mathfrak{p}_0 = \mathfrak{p}_1 = \mathfrak{p}_2 = \mathfrak{p}_{inj} = \emptyset$, i.e., $\mathfrak{m} \setminus \{0\} = \mathfrak{p}_{non}$;
Case (2) $\mathfrak{p}_0 = \mathfrak{p}_1 = \mathfrak{p}_{non} = \mathfrak{p}_{inj} = \emptyset$, i.e., $\mathfrak{m} \setminus \{0\} = \mathfrak{p}_2$.
We conjecture that in both cases (1) and (2) there are no such solutions $\mathbf{\Psi}$. In other words:

$$\mathrm{Crank}(P^n(\boldsymbol{C})) \geq 2n-1 \quad (n \geq 3);$$
$$\mathrm{Crank}(P^n(\boldsymbol{H})) \geq 4n-2 \quad (n \geq 3).$$

If this is true, then we obtain an improvement of Theorem 1:

$$\mathrm{class}(P^n(\boldsymbol{C})) \geq 2n-1 \quad (n \geq 3);$$
$$\mathrm{class}(P^n(\boldsymbol{H})) \geq 4n-2 \quad (n \geq 3).$$

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