

## Blowup for systems of semilinear wave equations in two space dimensions

(Dedicated to Professor Mitsuhiro Nakao on the occasion of his sixtieth birthday)

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**Abstract.** We consider semilinear systems of wave equations with multiple propagation speeds and find out the critical order of the nonlinearity which characterizes large time behavior of small amplitude solutions to the system by establishing blowup results. We also evaluate the lifespan of the solution in terms of the size of the initial data from above and below. We underline that not only the order of the nonlinearity but also the way of coupling among unknowns in it has a major effect on the lifespan.

*Key words:* blowup, semilinear wave equations, lifespan.

### 1. Introduction

We are concerned with the Cauchy problem for systems of semilinear wave equations of the form

$$(\partial_t^2 - c_i^2 \Delta)u_i = F_i(u), \quad (x, t) \in \mathbb{R}^n \times [0, \infty), \quad (1.1)$$

$$u_i(x, 0) = \varepsilon \varphi_i(x), \partial_t u_i(x, 0) = \varepsilon \psi_i(x), \quad x \in \mathbb{R}^n, \quad (1.2)$$

where  $i = 1, \dots, m$ ,  $c_i > 0$ ,  $u = (u_1, \dots, u_m)$  is a  $\mathbb{R}^m$ -valued unknown function of  $(x, t)$ , and  $\varepsilon > 0$  is a small parameter. A typical example of  $F_i$  is

$$F_i(u) = \sum_{1 \leq j < k \leq m} A_i^{jk} |u_j|^{p_{ijk}} |u_k|^{\alpha - p_{ijk}}, \quad (1.3)$$

where  $A_i^{jk} \in \mathbb{R}$ ,  $\alpha \geq 2$  and  $1 \leq p_{ijk} \leq \alpha - 1$ . We study small data global existence and blowup for (1.1) especially in two space dimensions  $n = 2$ . Here, we say that small data global existence holds for (1.1) if for any  $\varphi_i, \psi_i \in C_0^\infty(\mathbb{R}^n)$  ( $1 \leq i \leq m$ ) there exists a constant  $\varepsilon_0 > 0$  such that the Cauchy problem (1.1)–(1.2) admits a global solution for any  $\varepsilon \in (0, \varepsilon_0]$ .

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Moreover, we say that small data blowup occurs for (1.1) if small data global existence does not hold for (1.1).

Before we proceed to our problem, we briefly recall some known results. For details, see an expository article [15]. We begin with a single equation

$$(\partial_t^2 - \Delta)u = |u|^p, \quad (x, t) \in \mathbb{R}^n \times [0, \infty), \quad (1.4)$$

where  $n \geq 2$  and  $p > 1$ . Let  $p_0(n)$  be the positive root of the quadratic equation

$$p \left( \frac{n-1}{2}p - \frac{n+1}{2} \right) = 1.$$

Note that  $p_0(2) = (3 + \sqrt{17})/2$ ,  $p_0(3) = 1 + \sqrt{2}$ ,  $p_0(4) = 2$ . Then, for any  $n \geq 2$ , it is known that small data global existence holds for (1.4) if  $p > p_0(n)$ , while small data blowup occurs if  $1 < p < p_0(n)$ . This result was first proved by John [10] for  $n = 3$ , and the general case  $n \geq 2$  was conjectured by Strauss [20]. Glassey [8, 9] verified the conjecture for  $n = 2$ . For the critical case  $p = p_0(n)$ , Schaeffer [18] proved small data blowup when  $n = 2, 3$  (see also [24, 25, 21]). For  $n \geq 4$ , small data blowup was proved by Sideris [19] when  $1 < p < p_0(n)$ , and recently by Yordanov and Zhang [23] when  $p = p_0(n)$ . While, for  $n \geq 4$  and  $p > p_0(n)$ , small data global existence has been studied by many authors, and was finally proved by Georgiev, Lindblad and Sogge [7] (see also [26, 6, 22, 2] and references cited therein).

Next, we consider a weakly coupled system

$$\begin{cases} (\partial_t^2 - c_1^2 \Delta)u_1 = |u_2|^p, & (x, t) \in \mathbb{R}^n \times [0, \infty), \\ (\partial_t^2 - c_2^2 \Delta)u_2 = |u_1|^q, & (x, t) \in \mathbb{R}^n \times [0, \infty), \end{cases} \quad (1.5)$$

where  $n = 2, 3$  and  $1 < p \leq q$ . The system (1.5) was first studied by Del Santo, Georgiev and Mitidieri [3] for the case  $c_1 = c_2$ , and they found a critical curve  $\Gamma(p, q) = 0$  such that small data global existence holds for (1.5) if  $\Gamma(p, q) < 0$ , while small data blowup occurs if  $\Gamma(p, q) > 0$  (see also [5, 4, 1, 12]). The case  $c_1 \neq c_2$  was studied by the authors [13] for  $n = 3$  and [15] for  $n = 2, 3$ , and it was shown that the critical curve  $\Gamma(p, q) = 0$  does not change even if  $c_1 \neq c_2$ . Especially, for the case  $q = p$ , it is proved that small data global existence holds for (1.5) if  $p > p_0(n)$ , while small data blowup occurs if  $1 < p \leq p_0(n)$  for  $n = 2, 3$  even if  $c_1 \neq c_2$  (for the general case  $q \neq p$ , see [13] and [15, Section 3]).

Finally, we consider a strongly coupled system

$$\begin{cases} (\partial_t^2 - c_1^2 \Delta)u_1 = |u_1|^{p_1}|u_2|^{q_1}, & (x, t) \in \mathbb{R}^n \times [0, \infty), \\ (\partial_t^2 - c_2^2 \Delta)u_2 = |u_1|^{p_2}|u_2|^{q_2}, & (x, t) \in \mathbb{R}^n \times [0, \infty), \end{cases} \tag{1.6}$$

where  $n = 2, 3$ ,  $p_i, q_i \geq 1$  for  $i = 1, 2$ . For simplicity, we assume  $\alpha := p_1 + q_1 = p_2 + q_2$ . This is a special case of (1.1) with (1.3). In three space dimensions  $n = 3$ , it is proved that small data global existence holds for (1.6) if  $c_1 \neq c_2$  and  $\alpha \geq 2$  (see Kubo and Tsugawa [16] for  $\alpha > 2$ , and the authors [14] for  $\alpha \geq 2$ ). On the other hand, in two space dimensions  $n = 2$ , it is proved that small data global existence holds for (1.6) if  $c_1 \neq c_2$  and  $\alpha > 3$  (see Kubo and Kubota [11] for a special case  $p_1 = q_2 = 1$  and  $q_1 = p_2 = \alpha - 1$ , and the authors [15, Theorem 4.1] for general case). It is remarkable that  $2 < p_0(3) < 3 < p_0(2)$  and that for the case  $c_1 = c_2$  small data blowup occurs for (1.6) if  $n = 2, 3$  and  $\alpha \leq p_0(n)$ . The last fact follows immediately from the result for the single equation (1.4). This means that the interaction through the nonlinearity in (1.6) has much stronger effect on the behavior of the solution than that in (1.5). In this sense we call (1.6) a strongly coupled system, while (1.5) a weakly coupled system.

Our main goal in this paper is in essence to show that small data blowup occurs for (1.6) when  $n = 2$  and  $2 \leq \alpha \leq 3$  even if  $c_1 \neq c_2$ . Recall that when  $n = 3$  small data global existence holds for (1.6) for any  $\alpha \geq 2$  if  $c_1 \neq c_2$ . The difference between the cases  $n = 2$  and  $n = 3$  comes from the fact that Huygens' principle does not hold in two space dimensions unlike in three space dimensions.

To state our main result precisely, we give the assumptions on the nonlinearity  $F_i$  and initial data. We assume that  $F_i \in C(\mathbb{R}^m)$  and there exist constants  $A > 0$  and  $\alpha > 1$  such that

$$F_i(\lambda) \geq A (\min\{|\lambda_j| : 1 \leq j \leq m\})^\alpha, \quad \lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m \tag{1.7}$$

for all  $1 \leq i \leq m$ . Note that the example (1.3) satisfies (1.7) if

$$\min\{A_i^{jk} : 1 \leq j < k \leq m\} \geq 0, \quad \max\{A_i^{jk} : 1 \leq j < k \leq m\} > 0.$$

Indeed, taking  $(j_i, k_i)$  such that  $A_i^{j_i k_i} = \max\{A_i^{jk} : 1 \leq j < k \leq m\}$ , we have

$$F_i(\lambda) \geq A_i^{j_i k_i} |\lambda_{j_i}|^{p_{ij_i k_i}} |\lambda_{k_i}|^{\alpha - p_{ij_i k_i}} \geq A_i^{j_i k_i} (\min\{|\lambda_l| : 1 \leq l \leq m\})^\alpha$$

for  $\lambda \in \mathbb{R}^m$ , so (1.7) is satisfied with  $A = \min\{A_i^{j_i k_i} : 1 \leq i \leq m\}$ .

We consider initial conditions of the form

$$u_i(x, 0) = 0, \quad \partial_t u_i(x, 0) = \varepsilon \psi_i(x), \quad x \in \mathbb{R}^2 \quad (1.8)$$

for  $1 \leq i \leq m$ , and assume that for all  $1 \leq i \leq m$ ,  $\psi_i \in C(\mathbb{R}^2)$  satisfies

$$\psi_i(x) \geq 0 \text{ for } x \in \mathbb{R}^2, \quad \psi_i(0) > 0. \quad (1.9)$$

Then, the Cauchy problem (1.1) and (1.8) can be written in the integral form

$$u_i = \varepsilon J_{c_i}[\psi_i] + L_{c_i}[F_i(u)], \quad (x, t) \in \mathbb{R}^2 \times [0, \infty), \quad (1.10)$$

where  $i = 1, \dots, m$ , and

$$J_c[g](x, t) = \frac{t}{2\pi} \int_{|\xi| < 1} \frac{g(x + ct\xi)}{\sqrt{1 - |\xi|^2}} d\xi,$$

$$L_c[f](x, t) = \int_0^t \frac{t-s}{2\pi} \int_{|\xi| < 1} \frac{f(x + c(t-s)\xi, s)}{\sqrt{1 - |\xi|^2}} d\xi ds.$$

Our main result in this paper is as follows.

**Theorem 1.1** *Let  $n = 2$ ,  $m \in \mathbb{N}$ ,  $0 < \varepsilon \leq 1$ ,  $c_i > 0$ ,  $F_i \in C(\mathbb{R}^m)$ ,  $\psi_i \in C(\mathbb{R}^2)$  for  $1 \leq i \leq m$ . Assume that  $\psi_i$  satisfies (1.9) for all  $1 \leq i \leq m$ , and that there exist constants  $A > 0$  and  $1 < \alpha \leq 3$  such that (1.7) holds for all  $\lambda \in \mathbb{R}^m$  and  $1 \leq i \leq m$ . Then the solution of (1.10) blows up in a finite time  $T^*(\varepsilon)$ . Moreover, there exists a positive constant  $C^*$  independent of  $\varepsilon$  such that*

$$T^*(\varepsilon) \leq \begin{cases} \exp(C^* \varepsilon^{-2}) & \text{if } \alpha = 3, \\ C^* \varepsilon^{-(\alpha-1)/(3-\alpha)} & \text{if } 1 < \alpha < 3. \end{cases} \quad (1.11)$$

For completeness, we study the lower bound of the lifespan  $T^*(\varepsilon)$ . To do so, we consider the following integral equations associated with (1.1)–(1.2):

$$u_i = \varepsilon K_{c_i}[\varphi_i, \psi_i] + L_{c_i}[F_i(u)], \quad (x, t) \in \mathbb{R}^2 \times [0, T), \quad (1.12)$$

where  $i = 1, \dots, m$ ,  $T > 0$  and

$$K_c[\varphi, \psi](x, t) = J_c[\psi](x, t) + \partial_t J_c[\varphi](x, t)$$

with  $J_c[g]$  and  $L_c[f]$  defined in (1.10). If  $\varphi_i \in C_0^1(\mathbb{R}^2)$ ,  $\psi_i \in C_0(\mathbb{R}^2)$ , then

$$\|(\varphi, \psi)\| \equiv \sup_{x \in \mathbb{R}^2} \{ \langle x \rangle^\nu |\varphi(x)| + \langle x \rangle^{\nu+1} (|\nabla \varphi(x)| + |\psi(x)|) \} \quad (1.13)$$

is finite for any  $\nu > 1$ . Therefore, as is well-known, there is a positive constant  $C = C(c)$  such that

$$|K_c[\varphi, \psi](x, t)| \leq C(c) \|(\varphi, \psi)\| \langle t + |x| \rangle^{-1/2} \langle ct - |x| \rangle^{-1/2} \tag{1.14}$$

holds for all  $(x, t) \in \mathbb{R}^2 \times [0, \infty)$  (for the proof, see e.g. [17]). Here we have fixed  $\nu > 1$  and denoted  $\langle x \rangle = \sqrt{1 + |x|^2}$ . Bearing this in mind, we define

$$\|u\|_{c_i} = \sup_{(x,t) \in \mathbb{R}^2 \times [0, T]} \langle t + |x| \rangle^{1/2} \langle c_i t - |x| \rangle^{1/2} |u(x, t)|, \tag{1.15}$$

so that

$$\|K_{c_i}[\varphi_i, \psi_i]\|_{c_i} \leq C(c_i) \|(\varphi, \psi)\|. \tag{1.16}$$

Then we have the following.

**Theorem 1.2** *Let  $c_1, \dots, c_m$  be different from each other and let  $2 \leq \alpha \leq 3$  in (1.3). Assume that  $\varphi_i \in C_0^1(\mathbb{R}^2)$  and  $\psi_i \in C_0(\mathbb{R}^2)$ . Then there is a positive constant  $\varepsilon_0 = \varepsilon_0(c_j, p_{ijk}, \varphi_i, \psi_i)$  such that for  $0 < \varepsilon \leq \varepsilon_0$  there exists a unique solution  $(u_1, \dots, u_m) \in (C(\mathbb{R}^2 \times [0, T^*(\varepsilon)))^m$  to (1.12) with (1.3) satisfying*

$$\max_{1 \leq i \leq m} \|u_i\|_{c_i} \leq 2C_0 \varepsilon \|(\varphi, \psi)\|, \quad C_0 = \max_{1 \leq i \leq m} C(c_i). \tag{1.17}$$

Moreover, there exists a positive constant  $C$  independent of  $\varepsilon$  such that

$$T^*(\varepsilon) \geq C\varepsilon^{-(\alpha-1)/(3-\alpha)} \tag{1.18}$$

for  $2 \leq \alpha < 3$ , and

$$T^*(\varepsilon) \geq \begin{cases} \exp(C\varepsilon^{-2}) & \text{if } 1 < p_{ijk} < 2 \text{ for all } 1 \leq i, j, k \leq m, \\ \exp(C\varepsilon^{-1}) & \text{otherwise} \end{cases} \tag{1.19}$$

for  $\alpha = 3$ .

We see from Theorems 1.1 and 1.2 that the estimate given by (1.11) is sharp with respect to the order of  $\varepsilon$  except for the case where  $\alpha = 3$  and either  $p_{ijk} = 1$  or  $p_{ijk} = 2$  for some  $1 \leq i, j, k \leq m$ . The following result tells us that the case is actually exceptional and it is impossible to unify the lower bound of the lifespan in the case  $\alpha = 3$ . To be specific, we assume that  $F_k \in C(\mathbb{R}^m)$  is nonnegative for all  $1 \leq k \leq m$  and that there exist

$A > 0$  and a pair  $(i, j)$  such that  $c_i < c_j$  and

$$F_i(\lambda) \geq A|\lambda_i|^2|\lambda_j|, \quad \lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m. \quad (1.20)$$

Then we have the following.

**Theorem 1.3** *Let  $n = 2$ ,  $m \in \mathbb{N}$ ,  $0 < \varepsilon \leq 1$ ,  $c_i > 0$ ,  $\psi_i \in C(\mathbb{R}^2)$  and let  $F_i \in C(\mathbb{R}^m)$  be nonnegative for  $1 \leq i \leq m$ . Assume that  $\psi_i$  satisfies (1.9) for all  $1 \leq i \leq m$ , and there exist  $A > 0$  and a pair  $(i, j)$  such that  $c_i < c_j$  and (1.20) holds for all  $\lambda \in \mathbb{R}^m$ . Then the solution of (1.10) blows up in a finite time  $T^*(\varepsilon)$ . Moreover, there exists a positive constant  $C^*$  independent of  $\varepsilon$  such that*

$$T^*(\varepsilon) \leq \exp(C^* \varepsilon^{-1}). \quad (1.21)$$

This paper is organized as follows. In Section 2 we prove Theorem 1.1. As stated above, the proof of the theorem is based on the fact that Huygens' principle does not hold in two space dimensions unlike in three space dimensions. For estimating  $J_c$  and  $L_c$  from below, we use formulas (2.2) and (2.3) which are characteristic of two space dimensions (see Lemma 2.1 and Remark 2.1 below). In Section 3 we prove Theorem 1.3. Our starting point is (3.3) which involves the small parameter  $\varepsilon$ , unlike (2.7). Nevertheless we are able to establish (3.6) that is independent of  $\varepsilon$  by modifying the argument in Section 2 carefully. In Section 4 we show Theorem 1.2. Such a local existence result together with the lower bounds of the lifespan will be obtained by the standard contraction mapping argument in combination with weighted  $L^\infty$ - $L^\infty$  estimates introduced in Lemma 4.2 below. Especially, to establish (1.19) for the exceptional case, we need a refined estimate (4.5).

## 2. Proof of Theorem 1.1

For  $c > 0$  and  $y > 0$ , we put

$$\begin{aligned} \Sigma(c; y) &= \{(\lambda, s) \in [0, \infty)^2 : \lambda \leq c(s - y)\}, \\ \tilde{\Sigma}(c; y) &= \{(x, t) \in \mathbb{R}^2 \times [0, \infty) : (|x|, t) \in \Sigma(c; y)\}, \end{aligned}$$

and for  $(r, t) \in \Sigma(c; 1)$ , we put

$$E(c; r, t) = \{(\lambda, s) : 0 \leq \lambda \leq c(t - s) - r, 0 \leq cs \leq ct - r\}. \quad (2.1)$$

**Lemma 2.1** *Let  $c > 0$ , and put  $r = |x|$  for  $x \in \mathbb{R}^2$ . If  $G(x) \geq g(|x|) \geq 0$  for  $x \in \mathbb{R}^2$ , then*

$$J_c[G](x, t) \geq \frac{1}{c\sqrt{2(ct+r)(ct-r)}} \int_0^{ct-r} \lambda g(\lambda) d\lambda \tag{2.2}$$

*holds for any  $(x, t) \in \tilde{\Sigma}(c; 1)$ . Moreover, if  $F(x, t) \geq f(|x|, t) \geq 0$  for  $(x, t) \in \mathbb{R}^2 \times [0, \infty)$ , then*

$$L_c[F](x, t) \geq \frac{1}{c\sqrt{2(ct+r)(ct-r)}} \iint_{E(c;r,t)} \lambda f(\lambda, s) d\lambda ds \tag{2.3}$$

*holds for any  $(x, t) \in \tilde{\Sigma}(c; 1)$ .*

*Proof.* Let  $\tilde{g}(x) = g(|x|)$  for  $x \in \mathbb{R}^2$ . Then, it is well-known that

$$J_c[\tilde{g}](x, t) = \frac{2}{c\pi} \int_0^{ct} \frac{\rho}{\sqrt{(ct)^2 - \rho^2}} \left( \int_{|r-\rho|}^{r+\rho} \frac{\lambda g(\lambda)}{\sqrt{h(\lambda, \rho, r)}} d\lambda \right) d\rho$$

holds for  $(x, t) \in \mathbb{R}^2 \times [0, \infty)$ , where we put

$$h(\lambda, \rho, r) = \{(r + \rho)^2 - \lambda^2\} \{\lambda^2 - (r - \rho)^2\}.$$

Since  $h(\lambda, \rho, r) = h(\rho, \lambda, r)$  and  $g(\lambda) \geq 0$ , we have for  $r < ct$

$$\begin{aligned} J_c[G](x, t) &\geq \frac{2}{c\pi} \int_0^{ct-r} \left( \int_{|r-\lambda|}^{r+\lambda} \frac{\rho \lambda g(\lambda)}{\sqrt{(ct)^2 - \rho^2} \sqrt{h(\rho, \lambda, r)}} d\rho \right) d\lambda \\ &\geq \frac{2}{c\pi} \int_0^{ct-r} \frac{\lambda g(\lambda)}{\sqrt{(ct)^2 - (r - \lambda)^2}} \left( \int_{|r-\lambda|}^{r+\lambda} \frac{\rho}{\sqrt{h(\rho, \lambda, r)}} d\rho \right) d\lambda \\ &= \frac{1}{c} \int_0^{ct-r} \frac{\lambda g(\lambda)}{\sqrt{(ct)^2 - (r - \lambda)^2}} d\lambda. \end{aligned}$$

Here, for the last equality, we used the fact that

$$\int_a^b \frac{\rho}{\sqrt{(b^2 - \rho^2)(\rho^2 - a^2)}} d\rho = \frac{\pi}{2}, \quad 0 \leq a < b.$$

Since  $(ct)^2 - (r - \lambda)^2 = (ct - r + \lambda)(ct + r - \lambda) \leq 2(ct - r)(ct + r)$  for  $0 \leq \lambda \leq ct - r$ , we obtain (2.2). Finally, (2.3) follows from (2.2).  $\square$

**Remark 2.1** Let  $n = 2, 3$ . If  $F(x, t) \geq f(|x|, t) \geq 0$  for  $(x, t) \in \mathbb{R}^n \times [0, \infty)$ , then

$$L_c[F](x, t) \geq \frac{1}{2cr^{(n-1)/2}} \iint_{D(c;r,t)} \lambda^{(n-1)/2} f(\lambda, s) d\lambda ds \tag{2.4}$$

holds for any  $(x, t) \in \mathbb{R}^n \times [0, \infty)$ , where

$$D(c; r, t) = \{(\lambda, s) : 0 \leq s \leq t, |r - c(t - s)| \leq \lambda \leq r + c(t - s)\}.$$

The formula (2.4) has been used for proving small data blowup for the single equation (1.4) and the weakly coupled system (1.5) in a unified way for both  $n = 2$  and  $3$  (see, e.g., [21, 12, 15]). Moreover, in three space dimensions, the equality holds in (2.4) if  $F(x, t) = f(|x|, t)$  for  $(x, t) \in \mathbb{R}^3 \times [0, \infty)$ . Therefore, there is no formula corresponding to (2.3) in three space dimensions.

**Lemma 2.2** Let  $a, c > 0$  and  $\mu, \nu \geq 0$ . Then there exists a positive constant  $C$  depending only on  $a, c, \mu, \nu$  such that

$$\begin{aligned} & \iint_{E(a;r,t) \cap \Sigma(c;1)} \frac{\lambda^\nu}{(s + \lambda)^\mu} f\left(\frac{cs - \lambda}{c}\right) d\lambda ds \\ & \geq C \int_1^{(at-r)/a} \left(1 - \frac{a\eta}{at - r}\right)^{\nu+1} \frac{f(\eta)}{\eta^{\mu-\nu-1}} d\eta \end{aligned} \tag{2.5}$$

holds for any non-negative function  $f$  on  $[1, \infty)$  and  $(r, t) \in \Sigma(a; 1)$ .

*Proof.* Let  $(r, t) \in \Sigma(a; 1)$ . We denote the left-hand side of (2.5) by  $I(r, t)$ , and change the variables by  $\xi = as + \lambda, \eta = (cs - \lambda)/c$ . Then we have

$$I(r, t) \geq C \int_1^{(at-r)/a} \left( \int_{a\eta}^{at-r} \frac{(\xi - a\eta)^\nu}{\xi^\mu} d\xi \right) f(\eta) d\eta.$$

Thus, (2.5) follows from Lemma 2.3 below. □

**Lemma 2.3** Let  $\mu \geq 0$  and  $\nu \geq 0$ . Then there exists a constant  $C = C(\mu, \nu) > 0$  such that

$$\int_a^b \frac{(\rho - a)^\nu}{\rho^\mu} d\rho \geq \frac{C}{a^{\mu-\nu-1}} \left(1 - \frac{a}{b}\right)^{\nu+1}$$

for  $0 < a < b$ .



*Proof.* We distinguish two cases  $2a \leq b$  and  $b < 2a$ . When  $2a \leq b$ , we have

$$\begin{aligned} \int_a^b \frac{(\rho - a)^\nu}{\rho^\mu} d\rho &\geq \int_a^{2a} \frac{(\rho - a)^\nu}{\rho^\mu} d\rho \geq \frac{1}{(2a)^\mu} \int_a^{2a} (\rho - a)^\nu d\rho \\ &= \frac{a^{\nu+1}}{(\nu + 1)(2a)^\mu} \geq \frac{1}{(\nu + 1)2^\mu a^{\mu-\nu-1}} \left(1 - \frac{a}{b}\right)^{\nu+1}. \end{aligned}$$

While, if  $a < b < 2a$ , we have

$$\begin{aligned} \int_a^b \frac{(\rho - a)^\nu}{\rho^\mu} d\rho &\geq \frac{1}{b^\mu} \int_a^b (\rho - a)^\nu d\rho = \frac{(b - a)^{\nu+1}}{(\nu + 1)b^\mu} \\ &= \frac{b^{\nu+1}}{(\nu + 1)b^\mu} \left(1 - \frac{a}{b}\right)^{\nu+1} \geq \frac{1}{(\nu + 1)2^\mu a^{\mu-\nu-1}} \left(1 - \frac{a}{b}\right)^{\nu+1}. \end{aligned}$$

This completes the proof. □

The following lemma has been often used for proving small data blowup for semilinear wave equations (see, e.g., [24, 21, 12, 14, 15]).

**Lemma 2.4** *Let  $C_1, C_2 > 0, \alpha, \beta \geq 0, b \geq 0, p > 1, \kappa \leq 1$  and  $0 < \varepsilon \leq 1$ . Suppose that  $f(y)$  satisfies*

$$f(y) \geq C_1 \varepsilon^\alpha, \quad f(y) \geq C_2 \varepsilon^\beta \int_1^y \left(1 - \frac{\eta}{y}\right)^b \frac{f(\eta)^p}{\eta^\kappa} d\eta, \quad y \geq 1.$$

*Then,  $f(y)$  blows up in a finite time  $T^*(\varepsilon)$ . Moreover, there exists a constant  $C^* = C^*(C_1, C_2, b, p, \kappa) > 0$  such that*

$$T^*(\varepsilon) \leq \begin{cases} \exp(C^* \varepsilon^{-\{(p-1)\alpha+\beta\}}) & \text{if } \kappa = 1, \\ C^* \varepsilon^{-\{(p-1)\alpha+\beta\}/(1-\kappa)} & \text{if } \kappa < 1. \end{cases}$$

For the proof, see [14, Lemma 6.3] and [15, Lemma 2.3].

We are now in position to give the proof of Theorem 1.1. In what follows, we put  $r = |x|$  for  $x \in \mathbb{R}^2$ , and

$$\langle f \rangle_c(y) = \inf\{\sqrt{(ct + |x|)(ct - |x|)} |f(x, t)| : (x, t) \in \tilde{\Sigma}(c; y)\}.$$

*Proof of Theorem 1.1.* For the solution  $(u_1, \dots, u_m)$  of the system (1.10), we put

$$U(y) = \min\{\langle u_i \rangle_{c_i}(y) : 1 \leq i \leq m\}.$$

First, we show that there exists  $C_1 > 0$  such that

$$U(y) \geq C_1 \varepsilon, \quad y \geq 1. \tag{2.6}$$

Since  $\psi_i \in C(\mathbb{R}^2)$  satisfies (1.9), there exist  $\delta_i > 0$  and  $\phi_i \in C([0, \infty))$  such that  $\psi_i(x) \geq \phi_i(|x|) \geq 0$  for  $x \in \mathbb{R}^2$  and  $\phi_i(r) > 0$  for  $0 \leq r \leq \delta_i$ . By (2.2) in Lemma 2.1, we have

$$\begin{aligned} \sqrt{(c_i t + |x|)(c_i t - |x|)} J_{c_i}[\psi_i](x, t) &\geq C \int_0^{c_i t - r} \lambda \phi_i(\lambda) d\lambda \\ &\geq C \int_0^{c_i} \lambda \phi_i(\lambda) d\lambda \end{aligned}$$

for all  $(x, t) \in \tilde{\Sigma}(c_i; 1)$ . Thus, for any  $1 \leq i \leq m$ , we have

$$\langle u_i \rangle_{c_i}(y) \geq \varepsilon \langle J_{c_i}[\psi_i] \rangle_{c_i}(y) \geq C\varepsilon, \quad y \geq 1,$$

which implies (2.6).

Next, we show that there exists  $C_2 > 0$  such that

$$U(y) \geq C_2 \int_1^y \left(1 - \frac{\eta}{y}\right)^2 \frac{U(\eta)^\alpha}{\eta^{\alpha-2}} d\eta, \quad y \geq 1. \quad (2.7)$$

We put  $c_* = \min\{c_i : 1 \leq i \leq m\}$ . From the definition of  $\langle u_j \rangle_{c_j}(y)$ , for any  $1 \leq j \leq m$  and  $(x, t) \in \tilde{\Sigma}(c_*; 1)$ , we see that

$$\begin{aligned} \sqrt{(c_j t + |x|)(c_j t - |x|)} |u_j(x, t)| &\geq \langle u_j \rangle_{c_j} \left( \frac{c_j t - r}{c_j} \right) \\ &\geq \langle u_j \rangle_{c_j} \left( \frac{c_* t - r}{c_*} \right). \end{aligned}$$

Thus, by (1.7), for any  $1 \leq i \leq m$  and  $(x, t) \in \tilde{\Sigma}(c_*; 1)$ , we have

$$\begin{aligned} F_i(u(x, t)) &\geq A (\min\{|u_j(x, t)| : 1 \leq j \leq m\})^\alpha \\ &\geq \frac{A}{(c_* t + r)^\alpha} \left[ U \left( \frac{c_* t - r}{c_*} \right) \right]^\alpha, \end{aligned}$$

where  $c_* = \max\{c_i : 1 \leq i \leq m\}$ . By Lemmas 2.1 and 2.2, for any  $1 \leq i \leq m$  and  $(x, t) \in \tilde{\Sigma}(c_i; 1)$ , we have

$$\begin{aligned} &\sqrt{(c_i t + |x|)(c_i t - |x|)} L_{c_i}[F_i(u)](x, t) \\ &\geq C \iint_{E(c_i; r, t) \cap \Sigma(c_*, 1)} \frac{\lambda}{(s + \lambda)^\alpha} \left[ U \left( \frac{c_* s - \lambda}{c_*} \right) \right]^\alpha d\lambda ds \\ &\geq C \int_1^{(c_i t - r)/c_i} \left(1 - \frac{c_i \eta}{c_i t - r}\right)^2 \frac{U(\eta)^\alpha}{\eta^{\alpha-2}} d\eta. \end{aligned}$$

Since the function

$$y \mapsto \int_1^y \left(1 - \frac{\eta}{y}\right)^2 \frac{U(\eta)^\alpha}{\eta^{\alpha-2}} d\eta$$

is non-decreasing, for any  $1 \leq i \leq m$ , we have

$$\langle u_i \rangle_{c_i}(y) \geq \langle L_{c_i}[F_i(u)] \rangle_{c_i}(y) \geq C \int_1^y \left(1 - \frac{\eta}{y}\right)^2 \frac{U(\eta)^\alpha}{\eta^{\alpha-2}} d\eta, \quad y \geq 1,$$

which implies (2.7).

Finally, since we assume  $1 < \alpha \leq 3$ , applying Lemma 2.4 to (2.6) and (2.7), we see that  $U(y)$  blows up in a finite time  $T^*(\varepsilon)$  and that  $T^*(\varepsilon)$  satisfies the estimate (1.11). This completes the proof.  $\square$

### 3. Proof of Theorem 1.3

Throughout this section, we fix a pair  $(i, j)$  in such a way that  $c_i < c_j$  and  $F_i(u) \geq A|u_i|^2|u_j|$  holds according to (1.20). For the solution  $(u_1, \dots, u_m)$  of the system (1.10), we put

$$U(y) = \langle u_i \rangle_{c_i}(y).$$

First of all, seeing the proof of (2.6), we find that it is also valid under the assumptions of Theorem 1.3. In particular, we have

$$\langle u_i \rangle_{c_i}(y) \geq C\varepsilon, \quad \langle u_j \rangle_{c_j}(y) \geq C\varepsilon, \quad y \geq 1. \tag{3.1}$$

Hence there exists  $C_1 > 0$  such that

$$U(y) \geq C_1\varepsilon, \quad y \geq 1. \tag{3.2}$$

Next, we show that there exists  $C_2 > 0$  such that

$$U(y) \geq C_2\varepsilon \int_1^y \left(\log \frac{y}{\eta} - 1 + \frac{\eta}{y}\right) \frac{U(\eta)^2}{\eta} d\eta, \quad y \geq 1. \tag{3.3}$$

Since  $c_i < c_j$ , we see from (3.1) that for any  $(x, t) \in \tilde{\Sigma}(c_i; 1)$ ,

$$|u_j(x, t)| \geq \frac{C\varepsilon}{\sqrt{(c_j t + |x|)(c_j t - |x|)}} \geq \frac{C\varepsilon}{t + |x|}.$$

Thus, by (1.20), for any  $(x, t) \in \tilde{\Sigma}(c_i; 1)$ , we have

$$F_i(u(x, t)) \geq \frac{A}{(c_i t + |x|)(c_i t - |x|)} \left[ U \left( \frac{c_i t - r}{c_i} \right) \right]^2 \times \frac{C\varepsilon}{t + |x|}.$$

Using Lemma 2.1 and changing the variables by  $\xi = c_i s + \lambda$ ,  $\eta = (c_i s - \lambda)/c_i$ , for any  $(x, t) \in \tilde{\Sigma}(c_i; 1)$ , we have

$$\begin{aligned} & \sqrt{(c_i t + |x|)(c_i t - |x|)} L_{c_i}[F_i(u)](x, t) \\ & \geq C\varepsilon \int_1^{(c_i t - r)/c_i} \left( \int_{c_i \eta}^{c_i t - r} \frac{\xi - c_i \eta}{\xi^2} d\xi \right) \frac{U(\eta)^2}{\eta} d\eta \\ & = C\varepsilon \int_1^{(c_i t - r)/c_i} \left( \log \frac{c_i t - r}{c_i \eta} - 1 + \frac{c_i \eta}{c_i t - r} \right) \frac{U(\eta)^2}{\eta} d\eta. \end{aligned}$$

Since the function

$$y \mapsto \int_1^y \left( \log \frac{y}{\eta} - 1 + \frac{\eta}{y} \right) \frac{U(\eta)^2}{\eta} d\eta$$

is non-decreasing, we obtain (3.3).

If we set  $\varphi(z) = \varepsilon^{-1} U(e^{z/\varepsilon})$  in (3.2), (3.3), then for  $z \geq 0$  we have

$$\varphi(z) \geq C_1, \tag{3.4}$$

$$\varphi(z) \geq C_2 \varepsilon \int_0^z K \left( \frac{z - \zeta}{\varepsilon} \right) \varphi(\zeta)^2 d\zeta, \tag{3.5}$$

where

$$K(z) = z - 1 + e^{-z}.$$

In addition, introducing a function  $\tilde{K}(z)$  as

$$\tilde{K}(z) = \frac{1}{2e} \min\{z, z^2\},$$

we get  $K(z) \geq \tilde{K}(z)$  for  $z \geq 0$ . Note that  $\tilde{K}(\lambda z) \geq \lambda \tilde{K}(z)$  holds for  $\lambda \geq 1$ ,  $z \geq 0$ . Therefore, if  $0 < \varepsilon \leq 1$ ,  $z \geq 1$ ,  $1 \leq \zeta \leq z$ , then we find

$$\varepsilon K \left( \frac{z - \zeta}{\varepsilon} \right) \geq \varepsilon \tilde{K} \left( \frac{z - \zeta}{\varepsilon} \right) \geq \tilde{K} \left( 1 - \frac{\zeta}{z} \right) = \frac{1}{2e} \left( 1 - \frac{\zeta}{z} \right)^2.$$

Hence, (3.5) implies

$$\varphi(z) \geq C_3 \int_1^z \left( 1 - \frac{\zeta}{z} \right)^2 \varphi(\zeta)^2 d\zeta \tag{3.6}$$

for  $z \geq 1$ , where we put  $C_3 = C_2/(2e)$ . Now, applying Lemma 2.4 to (3.4) and (3.6), we see that  $\varphi(z)$  blows up in a finite time, so that  $T^*(\varepsilon)$  satisfies the estimate (1.21). This completes the proof.  $\square$

**4. Proof of Theorem 1.2**

To evaluate the operator  $L_c[f]$ , we shall make use of basic estimates given in Lemma 4.2 below. It is a generalization of [15, Propositions 2.3 and 2.4] with  $n = 2$ . First we prepare the following elementary inequalities.

**Lemma 4.1** *Let  $\kappa \in \mathbb{R}$ ,  $\ell \geq 0$  and  $(r, t) \in [0, \infty)^2$ . Then there exists a constant  $C = C(\kappa, \ell) > 0$  such that*

$$\begin{aligned} & \int_0^{[t-r]_+} \langle \rho \rangle^{-(1/2)-\kappa} (t-r-\rho)^{-1/2} [\log(1 + \langle \rho \rangle)]^\ell d\rho \tag{4.1} \\ & \leq \frac{C[\log(1 + \langle r-t \rangle)]^\ell \langle r-t \rangle^{[(1/2)-\kappa]_+}}{\langle r-t \rangle^{1/2}}. \end{aligned}$$

Here for  $A \geq 1$  we have denoted

$$A^{[a]_+} = A^a \quad \text{if } a > 0; \quad A^{[a]_+} = 1 \quad \text{if } a < 0; \quad A^{[0]_+} = 1 + \log A.$$

Moreover, if  $\kappa > 0$  or  $t \geq 2r$ , then there exists a constant  $C = C(\kappa, \ell) > 0$  such that

$$\begin{aligned} & \int_{|r-t|}^{r+t} \langle \rho \rangle^{-(1/2)-\kappa} (\rho+r-t)^{-1/2} [\log(1 + \langle \rho \rangle)]^\ell d\rho \tag{4.2} \\ & \leq \frac{C[\log(1 + \langle r-t \rangle)]^\ell}{\langle r-t \rangle^\kappa}. \end{aligned}$$

*Proof.* When  $\kappa > 0$ , the above estimates follow from [15, Lemmas 2.5 and 2.10], hence we assume  $\kappa \leq 0$ . Let  $0 < \delta < 1/2$ .

First we show (4.1). The left-hand side of (4.1) is estimated by

$$\langle r-t \rangle^{-\kappa+\delta} \int_0^{[t-r]_+} \langle \rho \rangle^{-(1/2)-\delta} (t-r-\rho)^{-1/2} [\log(1 + \langle \rho \rangle)]^\ell d\rho.$$

Now we can apply (4.1) with  $\kappa$  replaced by  $\delta > 0$  to get the needed estimate.

Similarly, we see that the left-hand side of (4.2) is estimated by

$$C \langle r+t \rangle^{-\kappa+\delta} \langle r-t \rangle^{-\delta} [\log(1 + \langle r-t \rangle)]^\ell,$$

which implies (4.2) by the assumption  $t \geq 2r$ . This completes the proof.  $\square$

**Lemma 4.2** *Let  $a, c, \mu, \kappa > 0$ . For  $f \in C(\mathbb{R}^2 \times [0, T])$  we put*

$$\|f\|_{L^\infty(\mu, \kappa, a)} = \sup_{(x, t) \in \mathbb{R}^2 \times [0, T]} \langle t + |x| \rangle^\mu \langle at - |x| \rangle^\kappa |f(x, t)|.$$

*If  $0 < \kappa < 1$  and  $\mu + \kappa < 3$ , then there is a positive constant  $C = C(c, a, \mu, \kappa)$  such that*

$$w_c(x, t) |L_c[f](x, t)| \leq C \langle t + |x| \rangle^{3-\mu-\kappa} \|f\|_{L^\infty(\mu, \kappa, a)} \tag{4.3}$$

*for  $(x, t) \in \mathbb{R}^2 \times [0, T]$ , where we set*

$$w_c(x, t) = \langle t + |x| \rangle^{1/2} \langle ct - |x| \rangle^{1/2}.$$

*Moreover, if  $0 < \kappa < 1$  and  $\mu + \kappa = 3$ , then we have*

$$w_c(x, t) |L_c[f](x, t)| \leq C \log(1 + \langle ct - |x| \rangle) \|f\|_{L^\infty(\mu, \kappa, a)} \tag{4.4}$$

*for  $(x, t) \in \mathbb{R}^2 \times [0, T]$ . While, if  $\kappa = 1$  and  $\mu = 2$ , then we have*

$$w_c(x, t) |L_c[f](x, t)| \leq C (\log(1 + \langle ct - |x| \rangle))^2 \|f\|_{L^\infty(\mu, \kappa, a)} \tag{4.5}$$

*for  $(x, t) \in \mathbb{R}^2 \times [0, T]$ .*

*Proof.* Since  $L_c[f](x, t) = L_1[f_c](x, ct)$  with  $f_c(x, t) = c^{-2}f(x, t/c)$ , it suffices to show the estimates for  $c = 1$ . It follows that

$$|L_1[f](x, t)| \leq (I_1 + I_2) \|f\|_{L^\infty(\mu, \kappa, a)}, \tag{4.6}$$

where we put

$$I_1 = \iint_{D(1; r, t)} \frac{\langle \lambda + s \rangle^{-\mu+1} \langle \lambda - as \rangle^{-\kappa}}{\sqrt{(\lambda - s + t + r)(\lambda + s + r - t)}} d\lambda ds,$$

$$I_2 = \iint_{E(1; r, t)} \frac{\langle \lambda + s \rangle^{-\mu+1} \langle \lambda - as \rangle^{-\kappa}}{\sqrt{(\lambda - s + t + r)(t - r - \lambda - s)}} d\lambda ds,$$

where  $D(1; r, t), E(1; r, t)$  are defined by (2.4), (2.1), respectively. Changing the variables by  $\alpha = \lambda + s, \beta = \lambda - s$ , we have

$$I_1 = \frac{1}{2} \int_{|t-r|}^{t+r} \frac{\langle \alpha \rangle^{-\mu+1}}{\sqrt{\alpha + r - t}} d\alpha \int_{r-t}^\alpha \frac{\langle \frac{1-a}{2}\alpha + \frac{1+a}{2}\beta \rangle^{-\kappa}}{\sqrt{\beta + t + r}} d\beta, \tag{4.7}$$

$$I_2 = \frac{1}{2} \int_0^{[t-r]^+} \frac{\langle \alpha \rangle^{-\mu+1}}{\sqrt{t - r - \alpha}} d\alpha \int_{-\alpha}^\alpha \frac{\langle \frac{1-a}{2}\alpha + \frac{1+a}{2}\beta \rangle^{-\kappa}}{\sqrt{\beta + t + r}} d\beta, \tag{4.8}$$

where we put  $[a]_+ = \max(a, 0)$ .

First we show (4.3) for  $0 < \kappa < 1$  and  $\mu + \kappa < 3$ . In this case, we may assume  $\mu < 2$ , because when  $\mu \geq 2$ , we can find  $\mu', \kappa'$  such that  $\mu' < 2$ ,  $0 < \kappa' < 1$  and  $\mu' + \kappa' = \mu + \kappa < 3$ . In fact, if we put

$$\mu' = \mu - \frac{\mu - \kappa - 1}{2}, \quad \kappa' = \kappa + \frac{\mu - \kappa - 1}{2},$$

then they satisfy the needed conditions, since  $\mu \geq 2 > \kappa + 1$  and  $2 < \mu + \kappa < 3$ . In what follows, we denote  $r = |x|$ .

Now we divide the argument into three cases. First suppose  $2r \geq t \geq 0$  and  $t + r \geq 1$ . Since  $\beta + t + r \geq 2r$  if either  $\beta > -\alpha$  and  $0 < \alpha < t - r$  or  $\beta > r - t$ , we have  $\beta + t + r \geq C\langle t + r \rangle$  in this case. Since  $\kappa < 1$ , we get

$$\begin{aligned} & \langle t + r \rangle^{1/2} (I_1 + I_2) \\ & \leq C \left\{ \int_{|t-r|}^{t+r} \frac{\langle \alpha \rangle^{-\mu-\kappa+2}}{\sqrt{\alpha+r-t}} d\alpha + \int_0^{[t-r]_+} \frac{\langle \alpha \rangle^{-\mu-\kappa+2}}{\sqrt{t-r-\alpha}} d\alpha \right\}. \end{aligned} \tag{4.9}$$

When  $\mu + \kappa > 5/2$ , by (4.2) and (4.1) with  $\ell = 0$ , we obtain

$$I_1 + I_2 \leq C \langle t + r \rangle^{-1/2} \langle r - t \rangle^{-(\mu+\kappa-5/2)}. \tag{4.10}$$

On the other hand, when  $\mu + \kappa \leq 5/2$ , (4.9) yields

$$\begin{aligned} & \langle t + r \rangle^{1/2} (I_1 + I_2) \\ & \leq C \left\{ \langle t + r \rangle^{(5/2)-\mu-\kappa} \langle r - t \rangle^{-1/2} \int_{|t-r|}^{t+r} \frac{1}{\sqrt{\alpha+r-t}} d\alpha \right. \\ & \quad \left. + \langle r - t \rangle^{(5/2)-\mu-\kappa} \int_0^{[t-r]_+} \frac{\langle \alpha \rangle^{-1/2}}{\sqrt{t-r-\alpha}} d\alpha \right\} \\ & \leq C \langle t + r \rangle^{3-\mu-\kappa} \langle r - t \rangle^{-1/2}. \end{aligned} \tag{4.11}$$

Next suppose  $2r \geq t \geq 0$  and  $t + r \leq 1$ . Then the both  $\beta$ -integrals in  $I_1$  and  $I_2$  are bounded. Therefore (4.9) is still valid. Hence (4.10) or (4.11) also holds in this case.

It remains to consider the case where  $t \geq 2r \geq 0$ . We begin with proving

$$\int_{-\alpha}^{\alpha} \frac{\langle \frac{1-a}{2}\alpha + \frac{1+a}{2}\beta \rangle^{-\kappa}}{\sqrt{\beta+t+r}} d\beta \leq C \langle t + r \rangle^{-1/2} \langle t + r \rangle^{[1-\kappa]_+} \tag{4.12}$$

for  $0 \leq \alpha \leq t + r$ . Denoting the left-hand side by  $I$  and setting  $2d = (1 +$

$a)(t+r) + (a-1)\alpha$ , we get

$$I = \left(\frac{2}{1+a}\right)^{1/2} \int_{-a\alpha}^{\alpha} \frac{\langle\sigma\rangle^{-\kappa}}{\sqrt{\sigma+d}} d\sigma \leq \left(\frac{2}{1+a}\right)^{1/2} \int_{-d}^{d/a} \frac{\langle\sigma\rangle^{-\kappa}}{\sqrt{\sigma+d}} d\sigma,$$

since  $d \geq a\alpha$  for  $0 \leq \alpha \leq t+r$ . At this point, we apply the following elementary inequality: For  $\kappa, a, d > 0$ , there exists a positive constant  $C = C(\kappa, a)$  such that

$$\int_{-d}^{d/a} \langle\sigma\rangle^{-\kappa} (d+\sigma)^{-1/2} d\sigma \leq C \langle d \rangle^{-1/2} \langle d \rangle^{[1-\kappa]_+} \quad (4.13)$$

(for the proof, see e.g. [15, Lemma 2.6]). Since  $d \geq \min\{1, a\}(t+r)$  for  $0 \leq \alpha \leq t+r$  in our case, the application of (4.13) gives (4.12).

Now it follows from (4.7), (4.12) with  $\kappa < 1$  and (4.2) with  $\ell = 0$  that

$$I_1 \leq C \langle t+r \rangle^{-1/2} \langle t+r \rangle^{1-\kappa} \langle r-t \rangle^{-\mu+3/2},$$

since  $t \geq 2r \geq 0$ . Moreover, (4.8), (4.12) with  $\kappa < 1$  and (4.1) with  $\ell = 0$  yield

$$I_2 \leq C \langle t+r \rangle^{-1/2} \langle t+r \rangle^{1-\kappa} \langle r-t \rangle^{-\mu+3/2},$$

since  $\mu < 2$ . For  $t \geq 2r \geq 0$ , we see that these estimates imply (4.10). Therefore we obtain (4.3) via (4.6).

Secondly we prove (4.4) for  $0 < \kappa < 1$  and  $\mu + \kappa = 3$ . When  $2r \geq t \geq 0$ , proceeding as before, we obtain

$$I_1 + I_2 \leq C \langle t+r \rangle^{-1/2} \langle r-t \rangle^{-1/2} \log(1 + \langle r-t \rangle). \quad (4.14)$$

since  $\mu + \kappa = 3$ . Therefore we have only to consider the case where  $t \geq 2r \geq 0$ . It follows from (4.7), (4.12) with  $\kappa < 1$  and (4.2) with  $\ell = 0$  that

$$\begin{aligned} I_1 &\leq C \langle t+r \rangle^{-1/2} \langle t+r \rangle^{1-\kappa} \langle r-t \rangle^{-\mu+(3/2)} \\ &\leq C \langle t+r \rangle^{-1/2} \langle r-t \rangle^{-1/2} \end{aligned}$$

since  $t \geq 2r \geq 0$  and  $\mu + \kappa = 3$ . To evaluate  $I_2$  for  $t > r$ , we further split it as follows:

$$\begin{aligned} J_1 &= \frac{1}{2} \int_0^{(t-r)/2} \frac{\langle\alpha\rangle^{-\mu+1}}{\sqrt{t-r-\alpha}} d\alpha \int_{-\alpha}^{\alpha} \frac{\langle\frac{1-a}{2}\alpha + \frac{1+a}{2}\beta\rangle^{-\kappa}}{\sqrt{\beta+t+r}} d\beta, \\ J_2 &= \frac{1}{2} \int_{(t-r)/2}^{t-r} \frac{\langle\alpha\rangle^{-\mu+1}}{\sqrt{t-r-\alpha}} d\alpha \int_{-\alpha}^{\alpha} \frac{\langle\frac{1-a}{2}\alpha + \frac{1+a}{2}\beta\rangle^{-\kappa}}{\sqrt{\beta+t+r}} d\beta. \end{aligned}$$



By (4.12) with  $\kappa < 1$  we have

$$\begin{aligned} J_2 &\leq C\langle t+r \rangle^{(1/2)-\kappa} \int_{(t-r)/2}^{t-r} \frac{\langle \alpha \rangle^{-\mu+1}}{\sqrt{t-r-\alpha}} d\alpha \\ &\leq C\langle t+r \rangle^{(3/2)-\kappa-\mu} \int_0^{t-r} \frac{1}{\sqrt{t-r-\alpha}} d\alpha \\ &\leq C\langle t+r \rangle^{-1}, \end{aligned}$$

since  $\mu + \kappa = 3$ . On the one hand, when  $0 < \alpha < (t-r)/2$ , we have

$$\int_{-\alpha}^{\alpha} \frac{\langle \frac{1-a}{2}\alpha + \frac{1+a}{2}\beta \rangle^{-\kappa}}{\sqrt{\beta+t+r}} d\beta \leq C\langle t+r \rangle^{-1/2} \langle \alpha \rangle^{[1-\kappa]_+},$$

since  $\beta + t + r \geq (t+r)/2$  for  $\beta > -\alpha > -(t+r)/2$ . Thus we get

$$\begin{aligned} J_1 &\leq C\langle t+r \rangle^{-1/2} \int_0^{t-r} \frac{\langle \alpha \rangle^{-1}}{\sqrt{t-r-\alpha}} d\alpha \\ &\leq C\langle t+r \rangle^{-1/2} \langle r-t \rangle^{-1/2} \log(1 + \langle r-t \rangle), \end{aligned}$$

by (4.1) with  $\ell = 0$ . Hence we find (4.14), so that (4.4) follows from (4.6).

Finally we prove (4.5) for  $\kappa = 1$  and  $\mu = 2$ . First suppose  $2r \geq t \geq 0$  and  $t+r \geq 1$ . Similarly to derive (4.9), we have

$$\begin{aligned} \langle t+r \rangle^{1/2} (I_1 + I_2) &\leq C \left\{ \int_{|t-r|}^{t+r} \frac{\langle \alpha \rangle^{-1} \log(1 + \langle \alpha \rangle)}{\sqrt{\alpha+r-t}} d\alpha \right. \\ &\quad \left. + \int_0^{[t-r]_+} \frac{\langle \alpha \rangle^{-1} \log(1 + \langle \alpha \rangle)}{\sqrt{t-r-\alpha}} d\alpha \right\}. \end{aligned}$$

By (4.2) and (4.1) with  $\ell = 1$  we obtain

$$I_1 + I_2 \leq C\langle t+r \rangle^{-1/2} \langle r-t \rangle^{-1/2} (\log(1 + \langle r-t \rangle))^2. \tag{4.15}$$

This estimate is valid also for the case where  $2r \geq t \geq 0$  and  $t+r \leq 1$ , as before.

On the other hand, when  $t \geq 2r \geq 0$ , it follows from (4.7), (4.8) with  $\mu = 2$  and  $\kappa = 1$ , (4.12) and (4.2) with  $\ell = 0$  that

$$I_1 + I_2 \leq C\langle t+r \rangle^{-1/2} \log(1 + \langle t+r \rangle) \langle r-t \rangle^{-1/2} \log(1 + \langle r-t \rangle),$$

hence (4.15) holds as well. In conclusion we obtain (4.5) via (4.6) and finish the proof.  $\square$

*Proof of Theorem 1.2.* First of all, we introduce the following type of weighted  $L^\infty$ -space:

$$X = \{u = (u_1, \dots, u_m) \in C(\mathbb{R}^2 \times [0, T])^m : \|u\| \leq 2C_0\varepsilon\}, \quad (4.16)$$

where  $C_0$  is the number from (1.17) and

$$\|u\| \equiv \max_{1 \leq i \leq m} \|u_i\|_{c_i} = \max_{1 \leq i \leq m} \sup_{(x,t) \in \mathbb{R}^2 \times [0, T]} w_{c_i}(x, t) |u_i(x, t)|.$$

Then we define a map

$$\Phi[u] = (\varepsilon K_{c_i}[\varphi_i, \psi_i] + L_{c_i}[F_i(u)]; i = 1, \dots, m). \quad (4.17)$$

Since  $\varphi_i \in C_0^1(\mathbb{R}^2)$ ,  $\psi_i \in C_0(\mathbb{R}^2)$ , we can take a positive number  $R$  such that

$$\varphi_i(x) = \psi_i(x) = 0 \quad \text{for } |x| \geq R. \quad (4.18)$$

In addition, we may assume that  $F_i$  consists of only one term, that is

$$F_i(u) = |u_j|^p |u_k|^q, \quad 1 \leq i, j, k \leq m, \quad j \neq k,$$

where  $p, q \geq 1$  and  $2 \leq p + q \leq 3$ , since the general case can be handled in a similar way. Because  $c_j \neq c_k$  for  $j \neq k$ , we have

$$|F_i(u)(x, t)| \leq C \|u\|^\alpha \{ \langle t + |x| \rangle^{-(\alpha+q)/2} \langle c_j t - |x| \rangle^{-p/2} + \langle t + |x| \rangle^{-(\alpha+p)/2} \langle c_k t - |x| \rangle^{-q/2} \}$$

for all  $(x, t) \in \mathbb{R}^2 \times [0, T]$ , where we put  $\alpha = p + q$ .

First suppose  $2 \leq \alpha < 3$ , so that  $0 < p/2, q/2 < 1$ . Then by (4.3) we get

$$w_{c_i}(x, t) |L_{c_i}[F_i(u)](x, t)| \leq C \|u\|^\alpha \langle T \rangle^{3-\alpha}$$

for all  $(x, t) \in \mathbb{R}^2 \times [0, T]$ , since  $|x| \leq t + R$  by (4.18) and the finite speeds of propagation. Combining this with (1.16), we obtain for  $T \geq 1$

$$\|\Phi[u]\| \leq C_0\varepsilon + C_1 T^{3-\alpha} \|u\|^\alpha, \quad (4.19)$$

where  $C_1 = C_1(c_i, p, q, R)$  is a positive constant. Similarly we see that there is a positive constant  $C_2 = C_2(c_i, p, q, R)$  such that

$$\|\Phi[u] - \Phi[v]\| \leq C_2 T^{3-\alpha} (\|u\|^{\alpha-1} + \|v\|^{\alpha-1}) \|u - v\|. \quad (4.20)$$

Now we take  $\varepsilon_0$  and  $T$  so that

$$2^\alpha C_1 (C_0 \varepsilon_0)^{\alpha-1} T^{3-\alpha} \leq 1, \quad 2^{\alpha+1} C_2 (C_0 \varepsilon_0)^{\alpha-1} T^{3-\alpha} \leq 1, \quad T \geq 1. \quad (4.21)$$

Then we find from (4.19) and (4.20) that  $\Phi[u]$  has a fixed point in  $X$  for  $0 < \varepsilon \leq \varepsilon_0$ . This implies the desired conclusion.

Next suppose  $\alpha = 3$ . Then we apply (4.4) if  $p > 1$  and  $q > 1$ , or (4.5) otherwise. Then we find the local existence and estimate (1.19), by proceeding as before. This completes the proof.  $\square$

**Remark 4.1** When  $\alpha > 3$ , one can show a global existence result by modifying a little bit the proof above. Namely, for sufficiently small  $\varepsilon$ , there is a unique solution  $u \in (C(\mathbb{R}^2 \times [0, \infty)))^m$  to (1.12) with (1.3) satisfying

$$\max_{1 \leq i \leq m} \sup_{(x,t) \in \mathbb{R}^2 \times [0, \infty)} \langle t + |x| \rangle^{1/2} \langle c_i t - |x| \rangle^\nu |u_i(x, t)| \leq 2C_0 \varepsilon$$

with  $0 < \nu < 1/2$  (for the detail, see [15, Theorem 4.1]).

## References

- [1] Agemi R., Kurokawa Y. and Takamura H., *Critical curve for p-q systems of nonlinear wave equations in three space dimensions*. J. Differential Equations **167** (2000), 87–133.
- [2] D’Ancona P., Georgiev V. and Kubo H., *Weighted decay estimates for the wave equation*. J. Differential Equations **163** (2001), 146–208.
- [3] Del Santo D., Georgiev V. and Mitidieri E., *Global existence of the solutions and formation of singularities for a class of hyperbolic systems*, in “Geometric Optics and Related topics” (F. Colombini and N. Lerner ed.), Progress in Nonlinear Differential Equations and Their Applications, vol. 32, Birkhäuser, Boston, 1997, pp. 117–140.
- [4] Del Santo D. and Mitidieri E., *Blow-up of solutions of a hyperbolic system: the critical case*. Differential Equations **34** (1998), 1157–1163.
- [5] Deng K., *Nonexistence of global solutions of a nonlinear hyperbolic system*. Trans. Amer. Math. Soc. **349** (1997), 1685–1696.
- [6] Georgiev V., “*Semilinear Hyperbolic equations*”. MSJ Memoirs, vol. 7, Math. Soc. of Japan, Tokyo, 2000.
- [7] Georgiev V., Lindblad H. and Sogge C., *Weighted Strichartz estimate and global existence for semilinear wave equation*. Amer. J. Math. **119** (1997), 1291–1319.
- [8] Glassey R.T., *Finite-time blow-up for solutions of nonlinear wave equations*. Math. Z. **177** (1981), 323–340.
- [9] Glassey R.T., *Existence in the large for  $\square u = F(u)$  in two space dimensions*. Math. Z. **178** (1981), 233–261.
- [10] John F., *Blow-up of solutions of nonlinear wave equations in three space dimensions*. Manuscripta Math. **28** (1979), 235–268.
- [11] Kubo H. and Kubota K., *Scattering for systems of semilinear wave equations with different speeds of propagation*. Adv. Differential Equations **7** (2002), 441–468.

- [12] Kubo H. and Ohta M., *Critical blowup for systems of semilinear wave equations in low space dimensions*. J. Math. Anal. Appl. **240** (1999), 340–360.
- [13] Kubo H. and Ohta M., *Small data blowup for systems of semilinear wave equations with different propagation speeds in three space dimensions*. J. Differential Equations **163** (2000), 475–492.
- [14] Kubo H. and Ohta M., *On systems of semilinear wave equations with unequal propagation speeds in three space dimensions*. Funkcial. Ekvac. **48** (2005), 65–98.
- [15] Kubo H. and Ohta M., *On the global behaviour of classical solutions to coupled systems of semilinear wave equations*, in “New Trends in the Theory of Hyperbolic Equations” (M. Reissig and B.-W. Schulze ed.), Birkhäuser, 2005.
- [16] Kubo H. and Tsugawa K., *Global solutions and self-similar solutions of the coupled system of semilinear wave equations in three space dimensions*. Discrete Contin. Dyn. Syst. **9** (2003), 471–482.
- [17] Kubota K., *Existence of a global solutions to a semi-linear wave equation with initial data of non-compact support in low space dimensions*. Hokkaido Math. J. **22** (1993), 123–180.
- [18] Schaeffer J., *The equation  $u_{tt} - \Delta u = |u|^p$  for the critical value of  $p$* . Proc. Roy. Soc. Edinburgh **101A** (1985), 31–44.
- [19] Sideris T.C., *Nonexistence of global solutions to semilinear wave equations in high dimensions*. J. Differential Equations **52** (1984), 378–406.
- [20] Strauss W., *Nonlinear scattering and low energy*. J. Func. Anal. **41** (1981) 110–133.
- [21] Takamura H., *An elementary proof of the exponential blow-up for semilinear wave equations*. Math. Meth. Appl. Sci. **17** (1994) 239–249.
- [22] Tataru D., *Strichartz estimates in the hyperbolic space and global existence for the semilinear wave equation*. Trans. Amer. Math. Soc. **353** (2000), 795–807.
- [23] Yordanov B. and Zhang Q., *Finite time blow up for critical wave equations in high dimensions*. J. Funct. Anal. **231** (2006), 361–374.
- [24] Zhou Y., *Blow up of classical solutions to  $\square u = |u|^{1+\alpha}$  in three space dimensions*. J. Partial Differential Equations **5** (1992) 21–32.
- [25] Zhou Y., *Life span of classical solutions to  $\square u = |u|^p$  in two space dimensions*. Chinese Ann. Math. **14B** (1993) 225–236.
- [26] Zhou Y., *Cauchy problem for semilinear wave equations in four space dimensions with small initial data*. J. Partial Differential Equations **8** (1995) 135–144.

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