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Two sides of probe method and obstacle with impedance boundary condition

(Dedicated to Professor Minoru Murata on the occasion of his 60th birthday)

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Abstract. An inverse boundary value problem for the Helmholtz equation in a bounded domain is considered. The problem is to extract information about an unknown obstacle embedded in the domain with unknown impedance boundary condition (the Robin condition) from the associated Dirichlet-to-Neumann map. The main result is a characterization of the unknown obstacle via the sequences that are constructed by the Dirichlet-to-Neumann map, under smallness conditions on the wave number and the upper bound of the impedance. Moreover two alternative simple proofs of a previous result of Cheng-Liu-Nakamura which are based on only some energy estimates, an analysis of the blowup of the energy of so-called reflected solutions and an application of the enclosure method to the problem are also given.

Key words: inverse obstacle scattering problem, probe method, Poincaré inequality, enclosure method, impedance boundary condition, blowup, obstacle, indicator function.

1. Introduction

In this paper, we consider an inverse boundary value problem for the Helmholtz equation in a bounded domain. The problem is to extract information about an unknown obstacle embedded in the domain with unknown impedance boundary condition from the associated Dirichlet-to-Neumann map.

Let Ω be a bounded domain in $\mathbf{R}^m$ $(m = 2, 3)$ with Lipschitz boundary. Let D be an open subset with Lipschitz boundary of Ω and satisfy that $\overline{D} \subset \Omega$; $\Omega \setminus \overline{D}$ is connected. We always assume that D is given by a union of finitely many bounded Lipschitz domains $D_1, \ldots, D_N$ such that $\overline{D}_j \cap \overline{D}_l = \emptyset$ if $j \neq l$.

We denote by ν the unit outward normal relative to D. Let $k \geq 0$. We always assume that 0 is not a Dirichlet eigenvalue of $\triangle + k^2$ in Ω. Let $\lambda \in L^\infty(\partial D)$.

Given $f \in H^{1/2}(\partial\Omega)$ we say that $u \in H^1(\Omega \setminus \overline{D})$ is a weak solution of

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the elliptic problem

$$\begin{aligned}&\triangle u + k^2 u = 0 \ \text{ in } \ \Omega \setminus \overline{D},\\ &\frac{\partial u}{\partial \nu} + \lambda u = 0 \ \text{ on } \ \partial D,\\ &u = f \ \text{ on } \ \partial\Omega\end{aligned} \tag{1.1}$$

if u satisfies $u = f$ on $\partial\Omega$ in the sense of trace and, for all $\varphi \in H^1(\Omega \setminus \overline{D})$ with $\varphi = 0$ on $\partial\Omega$ in the sense of trace

$$\int_{\partial D} \lambda u\varphi \, dS - \int_{\Omega\setminus\overline{D}} \nabla u \cdot \nabla\varphi \, dy + k^2 \int_{\Omega\setminus\overline{D}} u\varphi \, dy = 0.$$

We assume that: there exists a positive constant $C > 0$ such that $\operatorname{Im}\lambda(x) \geq C$ for almost all $x \in \partial D$. This assumption is motivated by a possibility of application to inverse scattering problems of electromagnetic/acoustic wave ([3, 4, 7]). It is really routine to see that the weak solution of (1.1) exists and is unique.

Define the bounded linear functional $\Lambda_D f$ on $H^{1/2}(\partial\Omega)$ by the formula

$$\langle \Lambda_D f, h\rangle = -\int_{\partial D} \lambda uv \, dS + \int_{\Omega\setminus\overline{D}} \nabla u \cdot \nabla v \, dy - k^2 \int_{\Omega\setminus\overline{D}} uv \, dy, \quad h \in H^{1/2}(\partial\Omega)$$

where u is the weak solution of (1.1) and $v \in H^1(\Omega \setminus \overline{D})$ is an arbitrary function with $v = h$ on $\partial\Omega$ in the sense of the trace. The map $\Lambda_D \colon f \mapsto \Lambda_D f$ is called the Dirichlet-to-Neumann map. We set $\Lambda_D = \Lambda_0$ in the case when $D = \emptyset$.

Here we consider the problem

Inverse Problem Extract information about the shape and location of D from Λ_D or its partial knowledge.

In [9] the author introduced the probe method which gives a general idea to obtain a reconstruction formula of unknown objects embedded in a known background medium from a mathematical counterpart (the Dirichlet-to-Neumann map) of the measured data of some physical quantity on the boundary of the medium. The method was applied to an inverse boundary value problem in elasticity [14] and mixed problems [5].

In [11] we considered *Inverse Problem* in the two extreme cases when $\lambda \to 0$ (sound-hard)/$\lambda \to \infty$ (sound-soft). Using the probe method, we established reconstruction formulae of ∂D by using $(\Lambda_0 - \Lambda_D)f$ for infinitely many f provided both $\partial\Omega$ and ∂D are C^2. In [10] we considered inverse obstacle scattering problems with a fixed frequency and established a way of reconstructing sound-hard/sound-soft obstacle from the scattering data.

Recently Erhard-Potthast ([6]) studied the probe method numerically. They considered *Inverse Problem* in the case when $\lambda \to \infty$ and computed an approximation of the corresponding *indicator function* by employing the techniques of the point source and singular sources methods by Potthast ([19, 20]). Applying the probe method to *Inverse Problem* for general λ, Cheng-Liu-Nakamura [1] gave a corresponding reconstruction formula of ∂D provided also both $\partial\Omega$ and ∂D are C^2 and λ is unknown. Quite recently a numerical testing of the probe method applied to *Inverse Problem* has been done in [2].

By the way, in order to give an explanation for an observation obtained in [6], quite recently the author gave a new formulation of the probe method, raised new questions about the probe method itself and gave answers to some of them in the two extreme cases [15].

The aim of this paper is to: reconsider the application of the previous version of the probe method to *Inverse Problem* and add a new result based on the new formulation of the probe method in [15]. We clarify that the probe method has two different sides. One side has a common character with the enclosure method which was introduced in [12] and is based on the blowup property of the exponentially growing solutions. Thus we apply also the enclosure method to *Inverse Problem* in the final section for comparison.

For application of the previous version of the probe method to inverse boundary value problems related to cracks (non volumetric discontinuity) see [17, 18].

2. Two sides of probe method

In [15] we introduced a simpler formulation of the probe method and further investigated the method itself in the two extreme cases when $\lambda \to 0$ or $\lambda \to \infty$. We start with describing the formulation.

Given a point $x \in \Omega$ let N_x denote the set of all piecewise linear curves $\sigma\colon [0,1] \mapsto \overline{\Omega}$ such that:

- $\sigma(0) \in \partial\Omega$, $\sigma(1) = x$ and $\sigma(t) \in \Omega$ for all $t \in]0,1[$;
- σ is injective.

We call $\sigma \in N_x$ a *needle with tip* at x.

We denote by $G_k(x)$ the standard fundamental solution of the Helmholtz equation. For the new formulation of the probe method we need the following.

Definition 2.1 Let $\sigma \in N_x$. We call the sequence $\xi = \{v_n\}$ of $H^1(\Omega)$ solutions of the Helmholtz equation a *needle sequence* for (x, σ) if it satisfies, for each fixed compact set K of $\mathbf{R}^m$ with $K \subset \Omega \setminus \sigma([0,1])$

$$\lim_{n\to\infty}\big(\|v_n(\cdot)-G_k(\cdot-x)\|_{L^2(K)}+\|\nabla\{v_n(\cdot)-G_k(\cdot-x)\}\|_{L^2(K)}\big)=0.$$

The existence of the needle sequence has been ensured in [11].

2.1. Side A of probe method—A review on Cheng-Liu-Nakamura's result

Definition 2.2 Given $x \in \Omega$, needle σ with tip x and needle sequence $\xi = \{v_n\}$ for (x, σ) define

$$I(x,\sigma,\xi)_n = \mathrm{Re}\langle(\Lambda_0-\Lambda_D)f_n, \overline{f}_n\rangle, \qquad n = 1,\, 2, \ldots$$

where

$$f_n(y) = v_n(y), \qquad y \in \partial\Omega.$$

$\{I(x,\sigma,\xi)_n\}_{n=1,2,\ldots}$ is a sequence depending on ξ and $\sigma \in N_x$. We call the sequence the *indicator sequence*.

Now we can say that the *probe method* is a method of *probing* inside a given medium by monitoring the behaviour of the indicator sequences for many needles. For the description of the behaviour we introduce a function defined outside the obstacle.

Definition 2.3 The *indicator function* I is defined by the formula

$$\begin{aligned}I(x) = &\int_{\Omega\setminus\overline{D}}|\nabla w_x|^2\,dy - k^2\int_{\Omega\setminus\overline{D}}|w_x|^2\,dy\\ &+\int_D|\nabla G_k(y-x)|^2\,dy-k^2\int_D|G_k(y-x)|^2\,dy\\ &+\int_{\partial D}\big\{(\mathrm{Re}\,\lambda)(|G_k(\,\cdot-x)|^2-|w_x|^2)\\ &-2(\mathrm{Im}\,\lambda)\,\mathrm{Im}\{w_x\overline{G}_k(\,\cdot-x)\}\big\}dS, \qquad x\in\Omega\setminus\overline{D}\end{aligned} \tag{2.1}$$

where $w_x \in H^1(\Omega \setminus \overline{D})$ is the unique weak solution of the problem

$$\begin{aligned}
&\triangle w + k^2 w = 0 \ \text{ in } \ \Omega \setminus \overline{D},\\
&\frac{\partial w}{\partial \nu} + \lambda w = -\left(\frac{\partial}{\partial \nu} G_k(\,\cdot - x) + \lambda G_k(\,\cdot - x)\right) \ \text{ on } \ \partial D,\\
&w = 0 \ \text{ on } \ \partial\Omega.
\end{aligned}$$

The function w_x is called the *reflected solution* by D.

The following theorem extends the result in [11] and is nothing but a result established in [1].

Theorem A *We have*:

- (A.1) *given* $x \in \Omega \setminus \overline{D}$ *and needle* σ *with tip at* x *if* $\sigma(]0,1]) \cap \overline{D} = \emptyset$, *then for any needle sequence* $\xi = \{v_n\}$ *for* (x, σ) *the sequence* $\{I(x, \sigma, \xi)_n\}$ *converges to the indicator function* $I(x)$;
- (A.2) *for each* $\epsilon > 0$

$$\sup_{dist(x,D) > \epsilon} I(x) < \infty;$$

- (A.3) *given point* $a \in \partial D$

$$\lim_{x \to a} I(x) = \infty$$

provided both $\partial\Omega$ *and* ∂D *are* C^2 *and* $\lambda \in C^1(\partial D)$.

(A.2) is trivial. (A.3) is the most important property of the indicator function. The indicator function blows up at the boundary of the obstacle. (A.1) gives a way of calculating the indicator function by using the indicator sequence and is a direct consequence of the integral identity (2.3) given below and the well posedness of the problem

$$\begin{aligned}
&\triangle w + k^2 w = 0 \ \text{ in } \ \Omega \setminus \overline{D},\\
&\frac{\partial w}{\partial \nu} + \lambda w = g \ \text{ on } \ \partial D,\\
&w = 0 \ \text{ on } \ \partial\Omega
\end{aligned} \tag{2.2}$$

where $g \in H^{-1/2}(\partial D)$.

Proposition 2.1 ([1]) *For all* $f \in H^{1/2}(\partial\Omega)$

$$\begin{aligned}\langle(\Lambda_0-\Lambda_D)f,\overline{f}\rangle &= \int_{\Omega\setminus\overline{D}}|\nabla(u-v)|^2\,dy - k^2\int_{\Omega\setminus\overline{D}}|u-v|^2\,dy\\ &+\int_D|\nabla v|^2\,dy - k^2\int_D|v|^2\,dy\\ &+\int_{\partial D}\bigl\{(\operatorname{Re}\lambda)(|v|^2-|u-v|^2)-2(\operatorname{Im}\lambda)\operatorname{Im}((u-v)\overline{v})\bigr\}dS\\ &+i\int_{\partial D}\operatorname{Im}\lambda|u|^2\,dS \end{aligned}\tag{2.3}$$

where u *solves* (1.1); v *solves*

$$(\triangle+k^2)v=0 \ \text{ in } \ \Omega,$$
$$v=f \ \text{ on } \ \partial\Omega.$$

Note that: this identity is a generalization of the corresponding one in the case when $\lambda\to 0$ which has been established in [11].

Using (A.1) to (A.3), one can define another indicator function which gives a previous formulation of the probe method, however, we do not repeat to do it (see [11] for the previous formulation of the probe method).

The proof of (A.3) given by Cheng-Liu-Nakamura is not trivial and quite involved. Here we describe the review on their proof of (A.3) by using the formulation given here and points out also the problem.

They start with deriving the estimate from (2.1):

$$\begin{aligned}I(x)\ge&\int_D|\nabla G_k(y-x)|^2\,dy-k^2\int_D|G_k(y-x)|^2\,dy-k^2\int_{\Omega\setminus\overline{D}}|w_x|^2\,dy\\ &-C_1\int_{\partial D}|G_k(y-x)|^2\,dS(y)-C_2\int_{\partial D}|w_x|^2\,dy\end{aligned}\tag{2.4}$$

where both $C_1\ge 0$ and $C_2\ge 0$ are independent of x. Note that both C_1 and C_2 vanish in the case when $\lambda\to 0$ and (2.4) becomes the inequality (25) in [11].

It is easy to see that the second term of (2.4) is bounded as $x\to a\in\partial D$. Using the completely same argument as done in [11] in the case when $\lambda\to 0$ and the regularity assumptions on $\partial\Omega$, ∂D and λ (say $\lambda\in C^1(\partial D)$) which ensure the $H^2(\Omega\setminus\overline{D})$ regularity of solutions $p\in H^1(\Omega\setminus\overline{D})$ of the problem

$$\triangle p+k^2p\in L^2(\Omega\setminus\overline{D}),$$

$$\frac{\partial p}{\partial \nu} + \lambda p = 0 \text{ on } \partial D,$$
$$p = 0 \text{ on } \partial\Omega,$$

one knows that the L^2-norm of the reflected solution w_x over $\Omega \setminus \overline{D}$ is bounded as $x \to a$ (Theorem 3 of [1]). Note that, in their proof, it is just assumed that $\lambda \in L^\infty(\partial D)$, however, we do not know how to deduce the desired regularity for p under the condition. This means that one may propose: *prove (A.3) under the regularity condition $\lambda \in L^\infty(\partial D)$.*

For the proof of (A.3) they showed that

$$\int_{\partial D} |w_x|^2 \, dS(y) = O\Big(\int_{\partial D} |G_k(y-x)|^2 \, dS(y)\Big) \tag{2.5}$$

as $x \to a$ (Theorem 4 of [1]). Since

$$\lim_{x \to a} \int_D |\nabla G_k(y-x)|^2 \, dy = \infty$$

and

$$\lim_{x \to a} \frac{\displaystyle\int_{\partial D} |G_k(y-x)|^2 \, dS(y)}{\displaystyle\int_D |\nabla G_k(y-x)|^2 \, dy} = 0,$$

from (2.4) one obtains the desired conclusion. The proof of (2.5) is done by extracting the leading term of w_x as $x \to a$ that contributes the blowup of $\|w_x|_{\partial D}\|^2_{L^2(\partial D)}$ as $x \to a$ and needs a careful analysis of w_x in a neighbourhood of a (4 pages in the journal!).

We point out that (2.5) is unnecessary for the proof of (A.3). In Section 3 we present two alternative simple proofs of (A.3). The proofs do not make use of the local expression of the function w_x in a neighbourhood of $a \in \partial D$ and are based on some energy estimates only. This means that it maybe possible to apply the method to more general cases (other inverse boundary value problems for elliptic equations and systems of equations) without essential changes and difficulty. Those belong to our future study.

2.2. Side B of probe method

The result in this subsection is new and, needless to say, not covered in [1]. It is related to another side of the probe method.

In order to describe another side of the probe method we introduce two positive constants appearing in two types of the Poincaré inequalities (e.g., see [21, 23] and also [15]). One is given in the following.

Proposition 2.2 *For all* $w \in H^1(\Omega \setminus \overline{D})$ *with* $w = 0$ *on* $\partial\Omega$

$$\int_{\Omega\setminus\overline{D}} |w|^2\,dy \le C_0(\Omega \setminus \overline{D})^2 \int_{\Omega\setminus\overline{D}} |\nabla w|^2\,dy$$

where $C_0(\Omega \setminus \overline{D})$ *is a positive constant independent of* w.

Another is given in the following.

Proposition 2.3 *Let* U *be a bounded Lipschitz domain of* $\mathbf{R}^m$. *For any* $v \in H^1(U)$ *and measurable* $A \subset U$ *with* $|A| > 0$ *we have*

$$\int_U |v - v_A|^2\,dy \le C(U)^2 \left(1 + \frac{|U|^{1/2}}{|A|^{1/2}}\right)^2 \int_U |\nabla v|^2\,dy$$

where $C(U)$ *is a positive constant independent of* v *and* A;

$$v_A = \frac{1}{|A|} \int_A v\,dy.$$

We make use of the property that $C(U)^2 \left(1 + \frac{|U|^{1/2}}{|A|^{1/2}}\right)^2$ depends on $|A|$ continuously for each fixed U.

The following lemma is a special case of Theorem 1.5.1.10 in page 41 of [8].

Lemma 2.4 *Let* W *be a bounded open subset of* $\mathbf{R}^m$ *with a Lipschitz boundary* Γ. *Then there exists a positive constant* $K(W)$ *such that*

$$\int_\Gamma |u|^2\,dS \le K(W)\left(\epsilon \int_W |\nabla u|^2\,dy + \epsilon^{-1} \int_W |u|^2\,dy\right)$$

for all $u \in H^1(W)$ *and* $\epsilon \in]0, 1[$.

Now we can state another side of the probe method.

Theorem B *Let* $k \ge 0$ *and* $L > 0$ *satisfy*

$$\|\mathrm{Re}\,\lambda\|_{L^\infty(\partial D)} + \|\mathrm{Im}\,\lambda\|_{L^\infty(\partial D)} \le L; \tag{2.6}$$

for some $\epsilon \in]0, 1[$

$$2K(\Omega \setminus \overline{D})L\epsilon + (k^2 + 2K(\Omega \setminus \overline{D})L\epsilon^{-1})C_0(\Omega \setminus \overline{D})^2 \le 1 \tag{2.7}$$

and

$$\min_j\left\{1-2K(D)L\epsilon-2(k^2+2K(D)L\epsilon^{-1})C(D_j)^2(1+1)^2\right\}>0. \quad (2.8)$$

Let $x\in\Omega$ and $\sigma\in N_x$. If $x\in\Omega\setminus\overline{D}$ and σ satisfies $\sigma(]0,1])\cap D\neq\emptyset$ or $x\in\overline{D}$, then for any needle sequence $\xi=\{v_n\}$ for (x,σ) we have $\lim_{n\to\infty}I(x,\sigma,\xi)_n=\infty$.

Note that both $\partial\Omega$ and ∂D are Lipschitz; $\lambda\in L^\infty(\partial D)$. (2.7) and (2.8) mean that both k and L are small.

Remarks are in order.

- This theorem does not cover the case when $x\in\Omega\setminus\overline{D}$ and σ satisfies both $\sigma(]0,1])\cap D=\emptyset$ and $\sigma(]0,1])\cap\overline{D}\neq\emptyset$. However, this is quite an exceptional case.
- At the present time we do not know how to drop the conditions (2.7) and (2.8). This remains open.

Here we make a definition. Let $\boldsymbol{b}$ be a nonzero vector in $\mathbf{R}^m$. Given $x\in\mathbf{R}^m$, $\rho>0$ and $\theta\in\,]0,\pi[$ the set

$$V=\left\{y\in\mathbf{R}^m\,|\,|y-x|<\rho\ \text{ and }\ (y-x)\cdot\boldsymbol{b}>|y-x|\,|\boldsymbol{b}|\cos(\theta/2)\right\}$$

is called a finite cone of height ρ, axis direction $\boldsymbol{b}$ and aperture angle θ with vertex at x. The two lemmas given below are the core of side B of the probe method and tell us that any needle sequence for any needle blows up on the needle. See [15] for the proof.

Lemma 2.5 *Let $x\in\Omega$ be an arbitrary point and σ be a needle with tip at x. Let $\xi=\{v_n\}$ be an arbitrary needle sequence for (x,σ). Then, for any finite cone V with vertex at x we have*

$$\lim_{n\to\infty}\int_{V\cap\Omega}|\nabla v_n(y)|^2\,dy=\infty.$$

Lemma 2.6 *Let $x\in\Omega$ be an arbitrary point and σ be a needle with tip at x. Let $\xi=\{v_n\}$ be an arbitrary needle sequence for (x,σ). Then for any point $z\in\sigma(]0,1[)$ and open ball B centered at z we have*

$$\lim_{n\to\infty}\int_{B\cap\Omega}|\nabla v_n(y)|^2\,dy=\infty.$$

2.3. Remark

Finally we point out that as a corollary of (A.1) and Theorem B we obtain a characterization of the obstacle by using the indicator sequence.

Corollary C *Under the same assumptions as those of Theorem B we have: a point $x \in \Omega$ belongs to $\Omega \setminus \overline{D}$ if and only if there exist a needle σ with tip at x and needle sequence ξ for (x, σ) such that the sequence $\{I(x, \sigma, \xi)_n\}$ is bounded from above.*

For applying side B of the probe method to non volumetric discontinuity (crack) one needs an idea of making use of the reflected solution only to show the blowup of the indicator sequence. We have already developed the idea in [16]. In Section 4 we present an application of the idea to the study of the reflected solutions.

3. Proof of Theorems A and B

3.1. Proof of Theorem B

Applying Lemma 2.4 for $W = \Omega\setminus\overline{D}$ or D to the right hand side of (2.3), we have

$$\begin{aligned}\mathrm{Re}\langle(\Lambda_0-\Lambda_D)f,\overline{f}\rangle \geq{}& (1-2K(\Omega\setminus\overline{D})L\epsilon)\int_{\Omega\setminus\overline{D}}|\nabla(u-v)|^2\,dy\\ &-(k^2+2K(\Omega\setminus\overline{D})L\epsilon^{-1})\int_{\Omega\setminus\overline{D}}|u-v|^2\,dy\\ &+(1-2K(D)L\epsilon)\int_D|\nabla v|^2\,dy\\ &-(k^2+2K(D)L\epsilon^{-1})\int_D|v|^2\,dy. \qquad (3.1)\end{aligned}$$

Applying Proposition 2.2 to the second term of (3.1), we have

$$\begin{aligned}&\mathrm{Re}\langle(\Lambda_0-\Lambda_D)f,\overline{f}\rangle\\ &\geq\left\{1-2K(\Omega\setminus\overline{D})L\epsilon-(k^2+2K(\Omega\setminus\overline{D})L\epsilon^{-1})C_0(\Omega\setminus\overline{D})^2\right\}\\ &\quad\times\int_{\Omega\setminus\overline{D}}|\nabla(u-v)|^2\,dy\\ &\quad+(1-2K(D)L\epsilon)\int_D|\nabla v|^2dy-(k^2+2K(D)L\epsilon^{-1})\int_D|v|^2dy. \qquad (3.2)\end{aligned}$$

On the other hand, from Proposition 2.3 we have

$$\begin{aligned}\int_D |v|^2\,dy &= \sum_j \int_{D_j} |v|^2\,dy \\ &\le \sum_j 2\int_{D_j} |v - v_{A_j}|^2\,dy + 2\int_{D_j} |v_{A_j}|^2\,dy \\ &\le \sum_j 2C(D_j)^2\left(1+\frac{|D_j|^{1/2}}{|A_j|^{1/2}}\right)^2 \int_{D_j} |\nabla v|^2\,dy + \sum_j 2|D_j|\,|v_{A_j}|^2\end{aligned}$$

where $A_j \subset D_j$ and satisfy $|A_j| > 0$.

Using this inequality, (2.6) and (2.7), from (3.2) we have the basic inequality

$$\begin{aligned}&\mathrm{Re}\langle(\Lambda_0 - \Lambda_D)f, \overline{f}\rangle \\ &\ge \sum_j \left\{1 - 2K(D)L\epsilon - 2(k^2 + 2K(D)L\epsilon^{-1})C(D_j)^2\left(1+\frac{|D_j|^{1/2}}{|A_j|^{1/2}}\right)^2\right\} \\ &\quad \times \int_{D_j} |\nabla v|^2\,dy - 2(k^2 + 2K(D)L\epsilon^{-1})|D|\sum_j |v_{A_j}|^2. \qquad (3.3)\end{aligned}$$

Hereafter we proceed along the same line as [15] and completes the proof. However, for reader's convenience we present the detail. Choose a sequence $\{K_l\}$ of compact sets of $\mathbf{R}^m$ in such a way that $K_l \subset \Omega \setminus \sigma(]0,1])$; $\overline{K_l} \subset K_{l+1}$ for $l = 1, \ldots$; $\Omega \setminus \sigma(]0,1]) = \bigcup_{l=1}^{\infty} K_l$. Then $|K_l \cap D_j| \to |D_j \setminus \sigma(]0,1])| = |D_j|$ as $l \to \infty$ uniformly with $j = 1, \ldots, N$. Thus one can take a large l_0 in such a way that the set $A_j \equiv K_{l_0} \cap D_j$ satisfies

$$\begin{aligned}&\max_j\left\{2(k^2 + 2K(D)L\epsilon^{-1})\left(C(D_j)^2\left(1+\frac{|D_j|^{1/2}}{|A_j|^{1/2}}\right)^2 - C(D_j)^2(1+1)^2\right)\right\} \\ &\quad < \min_j\left\{1 - 2K(D)L\epsilon - 2(k^2 + 2K(D)L\epsilon^{-1})C(D_j)^2(1+1)^2\right\}.\end{aligned}$$

Note that this right hand side is positive because of (2.8). We know that the sequences $\{(v_n)_{A_j}\}$ for each $j = 1, \ldots, N$ are always convergent since $\overline{A}_j \subset \Omega \setminus \sigma(]0,1])$. From (3.3) with $f = v_n|_{\partial\Omega}$ we have

$$I(x,\sigma,\xi)_n \ge NC\int_D |\nabla v_n|^2 dy - 2(k^2 + 2K(D)L\epsilon^{-1})|D|\sum_j |(v_n)_{A_j}|^2$$

where

$$C = \min_j\{1 - 2K(D)L\epsilon - 2(k^2 + 2K(D)L\epsilon^{-1})C(D_j)^2(1+1)^2\}$$
$$- \max_j\left\{2(k^2 + 2K(D)L\epsilon^{-1}) \times \left(C(D_j)^2\left(1 + \frac{|D_j|^{1/2}}{|A_j|^{1/2}}\right)^2 - C(D_j)^2(1+1)^2\right)\right\} > 0.$$

Then the blowup of $I(x,\sigma,\xi)_n$ comes from the blowup of the sequence

$$\int_D |\nabla v_n|^2\, dy. \tag{3.4}$$

If $x \in D$, then the blowup of the sequence given by (3.4) is a direct consequence of Lemma 2.5. If $x \in \partial D$, then the exists a finite cone V at vertex at x such that $V \subset D$. This is because of the Lipshitz regularity of ∂D. Then Lemma 2.5 gives the blowup of the sequence. Now consider the case when $x \in \Omega \setminus \overline{D}$ and σ satisfies $\sigma(]0,1]) \cap D \neq \emptyset$. Then, there exists a point z on $\sigma(]0,1[) \cap D$. Choose an open ball centered at z in such a way that $B \subset D$. Then from Lemma 2.6 we see the blowup of the sequence given by (3.4). □

In the following two subsections we always assume that $\lambda \in C^1(\partial D)$ by the reason described in subsection 2.1.

In the following let $L > 0$ satisfy (2.6).

3.2. First alternative proof of (A.3)

The proof is based on (3.1).

Choose $\epsilon \in]0,1[$ of (3.1) in such a way that

$$2K(\Omega \setminus \overline{D})L\epsilon \leq 1$$

and

$$2K(D)L\epsilon < 1.$$

Then (3.1) gives

$$\mathrm{Re}\langle(\Lambda_0 - \Lambda_D)f, \overline{f}\rangle \geq -(k^2 + 2K(\Omega\setminus\overline{D})L\epsilon^{-1})\int_{\Omega\setminus\overline{D}} |u-v|^2\, dy$$
$$+ (1 - 2K(D)L\epsilon)\int_D |\nabla v|^2\, dy - (k^2 + 2K(D)L\epsilon^{-1})\int_D |v|^2\, dy. \tag{3.5}$$

Now let $v = v_n$ and $x \in \Omega \setminus \overline{D}$. Then from (3.5) and (A.1), one has

$$\begin{aligned} I(x) \geq & -(k^2 + 2K(\Omega \setminus \overline{D})L\epsilon^{-1}) \int_{\Omega\setminus\overline{D}} |w_x|^2 \, dy \\ & + (1 - 2K(D)L\epsilon) \int_D |\nabla G_k(y-x)|^2 \, dy \\ & - (k^2 + 2K(D)L\epsilon^{-1}) \int_D |G_k(y-x)|^2 \, dy. \end{aligned} \tag{3.6}$$

The boundedness of L^2-norm of w_x over $\Omega \setminus \overline{D}$ has been established as described in subsection 2.1. From the boundedness of the last term of (3.6) and the blowup of the middle term we obtain the desired conclusion.

3.3. Second alternative proof of (A.3)

This proof does not make use of (3.1). The key point is a weaker version of (2.5). We prove that

Lemma 3.1 *There exists a positive constant* C *such that, for all* $y_0 \in \overline{\Omega} \setminus D$

$$\begin{aligned} \int_{\partial D} |w|^2 \, dS \leq C \|\nabla w\|_{L^2(\Omega\setminus\overline{D})} \bigg\{ & \int_{\partial D} |y - y_0|^{1/2} \left| \frac{\partial v}{\partial \nu} \right| dS(y) \\ & + k^2 \left| \int_D v \, dy \right| + L \int_{\partial D} |v| \, dS(y) \bigg\} \end{aligned} \tag{3.7}$$

where $v \in H^1(\Omega)$ *is a solution of the equation* $\triangle v + k^2 v = 0$ *in* Ω; *the function* $w \in H^1(\Omega \setminus \overline{D})$ *is the weak solution of the problem*

$$\begin{aligned} & \triangle w + k^2 w = 0 \ \text{ in } \ \Omega \setminus \overline{D}, \\ & \frac{\partial w}{\partial \nu} + \lambda w = -\left(\frac{\partial v}{\partial \nu} + \lambda v \right) \ \text{ on } \partial D, \\ & w = 0 \ \text{ on } \partial \Omega \end{aligned} \tag{3.8}$$

in the following sense: the trace of w *onto* $\partial\Omega$ *vanishes and, for all* $\varphi \in H^1(\Omega \setminus \overline{D})$ *with* $\varphi = 0$ *on* $\partial\Omega$

$$\begin{aligned} & \int_{\partial D} \lambda w \varphi \, dS - \int_{\Omega\setminus\overline{D}} \nabla w \cdot \nabla \varphi \, dy + k^2 \int_{\Omega\setminus\overline{D}} w \varphi \, dy \\ & \quad = - \int_{\partial D} \left(\frac{\partial v}{\partial \nu} + \lambda v \right) \varphi \, dS. \end{aligned} \tag{3.9}$$

Proof. Let $p \in H^1(\Omega \setminus \overline{D})$ be the weak solution of the problem:

$$\begin{aligned}&\triangle p + k^2 p = 0 \ \text{ in } \ \Omega \setminus \overline{D},\\ &\frac{\partial p}{\partial \nu} + \lambda p = -\overline{w} \ \text{ on } \ \partial D,\\ &p = 0 \ \text{ on } \ \partial\Omega.\end{aligned} \tag{3.10}$$

It is easy to see that we have

$$\begin{aligned}\|p\|_{H^1(\Omega\setminus\overline{D})} &\le C_3\|w\|_{H^{-1/2}(\partial D)}\\ &\le C_3\|w\|_{L^2(\partial D)}.\end{aligned}$$

Since $\lambda \in C^1(\partial D)$, we have $\partial p/\partial \nu \in H^{1/2}(\partial D)$. Thus a standard regularity result for the Laplacian yields $p \in H^2(\Omega \setminus \overline{D})$ and the estimate

$$\|p\|_{H^2(\Omega\setminus\overline{D})} \le C_4\|w\|_{H^1(\Omega\setminus\overline{D})}.$$

Then the Poincaré inequality gives

$$\|p\|_{H^2(\Omega\setminus\overline{D})} \le C_5\|\nabla w\|_{L^2(\Omega\setminus\overline{D})}. \tag{3.11}$$

Using (3.9) and (3.10), one can write

$$\begin{aligned}\int_{\partial D} |w|^2\, dS &= \int_{\partial D} \overline{w} w \, dS\\ &= -\int_{\partial D}\left(\frac{\partial p}{\partial \nu} + \lambda p\right) w\, dS\\ &= \int_{\Omega\setminus\overline{D}} (\nabla p \cdot \nabla w - k^2 p w)\, dy - \int_{\partial D} \lambda p w\, dS\\ &= -\int_{\partial D} p\left(\frac{\partial w}{\partial \nu} + \lambda w\right) dS\\ &= \int_{\partial D} p\left(\frac{\partial v}{\partial \nu} + \lambda v\right) dS.\end{aligned} \tag{3.12}$$

The Sobolev imbedding theorem ensures that p can be identified with a uniformly Hölder continuous function with exponent $1/2$ on $\overline{\Omega} \setminus D$ and the estimates

$$\begin{aligned}&|p(y) - p(y_0)| \le C_6|y - y_0|^{1/2}\|p\|_{H^2(\Omega\setminus\overline{D})}\\ &|p(y)| \le C_6\|p\|_{H^2(\Omega\setminus\overline{D})}\end{aligned} \tag{3.13}$$

where $C_6 > 0$ is independent of p, y and y_0, are valid. A combination of (3.12) and (3.13) yields

$$\begin{aligned}\int_{\partial D} |w|^2\,dS &\le \left|\int_{\partial D}(p(y)-p(y_0))\frac{\partial v}{\partial\nu}\,dS(y)\right| \\ &\quad + \left|p(y_0)\int_{\partial D}\frac{\partial v}{\partial\nu}\,dS(y)\right| + \left|\int_{\partial D} p\lambda v\,dS(y)\right| \\ &\le C_6\|p\|_{H^2(\Omega\setminus\overline{D})}\left\{\int_{\partial D}|y-y_0|^{1/2}\left|\frac{\partial v}{\partial\nu}\right|dS(y)\right. \\ &\quad \left. + k^2\left|\int_D v\,dy\right| + L\int_{\partial D}|v|\,dS(y)\right\}.\end{aligned}$$

From this and (3.11) we see that (3.7) is valid for $C = C_6C_5$. □

We call the function w in Lemma 3.1 the *reflected solution* of v by D. Let $v = v_n$ and w_n is the weak solution of (3.8). From (2.3) we have

$$\begin{aligned}I(x,\sigma,\xi)_n &\ge \int_{\Omega\setminus\overline{D}}|\nabla w_n|^2\,dy - k^2\int_{\Omega\setminus\overline{D}}|w_n|^2\,dy \\ &\quad + \int_D|\nabla v_n|^2\,dy - k^2\int_D|v_n|^2\,dy - 2L\int_{\partial D}(|v_n|^2+|w_n|^2)\,dS \\ &\ge (1-\epsilon CL)\int_{\Omega\setminus\overline{D}}|\nabla w_n|^2\,dy - k^2\int_{\Omega\setminus\overline{D}}|w_n|^2\,dy \\ &\quad + \int_D|\nabla v_n|^2\,dy - k^2\int_D|v_n|^2\,dy - 2L\int_{\partial D}|v_n|^2\,dS \\ &\quad - \epsilon^{-1}CL\left\{\int_{\partial D}|y-y_0|^{1/2}\left|\frac{\partial v_n}{\partial\nu}\right|dS(y)\right. \\ &\quad \left. + k^2\left|\int_D v_n\,dy\right| + L\int_{\partial D}|v_n|\,dS(y)\right\}^2 \end{aligned} \tag{3.14}$$

where $\epsilon > 0$ is arbitrary. Let $x \in \Omega\setminus\overline{D}$ and take $y_0 = x$ in Lemma 3.1. Choose ϵ in such a way that $1 \ge \epsilon CL$ and letting $n \to \infty$, from (3.14) and (A.1) we obtain

$$\begin{aligned}I(x) &\ge -k^2\int_{\Omega\setminus\overline{D}}|w_x|^2\,dy \\ &\quad + \int_D|\nabla G_k(y-x)|^2\,dy - k^2\int_D|G_k(y-x)|^2\,dy\end{aligned}$$

$$
\begin{aligned}
&- 2L\int_{\partial D}|G_k(y-x)|^2\,dS\\
&- \epsilon^{-1}CL\Big\{\int_{\partial D}|y-x|^{1/2}\Big|\frac{\partial}{\partial\nu}G_k(y-x)\Big|\,dS(y)\\
&+ k^2\Big|\int_D G_k(y-x)\,dy\Big| + L\int_{\partial D}|G_k(y-x)|\,dS(y)\Big\}^2.
\end{aligned}
$$

Now one can easily see that $\lim_{x\to a} I(x)=\infty$.

4. Blowup of reflected solution

In this section we study the blowup property of the energy of the reflected solutions. This is an application of the idea developed in [16]. The starting point is the following simple estimate.

Lemma 4.1 *Let $v\in H^1(\Omega)$ be a solution of the equation $\triangle v+k^2v=0$ in Ω. Let $w\in H^1(\Omega\setminus\overline{D})$ be the reflected solution of v by D. This means that the function w is the weak solution of* (3.8). *Then we have*

$$
\begin{aligned}
&\frac{\displaystyle\int_D|\nabla v|^2\,dy-k^2\int_D|v|^2\,dy+\int_{\partial D}(\operatorname{Re}\lambda)|v|^2\,dS}{\|v\|_{H^1(D)}}\\
&\le C\|\nabla w\|_{L^2(\Omega\setminus\overline{D})}
\end{aligned}
\tag{4.1}
$$

where C is a positive constant and independent of v and w.

Proof. From the trace theorem we know that there exists $p\in H^1(\Omega\setminus\overline{D})$ such that

$$
\begin{aligned}
&p=\overline{v}\ \text{ on }\ \partial D,\\
&p=0\ \text{ on }\ \partial\Omega.
\end{aligned}
$$

This p satisfies

$$
\|p\|_{H^1(\Omega\setminus\overline{D})}\le C_1\|v|_{\partial D}\|_{H^{1/2}(\partial D)}
\tag{4.2}
$$

where $C_1=C_1(\Omega\setminus\overline{D})>0$ and independent of v and k^2. Integration by parts gives

$$
\int_D|\nabla v|^2\,dy-k^2\int_D|v|^2\,dy=\int_{\partial D}\frac{\partial v}{\partial\nu}\,\overline{v}\,dS
$$

$$= \int_{\partial D} \left(\frac{\partial v}{\partial \nu} + \lambda v \right) \overline{v} \, dS - \int_{\partial D} \lambda |v|^2 \, dS. \tag{4.3}$$

Then a combination of (3.9) for $\varphi = p$ and (4.3) yields

$$\begin{aligned} &\int_D |\nabla v|^2 \, dy - k^2 \int_D |v|^2 \, dy + \int_{\partial D} \lambda |v|^2 \, dS \\ &= \int_{\Omega \setminus \overline{D}} \nabla w \cdot \nabla p - k^2 \int_{\Omega \setminus \overline{D}} wp \, dy - \int_{\partial D} \lambda w \overline{v} \, dS. \end{aligned} \tag{4.4}$$

Let $C_2 = C_2(D) > 0$ satisfy, for all $\Psi \in H^1(D)$

$$\|\Psi|_{\partial D}\|_{H^{1/2}(\partial D)} \leq C_2 \|\Psi\|_{H^1(D)}. \tag{4.5}$$

Taking the real part of (4.4), from (4.5) we have

$$\begin{aligned} &\int_D |\nabla v|^2 \, dy - k^2 \int_D |v|^2 \, dy + \int_{\partial D} (\operatorname{Re} \lambda) |v|^2 \, dS \\ &\leq \|\nabla w\|_{L^2(\Omega \setminus \overline{D})} \|\nabla p\|_{L^2(\Omega \setminus \overline{D})} + k^2 \|w\|_{L^2(\Omega \setminus \overline{D})} \|p\|_{L^2(\Omega \setminus \overline{D})} \\ &\quad + L \|w|_{\partial D}\|_{L^2(\partial D)} \|v|_{\partial D}\|_{L^2(\partial D)} \\ &\leq C_1 (\|\nabla w\|_{L^2(\Omega \setminus \overline{D})} + k^2 \|w\|_{L^2(\Omega \setminus \overline{D})}) \|v|_{\partial D}\|_{H^{1/2}(\partial D)} \\ &\quad + L \|w|_{\partial D}\|_{L^2(\partial D)} \|v|_{\partial D}\|_{L^2(\partial D)} \\ &\leq C_1 C_2 \Big(\|\nabla w\|_{L^2(\Omega \setminus \overline{D})} + k^2 \|w\|_{L^2(\Omega \setminus \overline{D})} \\ &\quad + \frac{L}{C_1} \|w|_{\partial D}\|_{L^2(\partial D)} \Big) \|v\|_{H^1(D)} \end{aligned} \tag{4.6}$$

Now applying Proposition 2.1, the trace theorem in the domain $\Omega \setminus \overline{D}$ to the factor of (4.6) involving w, we obtain (4.1). □

The point is: everything has been done in the context of the weak solution. We do not make use of higher regularity.

Let $L > 0$ satisfy (2.6). Then from Lemma 2.4 we have, for all $\epsilon \in]0, 1[$

$$\begin{aligned} &\int_D |\nabla v|^2 dy - k^2 \int_D |v|^2 \, dy + \int_{\partial D} (\operatorname{Re} \lambda) |v|^2 \, dS \\ &\geq (1 - K(D) L \epsilon) \int_D |\nabla v|^2 \, dy - (k^2 + K(D) L \epsilon^{-1}) \int_D |v|^2 \, dy. \end{aligned} \tag{4.7}$$

4.1. Blowup of energy of reflected solution by D

Given $x \in \Omega \setminus \overline{D}$ let w_x be the reflected solution by D. One can choose a needle σ with tip at x in such a way that $\sigma(]0, 1]) \cap \overline{D} = \emptyset$. Then any needle sequence $\xi = \{v_n\}$ for (x, σ) is convergent on D. Then the sequence $\{w_n\}$ of reflected solutions w_n of v_n by D converges to w_x. Thus from (4.1) and (4.7) we obtain the estimate

$$
\begin{aligned}
&C\|\nabla w_x\|_{L^2(\Omega\setminus\overline{D})}\\
&\ge \Big\{(1-K(D)L\epsilon)\int_D |\nabla G_k(y-x)|^2\,dy\\
&\quad -(k^2+K(D)L\epsilon^{-1})\int_D |G_k(y-x)|^2\,dy\Big\}\\
&\quad \times\left(\int_D |\nabla G_k(y-x)|^2\,dy+\int_D |G_k(y-x)|^2\,dy\right)^{-1/2}.
\end{aligned}
$$

Thus choosing ϵ in such a way that $1 - LK(D)\epsilon > 0$, one concludes that, for all $a \in \partial D$

$$\lim_{x\to a}\int_{\Omega\setminus\overline{D}} |\nabla w_x|^2\,dy = \infty.$$

4.2. Blowup of energy of reflected solution of v_n by D

The following theorem gives a characterization of the blowup of the energy of v_n on D by the blowup of the energy of the reflected solution of v_n by D.

Theorem D *Let k, L and $\epsilon \in]0,1[$ satisfy*

$$\min_j\big\{1-K(D)L\epsilon-2(k^2+K(D)L\epsilon^{-1})C(D_j)^2(1+1)^2\big\}>0. \quad (4.8)$$

Let $x \in \Omega$ and σ be a needle with tip at x. Let $\xi = \{v_n\}$ be a needle sequence for (x, σ). Let $\{w_n\}$ be the sequence of the reflected solutions of v_n by D. If

$$\lim_{n\to\infty}\int_D |\nabla v_n|^2\,dy = \infty,$$

then

$$\lim_{n\to\infty}\int_{\Omega\setminus\overline{D}} |\nabla w_n|^2\,dy = \infty.$$

Note that (2.8) in Theorem B implies (4.8).

Proof. Applying the argument after (3.2) in the proof of Theorem B to the right hand side of (4.7), we obtain the estimate

$$\begin{aligned}(1-K(D)L\epsilon)\int_D|\nabla v_n|^2\,dy-(k^2+K(D)L\epsilon^{-1})\int_D|v_n|^2\,dy\\ \geq C'\int_D|\nabla v_n|^2\,dy-C''\end{aligned}\tag{4.9}$$

where C' and C'' are positive constant independent of n.

Here we cite a lemma in [16] that says that ∇v_n dominates v_n.

Lemma 4.2 ([16]) *Let $x\in\Omega$ and σ be a needle with tip at x. Let $\xi=\{v_n\}$ be a needle sequence for (x,σ). If*

$$\lim_{n\to\infty}\int_D|\nabla v_n|^2\,dy=\infty,$$

then there exists a natural number n_0 such that the sequence

$$\left\{\frac{\displaystyle\int_D|v_n|^2\,dy}{\displaystyle\int_D|\nabla v_n|^2\,dy}\right\}_{n\geq n_0},$$

is bounded.

Now from (4.1), (4.7), (4.9) and Lemma 4.2 yields Theorem D. The converse of Theorem D is also true without the condition (4.8). It is an easy consequence of the well posedness of (2.2), the trace theorem in the domain D and Lemma 4.2.

5. Enclosure method

In this section we present an application of the enclosure method introduced in [12] to *Inverse Problem* in three-dimensions ($m=3$).

In [12] we considered the problem in the case when $\lambda\to\infty$. The case when $\lambda\to 0$ was studied in [13]. Therein we demonstrated how to use the exponentially growing solutions for elliptic equations constructed by Sylvester-Uhlmann [22] for extracting the convex hull of unknown obstacles from the Dirichlet-to-Neumann map. Here we apply the idea to *Inverse Problem.*

The convex hull of D is uniquely determined by the support function

$$h_D(\omega)=\sup_{x\in D}x\cdot\omega,\qquad \omega\in S^2.$$

The exponentially growing solutions for the Helmholtz equation have the form

$$v(x) = e^{x\cdot(\tau\omega + i\sqrt{\tau^2+k^2}\,\omega^\perp)} \tag{5.1}$$

where $\omega^\perp \in S^2$ and satisfies $\omega \cdot \omega^\perp = 0$; $\tau > 0$ is a large parameter.

Using the traces of these functions onto $\partial\Omega$, we define the indicator function

$$I_{\omega,\omega^\perp}(\tau, t) = e^{-2\tau t}\,\mathrm{Re}\langle(\Lambda_0 - \Lambda_D)f, \overline{f}\rangle, \qquad -\infty < t < \infty$$

where $f = v$ on $\partial\Omega$.

Here we prove the following.

Theorem E *Assume that both $\partial\Omega$ and ∂D are C^2 and $\lambda \in C^1(\partial D)$. Let ω and ∂D satisfy the condition (R): the set $\{x \in \partial D \,|\, x\cdot\omega = h_D(\omega)\}$ consists of a single point and the Gaussian curvature of ∂D does not vanish at the point.*

Then the formula

$$\lim_{\tau\to\infty} \frac{\log|I_{\omega,\omega^\perp}(\tau, 0)|}{2\tau} = h_D(\omega), \tag{5.2}$$

is valid. Moreover we have:

if $t > h_D(\omega)$, *then* $\lim_{\tau\to\infty}|I_{\omega,\omega^\perp}(\tau, t)| = 0$;
if $t = h_D(\omega)$, *then* $\liminf_{\tau\to\infty}|I_{\omega,\omega^\perp}(\tau, t)| > 0$;
if $t < h_D(\omega)$, *then* $\lim_{\tau\to\infty}|I_{\omega,\omega^\perp}(\tau, t)| = \infty$.

Proof. The proof can be done along the same line as Theorem 4.2 of [13]. Thus it suffices to prove that there exists a positive constant C independent of $\tau \gg 1$ such that, for all $\tau \gg 1$

$$|I_{\omega,\omega^\perp}(\tau, h_D(\omega))| \ge C. \tag{5.3}$$

This is proved as follows. From (3.1) for v given by (5.1) we obtain

$$\begin{aligned}|I_{\omega,\omega^\perp}(\tau, h_D(\omega))| &\ge -(k^2 + 2K(\Omega\setminus\overline{D})L\epsilon^{-1})e^{-2\tau h_D(\omega)}\int_{\Omega\setminus\overline{D}}|w|^2\,dy\\ &\quad + \left\{(1 - 2K(D)L\epsilon) - \frac{k^2 + 2K(D)L\epsilon^{-1}}{2\tau^2 + k^2}\right\}e^{-2\tau h_D(\omega)}\int_D|\nabla v|^2\,dy\end{aligned} \tag{5.4}$$

where $L > 0$ satisfy (2.6) and w is the weak solution of the problem (3.8).

We claim

$$\lim_{\tau\to\infty}\frac{e^{-2\tau h_D(\omega)}\displaystyle\int_{\Omega\setminus\overline{D}}|w|^2\,dy}{e^{-2\tau h_D(\omega)}\displaystyle\int_D|\nabla v|^2\,dy}=0. \tag{5.5}$$

This is proved as follows.

Let y_0 be the single point in the condition (R). Using the completely same argument as [13], one obtains the estimate

$$\begin{aligned}&e^{-2\tau h_D(\omega)}\int_{\Omega\setminus\overline{D}}|w|^2\,dy\\ &\quad\le C'\biggl\{\biggl(\int_{\partial D}|y-y_0|^{1/2}e^{-\tau h_D(\omega)}\biggl|\frac{\partial v}{\partial\nu}+\lambda v\biggr|\,dS(y)\biggr)^2\\ &\qquad+\biggl|k^2\int_D e^{-\tau h_D(\omega)}v\,dy-\int_{\partial D}\lambda e^{-\tau h_D(\omega)}v\,dS\biggr|^2\biggr\}\end{aligned} \tag{5.6}$$

where C' is a positive constant independent of τ. From (17) of [13] we have immediately, as $\tau\to\infty$

$$\biggl(\int_{\partial D}|y-y_0|^{1/2}e^{-\tau h_D(\omega)}\biggl|\frac{\partial v}{\partial\nu}+\lambda v\biggr|\,dS(y)\biggr)=O(\tau^{-1/2}). \tag{5.7}$$

Here we made use of the non vanishing of the Gaussian curvature of ∂D at y_0. From the Schwartz inequality we have, as $\tau\to\infty$

$$\frac{\biggl|\displaystyle\int_D e^{-\tau h_D(\omega)}v\,dy\biggr|^2}{e^{-2\tau h_D(\omega)}\displaystyle\int_D|\nabla v|^2\,dy}=O(\tau^{-2}) \tag{5.8}$$

and we have already known that

$$e^{-2\tau h_D(\omega)}\int_D|\nabla v|^2\,dy\ge C'',\qquad \tau\gg 1 \tag{5.9}$$

where C'' is a positive constant independent of τ. Here we made use of C^2-regularity of ∂D at y_0. Moreover we have

$$\biggl|\int_{\partial D}\lambda e^{-\tau h_D(\omega)}v\,dS\biggr|^2\le\|\lambda\|^2_{L^\infty(\partial D)}\biggl(\int_{\partial D}e^{\tau(y\cdot\omega-h_D(\omega))}\,dS(y)\biggr)^2$$

$$\leq \|\lambda\|^2_{L^\infty(\partial D)}|\partial D| \int_{\partial D} e^{2\tau(y\cdot\omega - h_D(\omega))}\, dS(y). \tag{5.10}$$

In the proof of Lemma 4.2 of [13] we have already proved that

$$\lim_{\tau\to\infty} \frac{\displaystyle\int_{\partial D} e^{2\tau(y\cdot\omega - h_D(\omega))}\, dS(y)}{\displaystyle e^{-2\tau h_D(\omega)} \int_D |\nabla v|^2\, dy} = 0. \tag{5.11}$$

Now from (5.6) to (5.11) one obtains (5.5). A combination of (5.4) for sufficiently small ϵ, (5.5) and (5.9) gives (5.3). □

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