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The invariant subspace structure of $L^2(\mathbb{T}^2)$ for certain von Neumann algebras

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Abstract. In this note, we study invariant subspaces of $L^2(\mathbb{T}^2)$ with respect to certain von Neumann algebras. We give a characterization of Beurling-type left-invariant subspaces of $L^2(\mathbb{T}^2)$. We also give a structure theorem of a non-trivial two-sided invariant subspace of $L^2(\mathbb{T}^2)$.

Key words: von Neumann algebras, invariant subspaces, Popovici's decomposition.

1. Introduction

Let $\mathbb{T}^2$ be the torus that is the cartesian product of two unit circles in $\mathbb{C}$. Let $L^2(\mathbb{T}^2)$ and $H^2(\mathbb{T}^2)$ be the usual Lebesgue and Hardy space in the torus $\mathbb{T}^2$, respectively.

A closed subspace $\mathfrak{M}$ of $L^2(\mathbb{T}^2)$ is said to be invariant if

$$z\mathfrak{M} \subset \mathfrak{M} \text{ and } w\mathfrak{M} \subset \mathfrak{M}.$$

There are many results about invariant subspaces of $L^2(\mathbb{T}^2)$ (cf. [2], [3], [5], [6], [7], [8], etc.). Especially, in [6] the invariant subspaces $\mathfrak{M}$ of $L^2(\mathbb{T}^2)$ of the Beurling form were characterized as the subspaces on which multiplication operators by z and w are doubly commuting. In [3] the structural theorem of the invariant subspaces of $L^2(\mathbb{T}^2)$ is given.

Let θ be an irrational number in $(0,1)$. We consider the unitary operators on $L^2(\mathbb{T}^2)$ satisfying:

$$L_z(z^m w^n) = z^{m+1}w^n, \quad L_w(z^m w^n) = e^{-2\pi i m\theta} z^m w^{n+1},$$
$$R_z(z^m w^n) = e^{-2\pi i n\theta} z^{m+1}w^n \text{ and } R_w(z^m w^n) = z^m w^{n+1},$$

where $z,\ w \in \mathbb{C}$ such that $|z| = |w| = 1$. By the simple calculation, we have

$$L_z L_w = e^{2\pi i\theta} L_w L_z \text{ and } R_w R_z = e^{2\pi i\theta} R_z R_w.$$

Let $\mathfrak{L}$ (resp. $\mathfrak{R}$) denote the von Neumann algebra generated by L_z and L_w

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(resp. R_z and R_w), then they are II_1-factors. Moreover, $\mathfrak{L}$ and $\mathfrak{R}$ are the commutant of each other. So $\mathfrak{L}$ and $\mathfrak{R}$ are called the left von Neumann algebra and the right von Neumann algebra, respectively. They are important classes of operator algebras. More generally, let U and V be unitary operators on a Hilbert space $\mathcal{H}$ satisfying $UV = e^{2\pi i\theta}VU$. The C^*-algebra $\mathfrak{A}_\theta$ generated by U and V is called an irrational rotation C^*-algebra. Then, up to unitary equivalence, there exists a unique $*$-representation π of $\mathfrak{A}_\theta$ onto $C^*(L_z, L_w)$ such that $\pi(U) = L_z$ and $\pi(V) = L_w$ (see [4]).

Let $\mathfrak{L}_+$ (resp. $\mathfrak{R}_+$) denote the σ-weakly closed subalgebra of $\mathfrak{L}$ (resp. $\mathfrak{R}$) generated by the positive powers of L_z and L_w (resp. R_z and R_w). A closed subspace $\mathfrak{M}$ of $L^2(\mathbb{T}^2)$ is said to be *left-invariant* (resp. *right-invariant*) if $\mathfrak{L}_+\mathfrak{M} \subset \mathfrak{M}$ (resp. $\mathfrak{R}_+\mathfrak{M} \subset \mathfrak{M}$). Moreover if $\mathfrak{L}\mathfrak{M} \subset \mathfrak{M}$ (resp. $\mathfrak{R}\mathfrak{M} \subset \mathfrak{M}$), then $\mathfrak{M}$ is said to be *left-reducing* (resp. *right-reducing*). If $\mathfrak{M}$ contains no left-reducing (resp. right-reducing) subspace, $\mathfrak{M}$ is said to be *left-pure* (resp. *right-pure*). If the smallest left-reducing (resp. right-reducing) subspace containing $\mathfrak{M}$ is $L^2(\mathbb{T}^2)$, $\mathfrak{M}$ is said to be *left-full* (resp. *right-full*). In this setting, we have an interest in the invariant subspace structure of $L^2(\mathbb{T}^2)$. If $\theta = 0$, then $\mathfrak{L}$ and $\mathfrak{R}$ are generated by the multiplication operators $\{M_f : f \in L^\infty(\mathbb{T}^2)\}$ on $L^2(\mathbb{T}^2)$, and the notion of invariant subspaces is that of usual invariant subspaces of $L^2(\mathbb{T}^2)$ satisfying $z\mathfrak{M} \subset \mathfrak{M}$ and $w\mathfrak{M} \subset \mathfrak{M}$.

In this paper we obtain necessary and sufficient conditions on the left-invariant subspace of $L^2(\mathbb{T}^2)$ to be of the form $VH^2(\mathbb{T}^2)$, where V is a unitary operator in $\mathfrak{R}$. A closed subspace $\mathfrak{M}$ of $L^2(\mathbb{T}^2)$ is said to be *two-sided invariant* if $\mathfrak{M}$ is both left-invariant and right-invariant. The concept of *two-sided pure* and *two-sided full* are defined similarly. We prove that a non-trivial two-sided invariant subspace of $L^2(\mathbb{T}^2)$ is two-sided pure and two-sided full.

Let $\mathfrak{M}$ be a non-trivial two-sided subspace of $L^2(\mathbb{T}^2)$. Put $U = (L_zL_w)|_{\mathfrak{M}}$ and $V = (R_zR_w)|_{\mathfrak{M}}$. Then $\mathfrak{M}$ is both U-invariant and V-invariant. The couple $W = (U, V)$ is a commuting pair of isometries on $\mathfrak{M}$. By Popovici's decomposition of $\mathfrak{M}$ with respect to W, we have

$$\mathfrak{M} = \mathfrak{M}_{uu} \oplus \mathfrak{M}_{us} \oplus \mathfrak{M}_{su} \oplus \mathfrak{M}_{ws}$$

(cf. [9, Theorem 2.8]). Then we show that W is a weak bi-shift on $\mathfrak{M}$, that is, $\mathfrak{M} = \mathfrak{M}_{ws}$.

In § 2 we study left-invariant subspaces of $L^2(\mathbb{T}^2)$. In particular we give the characterization of Beurling-type invariant subspaces of $L^2(\mathbb{T}^2)$.

In § 3 we introduce the notion of two-sided invariant, two-sided pure and two-sided full. We prove that a non-trivial two-sided invariant subspace of $L^2(\mathbb{T}^2)$ is two-sided pure and two-sided full. In § 4 we concern Popovici's decomposition of a non-trivial two-sided invariant subspace $\mathfrak{M}$ with respect to a commuting pair of isometries $W = ((L_zL_w)|_{\mathfrak{M}}, (R_zR_w)|_{\mathfrak{M}})$. We prove that W is a weak bi-shift on $\mathfrak{M}$.

2. Beurling-type invariant subspaces of $L^2(\mathbb{T}^2)$

In this section, we show certain properties of left-invariant subspaces of $L^2(\mathbb{T}^2)$. Let θ be an irrational number in $(0, 1)$. We consider the unitary operators on $L^2(\mathbb{T}^2)$ satisfying:

$$L_z(z^m w^n) = z^{m+1}w^n, \quad L_w(z^m w^n) = e^{-2\pi i m\theta} z^m w^{n+1},$$
$$R_z(z^m w^n) = e^{-2\pi i n\theta} z^{m+1} w^n \text{ and } R_w(z^m w^n) = z^m w^{n+1},$$

where $(z, w) \in \mathbb{T}^2$. By the simple calculation, we have

$$L_zL_w = e^{2\pi i\theta} L_wL_z \text{ and } R_wR_z = e^{2\pi i\theta} R_zR_w.$$

If we define $JA1 = A^*1$ for all $A \in \mathfrak{L}$, then J is a conjugate linear isometry from $L^2(\mathbb{T}^2)$ onto $L^2(\mathbb{T}^2)$ and

$$J(z^m w^n) = e^{-2\pi i mn\theta} z^{-m} w^{-n}.$$

Thus $JL_zJ = R_z{}^*$, $JL_wJ = R_w{}^*$. Let $\mathfrak{L}$ (resp. $\mathfrak{R}$) denote the von Neumann algebra generated by L_z and L_w (resp. R_z and R_w), then $J\mathfrak{L}J = \mathfrak{R}$ and $J\mathfrak{R}J = \mathfrak{L}$. If we define $\tau(A) = \langle A1, 1\rangle$ for all $A \in \mathfrak{L}$, then τ is a unique normal faithful tracial state on $\mathfrak{L}$. So we have

Proposition 2.1 *$\mathfrak{L}$ and $\mathfrak{R}$ are II_1-factors. Moreover, $\mathfrak{L} = \mathfrak{R}'$ and $\mathfrak{R} = \mathfrak{L}'$.*

Thus we shall call $\mathfrak{L}$ and $\mathfrak{R}$ the left von Neumann algebra and the right von Neuman algebra, respectively.

Definition 2.2 Let $\mathfrak{M}$ be a closed subspace of $L^2(\mathbb{T}^2)$. We shall say that $\mathfrak{M}$ is; *left-invariant*, if $\mathfrak{L}_+\mathfrak{M} \subset \mathfrak{M}$; *left-reducing*, if $\mathfrak{L}\mathfrak{M} \subset \mathfrak{M}$; *left-pure*, if $\mathfrak{M}$ contains no left-reducing subspace; *left-full*, if the smallest left-reducing subspace containing $\mathfrak{M}$ is all of $L^2(\mathbb{T}^2)$. The right-hand versions of these concepts are defined similarly.

Remark 2.3 Let $\mathfrak{M}$ be a closed subspace of $L^2(\mathbb{T}^2)$. Then $\mathfrak{M}$ is left-invariant if and only if $L_z\mathfrak{M} \subset \mathfrak{M}$ and $L_w\mathfrak{M} \subset \mathfrak{M}$, left-reducing if and only if there exists a projection $P \in \mathfrak{R}$ such that $\mathfrak{M} = PL^2(\mathbb{T}^2)$, left-pure if and only if $\bigcap_{m,n\geq 0} L_z^m L_w^n \mathfrak{M} = \{0\}$, and left-full if and only if $\overline{\bigcup_{m,n<0} L_z^m L_w^n \mathfrak{M}} = L^2(\mathbb{T}^2)$. The right-hand versions of this property hold similarly.

Our goal of this section is to characterize the Beurling-type left-invariant subspaces of $L^2(\mathbb{T}^2)$.

Lemma 2.4 *Let $\mathfrak{M}_0 = \sum \bigoplus_{m,n\geq 0} {L_z}^m {L_w}^n [q]$ for some norm one element q of $L^2(\mathbb{T}^2)$. Then there exists a unitary operator $V \in \mathfrak{R}$ such that $\mathfrak{M}_0 = VH^2(\mathbb{T}^2)$.*

Proof. Suppose that $\mathfrak{M}_0 = \sum \bigoplus_{m,n\geq 0} {L_z}^m {L_w}^n [q]$ for some norm one element q of $L^2(\mathbb{T}^2)$. Then we note that $\langle {L_z}^m {L_w}^n q, {L_z}^k {L_w}^l q\rangle = 0$ for all m, n, k, $l \in \mathbb{Z}$ such that $(m,n) \neq (k,l)$. Now we define an operator V by

$$V\left(\sum_{m,n\geq 0} \bigoplus \alpha_{m,n} {L_z}^m {L_w}^n 1\right) = \sum_{m,n\geq 0} \bigoplus \alpha_{m,n} {L_z}^m {L_w}^n q.$$

Then V is an isometry and $VL_z = L_z V$, $VL_w = L_w V$. Hence V is in the commutant of $\mathfrak{L}$. That is, V is in $\mathfrak{R}$. Since $\mathfrak{R}$ is a finite von Neumann algebra, V is unitary. Since $q = V1$, $\mathfrak{M}_0 = \sum \bigoplus_{m,n\geq 0} {L_z}^m {L_w}^n [V1] = VH^2(\mathbb{T}^2)$. This completes the proof. □

We note that subspaces of the form $VH^2(\mathbb{T}^2)$ can be represented:

$$VH^2(\mathbb{T}^2) = \sum_{m,n\geq 0} \bigoplus {L_z}^m {L_w}^n [V1] \tag{2.1}$$

where V is a partial isometry in the commutant $\mathfrak{R}$ of $\mathfrak{L}$. From above lemmas we now get the following Beurling-type theorem.

Let $\mathfrak{F}$ be a closed subspace of $L^2(\mathbb{T}^2)$. We shall say that $\mathfrak{F}$ is *a wandering subspace*, if $L_z^m L_w^n \mathfrak{F}$ and $L_z^{m'} L_w^{n'} \mathfrak{F}$ are orthogonal for any different (m,n) and (m',n') in $\mathbb{Z}^2$.

Theorem 2.5 *Let $\mathfrak{M}$ be a left-invariant subspace of $L^2(\mathbb{T}^2)$ and put $V_z = L_z|_{\mathfrak{M}}$, $V_w = L_w|_{\mathfrak{M}}$, $\mathfrak{F}_z = \mathfrak{M} \ominus V_z\mathfrak{M}$ and $\mathfrak{F}_w = \mathfrak{M} \ominus V_w\mathfrak{M}$. Then the following statements are equivalent:*

(1) *There exists a wandering subspace $\mathfrak{F}$ such that $\mathfrak{M} = \sum \bigoplus_{m,n\geq 0} {V_z}^m {V_w}^n \mathfrak{F}$,*

(2) *V_z, V_w are shift operatorson $\mathfrak{M}$ and $V_wV_z^* = e^{2\pi i\theta}V_z^*V_w$,*

(3) *V_w is a shift operator on $\mathfrak{M}$ and $\mathfrak{F}_w = \sum\bigoplus_{n\geq 0} V_z{}^n(\mathfrak{F}_z \cap \mathfrak{F}_w)$, or V_z is a shift operator on $\mathfrak{M}$ and $\mathfrak{F}_z = \sum\bigoplus_{m\geq 0} V_w{}^m(\mathfrak{F}_z \cap \mathfrak{F}_w)$,*

(4) *$\mathfrak{F}_z\cap\mathfrak{F}_w$ is a wandering subspace and $\mathfrak{M}=\sum\bigoplus_{m,n\geq 0}V_z{}^mV_w{}^n(\mathfrak{F}_z\cap\mathfrak{F}_w)$,*

(5) *$\mathfrak{M}$ is of the form $VH^2(\mathbb{T}^2)$, where V is a unitary operator in $\mathfrak{R}$.*

Proof. (1) ⇒ (2). Let $\mathfrak{F}$ be a wandering subspace such that $\mathfrak{M} = \sum\bigoplus_{m,n\geq 0} V_z{}^mV_w{}^n\mathfrak{F}$. We define

$$\mathfrak{F}_z{}' = \sum_{m\geq 0}\bigoplus V_w{}^m\mathfrak{F} \text{ and } \mathfrak{F}_w{}' = \sum_{n\geq 0}\bigoplus V_z{}^n\mathfrak{F}.$$

Since

$$\mathfrak{M} = \sum_{n\geq 0}\bigoplus V_z{}^n\mathfrak{F}_z{}' = \sum_{m\geq 0}\bigoplus V_w{}^m\mathfrak{F}_w{}',$$

V_z and V_w are shift operators. It follows that $\mathfrak{F}_z = \mathfrak{F}_z{}'$ and $\mathfrak{F}_w = \mathfrak{F}_w{}'$. Now we shall show $V_wV_z^* = e^{2\pi i\theta}V_z^*V_w$. If $x \in \mathfrak{M}$, then $x = \sum_{m\geq 0} V_z{}^mx_m$, where $x_m \in \mathfrak{F}_z$. Then we have

$$\begin{aligned} V_z{}^*V_wx &= \sum_{m\geq 0} V_z{}^*V_wV_z{}^mx_m = \sum_{m\geq 0} e^{-2\pi i m\theta}V_z{}^*V_z{}^mV_wx_m \\ &= \sum_{m\geq 1} e^{-2\pi i m\theta}V_z{}^{m-1}V_wx_m + V_z{}^*V_wx_0. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} V_wV_z{}^*x &= \sum_{m\geq 0} V_wV_z{}^*V_z{}^mx_m = \sum_{m\geq 1} V_wV_z{}^{m-1}x_m + V_wV_z{}^*x_0 \\ &= e^{2\pi i\theta}\sum_{m\geq 1} e^{-2\pi i m\theta}V_z{}^{m-1}V_wx_m + V_wV_z{}^*x_0. \end{aligned}$$

Since $V_z{}^*V_wx_0 = 0$ and $V_wV_z{}^*x_0 = 0$, we have $V_wV_z{}^* = e^{2\pi i\theta}V_z{}^*V_w$.
(2) ⇒ (3). We shall prove that $\mathfrak{F}_w = \sum\bigoplus_{n\geq 0} V_z{}^n(\mathfrak{F}_z\cap\mathfrak{F}_w)$. The second assertion can be obtained in the same way. First we notice that $\mathfrak{F}_w$ reduces V_z. Hence for all $n \geq 0$,

$$V_z{}^n(\mathfrak{F}_z \cap \mathfrak{F}_w) \subset \mathfrak{F}_w.$$

Evidently $\mathfrak{F}_z \cap \mathfrak{F}_w$ is a wandering subspace for V_z. Then we have

$$\sum_{n\geq 0}\bigoplus V_z{}^n(\mathfrak{F}_z \cap \mathfrak{F}_w) \subset \mathfrak{F}_w.$$

Let $\mathfrak{F}_0 = \mathfrak{F}_w \ominus V_z\mathfrak{F}_w$. If we prove that $\mathfrak{F}_0 \subset \mathfrak{F}_z \cap \mathfrak{F}_w$, then we get

$$\mathfrak{F}_w = \sum_{n\geq 0} \bigoplus {V_z}^n \mathfrak{F}_0 \subset \sum_{n\geq 0} \bigoplus {V_z}^n (\mathfrak{F}_z \cap \mathfrak{F}_w) \subset \mathfrak{F}_w,$$

which finishes this part of the proof. Suppose that $x \in \mathfrak{F}_0$. Then $x \perp V_z\mathfrak{F}_w$ and consequently ${V_z}^*x \perp \mathfrak{F}_w$. On the other hand $x \in \mathfrak{F}_w$. Since $\mathfrak{F}_w$ reduces V_z, we have ${V_z}^*x \in \mathfrak{F}_w$. This implies that ${V_z}^*x = 0$ and so $x \in \mathfrak{F}_z$. Since $x \in \mathfrak{F}_w$, our proof is complete.
(3) $\Rightarrow$ (4). Suppose that the first condition of (3) is fullfilled. Since V_w is a shift, we have $\mathfrak{M} = \sum_{m\geq 0} \bigoplus {V_w}^m \mathfrak{F}_w$. Then

$$\begin{aligned}\mathfrak{M} &= \sum_{m\geq 0} \bigoplus {V_w}^m \left(\sum_{n\geq 0} \bigoplus {V_z}^n (\mathfrak{F}_z \cap \mathfrak{F}_w) \right) \\ &= \sum_{m,n\geq 0} \bigoplus {V_z}^m {V_w}^n (\mathfrak{F}_z \cap \mathfrak{F}_w).\end{aligned}$$

In the second case the proof is the same.
(4) $\Rightarrow$ (1). (1) follows (4) immediately.
(5) $\Rightarrow$ (2). It is clear from (2.1).
(4) $\Rightarrow$ (5). Suppose

$$\mathfrak{M} = \sum_{m,n\geq 0} \bigoplus {V_z}^m {V_w}^n (\mathfrak{F}_z \cap \mathfrak{F}_w).$$

We shall now prove that $\mathfrak{F}_z \cap \mathfrak{F}_w$ is one-dimensional. Suppose $\dim(\mathfrak{F}_z \cap \mathfrak{F}_w) > 1$, and fix norm one orthogonal elements q_1, q_2 in $\mathfrak{F}_z \cap \mathfrak{F}_w$. Let

$$\mathfrak{M}_1 = \sum_{m,n\geq 0} \bigoplus {L_z}^m {L_w}^n [q_1] \text{ and } \mathfrak{M}_2 = \sum_{m,n\geq 0} \bigoplus {L_z}^m {L_w}^n [q_2].$$

By Lemma 2.4 there exists unitary operators U_1 and U_2 in $\mathfrak{R}$ such that

$$\mathfrak{M}_1 = U_1 H^2(\mathbb{T}^2) \text{ and } \mathfrak{M}_2 = U_2 H^2(\mathbb{T}^2).$$

Since $q_1 \perp q_2$, we have

$$U_1 H^2(\mathbb{T}^2) \perp U_2 H^2(\mathbb{T}^2).$$

Putting $U_0 = {U_1}^* U_2$, then U_0 is a unitary operator in $\mathfrak{R}$. Moreover we have

$$H^2(\mathbb{T}^2) \perp U_0 H^2(\mathbb{T}^2).$$

So we see that ${L_z}^m {L_w}^n 1 \perp U_0 H^2(\mathbb{T}^2)$ for all m, $n \in \mathbb{Z}$. Therefore we see

$$L^2(\mathbb{T}^2) \perp U_0 H^2(\mathbb{T}^2).$$

That is $U_0 H^2(\mathbb{T}^2) = \{0\}$, a contradiction. So we have norm one element q in $\mathfrak{F}_z \cap \mathfrak{F}_w$. Again from Lemma 2.4 we have $\mathfrak{M} = V H^2(\mathbb{T}^2)$ for some unitary operator $V \in \mathfrak{R}$. This completes the proof. □

3. Two-sided invariant subspaces of $L^2(\mathbb{T}^2)$

In this section we shall study about two-sided invariant subspaces of $L^2(\mathbb{T}^2)$.

Definition 3.1 Let $\mathfrak{M}$ be a closed subspace of $L^2(\mathbb{T}^2)$. We shall say that $\mathfrak{M}$ is; *two-sided invariant*, if $\mathfrak{M}$ is both left-invariant and right-invariant; *two-sided reducing*, if $\mathfrak{M}$ is both left-reducing and right-reducing, *two-sided pure*, if $\mathfrak{M}$ is both left-pure and right-pure; *two-sided full*, if $\mathfrak{M}$ is both left-full and right-full.

To prove the theorem about two-sided invariant subspaces of $L^2(\mathbb{T}^2)$, we need the following lemma.

Lemma 3.2 *If $\mathfrak{M}$ is a left-invariant (resp. right-invariant) subspace of $L^2(\mathbb{T}^2)$ and a right-reducing (resp. left-reducing) subspace of $L^2(\mathbb{T}^2)$, then $\mathfrak{M}$ is either $\{0\}$ or $L^2(\mathbb{T}^2)$.*

Proof. Let P be the projection with range $\mathfrak{M}$. Then since $\mathfrak{M}$ is right reducing, P belongs to $\mathfrak{L}$. Since $\mathfrak{M}$ is left-invariant, we have $L_z P L_z^* \leq P$. It is easy to see $L_z P L_z^* \sim P$. Since $\mathcal{L}$ is a finite von Neumann algebra, we have $L_z P L_z^* = P$, that is, $L_z P = P L_z$. Similarly, we have $L_w P = P L_w$. Hence P lies in $\mathfrak{L}'$. Therefore P belongs to the center of $\mathfrak{L}$. Since $\mathfrak{L}$ is a factor, P is either 0 or 1. This completes the proof. □

Remark 3.3 If $\theta = 0$, then the assumption of the above lemma is that $\mathfrak{M}$ is reducing ($z\mathfrak{M} = \mathfrak{M}$, $w\mathfrak{M} = \mathfrak{M}$). In this case $\mathfrak{M}$ is of the form $\chi_E L^2(\mathbb{T}^2)$.

Theorem 3.4 *A non-trivial two-sided invariant subspace of $L^2(\mathbb{T}^2)$ is two-sided pure and two-sided full.*

Proof. Let $\mathfrak{M}$ be a non-trivial two-sided invariant subspace of $L^2(\mathbb{T}^2)$. Put $\mathfrak{M}_\infty = \bigcap_{m,n \geq 0} L_z^m L_w^n \mathfrak{M}$ and let P_∞ be the projection from $L^2(\mathbb{T}^2)$

onto $\mathfrak{M}_\infty$. Then we have that $P_\infty \neq I$ and $\mathfrak{M}_\infty$ is right-invariant and left-reducing. From Lemma 3.2 we have $\mathfrak{M}_\infty = \{0\}$. Thus $\mathfrak{M}$ is left-pure.

The right-pureness is similarly proved by considering a projection from $L^2(\mathbb{T}^2)$ onto $\bigcap_{m,n\geq 0} R_z^m R_w^n \mathfrak{M}$. The left-fullness and the right-fullness is similarly proved by cosidering projections onto $\overline{\bigcup_{m,n<0} L_z^m L_w^n \mathfrak{M}}$ and onto $\overline{\bigcup_{m,n<0} R_z^m R_w^n \mathfrak{M}}$ respectively. This completes the proof. □

4. Popovici's decomposition

In this section we shall characterize two-sided invariant subspaces of $L^2(\mathbb{T}^2)$ by using Popovici's decomposition with respect to a bi-isometry (a commuting pair of isometries).

Definition 4.1 Let S be an isometry on $L^2(\mathbb{T}^2)$ and $\mathfrak{M}$ be a closed subspace of $L^2(\mathbb{T}^2)$. We shall say that $\mathfrak{M}$ is; *S-invariant*, if $S\mathfrak{M} \subset \mathfrak{M}$.

Let $\mathfrak{M}$ be a non-trivial two-sided invariant subspace of $L^2(\mathbb{T}^2)$. Put $U = (L_z L_w)|_{\mathfrak{M}}$ and $V = (R_z R_w)|_{\mathfrak{M}}$. Then we note that $\mathfrak{M}$ is both U-invariant and V-invariant. The couple $W = (U, V)$ is a bi-isometry on $\mathfrak{M}$.

By Popovici's decomposition of $\mathfrak{M}$ with respect to W, we have

$$\mathfrak{M} = \mathfrak{M}_{uu} \oplus \mathfrak{M}_{us} \oplus \mathfrak{M}_{su} \oplus \mathfrak{M}_{ws}$$

such that $W|_{\mathfrak{M}_{uu}}$ is a bi-unitary (that is, both $U|_{\mathfrak{M}_{uu}}$ and $V|_{\mathfrak{M}_{uu}}$ are unitary operators), $W|_{\mathfrak{M}_{us}}$ is a unitary-shift (that is, $U|_{\mathfrak{M}_{us}}$ is a unitary and $V|_{\mathfrak{M}_{us}}$ is a shift), $W|_{\mathfrak{M}_{su}}$ is a shift-unitary (that is, $U|_{\mathfrak{M}_{su}}$ is a shift and $V|_{\mathfrak{M}_{su}}$ is a unitary) and $W|_{\mathfrak{M}_{ws}}$ is a weak bi-shift (that is, $U|_{\cap_{i\geq 0} \ker V^* U^i}$, $V|_{\cap_{j\geq 0} \ker U^* V^j}$ and $(U|_{\mathfrak{M}_{ws}})(V|_{\mathfrak{M}_{ws}})$ are shift operators).

We have the following:

Theorem 4.2 *Let $\mathfrak{M}$ be a non-trivial two-sided invariant subspace of $L^2(\mathbb{T}^2)$. Then the couple $W = (U, V)$ is a weak bi-shift on $\mathfrak{M}$, that is, $\mathfrak{M} = \mathfrak{M}_{ws}$.*

Proof. Both U and V are unitary on $\mathfrak{M}_{uu}$, thus $\mathfrak{M}_{uu}$ is two-sided reducing by [5, Proposition 1]. By Lemma 3.2, we have that $\mathfrak{M}_{uu} = \{0\}$. Since $\mathfrak{M}$ is U-invariant, we have the Wold-type decomposition of $\mathfrak{M}$ with respect to U as follows;

$$\mathfrak{M} = \bigcap_{n\in\mathbb{Z}} U^n \mathfrak{M} \oplus \sum_{n\geq 0} \bigoplus U^n \mathfrak{F}^U,$$

where $\mathfrak{F}^U = \mathfrak{M} \ominus U\mathfrak{M}$. Define $\mathfrak{M}_u^U = \bigcap_{n\in\mathbb{Z}} U^n\mathfrak{M}$ and $\mathfrak{M}_s^U = \sum \bigoplus_{n\geq 0} U^n \mathfrak{F}^U$. Then it is clear that $\mathfrak{M}_u^U$ is right-invariant.

For each $n \in \mathbb{Z}$, we have

$$\begin{aligned} L_z(L_zL_w)^n &= L_z(e^{2\pi i\theta}L_wL_z)^n \\ &= e^{2\pi in\theta}(L_zL_w)^nL_z. \end{aligned}$$

Since $\mathfrak{M}$ is two-sided invariant, we have

$$\begin{aligned} L_z\mathfrak{M}_u^U &= \bigcap_{n\in\mathbb{Z}} L_z(L_zL_w)^n\mathfrak{M} \\ &= \bigcap_{n\in\mathbb{Z}} (L_zL_w)^nL_z\mathfrak{M} \\ &\subset \bigcap_{n\in\mathbb{Z}} U^n\mathfrak{M} \\ &= \mathfrak{M}_u^U. \end{aligned}$$

Similarly we see $L_w\mathfrak{M}_u^U \subset \mathfrak{M}_u^U$. On the other hand, for each $n \in \mathbb{Z}$, we have

$$\begin{aligned} L_z^*(L_zL_w)^n &= L_z^*(L_zL_w)(L_zL_w)^{n-1} \\ &= L_w(L_zL_w)^{n-1}. \end{aligned}$$

Thus we have

$$\begin{aligned} L_z^*\mathfrak{M}_u^U &= \bigcap_{n\in\mathbb{Z}} L_z^*(L_zL_w)^n \\ &= L_w \bigcap_{n\in\mathbb{Z}} (L_zL_w)^{n-1}\mathfrak{M} \\ &= L_w\mathfrak{M}_u^U \\ &\subset \mathfrak{M}_u^U. \end{aligned}$$

Moreover we have

$$\begin{aligned} L_w^*(L_zL_w)^n &= L_w^*(L_zL_w)(L_zL_w)^{n-1} \\ &= L_w^*(e^{2\pi i\theta}L_wL_z)(L_zL_w)^{n-1} \\ &= e^{2\pi i\theta}L_z(L_zL_w)^{n-1}. \end{aligned}$$

It follows $L_w^*\mathfrak{M}_u^U \subset \mathfrak{M}_u^U$. Thus $\mathfrak{M}_u^U$ is right-invariant and left-reducing. By Lemma 3.2 and the assumption, $\mathfrak{M}_u^U = \{0\}$. Similarly, if we consider the Wold-type decomposition $\mathfrak{M} = \mathfrak{M}_u^V \oplus \mathfrak{M}_s^V$ of $\mathfrak{M}$ with respect to V, then we have $\mathfrak{M}_u^V = \{0\}$. As in the proof of [9, Theorem 2.8], we have

$$\mathfrak{M}_{us} \subset \mathfrak{M}_u^U \cap \mathfrak{M}_s^V \text{ and } \mathfrak{M}_{su} \subset \mathfrak{M}_s^U \cap \mathfrak{M}_u^V.$$

It follows $\mathfrak{M}_{uu} \oplus \mathfrak{M}_{us} \oplus \mathfrak{M}_{su} = \{0\}$ and so $\mathfrak{M} = \mathfrak{M}_{ws}$. This completes the proof. □

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References

[1] Davidson K.R., *C^*-Algebras by example.* The Fields Institute for Research in Mathematical Sciences **6**, Amer. Math. Soc. (1996).

[2] Gaspar D. and Suciu N., *On invariant subspaces in the bitorus.* J. Operator Theory **30** (1933), 227–241.

[3] Ghatage P. and Mandrekar V., *On Beurling type invariant subspaces of $L^2(\mathbb{T}^2)$ and their equivalence.* J. Operator Theory **20** (1988), 31–38.

[4] Han D., *Wandering vectors for irrational rotation unitary systems.* Trans. Amer. Math. Soc. **350** (1998), 309–320.

[5] Ji G., Ohwada T. and Saito K.-S., *Certain invariant subspace structure of $L^2(\mathbb{T}^2)$.* Proc. Amer. Math. Soc. **126** (1998), 2361–2368.

[6] Mandrekar R., *The validty of Beurling theorems in polidisc.* Proc. Amer. Math. Soc. **103** (1988), 145–148.

[7] Nakazi T., *Certain invariant subspaces of H^2 and L^2 on a bidisc.* Canadian J. Math. **40** (1988), 1272–1280.

[8] Nakazi T., *Invariant subspaces in the bidisc and commutators.* J. Austral. Math. Soc. **56** (1994), 232–242.

[9] Popovici D., *A Wold-type decomposition for commuting isometric pairs.* Proc. Amer. Math. Soc. **132** (2004), 2303–2314.

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