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Some sequence spaces which include c_0 and c

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Abstract. It is first shown that a certain class of regular factorable matrices is equivalent to C, the Cesáro matrix of order one. As a corollary it is shown that the matrix A of [1] is equivalent to C. Then most of the results of [1] and [3] are extended to regular factorable matrices.

Key words: α dual, β dual, Cesáro summability, γ dual, factorable matrices, ℓ_p spaces, weighted mean matrices.

1. Introduction

Let A be an infinite matrix, c_0 and c denote, respectively, the space of null sequences and the space of convergent sequences. Define

$$c_A = \left\{x : \left\{\sum_k a_{nk}x_k\right\} \in c\right\},$$

$$c_{0_A} = \left\{x : \left\{\sum_k a_{nk}x_k\right\} \in c_0\right\}.$$

Let X and Y be sequence spaces. The notation $A \in (X : Y)$ will mean that $A\colon X \to Y$.

In a recent paper [1] it was shown that c_A and c_{0_A} are isomorphic to c and c_0, respectively, for a certain matrix A. The α-, β-, and γ-duals of c_A and c_{0_A} were computed, and necessary and sufficient conditions were obtained for a matrix $D \in (c_A : \ell_p)$ and $\in (c_A : c)$.

Corollary 2.2 of this paper shows that the matrix A of [1] is equivalent to C, the Cesáro matrix of order one.

Then most of the results of [1] and [3] are extended to the larger class of regular factorable matrices.

A discussion of some sequence spaces which include c and c_0 appears in [3].

A lower triangular matrix is called factorable if one can write each

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$a_{nk} = a_n b_k$, where a_n depends only on n and b_k depends only on k, $0 \leq k \leq n$. A triangle is a lower triangluar matrix with no zeros on the principal diagonal.

A matrix A is called regular if A is limit preserving over c. Define $t_n = \sum_{k=0}^{n} a_{nk}$. One of the conditions of regularity is that $\lim t_n = 1$. A matrix B is said to be stronger than a matrix A if $c_A \subset c_B$. If A and B are triangles, then $c_A \subset c_B$ iff BA^{-1} is regular. The symbol e means the sequence of all 1's, and $e^{(k)}$ denotes the coordinate sequence with a 1 in the k-th position and all other terms zero. Let $\{w_n\}$ denote any sequence. The notation $w_n \asymp O(1)$ means that $w_n = O(1)$ and that $1/w_n = O(1)$.

2. Matrices equivalent to $(C, 1)$

Theorem 2.1 *Let A be a regular factorable triangle with nonnegative entries satisfying*

$$(n+1)a_n b_n \asymp O(1) \tag{2.1}$$

and $\{b_n\}$ monotone. Then $c_A = c_C$.

Proof. For $k < n$,

$$\begin{aligned}(AC^{-1})_{nk} &= \sum_{j=k}^{n} a_{nj} c_{jk}^{-1} = a_{nk} c_{kk}^{-1} + a_{n,k+1} c_{k+1,k}^{-1} \\ &= a_n b_k (k+1) + a_n b_{k+1}(-1)(k+1) \\ &= (k+1) a_n (b_k - b_{k+1}). \\ (AC^{-1})_{nn} &= (n+1) a_n b_n.\end{aligned}$$

Thus, if we define $B = AC^{-1}$, then

$$b_{nk} = \begin{cases} (k+1)a_n(b_k - b_{k+1}), & 0 \leq k < n, \\ (n+1)a_n b_n, & k = n, \\ \quad 0, & k > n. \end{cases}$$

Case I. $\{b_n\}$ nonincreasing. Then

$$\sum_{k=0}^{n} |b_{nk}| = (n+1)a_n b_n + a_n \sum_{k=0}^{n-1} (k+1)(b_k - b_{k+1})$$

$$= (n+1)a_nb_n + a_n\left[\sum_{k=0}^{n-1}(k+1)b_k - \sum_{k=0}^{n-1}(k+1)b_{k+1}\right]$$

$$= (n+1)a_nb_n + a_n\left[b_0 - nb_n + \sum_{k=0}^{n-1}[(k+1)b_k - kb_k]\right]$$

$$= a_nb_n + a_nb_0 + a_n\sum_{k=0}^{n-1}b_k = a_nb_0 + t_n \to 1.$$

Therefore $\|B\| < \infty$, and the limit of the row sums is one. Also B has zero column limits. Therefore B is regular.

Case II. $\{b_n\}$ is nondecreasing. Then

$$\sum_{k=0}^{n}|b_{nk}| = (n+1)a_nb_n + a_n\sum_{k=0}^{n-1}(k+1)(b_{k+1} - b_k)$$

$$= (n+1)a_nb_n + a_n\left[\sum_{k=0}^{n-1}(k+1)b_{k+1} - \sum_{k=0}^{n-1}(k+1)b_k\right]$$

$$= (n+1)a_nb_n + a_nnb_n - a_nb_0 + a_n\sum_{k=0}^{n-1}(kb_k - (k+1)b_k)$$

$$= O(1) + o(1) - a_n\sum_{k=0}^{n-1}b_k \le O(1),$$

and again $\|B\| < \infty$ and the limit of the row sums of B exists. With $t := \{t_n\}$, $t = Be = AC^{-1}(e) = Ae$, so the limit of the row sums of B is one. Also B has zero column limits. Therefore B is regular.

To show that B^{-1} is regular, it will be sufficient to show that $\|B^{-1}\| < \infty$.

A direct calculation verifies that, for a factorable triangle,

$$a_{nk}^{-1} = \begin{cases} \dfrac{1}{a_nb_n}, & k = n, \\ -\dfrac{1}{a_{n-1}b_n}, & k = n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, for $k < n$,

$$(B^{-1})_{nk} = (CA^{-1})_{nk} = \sum_{i=k}^{n} c_{ni}a_{ik}^{-1} = c_{nk}a_{kk}^{-1} + c_{n,k+1}a_{k+1,k}^{-1}$$

$$= \frac{1}{n+1}\Big(\frac{1}{a_k b_k}\Big) + \frac{1}{n+1}\Big(-\frac{1}{a_k b_{k+1}}\Big)$$
$$= \frac{1}{(n+1)a_k}\Big(\frac{1}{b_k} - \frac{1}{b_{k+1}}\Big),$$

and

$$(CA^{-1})_{nn} = \frac{1}{a_n b_n (n+1)}.$$
$$\sum_{k=0}^{n} |b_{nk}^{-1}| = \frac{1}{a_n b_n (n+1)} + \sum_{k=0}^{n-1} \frac{1}{(n+1)a_k}\Big|\frac{1}{b_k} - \frac{1}{b_{k+1}}\Big|.$$

Case I. $\{b_n\}$ nondecreasing. Then

$$\frac{1}{n+1}\sum_{k=0}^{n-1}\frac{1}{a_k}\Big|\frac{1}{b_k} - \frac{1}{b_{k+1}}\Big|$$
$$= \frac{1}{n+1}\sum_{k=0}^{n-1}\frac{1}{a_k}\Big(\frac{1}{b_k} - \frac{1}{b_{k+1}}\Big)$$
$$= \frac{1}{n+1}\sum_{k=0}^{n-1}\frac{\sum_{j=0}^{k} b_j}{t_k}\Big(\frac{1}{b_k} - \frac{1}{b_{k+1}}\Big).$$

Since $\lim t_n = 1$, $\inf t_k > 0$. Therefore

$$\sum_{k=0}^{n} |b_{nk}^{-1}| = O(1) + \frac{O(1)}{n+1}\left[\sum_{k=0}^{n-1}\frac{\sum_{j=0}^{k} b_j}{b_k} - \sum_{k=0}^{n-1}\frac{\sum_{j=0}^{k} b_j}{b_{k+1}}\right]$$
$$= O(1) + \frac{O(1)}{n+1}\Bigg[\frac{b_0}{b_0} - \frac{1}{b_n}\sum_{j=0}^{n-1} b_j$$
$$+ \sum_{k=1}^{n-1}\frac{1}{b_k}\Bigg(\sum_{k=0}^{k} b_j - \sum_{j=0}^{k-1} b_j\Bigg)\Bigg]$$
$$\le O(1) + o(1) + \frac{O(1)}{n+1}\sum_{k=1}^{n-1} 1 = O(1).$$

Case II. $\{b_n\}$ nonincreasing. Then

$$\sum_{k=0}^{n} |b_{nk}^{-1}| = O(1) + \frac{1}{n+1}\sum_{k=0}^{n-1}\frac{\sum_{j=0}^{k} b_j}{t_k}\Big(\frac{1}{b_{k+1}} - \frac{1}{b_k}\Big)$$

$$= O(1) + \frac{O(1)}{n+1}\left[\sum_{k=0}^{n-1}\frac{\sum_{j=0}^{k} b_j}{b_{k+1}} - \sum_{k=0}^{n-1}\frac{\sum_{j=0}^{k} b_j}{b_k}\right]$$

$$= O(1) + \frac{O(1)}{n+1}\left[\sum_{j=0}^{n-1}\frac{b_j}{b_n} - \frac{b_0}{b_0} + \sum_{k=1}^{n-1}\frac{1}{b_k}\left(\sum_{j=0}^{k-1} b_j - \sum_{j=0}^{k} b_j\right)\right]$$

$$\leq O(1) + \frac{O(1)t_n}{(n+1)a_n b_n} = O(1),$$

and $\|B^{-1}\| < \infty$. □

A weighted mean matrix, denoted by $(\overline{N}, p)$, is a lower triangular matrix with entries p_k/P_n, where $\{p_k\}$ is a nonnegative sequence with $p_0 > 0$ and $P_n := \sum_{k=0}^{n} p_k$.

Corollary 2.1 *Let $A = (\overline{N}, p)$ with $\{p_n\}$ monotone and satisfying*

$$\frac{(n+1)p_n}{P_n} \asymp O(1). \tag{2.2}$$

Then $(\overline{N}, p)$ and C are equivalent.

Proof. Note that (2.2) implies (2.1). The result now follows from Theorem 2.1. □

Corollary 2.1 appears in [2].

Corollary 2.2 *Let A be a triangle with entries $a_{nk} = (1 + r^k)/(n + 1)$, $0 \leq k \leq n$, $0 < r < 1$. Then A is equivalent to C.*

Proof. Note that A is a factorable matrix with $a_n = 1/(n+1)$, $b_k = 1+r^k$, and $\{b_k\}$ is monotone decreasing.

$$(n+1)a_n b_n = (n+1)\frac{1+r^n}{(n+1)} \to 1.$$

Therefore A satisfies (2.1), and the result follows from Theorem 2.1. □

The matrix A of Corollary 2.2 is the matrix that appears in [1].

3. The sequence spaces c_A and c_{0_A}

Theorem 3.1 *Let A be a factorable triangle. Then the sequence spaces c_A and c_{0_A} are linearly isomorphic to c and c_0, respectively.*

Proof. From Example 7, page 76 of [5], $c_A \simeq c$. We shall now show that $A\colon c_{0_A} \to c_0$ is 1-1 and onto. Let $y \in c_0$. Consider $Ax = y$. Solving for x one obtains $x = A^{-1}y$. Since A and A^{-1} are triangles, the associativity of multiplication holds and $Ax = A(A^{-1}y) = y \in c_0$. Therefore $x \in c_{0_A}$ and A is onto. Since A is a triangle, it is clearly 1-1 on any domain. □

Theorem 2.2 of [1] is the special case of Theorem 3.1 with A as in Corollary 2.2.

For any lower triangular matrix A, and sequence x, $(Ax)_n := \sum_{k=0}^{n} a_{nk}x_k$, and $\lim_n x := \lim_n (Ax)_n$, if it exists.

Theorem 3.2 *Let A be a regular factorable matrix satisfying*

$$\sup_n \frac{1}{b_n}\left|\frac{1}{a_n} - \frac{1}{a_{n-1}}\right| = O(1). \tag{3.1}$$

Define $b^{(k)}$ by

$$b_n^{(k)} = \begin{cases} \dfrac{(-1)^{n-k}}{a_k b_n}, & k \le n \le k+1, \\ 0, & \text{otherwise.} \end{cases}$$

Then
(a) *$b^{(k)}$ is a basis for c_{0_A} and, any $x \in c_{0_A}$, has the unique representation*

$$x = \sum_k \lambda_k b^{(k)},$$

where $\lambda_k := (Ax)_k$.
(b) *The set $\{e, b^{(k)}\}$ forms a basis for c_A and each $x \in c_A$ has the unique representation*

$$x = \ell e + \sum_k [\lambda_k - \ell] b^{(k)}.$$

Proof. $b^{(k)} \in c_{0_A}$, since $Ab^{(k)} = e^{(k)} \in c_0$. Let $x \in c_A$. Define

$$x^{[m]} = \sum_{k=0}^{m} \lambda_k b^{(k)},$$

where $\lambda_k := (Ax)_k$. Then

$$Ax^{[m]} = \sum_{k=0}^{m} \lambda_k A b^{(k)} = \sum_{k=0}^{m} (Ax)_k e^{(k)}.$$

Therefore

$$\{A(x-x^{[m]})\}_i = \begin{cases} 0, & 0 \le i \le m, \\ (Ax)_i, & i > m. \end{cases}$$

Since $x \in c_{0_A}$, $Ax \in c_0$. Then there exists a positive integer m_0 such that $m \ge m_0$ implies that $|(Ax)_n| \le \epsilon/2$, and, for $n \ge m_0$,

$$\begin{aligned} \left\|x - x^{[m]}\right\|_{c_{0_A}} &= \sup_{n \ge m} |(Ax)_m| \le \sup_{n \ge m_0} |(Ax)_n| \\ &\le \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

To show uniqueness, suppose that there exists another representation

$$x = \sum_k \mu_k b^{(k)}.$$

Then

$$(Ax)_n = \sum_k \mu_k (Ab^{(k)})_n = \sum_k \mu_k b_n^{(k)} = \mu_n.$$

But $(Ax)_n = \lambda_n$. Therefore $\lambda_n = \mu_n$ and the respresentaton is unique.

Condition (3.1) guarantees that $x \in c$. Since $b^{(k)} \subset c_0$, $\{e, b^{(k)}\} \subset c_A$.

Let $x \in c_A$ and define $u = x - \ell t$, where $t := \{t_n\}$, $\ell = \lim_A x$. Then $\lim_A u = \lim_A x - \ell \lim t_n = 0$, and $u \in c_{0_A}$. From (a), u has a unique representation. Thus the stated representation for x is unique . □

Theorem 3.1 of [1] is the special case of Theorem 3.2, using the A of Corollary 2.2, since, for that A, $x_n = 1/(1 + r^n)$.

4. Duality results

Let X be a sequence space, ω the set of all sequences, ℓ_1 the space of all absolutely convergent series, cs the space of convergent series, and bs the space of bounded series. Then

$$\begin{aligned} X^\alpha &:= \{z \in \omega \,|\, \forall x \in X,\ zx \in \ell_1\} && (\alpha\text{-dual of } X), \\ X^\beta &:= \{z \in \omega \,|\, \forall x \in X zx \in cs\} && (\beta\text{-dual of } X), \\ X^\gamma &:= \{zx \in X \,|\, \forall x \in X, zx \in bs\} && (\gamma\text{-dual of } X). \end{aligned}$$

Theorem 4.1 *Let A be a factorable matrix. The α-dual of the spaces c_{0_A} and c_A is*

$$d_1 = \left\{ z \in \omega : \sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} \frac{(-1)^{n-k} z_n}{a_k b_n} \right| < \infty \right\}. \tag{4.1}$$

Proof. We shall need the following Lemma from [4].

Lemma 4.1 $A \in (c_0 : \ell_1)$ *iff*

$$\sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} a_{nk} \right| < \infty.$$

Let $z \in \omega$ and define B by

$$b_{nk} = \begin{cases} \dfrac{(-1)^{n-k} z_n}{a_k b_n}, & n-1 \le k \le n, \\ 0, & \text{otherwise.} \end{cases}$$

Then, with $y := Ax$,

$$z_n x_n = z_n \sum_{k=n-1}^{n} \frac{(-1)^{n-k} y_k}{a_k b_n} = (By)_n.$$

Therefore $zx \in \ell_1$ whenever $x \in c_{0_A} \cap c_A$ iff $By \in \ell_1$ whenever $y \in c_0$ or c. By Lemma 4.1, the condition becomes (4.1). □

The special case of Theorem 4.1 with A as defined in Corollary 2.2 is Theorem 4.3 of [1].

Theorem 4.2 *Let A be a factorable matrix and define d_i, $i = 2, 3, 4$, by*

$$d_2 = \left\{ z \in \omega : \sum_k \left| \frac{1}{a_k} \Delta\left(\frac{z_k}{b_k}\right) \right| < \infty \right\}, \tag{4.2}$$

$$d_3 = \left\{ z \in \omega : \left\{ \frac{z_n}{a_n b_n} \right\} \in \ell_\infty \right\}, \quad \textit{and} \tag{4.3}$$

$$d_4 = \left\{ z \in \omega : \left\{ \left(\frac{1}{a_k} - \frac{1}{a_{k+1}} \right) \frac{z_k}{b_k} \right\} \in cs \right\}. \tag{4.4}$$

Then $\{c_{0_A}\}^\beta = d_2 \cap d_3$ *and* $\{c_A\}^\beta = d_2 \cap d_4$.

Proof. Consider

$$\begin{aligned}
\sum_{k=0}^{n} z_k x_k &= \sum_{k=0}^{n}\left[\sum_{j=k-1}^{k}\frac{(-1)^{j-k}y_j}{a_j b_k}\right]z_k \\
&= \sum_{k=0}^{n}\frac{z_k}{b_k}\left[-\frac{y_{k-1}}{a_{k-1}}+\frac{y_k}{a_k}\right] \\
&= -\sum_{k=0}^{n}\frac{z_k y_{k-1}}{a_{k-1}b_k}+\sum_{k=0}^{n}\frac{y_k z_k}{a_k b_k} \\
&= \frac{y_n z_n}{a_n b_n}+\sum_{k=0}^{n-1}\Delta\Big(\frac{z_k}{b_k}\Big)\frac{y_k}{a_k} \\
&= (Ty)_n,
\end{aligned}$$

where

$$t_{nk} = \begin{cases} \dfrac{1}{a_k}\Delta\Big(\dfrac{z_k}{b_k}\Big), & 0 \le k < n, \\ \dfrac{z_n}{a_n b_n}, & k = n, \\ 0, & k > n. \end{cases}$$

Thus $zx \in cs$ whenever $x \in c_{0_A}$ iff $Ty \in c$ whenever $y \in c_0$; i.e., T must have finite norm and column limits.

$$\|T\|_\infty = \sup_n \left[\sum_{k=0}^{n-1}\frac{1}{a_k}\Big|\Delta\Big(\frac{z_k}{b_k}\Big)\Big| + \Big|\frac{z_n}{a_n b_n}\Big|\right] < \infty,$$

which implies d_2 and d_3. Since the converse is trivial, $\{c_{0_A}\}^\beta = d_2 \cap d_3$.

The condition $zx \in cs$ whenever $x \in c_A$ iff $Ty \in c$ whenever $y \in c$; i.e., T must satisfy the Silverman-Toeplitz conditions. That T has finite norm implies condition d_2. Since the sum of the column limits of T must exist we have

$$\sum_k \frac{1}{a_k}\Delta\Big(\frac{z_k}{b_k}\Big) < \infty.$$

But

$$\sum_k \frac{1}{a_k}\Delta\Big(\frac{z_k}{b_k}\Big) = \sum_k \frac{1}{a_k}\frac{z_k}{b_k} - \sum_k \frac{1}{a_k}\frac{z_{k+1}}{b_{k+1}}$$

$$= \frac{z_0}{a_0 b_0} + \sum_{k=1}^{\infty} \Big(\frac{1}{a_k} - \frac{1}{a_{k+1}} \Big) \frac{z_k}{b_k}.$$

Since the row sums of T exist, condition d_4 satisfied.

Note that conditions d_2 and d_4 imply d_3. Therefore the converse is true, and $\{c_A\}^\beta = d_2 \cap d_4$. □

Theorem 4.4 of [1] is the special case of Theorem 4.2 with A as in Corollary 2.2.

Theorem 4.3 *Let A be a factorable matrix. Then $\{c_{0_A}\}^\gamma = \{c_A\}^\gamma = d_2 \cap d_3$.*

Proof. From the proof of Theorem 4.2, $zx \in bs$ for each $x \in c_{0_A}$ iff Ty is bounded for each $y \in c_0$; i.e., $T\colon c_0 \to m$; i.e., T has finite norm. (See, e.g., [4].)

Also $zx \in bs$ for each $x \in c_A$ iff Ty is bounded for each $y \in c$; i.e., $T\colon c \to m$; i.e., T has finite norm. (See, e.g., [4].)

The norm of T being finite is equivalent to conditions d_2 and d_3. □

5. Mappings with domain c_A

Theorem 5.1 *Let A be a regular factorable matrix, B an infinite matrix. Then $B \in (c_A : \ell_p)$ iff*

(i) *For $1 \le p < \infty$,*

$$\sup_{F \in \mathcal{F}} \sum_n \Big| \sum_{k \in \mathcal{F}} \tilde{a}_{nk} \Big|^p < \infty, \tag{5.1}$$

$$\sum_k |\tilde{a}_{nk}| < \infty, \tag{5.2}$$

$$\Big\{ \frac{b_{nk}}{a_k b_k} \Big\} \in cs \quad \text{for each} \quad n. \tag{5.3}$$

(ii) *For $p = \infty$, condition (5.3) and*

$$\sup_n \sum_k |\tilde{a}_{nk}| < \infty \tag{5.4}$$

must be satisfied, where $\tilde{A}$ is as defined in the proof.

Proof. Suppose that $B \in (c_A : \ell_p)$. Then there exists a positive constant K such that

$$\|Bx\|_{\ell_p} \le K\|x\|_{c_A}, \qquad \text{for each} \quad x \in c_A. \tag{5.5}$$

Inequality (5.5) is also satisfied for any sequence $x = \sum_{k\in\mathcal{F}} b^{(k)}$ as defined in Theorem 3.2. Thus, for any $F \in \mathcal{F}$,

$$\|Bx\|_{\ell_p} = \left(\sum_n \left|\sum_{k\in\mathcal{F}} \tilde{a}_{nk}x_k\right|^p\right)^{1/p} \le K\|x\|_{c_A},$$

and (5.1) is necessary, where $\mathcal{F}$ denotes the collection of all finite subsets of $\mathbb{N}$.

$$\begin{aligned}
\sum_{k=0}^{m} b_{nk}x_k &= \sum_{k=0}^{m} b_{nk}\left[\sum_{j=k-1}^{k} \frac{(-1)^{k-j}y_j}{a_{j-1}b_j}\right] \\
&= \sum_{k=0}^{m} b_{nk}\Big(-\frac{y_{k-1}}{a_{k-1}b_k} + \frac{y_k}{a_kb_k}\Big) \\
&= -\sum_{k=0}^{m} \frac{b_{nk}y_{k-1}}{a_{k-1}b_k} + \sum_{k=0}^{m} \frac{y_kb_{nk}}{a_kb_k} \\
&= \frac{y_mb_{nm}}{a_mb_m} + \sum_{k=0}^{m-1} \Delta_k\Big(\frac{b_{nk}}{b_k}\Big)\frac{y_k}{a_k} \\
&= \sum_{k=0}^{m} \tilde{a}_{nk}y_k,
\end{aligned}$$

where

$$\tilde{a}_{nk} = \begin{cases} \Delta_k\Big(\dfrac{b_{nk}}{b_k}\Big)\dfrac{1}{a_k}, & 0 \le k < n, \\ \dfrac{b_{nn}}{a_nb_n}, & k = n, \\ 0, & k > n. \end{cases}$$

Since B applies to c_A, the necessity of conditions (5.2) and (5.3) is trivial.

Suppose that conditions (5.1)–(5.3) hold. Let $x \in c_A$. Then $\{b_{nk}\}_{k\in\mathbb{N}} \in \{c_A\}^\beta$. Hence Ax exists. Define $D = \tilde{A}$. Since (5.1) is satisfied for D, $D \in (c : \ell_p)$.

Note that

$$\sum_{k=0}^{m} b_{nk}x_k = \sum_{k=0}^{m-1} \Delta_k\Big(\frac{b_{nk}}{b_k}\Big)\frac{y_k}{a_k} + \frac{b_{mm}y_m}{a_m b_m}. \tag{5.6}$$

Since conditions (5.3) and (5.4) hold, it follows, as in the proof of Theorem 4.2 that

$$\Big\{\frac{b_{mm}}{a_m b_m}\Big\}_m \in c_0.$$

Taking the limit of (5.5) as $m \to \infty$ yields

$$\sum_{k=0}^{\infty} b_{nk}x_k = \sum_{k=0}^{\infty} \tilde{a}_{nk}y_k. \tag{5.7}$$

Thus

$$\|Bx\|_{\ell_p} = \|Dy\|_{\ell_p} < \infty.$$

Therefore $B \in (c_A : \ell_p)$.

Part (ii) is proved in a similar manner. □

Theorem 5.2 *Let A be a regular factorable matrix. Then an infinite matrix $B \in (c_A : c)$ iff (5.3) and (5.4) hold,*

$$\lim_n \tilde{a}_{nk} = \alpha_k \quad \text{for each} \quad k \in \mathbb{N}, \tag{5.8}$$

and

$$\lim_n \sum_k \tilde{a}_{nk} = \alpha. \tag{5.9}$$

Proof. Suppose that B satisfies (5.3), (5.4), (5.7), and (5.8). Let $x \in c_A$. Then $y = Ax$ exists and $\lim y_n = \ell$ for some number ℓ.

From (5.7) and (5.4),

$$\sum_{j=0}^{k} |\alpha_j| \le \sup_n \sum_j |\tilde{a}_{nj}| < \infty.$$

Therefore $\{\alpha_k\} \in \ell_1$.

From (5.6),

$$\sum_k b_{nk}x_k = \sum_k \tilde{a}_{nk}y_k$$

$$= \sum_k \tilde{a}_{nk}(y_k - \ell) + \ell \sum_k \tilde{a}_{nk}.$$

Therefore

$$\lim_n (Bx)_n = \sum_k \alpha_k (y_k - \ell) + \ell\alpha,$$

and $B \in (c_A : c)$.

Conversely, suppose that $B \in (c_A : c)$. Since $c \subset \ell_\infty$, it follows from Theorem 5.1 that (5.4) and (5.5) hold. Define $\{x^{(k)}\} \in c_A$ by

$$x_n^{(k)} = \begin{cases} \dfrac{(-1)^{n-k}}{a_k b_n}, & k \le n \le k+1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $Ax^{(k)} = \{\tilde{a}_{nk}\} \in c$ for each k.

Now set $x = e$ in (5.6) to obtain

$$Bx = \left\{ \sum_k \tilde{a}_{nk} \right\}_n \in c. \qquad \square$$

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