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The space of bilinear Fourier multipliers as a dual space

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Abstract. Figà-Talamanca characterized the space of Fourier multipliers as a dual space of a certain Banach space. In this paper, we give the similar result for bilinear Fourier multipliers.

Key words: Fourier multipliers, multilinear operators, translation invariant operators.

1. Introduction

To describe the result given by Figà-Talamanca for Fourier multipliers (in the single case), we first give some definitions. $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz class. $\mathcal{S}'(\mathbb{R}^n)$ is the dual space of $\mathcal{S}(\mathbb{R}^n)$. The space $M_p(\mathbb{R}^n)$ of Fourier multipliers consists of all $m \in \mathcal{S}'(\mathbb{R}^n)$ such that T_m is bounded on $L^p(\mathbb{R}^n)$, where T_m is defined by $T_m f = [\mathcal{F}^{-1}m] * f$ for all $f \in \mathcal{S}(\mathbb{R}^n)$. Let $1 < p < \infty$ and p' be the conjugate exponent of p (that is, $1/p + 1/p' = 1$). The space $A_p(\mathbb{R}^n)$ consists of all $f \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ which can be written as $f = \sum_{i=1}^\infty f_i * g_i$ in $L^\infty(\mathbb{R}^n)$, where $\{f_i\} \subset L^p(\mathbb{R}^n)$, $\{g_i\} \subset L^{p'}(\mathbb{R}^n)$ and $\sum_{i=1}^\infty \|f_i\|_p \|g_i\|_{p'} < \infty$. Then the norm $\|f\|_{A_p}$ is the infimum of the sums $\sum_{i=1}^\infty \|f_i\|_p \|g_i\|_{p'}$ corresponding to the representations for f. In [3], Figà-Talamanca proved that $M_p(\mathbb{R}^n) = A_p(\mathbb{R}^n)^*$, where $A_p(\mathbb{R}^n)^*$ is the dual space of $A_p(\mathbb{R}^n)$ (see also [7]).

Bilinear Fourier multipliers were studied by, for example, Coifman and Meyer [2], Grafakos and Torres [4] and Lacey and Thiele [6]. The purpose of the paper is to find Figà-Talamanca's theorem for bilinear Fourier multipliers. The space $M_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})$ of bilinear Fourier multipliers consists of all $m \in \mathcal{S}'(\mathbb{R}^{2n})$ such that T_m is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^{p_3}(\mathbb{R}^n)$, where T_m is defined by $T_m(f_1, f_2)(x) = [\mathcal{F}^{-1}m] * [f_1 \otimes f_2](x, x)$ for all f_1, $f_2 \in \mathcal{S}(\mathbb{R}^n)$ and $f_1 \otimes f_2(x_1, x_2) = f_1(x_1)\, f_2(x_2)$ (for multilinear Fourier multipliers, see [4]). We also denote the unique bounded extension of T_m by T_m and define the norm on $M_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})$ by

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$$\|m\|_{M^{p_3}_{p_1,p_2}} = \sup\{\|T_m(f_1, f_2)\|_{p_3} : \|f_1\|_{p_1} = \|f_2\|_{p_2} = 1\}.$$

For appropriate functions f on $\mathbb{R}^{2n}$ and g on $\mathbb{R}^n$, we define the function $f *_2 g$ on $\mathbb{R}^{2n}$ by

$$f *_2 g(x_1, x_2) = \int_{\mathbb{R}^n} f(x_1 - y, x_2 - y)\, g(y)\, dy \qquad (x_1,\, x_2 \in \mathbb{R}^n).$$

Let $1 < p_1, p_2, p_3 < \infty$ and $1/p_3 = 1/p_1 + 1/p_2$. The space $A^{p_3}_{p_1,p_2}(\mathbb{R}^{2n})$ consists of all $f \in C(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n})$ which can be written as $f = \sum_{i=1}^\infty [f_{1,i} \otimes f_{2,i}] *_2 g_i$ in $L^\infty(\mathbb{R}^{2n})$, where $\{f_{1,i}\} \subset L^{p_1}(\mathbb{R}^n)$, $\{f_{2,i}\} \subset L^{p_2}(\mathbb{R}^n)$, $\{g_i\} \subset L^{p_3'}(\mathbb{R}^n)$ and $\sum_{i=1}^\infty \|f_{1,i}\|_{p_1} \|f_{2,i}\|_{p_2} \|g_i\|_{p_3'} < \infty$. Since $\|[f_1 \otimes f_2] *_2 g\|_\infty \le \|f_1\|_{p_1} \|f_2\|_{p_2} \|g\|_{p_3'}$ and $[f_1 \otimes f_2] *_2 g \in C(\mathbb{R}^{2n})$ for all $f_1 \in L^{p_1}(\mathbb{R}^n)$, $f_2 \in L^{p_2}(\mathbb{R}^n)$ and $g \in L^{p_3'}(\mathbb{R}^n)$, we note that, if $\sum_{i=1}^\infty \|f_{1,i}\|_{p_1} \|f_{2,i}\|_{p_2} \|g_i\|_{p_3'} < \infty$, then $\sum_{i=1}^\infty [f_{1,i} \otimes f_{2,i}] *_2 g_i \in C(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n})$. We define the norm on $A^{p_3}_{p_1,p_2}(\mathbb{R}^{2n})$ by

$$\|f\|_{A^{p_3}_{p_1,p_2}} = \inf\left\{\sum_{i=1}^\infty \|f_{1,i}\|_{p_1} \|f_{2,i}\|_{p_2} \|g_i\|_{p_3'} : f = \sum_{i=1}^\infty [f_{1,i} \otimes f_{2,i}] *_2 g_i\right\}.$$

Then $A^{p_3}_{p_1,p_2}(\mathbb{R}^{2n})$ is a Banach space (Lemma 3.1). Given $m \in M^{p_3}_{p_1,p_2}(\mathbb{R}^{2n})$, we define the linear functional φ_m on $A^{p_3}_{p_1,p_2}(\mathbb{R}^{2n})$ by

$$\varphi_m(f) = \sum_{i=1}^\infty T_m(f_{1,i}, f_{2,i}) * g_i(0) \tag{1.1}$$

for $f = \sum_{i=1}^\infty [f_{1,i} \otimes f_{2,i}] *_2 g_i \in A^{p_3}_{p_1,p_2}(\mathbb{R}^{2n})$. We note that the value $\sum_{i=1}^\infty T_m(f_{1,i}, f_{2,i}) * g_i(0)$ is independent of the representations for f (Lemma 3.8). Our main result is the following.

Theorem *Let* $1 < p_1, p_2, p_3 < \infty$ *and* $1/p_3 = 1/p_1 + 1/p_2$. *If* $m \in M^{p_3}_{p_1,p_2}(\mathbb{R}^{2n})$, *then* $\varphi_m \in A^{p_3}_{p_1,p_2}(\mathbb{R}^{2n})^*$ *and* $\|\varphi_m\|_{(A^{p_3}_{p_1,p_2})^*} = \|m\|_{M^{p_3}_{p_1,p_2}}$. *Conversely, if* $\varphi \in A^{p_3}_{p_1,p_2}(\mathbb{R}^{2n})^*$, *then there exists* $m \in M^{p_3}_{p_1,p_2}(\mathbb{R}^{2n})$ *such that* $\varphi = \varphi_m$. *In this sense,* $M^{p_3}_{p_1,p_2}(\mathbb{R}^{2n}) = A^{p_3}_{p_1,p_2}(\mathbb{R}^{2n})^*$.

We point out that Berkson, Paluszyński and Weiss applied Figà-Talamanca's theorem to wavelet theory [1].

2. Preliminaries

We define the Fourier transform $\mathcal{F}f$ and the inverse Fourier transform $\mathcal{F}^{-1}f$ of $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\, dx,$$
$$\mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi)\, d\xi.$$

We also define the Fourier transform $\mathcal{F}u$ and the inverse Fourier transform $\mathcal{F}^{-1}u$ of $u \in \mathcal{S}'(\mathbb{R}^n)$ by

$$\langle \mathcal{F}u, \psi\rangle = \langle u, \mathcal{F}\psi\rangle \text{ and } \langle \mathcal{F}^{-1}u, \psi\rangle = \langle u, \mathcal{F}^{-1}\psi\rangle \quad \text{for all } \psi \in \mathcal{S}(\mathbb{R}^n).$$

We note that, if $u \in \mathcal{S}'(\mathbb{R}^n)$ is a function, then $\langle u, \psi\rangle = \int_{\mathbb{R}^n} u(x)\,\psi(x)\,dx$. For $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$, the convolution $u*\psi$ is defined by $u * \psi(x) = \langle u, \tau_x\check{\psi}\rangle$, where $\tau_x\check{\psi}(y) = \check{\psi}(y-x)$ and $\check{\psi}(y) = \psi(-y)$.

3. Proofs

Throughout the rest of the paper, we always assume that $1 < p_1, p_2, p_3 < \infty$ and $1/p_3 = 1/p_1 + 1/p_2$.

Lemma 3.1 $A^{p_3}_{p_1,p_2}(\mathbb{R}^{2n})$ *is a Banach space.*

Proof. The proof of Lemma 3.1 is similar to one of [9, Proposition 6.14]. Using that $\|f\|_\infty \le \|f\|_{A^{p_3}_{p_1,p_2}}$ for all $f \in A^{p_3}_{p_1,p_2}(\mathbb{R}^{2n})$, we see that $\|\cdot\|_{A^{p_3}_{p_1,p_2}}$ is a norm. To check that $A^{p_3}_{p_1,p_2}(\mathbb{R}^{2n})$ is complete, it is enough to show that, if $\{h_j\} \subset A^{p_3}_{p_1,p_2}(\mathbb{R}^{2n})$ and $\sum_{j=1}^\infty \|h_j\|_{A^{p_3}_{p_1,p_2}} < \infty$, then $\sum_{j=1}^\infty h_j \in A^{p_3}_{p_1,p_2}(\mathbb{R}^{2n})$. Then we can represent each h_j as $\sum_{i=1}^\infty \big[f^{(j)}_{1,i} \otimes f^{(j)}_{2,i}\big] *_2 g^{(j)}_i$, where $\sum_{i=1}^\infty \big\|f^{(j)}_{1,i}\big\|_{p_1}\big\|f^{(j)}_{2,i}\big\|_{p_2}\big\|g^{(j)}_i\big\|_{p_3'} \le 2\|h_j\|_{A^{p_3}_{p_1,p_2}}$. Thus, the representation $\sum_{j=1}^\infty\sum_{i=1}^\infty \big[f^{(j)}_{1,i} \otimes f^{(j)}_{2,i}\big] *_2 g^{(j)}_i$ implies that $\sum_{j=1}^\infty h_j \in A^{p_3}_{p_1,p_2}(\mathbb{R}^{2n})$. □

Lemma 3.2 *Let* $m \in M^{p_3}_{p_1,p_2}(\mathbb{R}^{2n})$, $f_1 \in L^{p_1}(\mathbb{R}^n)$, $f_2 \in L^{p_2}(\mathbb{R}^n)$ *and* $g \in L^{p_3'}(\mathbb{R}^n)$. *If* $\eta(x_1, x_2) = T_m(\tau_{-x_1}f_1, \tau_{-x_2}f_2) * g(0)$, *then* $\eta \in C(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n})$ *and* $\|\eta\|_\infty \le \|m\|_{M^{p_3}_{p_1,p_2}}\|f_1\|_{p_1}\|f_2\|_{p_2}\|g\|_{p_3'}$.

Proof. Using that

$$T_m(\tau_{-x_1}f_1, \tau_{-x_2}f_2) - T_m(\tau_{-x_1'}f_1, \tau_{-x_2'}f_2)$$
$$= T_m(\tau_{-x_1}f_1 - \tau_{-x_1'}f_1, \tau_{-x_2}f_2) + T_m(\tau_{-x_1'}f_1, \tau_{-x_2}f_2 - \tau_{-x_2'}f_2),$$

we have that

$$\begin{aligned}
&|\eta(x_1, x_2) - \eta(x_1', x_2')| \\
&\le \big(\|T_m(\tau_{-x_1}f_1 - \tau_{-x_1'}f_1, \tau_{-x_2}f_2)\|_{p_3} \\
&\quad + \|T_m(\tau_{-x_1'}f_1, \tau_{-x_2}f_2 - \tau_{-x_2'}f_2)\|_{p_3}\big)\|g\|_{p_3'} \\
&\le \|m\|_{M_{p_1,p_2}^{p_3}}\big(\|\tau_{-x_1}f_1 - \tau_{-x_1'}f_1\|_{p_1}\|f_2\|_{p_2} \\
&\quad + \|f_1\|_{p_1}\|\tau_{-x_2}f_2 - \tau_{-x_2'}f_2\|_{p_2}\big)\|g\|_{p_3'}.
\end{aligned}$$

This gives $\eta \in C(\mathbb{R}^{2n})$. On the other hand, by Hölder's inequality, we see that $\|\eta\|_\infty \le \|m\|_{M_{p_1,p_2}^{p_3}}\|f_1\|_{p_1}\|f_2\|_{p_2}\|g\|_{p_3'}$. □

Lemma 3.3 *Let $m \in M_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})$ be a $C^\infty(\mathbb{R}^{2n})$-function such that all its derivatives are slowly increasing. Then we have that*

$$\int_{\mathbb{R}^{2n}} T_m(\tau_{-x_1}f_1, \tau_{-x_2}f_2) * g(0)\,\psi(x_1, x_2)\,dx_1\,dx_2$$
$$= \int_{\mathbb{R}^{2n}} [\mathcal{F}^{-1}m] * \check{\psi}(-x_1, -x_2)\,[f_1 \otimes f_2] *_2 g(x_1, x_2)\,dx_1\,dx_2$$

for all f_1, f_2, $g \in \mathcal{S}(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^{2n})$.

Proof. By the assumption of m, we see that $[\mathcal{F}^{-1}m] * [f_1 \otimes f_2] \in \mathcal{S}(\mathbb{R}^{2n})$ for f_1, $f_2 \in \mathcal{S}(\mathbb{R}^n)$. Since $T_m(\tau_{-x_1}f_1, \tau_{-x_2}f_2)(-y) = [\mathcal{F}^{-1}m] * [f_1 \otimes f_2](x_1 - y, x_2 - y)$, we have that $\sup_{x_1,x_2,y} |T_m(\tau_{-x_1}f_1, \tau_{-x_2}f_2)(-y)| < \infty$. Hence, by Fubini's theorem, we see that

$$\int_{\mathbb{R}^{2n}} T_m(\tau_{-x_1}f_1, \tau_{-x_2}f_2) * g(0)\,\psi(x_1, x_2)\,dx_1\,dx_2$$
$$= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^{2n}} T_m(\tau_{-x_1}f_1, \tau_{-x_2}f_2)(-y)\,\psi(x_1, x_2)\,dx_1\,dx_2 \right) g(y)\,dy.$$

Using that $T_m(\tau_{-x_1}f_1, \tau_{-x_2}f_2)(-y) = [\mathcal{F}^{-1}m] * [\tau_y f_1 \otimes \tau_y f_2](x_1, x_2)$, we have that

$$\int_{\mathbb{R}^{2n}} T_m(\tau_{-x_1}f_1, \tau_{-x_2}f_2)(-y)\,\psi(x_1, x_2)\,dx_1\,dx_2$$
$$= \langle [\mathcal{F}^{-1}m] * [\tau_y f_1 \otimes \tau_y f_2], \psi \rangle = \langle [\mathcal{F}^{-1}m] * \check{\psi}, [\check{\tau_y f_1} \otimes \check{\tau_y f_2}] \rangle$$

$$= \int_{\mathbb{R}^{2n}} [\mathcal{F}^{-1}m] * \check{\psi}(-x_1, -x_2)\, f_1(x_1 - y)\, f_2(x_2 - y)\, dx_1\, dx_2.$$

Since $[\mathcal{F}^{-1}m] * \check{\psi} \in \mathcal{S}(\mathbb{R}^{2n})$, by Fubini's theorem, we get that

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^{2n}} T_m(\tau_{-x_1}f_1, \tau_{-x_2}f_2)(-y)\, \psi(x_1, x_2)\, dx_1\, dx_2\right) g(y)\, dy \\
&= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^{2n}} [\mathcal{F}^{-1}m] * \check{\psi}(-x_1, -x_2)\, f_1(x_1 - y)\, f_2(x_2 - y)\, dx_1\, dx_2\right) g(y)\, dy \\
&= \int_{\mathbb{R}^{2n}} [\mathcal{F}^{-1}m] * \check{\psi}(-x_1, -x_2)\left(\int_{\mathbb{R}^n} f_1(x_1 - y)\, f_2(x_2 - y)\, g(y)\, dy\right) dx_1\, dx_2 \\
&= \int_{\mathbb{R}^{2n}} [\mathcal{F}^{-1}m] * \check{\psi}(-x_1, -x_2)\, [f_1 \otimes f_2] *_2 g(x_1, x_2)\, dx_1\, dx_2
\end{aligned}$$

The proof is complete. □

Lemma 3.4 *Let $m \in M_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})$ be a $C^\infty(\mathbb{R}^{2n})$-function such that all its derivatives are slowly increasing. If $\{f_{1,i}\} \subset L^{p_1}(\mathbb{R}^n)$, $\{f_{2,i}\} \subset L^{p_2}(\mathbb{R}^n)$ and $\{g_i\} \subset L^{p_3'}(\mathbb{R}^n)$ satisfy $\sum_{i=1}^{\infty}[f_{1,i} \otimes f_{2,i}] *_2 g_i = 0$ in $L^\infty(\mathbb{R}^{2n})$ and $\sum_{i=1}^{\infty} \|f_{1,i}\|_{p_1} \|f_{2,i}\|_{p_2} \|g_i\|_{p_3'} < \infty$, then $\sum_{i=1}^{\infty} T_m(f_{1,i}, f_{2,i}) * g_i(0) = 0$.*

Proof. We define the function σ on $\mathbb{R}^{2n}$ by

$$\sigma(x_1, x_2) = \sum_{i=1}^{\infty} T_m(\tau_{-x_1}f_{1,i}, \tau_{-x_2}f_{2,i}) * g_i(0) \qquad (x_1,\, x_2 \in \mathbb{R}^n).$$

Then, from Lemma 3.2, we see that $\sigma \in C(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n})$. Hence, if

$$\int_{\mathbb{R}^{2n}} \sigma(x_1, x_2)\, \psi(x_1, x_2)\, dx_1\, dx_2 = 0 \quad \text{for all } \psi \in \mathcal{S}(\mathbb{R}^{2n}), \tag{3.1}$$

then we get that

$$\sigma(0, 0) = \sum_{i=1}^{\infty} T_m(f_{1,i}, f_{2,i}) * g_i(0) = 0.$$

We prove (3.1). Let $\psi \in \mathcal{S}(\mathbb{R}^{2n})$. By Lemma 3.2, we have that

$$\begin{aligned}
&\sum_{i=1}^{\infty} \int_{\mathbb{R}^{2n}} \left|T_m(\tau_{-x_1}f_{1,i}, \tau_{-x_2}f_{2,i}) * g_i(0)\, \psi(x_1, x_2)\right| dx_1\, dx_2 \\
&\qquad \le \|m\|_{M_{p_1,p_2}^{p_3}} \|\psi\|_1 \sum_{i=1}^{\infty} \|f_{1,i}\|_{p_1} \|f_{2,i}\|_{p_2} \|g_i\|_{p_3'} < \infty.
\end{aligned}$$

Thus, we see that

$$\begin{aligned}\langle\sigma,\psi\rangle &= \int_{\mathbb{R}^{2n}}\left(\sum_{i=1}^{\infty} T_m(\tau_{-x_1}f_{1,i},\tau_{-x_2}f_{2,i}) * g_i(0)\right)\psi(x_1,x_2)\,dx_1\,dx_2\\ &= \sum_{i=1}^{\infty}\int_{\mathbb{R}^{2n}} T_m(\tau_{-x_1}f_{1,i},\tau_{-x_2}f_{2,i}) * g_i(0)\,\psi(x_1,x_2)\,dx_1\,dx_2.\end{aligned}$$

For $f_{1,i}$, $f_{2,i}$ and g_i, we take $\{f_{1,i,j}\}_j$, $\{f_{2,i,j}\}_j$, $\{g_{i,j}\}_j \subset \mathcal{S}(\mathbb{R}^n)$ such that $f_{1,i,j} \to f_{1,i}$ in $L^{p_1}(\mathbb{R}^n)$, $f_{2,i,j} \to f_{2,i}$ in $L^{p_2}(\mathbb{R}^n)$ and $g_{i,j} \to g_i$ in $L^{p_3'}(\mathbb{R}^n)$ as $j \to \infty$. Then, by Lemma 3.3 and $[\mathcal{F}^{-1}m] * \check{\psi} \in L^1(\mathbb{R}^n)$, we have that

$$\begin{aligned}&\int_{\mathbb{R}^{2n}} T_m(\tau_{-x_1}f_{1,i},\tau_{-x_2}f_{2,i}) * g_i(0)\,\psi(x_1,x_2)\,dx_1\,dx_2\\ &= \lim_{j\to\infty}\int_{\mathbb{R}^{2n}} T_m(\tau_{-x_1}f_{1,i,j},\tau_{-x_2}f_{2,i,j}) * g_{i,j}(0)\,\psi(x_1,x_2)\,dx_1\,dx_2\\ &= \lim_{j\to\infty}\int_{\mathbb{R}^{2n}} [\mathcal{F}^{-1}m] * \check{\psi}(-x_1,-x_2)\,[f_{1,i,j}\otimes f_{2,i,j}] *_2 g_{i,j}(x_1,x_2)\,dx_1\,dx_2\\ &= \int_{\mathbb{R}^{2n}} [\mathcal{F}^{-1}m] * \check{\psi}(-x_1,-x_2)\,[f_{1,i}\otimes f_{2,i}] *_2 g_i(x_1,x_2)\,dx_1\,dx_2.\end{aligned}$$

Therefore, using that $[\mathcal{F}^{-1}m] * \check{\psi} \in L^1(\mathbb{R}^{2n})$ and $\sum_{i=1}^{\infty}[f_{1,i}\otimes f_{2,i}] *_2 g_i = 0$ in $L^{\infty}(\mathbb{R}^{2n})$, we get that

$$\begin{aligned}\langle\sigma,\psi\rangle &= \sum_{i=1}^{\infty}\int_{\mathbb{R}^{2n}} [\mathcal{F}^{-1}m] * \check{\psi}(-x_1,-x_2)\,[f_{1,i}\otimes f_{2,i}] *_2 g_i(x_1,x_2)\,dx_1\,dx_2\\ &= \int_{\mathbb{R}^{2n}} [\mathcal{F}^{-1}m] * \check{\psi}(-x_1,-x_2)\left(\sum_{i=1}^{\infty}[f_{1,i}\otimes f_{2,i}] *_2 g_i(x_1,x_2)\right)dx_1\,dx_2\\ &= 0.\end{aligned}$$

The proof is complete. □

The following lemma in the single case is given as [5, (1.2)].

Lemma 3.5 *If* $m \in M_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})$ *and* $\psi \in \mathcal{S}(\mathbb{R}^{2n})$, *then* $\psi * m \in M_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})$ *and* $\|\psi * m\|_{M_{p_1,p_2}^{p_3}} \le \|\psi\|_1 \|m\|_{M_{p_1,p_2}^{p_3}}$.

Proof. By duality, we have that

$$\|\psi * m\|_{M_{p_1,p_2}^{p_3}} = \sup\left|\int_{\mathbb{R}^n} T_{\psi * m}(f_1,f_2)(x)\,g(x)\,dx\right|, \tag{3.2}$$

where the supremum is taken over all $f_1,\ f_2,\ g \in \mathcal{S}(\mathbb{R}^n)$ such that $\|f_1\|_{p_1} = \|f_2\|_{p_2} = \|g\|_{p_3'} = 1$. For $f_1,\ f_2 \in \mathcal{S}(\mathbb{R}^n)$, we have that

$$\begin{aligned}
&T_{\psi * m}(f_1, f_2)(x) = \langle \mathcal{F}^{-1}[\psi * m], f_1(x-y_1)\, f_2(x-y_2)\rangle_{y_1,y_2}\\
&= (2\pi)^{2n}\langle [\mathcal{F}^{-1}m]\,[\mathcal{F}^{-1}\psi], \tau_x \check{f}_1 \otimes \tau_x \check{f}_2\rangle = \langle [\mathcal{F}^{-1}m]\,[\tau_x \check{f}_1 \otimes \tau_x \check{f}_2], \mathcal{F}\check{\psi}\rangle\\
&= \frac{1}{(2\pi)^{2n}}\langle m * \mathcal{F}[\tau_x \check{f}_1 \otimes \tau_x \check{f}_2], \check{\psi}\rangle.
\end{aligned}$$

Let M_y be the modulation operator defined by $M_y h(\xi) = e^{iy\cdot\xi}h(\xi)$. Using that $M_{-y}\tau_x \check{h}(\xi) = e^{-ix\cdot y}[M_y h](x-\xi)$, we see that

$$\begin{aligned}
&m * \mathcal{F}[\tau_x \check{f}_1 \otimes \tau_x \check{f}_2](y_1, y_2) = \langle m, \mathcal{F}[\tau_x \check{f}_1 \otimes \tau_x \check{f}_2](y_1-\xi_1, y_2-\xi_2)\rangle_{\xi_1,\xi_2}\\
&\quad = (2\pi)^{2n}\langle m, \mathcal{F}^{-1}[\tau_x \check{f}_1 \otimes \tau_x \check{f}_2](\xi_1-y_1, \xi_2-y_2)\rangle_{\xi_1,\xi_2}\\
&\quad = (2\pi)^{2n}\langle m, \mathcal{F}^{-1}[(M_{-y_1}\tau_x \check{f}_1) \otimes (M_{-y_2}\tau_x \check{f}_2)](\xi_1, \xi_2)\rangle_{\xi_1,\xi_2}\\
&\quad = (2\pi)^{2n}e^{-ix\cdot(y_1+y_2)}\langle \mathcal{F}^{-1}m, [M_{y_1}f_1](x-\xi_1)\,[M_{y_2}f_2](x-\xi_2)\rangle_{\xi_1,\xi_2}\\
&\quad = (2\pi)^{2n}e^{-ix\cdot(y_1+y_2)}T_m(M_{y_1}f_1, M_{y_2}f_2)(x).
\end{aligned}$$

Hence, by Hölder's inequality, we get that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} T_{\psi * m}(f_1, f_2)(x)\, g(x)\, dx\right|\\
&\quad = \left|\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^{2n}} e^{-ix\cdot(y_1+y_2)}T_m(M_{y_1}f_1, M_{y_2}f_2)(x)\right.\right.\\
&\qquad\qquad \left.\left.\times\, \psi(-y_1,-y_2)\, dy_1\, dy_2\right) g(x)\, dx\right|\\
&\quad = \left|\int_{\mathbb{R}^{2n}} \psi(-y_1,-y_2)\right.\\
&\qquad\qquad \left.\times\left(\int_{\mathbb{R}^n} e^{-ix\cdot(y_1+y_2)}T_m(M_{y_1}f_1, M_{y_2}f_2)(x)\, g(x)\, dx\right) dy_1\, dy_2\right|\\
&\quad \le \|\psi\|_1 \|m\|_{M^{p_3}_{p_1,p_2}} \|f_1\|_{p_1}\|f_2\|_{p_2}\|g\|_{p_3'}.
\end{aligned}\tag{3.3}$$

(3.2) and (3.3) prove Lemma 3.5. □

Lemma 3.6 *Let $m \in M^{p_3}_{p_1,p_2}(\mathbb{R}^{2n})$ and ρ be a radial $C^\infty(\mathbb{R}^{2n})$-function such that $\rho \ge 0$, $\operatorname{supp}\rho \subset \overline{B(0,1)}$ and $\int \rho(x)\, dx = 1$. Then for all $f_1 \in L^{p_1}(\mathbb{R}^n)$, $f_2 \in L^{p_2}(\mathbb{R}^n)$ and $g \in L^{p_3'}(\mathbb{R}^n)$ we have that*

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^n} T_{\rho_\epsilon * m}(f_1, f_2)(x)\, g(x)\, dx = \int_{\mathbb{R}^n} T_m(f_1, f_2)(x)\, g(x)\, dx,$$

where $\rho_\epsilon(x) = \epsilon^{-2n}\rho(x/\epsilon)$.

Proof. From Lemma 3.5, it is enough to prove Lemma 3.6 when $f_1,\ f_2,\ g \in \mathcal{S}(\mathbb{R}^n)$. Let $f_1,\ f_2,\ g \in \mathcal{S}(\mathbb{R}^n)$. $\rho_\epsilon * m \to m$ in $\mathcal{S}'(\mathbb{R}^{2n})$ as $\epsilon \to 0$ gives $T_{\rho_\epsilon * m}(f_1, f_2)(x) \to T_m(f_1, f_2)(x)$ as $\epsilon \to 0$ for all $x \in \mathbb{R}^n$. On the other hand, since $m \in \mathcal{S}'(\mathbb{R}^{2n})$, there exist $C > 0$ and $N \in \mathbb{Z}_+$ such that

$$|\langle m, \psi\rangle| \le C p_N(\psi) \qquad (\psi \in \mathcal{S}(\mathbb{R}^{2n})),$$

where $p_N(\psi) = \sum_{|\alpha|+k\le N} \sup_{y_1,y_2\in\mathbb{R}^n}(1 + |y_1| + |y_2|)^k |\partial^\alpha\psi(y_1, y_2)|$. Hence, we see that

$$|T_{\rho_\epsilon * m}(f_1, f_2)(x)| = |\langle m, \rho_\epsilon * \mathcal{F}^{-1}(\tau_x \check{f}_1 \otimes \tau_x \check{f}_2)\rangle| \le C_{f_1,f_2}(1 + |x|)^N,$$

where C_{f_1,f_2} is independent of $0 < \epsilon < 1$. By Lebesgue's theorem, we get Lemma 3.6 when $f_1,\ f_2,\ g \in \mathcal{S}(\mathbb{R}^n)$. □

Lemma 3.7 *Let* $m \in M^{p_3}_{p_1,p_2}(\mathbb{R}^{2n})$. *If* $\{f_{1,i}\} \subset L^{p_1}(\mathbb{R}^n)$, $\{f_{2,i}\} \subset L^{p_2}(\mathbb{R}^n)$ *and* $\{g_i\} \subset L^{p_3'}(\mathbb{R}^n)$ *satisfy* $\sum_{i=1}^\infty \|f_{1,i}\|_{p_1} \|f_{2,i}\|_{p_2} \|g_i\|_{p_3'} < \infty$ *and* $\sum_{i=1}^\infty [f_{1,i} \otimes f_{2,i}] *_2 g_i = 0$ *in* $L^\infty(\mathbb{R}^{2n})$, *then* $\sum_{i=1}^\infty T_m(f_{1,i}, f_{2,i}) * g_i(0) = 0$.

Proof. By Lemmas 3.5 and 3.6, for each i, we have that

$$|T_{\rho_\epsilon * m}(f_{1,i}, f_{2,i}) * g_i(0)| \le \|m\|_{M^{p_3}_{p_1,p_2}} \|f_{1,i}\|_{p_1} \|f_{2,i}\|_{p_2} \|g_i\|_{p_3'} \qquad (\epsilon > 0)$$

and

$$\lim_{\epsilon\to 0} T_{\rho_\epsilon * m}(f_{1,i}, f_{2,i}) * g_i(0) = T_m(f_{1,i}, f_{2,i}) * g_i(0),$$

where ρ_ϵ is given in Lemma 3.6. Hence, by Lebesgue's theorem, we get that

$$\lim_{\epsilon\to 0} \sum_{i=1}^\infty T_{\rho_\epsilon * m}(f_{1,i}, f_{2,i}) * g_i(0) = \sum_{i=1}^\infty T_m(f_{1,i}, f_{2,i}) * g_i(0).$$

On the other hand, since $\rho_\epsilon * m$ is a $C^\infty(\mathbb{R}^{2n})$-function such that all its derivatives are slowly increasing for each $\epsilon > 0$ ([8, Chapter 1, Theorem 3.13]), $\rho_\epsilon * m$ satisfies the assumption of Lemma 3.4. Therefore, by Lemma 3.4, we see that

$$\sum_{i=1}^\infty T_{\rho_\epsilon * m}(f_{1,i}, f_{2,i}) * g_i(0) = 0 \qquad (\epsilon > 0).$$

This proves Lemma 3.7. □

Lemma 3.8 *Let $m \in M_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})$. Then we can define the linear functional φ_m on $A_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})$ by (1.1).*

Proof. To define φ_m, we need to show that, if $\{f_{1,i}^{(1)}\}$, $\{f_{1,i}^{(2)}\} \subset L^{p_1}(\mathbb{R}^n)$, $\{f_{2,i}^{(1)}\}$, $\{f_{2,i}^{(2)}\} \subset L^{p_2}(\mathbb{R}^n)$ and $\{g_i^{(1)}\}$, $\{g_i^{(2)}\} \subset L^{p_3'}(\mathbb{R}^n)$ satisfy $\sum_{i=1}^{\infty}\|f_{1,i}^{(1)}\|_{p_1}\|f_{2,i}^{(1)}\|_{p_2}\|g_i^{(1)}\|_{p_3'}$, $\sum_{i=1}^{\infty}\|f_{1,i}^{(2)}\|_{p_1}\|f_{2,i}^{(2)}\|_{p_2}\|g_i^{(2)}\|_{p_3'} < \infty$ and $\sum_{i=1}^{\infty}[f_{1,i}^{(1)} \otimes f_{2,i}^{(1)}] *_2 g_i^{(1)} = \sum_{i=1}^{\infty}[f_{1,i}^{(2)} \otimes f_{2,i}^{(2)}] *_2 g_i^{(2)}$ in $L^{\infty}(\mathbb{R}^{2n})$, then

$$\sum_{i=1}^{\infty} T_m(f_{1,i}^{(1)}, f_{2,i}^{(1)}) * g_i^{(1)}(0) = \sum_{i=1}^{\infty} T_m(f_{1,i}^{(2)}, f_{2,i}^{(2)}) * g_i^{(2)}(0).$$

To do this, we define $\{f_{1,i}^{(3)}\} \subset L^{p_1}(\mathbb{R}^n)$, $\{f_{2,i}^{(3)}\} \subset L^{p_2}(\mathbb{R}^n)$ and $\{g_i^{(3)}\} \subset L^{p_3'}(\mathbb{R}^n)$ by $\{f_{1,i}^{(3)}\} = \{f_{1,1}^{(1)}, f_{1,1}^{(2)}, f_{1,2}^{(1)}, f_{1,2}^{(2)}, \ldots\}$, $\{f_{2,i}^{(3)}\} = \{f_{2,1}^{(1)}, f_{2,1}^{(2)}, f_{2,2}^{(1)}, f_{2,2}^{(2)}, \ldots\}$ and $\{g_i^{(3)}\} = \{g_1^{(1)}, -g_1^{(2)}, g_2^{(1)}, -g_2^{(2)}, \ldots\}$. Then we have that

$$\begin{aligned}&\sum_{i=1}^{\infty}\|f_{1,i}^{(3)}\|_{p_1}\|f_{2,i}^{(3)}\|_{p_2}\|g_i^{(3)}\|_{p_3'} \\ &= \sum_{i=1}^{\infty}\|f_{1,i}^{(1)}\|_{p_1}\|f_{2,i}^{(1)}\|_{p_2}\|g_i^{(1)}\|_{p_3'} + \sum_{i=1}^{\infty}\|f_{1,i}^{(2)}\|_{p_1}\|f_{2,i}^{(2)}\|_{p_2}\|g_i^{(2)}\|_{p_3'} < \infty\end{aligned}$$

and

$$\begin{aligned}&\sum_{i=1}^{\infty}[f_{1,i}^{(3)} \otimes f_{2,i}^{(3)}] *_2 g_i^{(3)} \\ &\quad = \sum_{i=1}^{\infty}[f_{1,i}^{(1)} \otimes f_{2,i}^{(1)}] *_2 g_i^{(1)} - \sum_{i=1}^{\infty}[f_{1,i}^{(2)} \otimes f_{2,i}^{(2)}] *_2 g_i^{(2)} = 0.\end{aligned}$$

Hence, by Lemma 3.7, we get that

$$\begin{aligned}&\sum_{i=1}^{\infty} T_m(f_{1,i}^{(1)}, f_{2,i}^{(1)}) * g_i^{(1)}(0) - \sum_{i=1}^{\infty} T_m(f_{1,i}^{(2)}, f_{2,i}^{(2)}) * g_i^{(2)}(0) \\ &\quad = \sum_{i=1}^{\infty} T_m(f_{1,i}^{(3)}, f_{2,i}^{(3)}) * g_i^{(3)}(0) = 0.\end{aligned}$$

Thus, we can see that the value $\sum_{i=1}^{\infty} T_m(f_{1,i}, f_{2,i}) * g_i(0)$ is independent of the representations for f. In the same way, we can prove the linearity of φ_m. □

We are now ready to prove Theorem given in the introduction.

Proof of Theorem. We first show that, if $m \in M_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})$, then $\varphi_m \in A_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})^*$ and $\|\varphi_m\|_{(A_{p_1,p_2}^{p_3})^*} = \|m\|_{M_{p_1,p_2}^{p_3}}$. Let $m \in M_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})$. Then, from Lemma 3.8, we have that φ_m is a linear functional on $A_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})$. Let $f = \sum_{i=1}^{\infty}[f_{1,i} \otimes f_{2,i}] *_2 g_i \in A_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})$. Since

$$|\varphi_m(f)| \le \|m\|_{M_{p_1,p_2}^{p_3}} \sum_{i=1}^{\infty} \|f_{1,i}\|_{p_1} \|f_{2,i}\|_{p_2} \|g_i\|_{p_3'},$$

taking the infimum over all the representations for f, we have that $|\varphi_m(f)| \le \|m\|_{M_{p_1,p_2}^{p_3}} \|f\|_{A_{p_1,p_2}^{p_3}}$, that is, $\varphi_m \in A_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})^*$ and $\|\varphi_m\|_{(A_{p_1,p_2}^{p_3})^*} \le \|m\|_{M_{p_1,p_2}^{p_3}}$. We prove $\|\varphi_m\|_{(A_{p_1,p_2}^{p_3})^*} \ge \|m\|_{M_{p_1,p_2}^{p_3}}$. Using the formula (3.2), for $\epsilon > 0$ we can take $f_{1,\epsilon}$, $f_{2,\epsilon}$, $g_\epsilon \in \mathcal{S}(\mathbb{R}^n)$ such that $\|f_{1,\epsilon}\|_{p_1} = \|f_{2,\epsilon}\|_{p_2} = \|g_\epsilon\|_{p_3'} = 1$ and

$$\|m\|_{M_{p_1,p_2}^{p_3}} - \epsilon < \left| \int_{\mathbb{R}^n} T_m(f_{1,\epsilon}, f_{2,\epsilon})(x)\, g_\epsilon(x)\, dx \right|.$$

Since $[f_{1,\epsilon} \otimes f_{2,\epsilon}] *_2 \check{g}_\epsilon \in A_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})$ and $\|[f_{1,\epsilon} \otimes f_{2,\epsilon}] *_2 \check{g}_\epsilon\|_{A_{p_1,p_2}^{p_3}} \le \|f_{1,\epsilon}\|_{p_1} \|f_{2,\epsilon}\|_{p_2} \|g_\epsilon\|_{p_3'}$, we see that

$$\begin{aligned}
\|m\|_{M_{p_1,p_2}^{p_3}} &< \left| \int_{\mathbb{R}^n} T_m(f_{1,\epsilon}, f_{2,\epsilon})(x)\, g_\epsilon(x)\, dx \right| + \epsilon \\
&= |T_m(f_{1,\epsilon}, f_{2,\epsilon}) * \check{g}_\epsilon(0)| + \epsilon = \left|\varphi_m[(f_{1,\epsilon} \otimes f_{2,\epsilon}) *_2 \check{g}_\epsilon]\right| + \epsilon \\
&\le \|\varphi_m\|_{(A_{p_1,p_2}^{p_3})^*} \|[f_{1,\epsilon} \otimes f_{2,\epsilon}] *_2 \check{g}_\epsilon\|_{A_{p_1,p_2}^{p_3}} + \epsilon \\
&\le \|\varphi_m\|_{(A_{p_1,p_2}^{p_3})^*} \|f_{1,\epsilon}\|_{p_1} \|f_{2,\epsilon}\|_{p_2} \|g_\epsilon\|_{p_3'} + \epsilon = \|\varphi_m\|_{(A_{p_1,p_2}^{p_3})^*} + \epsilon.
\end{aligned}$$

Hence, the arbitrariness of ϵ gives $\|\varphi_m\|_{(A_{p_1,p_2}^{p_3})^*} \ge \|m\|_{M_{p_1,p_2}^{p_3}}$.

We next prove the converse. Let $\varphi \in A_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})^*$. Since $[f_1 \otimes f_2] *_2 g \in A_{p_1,p_2}^{p_3}(\mathbb{R}^{2n})$ for $f_1 \in L^{p_1}(\mathbb{R}^n)$, $f_2 \in L^{p_2}(\mathbb{R}^n)$ and $g \in L^{p_3'}(\mathbb{R}^n)$, fixing $f_1 \in L^{p_1}(\mathbb{R}^n)$ and $f_2 \in L^{p_2}(\mathbb{R}^n)$, we can define the linear functional F_{f_1,f_2} on $L^{p_3'}(\mathbb{R}^n)$ by

$$F_{f_1,f_2}(g) = \varphi[(f_1 \otimes f_2) *_2 g] \qquad (g \in L^{p_3'}(\mathbb{R}^n)).$$

By the boundedness of φ, we see that

$$\begin{aligned}
|F_{f_1,f_2}(g)| &\le \|\varphi\|_{(A_{p_1,p_2}^{p_3})^*} \|[f_1 \otimes f_2] *_2 g\|_{A_{p_1,p_2}^{p_3}} \\
&\le \|\varphi\|_{(A_{p_1,p_2}^{p_3})^*} \|f_1\|_{p_1} \|f_2\|_{p_2} \|g\|_{p_3'} \qquad (g \in L^{p_3'}(\mathbb{R}^n)),
\end{aligned}$$

that is, $F_{f_1,f_2} \in L^{p_3'}(\mathbb{R}^n)^*$ and $\|F_{f_1,f_2}\|_{(L^{p_3'})^*} \le \|\varphi\|_{(A^{p_3}_{p_1,p_2})^*}\|f_1\|_{p_1}\|f_2\|_{p_2}$. By $L^{p_3'}(\mathbb{R}^n)^* = L^{p_3}(\mathbb{R}^n)$, we can find $h \in L^{p_3}(\mathbb{R}^n)$ such that $\|h\|_{p_3} = \|F_{f_1,f_2}\|_{(L^{p_3'})^*}$ and

$$F_{f_1,f_2}(g) = \int_{\mathbb{R}^n} h(x)\, g(x)\, dx \qquad (g \in L^{p_3'}(\mathbb{R}^n)).$$

Then we define the bilinear operator T by $T(f_1, f_2) = \check{h}$. By the definition of T, we have that

$$\|T(f_1, f_2)\|_{p_3} = \|\check{h}\|_{p_3} \le \|\varphi\|_{(A^{p_3}_{p_1,p_2})^*}\|f_1\|_{p_1}\|f_2\|_{p_2} \tag{3.4}$$

for all $f_1 \in L^{p_1}(\mathbb{R}^n)$ and $f_2 \in L^{p_2}(\mathbb{R}^n)$. We show that T commutes with translations. Since $[(\tau_x f_1) \otimes (\tau_x f_2)] *_2 g = [f_1 \otimes f_2] *_2 (\tau_x g)$, the equations

$$\varphi[(\tau_x f_1)\otimes(\tau_x f_2)*_2 g] = F_{\tau_x f_1,\tau_x f_2}(g) = \int_{\mathbb{R}^n} T(\tau_x f_1, \tau_x f_2)(y)\, g(-y)\, dy$$

and

$$\varphi[(f_1 \otimes f_2) *_2 (\tau_x g)] = F_{f_1,f_2}(\tau_x g) = \int_{\mathbb{R}^n} \tau_x T(f_1, f_2)(y)\, g(-y)\, dy$$

give $T(\tau_x f_1, \tau_x f_2) = \tau_x T(f_1, f_2)$. Since the bilinear operator T is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^{p_3}(\mathbb{R}^n)$ and commutes with translations, by [4, Proposition 3], we can find $m\in\mathcal{S}'(\mathbb{R}^{2n})$ such that $T(f_1, f_2)(x)=[\mathcal{F}^{-1}m]*[f_1\otimes f_2](x, x)$ for all $f_1,\ f_2 \in \mathcal{S}(\mathbb{R}^n)$. (3.4) implies $m \in M^{p_3}_{p_1,p_2}(\mathbb{R}^{2n})$. Finally, we prove that $\varphi = \varphi_m$. Let $f = \sum_{i=1}^{\infty}[f_{1,i} \otimes f_{2,i}] *_2 g_i \in A^{p_3}_{p_1,p_2}(\mathbb{R}^{2n})$. Since $\sum_{i=1}^{N}[f_{1,i} \otimes f_{2,i}] *_2 g_i \to f$ in $A^{p_3}_{p_1,p_2}(\mathbb{R}^{2n})$ as $N \to \infty$, by the continuity and linearity of φ and $\varphi[(f_1 \otimes f_2) *_2 g] = T_m(f_1, f_2) * g(0)$ for all $f_1 \in L^{p_1}(\mathbb{R}^n)$, $f_2 \in L^{p_2}(\mathbb{R}^n)$ and $g \in L^{p_3'}(\mathbb{R}^n)$, we see that

$$\begin{aligned}\varphi(f) &= \lim_{N\to\infty}\sum_{i=1}^{N}\varphi[(f_{1,i} \otimes f_{2,i}) *_2 g_i]\\ &= \lim_{N\to\infty}\sum_{i=1}^{N} T_m(f_{1,i}, f_{2,i}) * g_i(0) = \varphi_m(f).\end{aligned}$$

The proof is complete. □

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