Hokkaido Mathematical Journal Vol. 35 (2006) p. 497–545

Propagation of microlocal regularities in Sobolev spaces to solutions of boundary value problems for elastic equations

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(Received August 9, 2004; Revised May 17, 2006)

Abstract. We study propagation of microlocal regularities in the Sobolev space of solutions to boundary value problems for the isotropic elastic equation. We assume that the solutions microlocally belong to the Sobolev space of order s on the incident generalized bicharacteristic to the boundary. Then we discuss that whether the solutions have the same microlocal regularities in the Sobolev space on the reflected generalized bicharacteristic or not. Our results depend on the condition that how the incident generalized bicharacteristic attaches to the boundary. In this paper we only consider the boundary value problems for the isotropic elastic equation, however our method is valid for these of higher order hyperbolic equations and generalized elastic equations.

Key words: elastic equation, propagation of sigularities, Sobolev space.

1. Introduction

It is well known that a solution u of $Pu = f$, where P is a strictly hyperbolic single differential operator of order m, has the following property on a propagation of regularities: Under the assumption that f microlocally belongs to the Sobolev space H_s on a null bicharacteristic strip γ defined from the principal symbol of P, if the solution u microlocally belongs to H_{s+m-1} at a point on the null bicharacteristic strip γ, then the solution u has the same property at all points on γ (see Theorem 2.1 of Chapter VI in [15]). If we consider the boundary value problem for P, then the behavior of γ at the boundary point is very complicate depending on the shape of the boundary. In Definition 3.1 of [12] (see also Definition 9.1) they define generalized bicharacteristics for the boundary value problem and show that a solution of the boundary value problem with the condition $(2.2)_\pm$ in [12] for a strictly hyperbolic second order single differential operator has the same property on a propagation of microlocal regularities in C^∞ space along generalized bicharacteristics (see Theorem 5.10 in [12]).

2000 Mathematics Subject Classification : 35L55, 58J47.

The purpose of this paper is to study that in the Sobolev space H_s a solution of boundary value problem for a hyperbolic system whose characteristic roots have constant multiplicities has the same property. One of the important and typical boundary value problems of such hyperbolic operators is the elastic equation, that is,

$$\partial^2 u/\partial t^2 - (\lambda+\mu)\,\mathrm{grad}(\mathrm{div}\, u) - \mu\,\Delta u = f \qquad \text{in} \quad \mathbf{R}\times\Omega, \tag{1}$$

$$Bu = g \qquad \text{on} \quad \mathbf{R}\times\partial\Omega, \tag{2}$$

where $u = {}^t(u_1,\ldots,u_n)$ is the displacement, λ and μ are Lamé constants such that $\lambda+2\mu>0$, $\mu>0$ and $\lambda\neq 0$. Here Ω is an open set in $\mathbf{R}^n$ $(n\geq 2)$ with the smooth boundary $\partial\Omega$. The boundary condition is the Dirichlet condition $Bu = u$ or the free boundary condition

$$(Bu)_j = \sum_{i=1}^{n} \nu_i \sigma_{ij}(u) \qquad (j=1,\ldots,n), \tag{3}$$

where

$$\sigma_{ij}(u) = \lambda(\mathrm{div}\, u)\delta_{ij} + \mu(\partial u_i/\partial x_j + \partial u_j/\partial x_i) \tag{4}$$

and $\nu = (\nu_1,\ldots,\nu_n)$ is the unit normal vector to $\partial\Omega$.

In this paper we shall show that a solution of (1) and (2) has the property on propagations of microlocal regularities in the Sobolev space H_s along generalized bicharacteristics. Here we only consider the problem (1) and (2). However we remark that our method is valid for boundary value problems of higher order single hyperbolic differential operators with constant multiple characteristic roots considered in [4] and for these of generalized elastic equations appeared in [14]. In particular if we assume that Lamé constants λ and μ are functions of x satisfying the same condition in (1), all theorems in this paper are hold without any changes. We also remark that a generalized condition of $(2.2)_\pm$ in [12] appears in (41) of Section 7 in this paper.

In the previous paper [19] we consider the same problem in C^∞ category. However in this paper we are interested in one of the following problems: Let γ_+ be one of null bicharacteristic strips defined from the elastic equation belonging to $T^*(\mathbf{R}\times\Omega)\setminus 0$ and incoming to the boundary, and γ_- be the generalized bicharacteristic defined from γ_+ after γ_+ touches the boundary, which means that if γ_+ transversely hits the boundary, then γ_- is the reflected null bicharacteristic strip of γ_+. We assume that a solution of (1) and (2) microlocally belongs to H_s at a point of γ_+ and that date f and

g belong to the suitably good the Sobolev space. Then does the solution microlocally belong to the same space H_s at points on γ_-, or does the solution lose the microlocal regularity in the Sobolev space on γ_-? The answers depend on the condition that how γ_+ attaches to the boundary, and are stated as theorems of this paper. In the hyperbolic case, which means that γ_+ transversely hits the boundary, a solution of (1) and (2) does not lose the regularity on γ_- (see Theorem 4.5). In diffractive case, which is defined in the first part of Section 6, the statement ii) of Theorem 6.5 says that a solution for the Dirichlet boundary condition B loses $1/2$ the regularity on γ_-. We believe that this result comes from technical problems to prove the theorem. However the statement i) in Theorem 6.5 says that a solution for the free boundary condition B loses more $1/3$ the regularity on γ_-. We believe that this phenomenon is natural, because the free boundary condition does not satisfy the mathematically good condition at a diffractive point, that is the uniform Lopatinski condition. Thus there is a possibility that a solution of (1) and (2) loses microlocal regularities.

In the Sobolev space analogue problems are considered in [2] and [9]. In [2] they consider a microlocal regularity theorem of solutions to boundary value problems for second order single hyperbolic differential operators. Under the regularity conditions on the boundary of the solution and their first order derivatives, they obtain the microlocal regularity of solutions near a point belonging to $\Sigma^{2k,+} \cup \Sigma^{2k+1}$ (see the forward part of Lemma 9.6 on the definition of Σ^k). However it seems very difficult to extend their method to hyperbolic systems. In Appendix B of [9] for the elastic equations with the Dirichlet boundary condition they get similar theorems to these of this paper. However they only consider S waves and assume the strong condition that the divergence of the solution is regular in $\Omega \times \mathbf{R}$. Since they essentially use the special form of the elastic equation and the Dirichlet condition, it is also difficult to analyze general reflective phenomena appeared in our paper by their method.

We may assume that 0 belongs to $\partial\Omega$ and $\partial\Omega$ is defined by the equation $x_n - g(x') = 0$ in a neighbourhood U_1 of $0 \in \mathbf{R}^n$, where $(\operatorname{grad} g)(0) = 0$. By the coordinate transform; $y_0 = t$, $y_j = x_j$ $(j = 1, \ldots, n-1)$, $y_n = x_n - g(x')$ the problem (1) and (2) is reduced to the following one:

$$L(y, D_y)u = f \quad \text{in} \quad U_0 \cap \{y_n > 0\}, \tag{5}$$

$$B_0(y, D_y)u = g \quad \text{on} \quad U_0 \cap \{y_n = 0\}. \tag{6}$$

Here $U_0 = [-c, c] \times U_1$ for $c > 0$ and the principal symbol $L_2(y, \eta)$ of $L(y, D_y)$ is

$$\eta_0^2 E_n - (\lambda+\mu)(\bar{\eta}+G\eta_n)^t(\bar{\eta}+G\eta_n) - \mu|\bar{\eta}+G\eta_n|^2 E_n,$$

where $G = {}^t(-(\operatorname{grad} g)(y), 1)$, $\bar{\eta} = {}^t({}^t\eta'', 0) = {}^t(\eta_1, \ldots, \eta_{n-1}, 0)$, E_n is the $n \times n$ identity matrix. If B in (2) is the free boundary condition, then the principal symbol of $B_0(y, D_y)$ is

$$\lambda G^t(\bar{\eta} + G\eta_n) + \mu G \cdot (\bar{\eta} + G\eta_n) + \mu(\bar{\eta} + G\eta_n)^t G, \tag{7}$$

where $\cdot$ means the inner product in $\mathbf{R}^n$. From Lemma 1.1 of [19]

$$\operatorname{Det} L_2(y, \eta) = (\eta_0^2 - \mu|\bar{\eta}+G\eta_n|^2)^{n-1}(\eta_0^2 - (\lambda+2\mu)|\bar{\eta}+G\eta_n|^2). \tag{8}$$

Since $\eta_0^2 - \rho|\bar{\eta} + G\eta_n|^2 = -\rho|G|^2\{(\eta_n - a(y, \eta'))^2 + r_\rho(y, \eta_0, \eta'')\}$, where

$$a(y, \eta') = \eta'' \cdot (\operatorname{grad} g)(y)/|G|^2 \tag{9}$$

and

$$r_\rho(y, \eta_0, \eta'') = \big((\rho|\eta''|^2 - \eta_0^2)|G|^2 - \rho(\eta'' \cdot (\operatorname{grad} g(y)))^2\big)/\rho|G|^4, \tag{10}$$

we have the following five cases;

$$\text{i) } r_{\lambda+2\mu} \geq r_\mu > 0, \quad \text{ii) } r_{\lambda+2\mu} > 0 > r_\mu, \quad \text{iii) } 0 > r_{\lambda+2\mu} > r_\mu$$
$$\text{iv) } r_{\lambda+2\mu} > r_\mu = 0, \quad \text{v) } 0 = r_{\lambda+2\mu} > r_\mu.$$

In Section 2 we shall state function spaces used in this paper. In Section 3 we shall show Theorem 3.1 on a propagation of regularities in the interior of the domain. Since $L(y, D_y)$ is a system and from (8) all characteristic roots of $L_2(y, D_y)$ are not simple, we need to consider propagations of regularities in the interior of the domain. In Section 4 we shall consider hyperbolic problems, which is the case iii), and state Theorem 4.5 on reflection phenomena of regularities in the the Sobolev space. In Section 5 we shall consider the problem near elliptic points, which are the case i) and ii). Theorem 5.1 for the case ii) is not complicate. However in the case i) the existence of Rayleigh waves makes difficulties to propagations of regularities. This fact is stated in the later half of Section 5. In Section 6 we shall consider aq propagation of regularities near a diffractive point ρ_0 which is defined by the conditions $r_\rho(\rho_0) = 0$ and $\{\eta_n - a, r_\rho\}(\rho_0) < 0$. In [20] we construct a microlocal parametrix near the diffractive point ρ_0

for the boundary value problem (5) and (6). Making use of the parametrix, we shall show Theorem 6.5 and Theorem 6.6 on propagations of regularities near diffractive points.

After section 7 we shall analyze the problem near glancing point ρ_0, that is, $r_\rho(\rho_0) = 0$. One of the aims in Section 7 and 8 is to show the corresponding theorem to Theorem 2.3 in [12] for our problem in the Sobolev space. However their argument to prove the theorem does not work in the Sobolev space. So we shall improve and expand the argument used in [11], where they consider simple boundary conditions for single second order differential operators. In Section 7 as preliminaries we consider a boundary value problem of a first order system, which is the microlocally reduced form of (5) and (6) near a glancing point ρ_0. The key theorem on propagations of regularities is stated in Theorem 7.1, which is proved in Section 8. In Section 9 we shall state theorems on propagations of regularities near gliding points to solutions of our considered boundary value problem, which are Theorem 9.5 and Theorem 9.7.

2. Function spaces

In this paper we use the function spaces introduced in Chapter II of [5]. Let $H_{(m,s)}(\mathbf{R}^{n+1})$ be the function space defined in Definition 2.3.1 of [5]. Its localized space $H^{\text{loc}}_{(m,s)}(\mathbf{R}^{n+1})$ is denoted in Definition 2.3.1 of [5]. In Section 3 we shall use this space as $k_{(m,s)}(\xi) = (1+|\xi|^2)^{m/2}(1+|\xi'|^2)^{s/2}$, where ξ is the dual variable of $x = (x_0, \ldots, x_n)$ as $x_0 = t$ and $\xi' = (\xi_1, \ldots, \xi_n)$. Let $a(x, D_x)$ be a pseudodifferential operator of order p on $\mathbf{R}^{n+1}$ and $b(x, D_{x'})$ be a pseudodifferential operator of order q on $\mathbf{R}^n$ with a parameter x_0. Then we have the following properties, which are proved by the same argument to one used in the proof of Proposition 2.5 in [1]:

Proposition 2.1 *We have the following two statements*:
i) $a(x, D_x)$ *is a continuous linear operator from* $H_{(m,s)}(\mathbf{R}^{n+1}) \cap \mathcal{E}'(\mathbf{R}^{n+1})$ *to* $H^{\text{loc}}_{(m-p,s)}(\mathbf{R}^{n+1})$.
ii) $b(x, D_{x'})$ *is a continuous linear operator from* $H_{(m,s)}(\mathbf{R}^{n+1}) \cap \mathcal{E}'(\mathbf{R}^{n+1})$ *to* $H^{\text{loc}}_{(m,s-q)}(\mathbf{R}^{n+1})$.

After Section 4 we shall use the function space $H_{(m,s)}(\bar{\mathbf{R}}^{n+1}_+)$ defined in Definition 2.5.1 of [5] as $k_{(m,s)}(\eta) = (1+|\eta|^2)^{m/2}(1+|\eta'|^2)^{s/2}$, where $\bar{\mathbf{R}}^{n+1}_+ = \{(y_0, y_1 \cdots y_n) \in \mathbf{R}^{n+1};\ y_n \geq 0\}$, η is the dual variable of y appeared in (5) and $\eta' = (\eta_0, \dots, \eta_{n-1})$. Its localized space is denoted by $H^{\mathrm{loc}}_{(m,s)}(\bar{\mathbf{R}}^{n+1}_+)$. Let $b(y, D_{y'})$ be a pseudodifferential operator of order q with a smooth parameter y_n. Then from Theorem 2.5.1 in [5] and the argument used in the proof of Proposition 2.5 in [1]. We have the following:

Proposition 2.2 *The pseudodifferential operator $b(y, D_{y'})$ is a continuous linear operator from $H^c_{(m,s)}(\bar{\mathbf{R}}^{n+1}_+)$ to $H^{\mathrm{loc}}_{(m,s-q)}(\bar{\mathbf{R}}^{n+1}_+)$. Here by $H^c_{(m,s)}(\bar{\mathbf{R}}^{n+1}_+)$ we mean the set of all $u \in \mathcal{D}'(\mathbf{R}^{n+1}_+)$ such that there exists a distribution $U \in H_{(m,s)}(\mathbf{R}^{n+1}) \cap \mathcal{E}'(\mathbf{R}^{n+1})$ with $U = u$ in $\mathbf{R}^{n+1}_+ = \{y \in \mathbf{R}^{n+1};\ y_n > 0\}$.*

For simplicity we denote $H_{(m,0)}(\Omega)$ by $H_m(\Omega)$, where $\Omega = \mathbf{R}^{n+1}$ or $\bar{\mathbf{R}}^{n+1}_+$.

The composition of $a(x, D_x)$ and $b(x, D_{x'})$ appeared in Proposition 2.1 is well defined, if $WF(a(x, D_x)) \subset \{(x, \xi) \in T^*(\mathbf{R}^{n+1}) \setminus 0;\ |\xi_0| < C|\xi'|\}$. We have the following

Proposition 2.3 *Let $a(x, D_x)$ be a pseudodifferential operator of order p in $\mathbf{R}^{n+1}$ and $b(x, D_{x'})$ be a pseudodifferential operator of order q in $\mathbf{R}^n$. If $WF(a(x, D_x))$ satisfies the above condition, then the composition $a(x, D_x) \circ b(x, D_{x'})$ and $b(x, D_{x'}) \circ a(x, D_x)$ are pseudodifferential operators in $\mathbf{R}^{n+1}$ of order $p + q$, and their symbols have the same asymptotic expansions to these of usual composition formulas of pseudodifferential operators in $\mathbf{R}^{n+1}$.*

The above statement is proved by the same argument to one of Theorem 4.3 in Chapter II of [16].

Let $P(y, D_y)$ be a pseudodifferential operator with a form

$$A(y) D^p_{y_n} + \sum_{j=0}^{p-1} a_j(y, D_{y'}) D^j_{y_n},$$

where $A(y)$ is not zero and $a_j(y, D_{y'})$ is a properly supported pseudodifferential operator of order $p - j$. We have the following

Proposition 2.4 *Let us consider the equation $P(y, D_y)u = f$ in $\{y_n > 0\}$, where $f \in H_{(m,s)}(\bar{\mathbf{R}}^{n+1}_+)$ and $u \in H_{(m_1,s_1)}(\bar{\mathbf{R}}^{n+1}_+)$. We assume that there exists properly supported pseudodifferential operator $\phi(y, D_{y'})$ of order 0, which is elliptic at $(0, \eta^{0\prime})$, such that $\phi(y, D_{y'})f \in H_m(\bar{\mathbf{R}}^{n+1}_+)$ and $\phi(y, D_{y'})u \in H_r(\bar{\mathbf{R}}^{n+1}_+)$ with $r \le m + p$. Then for any properly supported pseudodifferential operator $\phi_0(y, D_{y'})$ such that the essential support Γ of the symbol $\phi_0(y, D_{y'})$ is contained in the set consisting of elliptic points of $\phi(y, D_{y'})$, we see that $\phi_0(y, D_{y'})f \in H_m(\bar{\mathbf{R}}^{n+1}_+)$ and $\phi_0(y, D_{y'})u \in H_r(\bar{\mathbf{R}}^{n+1}_+)$.*

Proof. Let $\Phi(y, D_{y'})$ be an elliptic properly supported pseudodifferential operator of order 0 such that the symbol $\Phi(y, \eta')$ of $\Phi(y, D_{y'})$ is equal to $\phi(y, \eta')$ for $(y, \eta') \in \Gamma$, where $\phi(y, \eta')$ is the symbol of $\phi(y, D_{y'})$. Then we have

$$\phi_0 v = \phi_0 \Phi^{-1} \phi v + \phi_0 \Phi^{-1} (\Phi - \phi) v + \phi_0 (1 - \Phi^{-1} \Phi) v. \tag{11}$$

It implies that $\phi_0(y, D_{y'}) f \in H_m(\bar{\mathbf{R}}^{n+1}_+)$ and $\phi_0(y, D_{y'}) u \in H_{(m_2, 0)}(\bar{\mathbf{R}}^{n+1}_+)$, where $m_2 = \min(m_1, r)$. Thus we may assume $r > m_1$. First we shall show that for any $\phi_1(y, D_{y'})$ which satisfies the same conditions to these of ϕ_0 there exists $t_2 \in \mathbf{R}$ such that $\phi_1(y, D_{y'}) u \in H_{(m+p, t_2)}(\bar{\mathbf{R}}^{n+1}_+)$, where t_2 does not depend on ϕ_1. The right hand side of $P(\phi_1 u) = \phi_1 f + [P, \phi_1] u$ belongs to $H_{(m_2, s_2)}(\bar{\mathbf{R}}^{n+1}_+)$, where $m_2 = \min(m, m_1 - p + 1)$ and $s_2 = \min(0, s_1)$. From Theorem 4.3.1 of [5] it implies that for some s_3 $\phi_1 u$ belongs to $H_{(m_3, s_3)}(\bar{\mathbf{R}}^{n+1}_+)$, where $m_3 = \min(m+p, m_1+1)$. It follows that for some s_4 $\phi_1 f + [P, \phi_1] u \in H_{(m_4, s_4)}(\bar{\mathbf{R}}^{n+1}_+)$, where $m_4 = \min(m+1, m_1 - p + 2)$. Repeatedly making use of this argument, we may assume that for some s_5 $P(\phi_1 u)$ belongs to $H_{(m, s_5)}(\bar{\mathbf{R}}^{n+1}_+)$, if ϕ_1 satisfies the conditions stated in Proposition 2.4. Theorem 4.3.1 of [5] says that $\phi_1(y, D_{y'}) u \in H_{(m+p, t_2)}(\bar{\mathbf{R}}^{n+1}_+)$ for some t_2. Next we shall use (11) as $v = u$. The second term of the right hand side of (11) $\Lambda_1^{t_2} \phi_0 \Phi^{-1} (\Phi - \phi)$ satisfies the conditions imposed on ϕ_1, where $\Lambda_1^{t_2} = (1 + |D_{y'}|)^{t_2/2}$. Thus $\phi_0 \Phi^{-1} (\Phi - \phi) u$ belongs to $H_{m+p}(\bar{\mathbf{R}}^{n+1}_+)$. Similarly we have the third term of the right hand side of (11) belongs to $H_{m+p}(\bar{\mathbf{R}}^{n+1}_+)$. The proof is completed. □

3. Propagation of singularities in interior of the domain

In this section we shall consider the problem (1) in interior of the domain. Let $u(x) \in \mathcal{D}'(\mathbf{R} \times \Omega)$ be a solution of (1) for $f(x) \in \mathcal{D}'(\mathbf{R} \times \Omega)$. We denote t by x_0 and the dual variable of t by ξ_0. We shall use the following notion:

Definition 3.1 For $(x^0, \xi^0) \in T^*(\mathbf{R} \times \Omega) \setminus 0$ we say that $u \in H_s$ at (x^0, ξ^0) if there exists a pseudodifferential operator $\phi(x, D_x)$ of order 0 such that $\phi(x, D_x)$ is elliptic at (x^0, ξ^0) and $\phi(x, D_x) u \in H_s(\mathbf{R}^{n+1})$.

Let (x^0, ξ^0) be a point such that $(\xi^0_0)^2 - \rho |\xi'^0|^2 = 0$, where $\xi' = (\xi_1, \ldots, \xi_n)$, ρ is μ or $\lambda + 2\mu$, and γ: $(-a, a) \longrightarrow T^*(\mathbf{R}^{n+1}) \setminus 0$ be a null bicharacteristic strip for $\xi_0^2 - \rho |\xi'|^2$ such that $\gamma(0) = (x^0, \xi^0)$. Then we have the following

Theorem 3.1 *Let u be a solution of* (1). *If $f \in H_s$ on γ and $u \in H_{s+1}$ at (x^0, ξ^0), then $u \in H_{s+1}$ at all points of γ.*

Proof. Let $\varphi(x) \in C_0^\infty(\mathbf{R} \times \Omega)$ be a smooth function with a compact support near x^0 such that $\varphi = 1$ in a small neighbourhood U_1 of x^0. Then $(D_{x_0}^2 - L_0(D_{x'}))(\varphi u) = \varphi f + f_1$, where $D_{x_0}^2 - L_0(D_{x'})$ is the linear elastic equation of (1) and $f_1 = [D_{x_0}^2 - L_0(D_{x'}), \varphi]u$ is zero in U_1. Let Λ be the pseudodifferential operator with the symbol $(1 + |\xi'|^2)^{1/2}$ with $\xi' = (\xi_1, \ldots, \xi_n)$. Then from Proposition 2.1 $u_1 = \Lambda(\varphi u)$ is well define, and $U = {}^t({}^tu_1, {}^tu_2)$ with $u_2 = D_{x_0}(\varphi u)$ satisfies the following equation

$$(D_{x_0} - M(D_{x'}))U = {}^t(0, {}^t(\varphi f + f_1)), \tag{12}$$

where

$$M(D_{x'}) = \begin{pmatrix} 0 & \Lambda \\ L_0(D_{x'})\Lambda^{-1} & 0 \end{pmatrix}.$$

By Lemma 1.1 and the argument in Section 2 of [19] there exist an elliptic pseudodifferential operator $S(D_{x'})$ of order 0 and a pseudodifferential operator $K(D_{x'})$ of order -1 such that $A(D_{x'}) = (D_{x_0} - \tilde{M})(1 + K)S^{-1} - (1 + K)S^{-1}(D_{x_0} - M)$ belongs to $L^{-\infty}(\mathbf{R}^n)$ near $(x^0, \xi^{0\prime})$, where $\tilde{M}$ is of order 1 and have the form

$$\begin{pmatrix} \tilde{M}_\mu & 0 \\ 0 & \tilde{M}_{\lambda+2\mu} \end{pmatrix} \quad \text{with} \quad \tilde{M}_\rho = \begin{pmatrix} \tilde{M}_\rho^+ & 0 \\ 0 & \tilde{M}_\rho^- \end{pmatrix}$$

for $\rho = \mu$ or $\lambda + 2\mu$. Here the principal symbol of $\tilde{M}_\mu^\pm(D_{x'})$ is $\pm\mu^{1/2}|\xi'|E_{k-1}$, and the principal symbol of $\tilde{M}_{\lambda+2\mu}^\pm(D_{x'})$ is $\pm(\lambda + 2\mu)^{1/2}|\xi'|$. From these observations we have that $V = S(1 + K)^{-1}U$ satisfies the equation

$$(D_{x_0} - \tilde{M}(D_{x'}))V = {}^t(0, {}^t(\varphi f + f_1)) + A_1(D_{x'})U, \tag{13}$$

where $A_1(D_{x'})$ belongs to $L^{-\infty}(\mathbf{R}^n)$ near $(x^0, \xi^{0\prime})$. Here we remark that from Proposition 2.3 the right hand side of (13) belongs to H_s on γ. Now we can use the argument in the proof of Theorem 2.1 in Chapter VI of [16], because in our proof we can take the operator $c(x, D_x)$ appeared in the proof of Theorem 2.1 in [16] as $c_1(x, D_x)E$, where $c_1(x, D_x)$ is a scalar pseudodifferential operator. It implies that $V \in H_s$ on γ. Again making use of Proposition 2.3, we see that $U \in H_s$ on γ, which means that $u \in H_{s+1}$ on γ. The proof is complete. $\square$

4. Hyperbolic cases

In the boundary value problem (5) and (6) we suppose that f belongs to $H^{\mathrm{loc}}_{(0,s_1)}(U_0 \cap \bar{\mathbf{R}}^{n+1}_+)$ for some $s_1 \in \mathbf{R}$. We also assume that a solution u belonging to $\mathcal{D}'(U_0 \cap \{y_n > 0\})$ is an extensible distribution to U_0. Then from Theorem 4.3.1 in [5] we see that u belongs to $H^{\mathrm{loc}}_{(2,s_2)}(U_0 \cap \bar{\mathbf{R}}^{n+1}_+)$ for some $s_2 \in \mathbf{R}$. It implies that we can take the meaning of the boundary condition $B_0 u$ on $U_0 \cap \{y_n = 0\}$ from Theorem 2.5.6 in [5]. We shall use the following similar definition to Definition 3.1 appeared in Section 3.

Definition 4.1 For a subset $\Gamma^{(1)}$ contained in $T^*\{y_n > 0\}\backslash 0$ and a subset $\Gamma^{(0)}$ contained in $T^*\{y_n = 0\}\backslash 0$ we say that $u(y) \in C(\bar{\mathbf{R}}_+;\ \mathcal{D}'(\mathbf{R}^n))$ belongs to H_s on $\Gamma^{(0)} \cup \Gamma^{(1)}$, if u belongs to H_s at each point of $\Gamma^{(1)}$ and for any point $\rho \in \Gamma^{(0)}$ there exists a pseudodifferential operator $\phi(y, D_{y'})$ of order 0, which is elliptic at ρ, such that $\phi(y, D_{y'})u \in H_s(\bar{\mathbf{R}}^{n+1}_+)$. We remark that Definition 3.1 in Section 3 is invariant under coordinate transform. However this definition is not invariant under the general coordinate transform preserving the set $\{y_n = 0\}$.

First we shall consider the following hyperbolic equation

$$(D_{y_n} - H(y, D_{y'}))V = F \quad \text{in} \quad \mathbf{R}^n \times \{0 < y_n < T\}, \tag{14}$$

where $H(y, D_{y'})$ is a pseudodifferential operator of order 1 such that its symbol $H(y, \eta')$ is zero, if $|y'|$ is sufficiently large, and its principal symbol has the form $\lambda(y, \eta')E_k$ with real valued $\lambda(y, \eta')$ and the $k \times k$ unit matrix E_k. Here we assume that $F \in L^2([0,T];\ H_{s_1}(\mathbf{R}^n))$ for some $s_1 \in \mathbf{R}$, and $V_+ \in H_{(1,s_2)}(\bar{\mathbf{R}}^{n+1}_+)$ for some $s_2 \in \mathbf{R}$. Let γ be the null bicharacteristic strip of $\eta_n - \lambda(y, \eta')$ passing through $(0, \eta^{0\prime}, \lambda(0, \eta^{0\prime}))$, where $\eta^{0\prime} \neq 0$, and γ^+ be $\gamma \cap T^*\{0 < y_n < T\}$. We have the following

Proposition 4.1 *We assume that for $s \geq 0$ F and V belongs to H_s on γ^+, and that there exists a pseudodifferential operator $\phi_1(y, D_{y'})$ of order 0, which is elliptic at $(0, \eta^{0\prime})$, such that $\phi_1(y, D_{y'})F \in L^2([0,T]; H_s(\mathbf{R}^n))$. Then we have the following two statements*:
i) $V(y', 0)$ *belongs to H_s at $(0, \eta^{0\prime})$.*
ii) *There exists a pseudodifferential operator $\phi(y, D_{y'})$ of order 0, which is elliptic at $(0, \eta^{0\prime})$, such that $\phi(y, D_{y'})V \in C([0,T];\ H_s(\mathbf{R}^n))$.*

In order to prove the above proposition we need several preparations.

We shall consider the following an initial boundary value problem for first order hyperbolic system

$$(D_{y_n} - H(y, D_{y'}))V = F \quad \text{in} \quad \mathbf{R}^n \times \{0 < y_n < T\}, \tag{15}$$

$$V = G \quad \text{on} \quad \mathbf{R}^n \times \{y_n = z_n\}, \tag{16}$$

where $z_n \in [0, T]$. The symbol $H(y, \eta')$ of $H(y, D_{y'})$ is zero, if $|y'|$ is sufficiently large, and $H(y, D_{y'}) - H^*(y, D_{y'})$ is of order 0. Then we have the following

Lemma 4.2 *Let r be an arbitrary real number. Then for any $G \in H_s(\mathbf{R}^n)$ and $F \in L^2([0,T];\ H_s(\mathbf{R}^n))$ the Cauchy problem* (15), (16) *has the unique solution $V(x_n, z_n) \in C([0,T]^2;\ H_s(\mathbf{R}^n))$. Moreover the following estimate holds. For any y_n, $z_n \in [0, T]$*

$$\|V(\,\cdot\,, y_n)\|_s^2 \leq C\bigg(\|G\|_s^2 + \bigg|\int_{z_n}^{y_n} \|F(\,\cdot\,, \tau)\|_s^2\, d\tau\bigg|\bigg). \tag{17}$$

The proof of this lemma is described in p.75 of [16] in the space $H_s(M)$, where M is a compact manifold. In our case we can not use the Ascoli's theorem, however we may use the weak compactness of the Hilbert space $H_s(\mathbf{R}^n)$. We may assume that the symbol $H(y, \eta')$ has an asymptotic expansion $\sum_{j=0}^{\infty} H_{1-j}(y, \eta')$ such that $H_{1-j}(y, \eta')$ is positively homogeneous of degree $1 - j$ with respect to η', $H_{1-j}(y, \eta') = 0$, if $|y'| \geq M_0 > 0$, and $H_1(y, \eta') = \lambda(y, \eta')E_k$, where E_k is the unit matrix. From Lemma 4.2 we can define the operator $E(y_n, z_n)$, which is a continuous operator on $[0,T]^2$ from $H_s(\mathbf{R}^n)$ to itself, such that $V(y) = E(y_n, z_n)G$ is the unique solution of (15), (16) as $G \in H_s(\mathbf{R}^n)$ and $F = 0$. From the uniqueness of the solution of the problem (15), (16) it follows that $E(t_1, z_n)E(z_n, t_2) = E(t_1, t_2)$ for all t_1, z_n, $t_2 \in [0, T]$ and the solution V of (15), (16) is equal to

$$V(\,\cdot\,, y_n) = E(y_n, z_n)G + \int_{z_n}^{y_n} E(y_n, \tau)F(\,\cdot\,, \tau)\, d\tau. \tag{18}$$

Let $\varphi(y', y_n, z_n, \eta')$ be the solution of the following eikonal equation

$$\frac{\partial \varphi}{\partial y_n} - \lambda\Big(y, \frac{\partial \varphi}{\partial y'}\Big) = 0 \tag{19}$$

$$\varphi(y', z_n, z_n, \eta') = \langle y', \eta' \rangle. \tag{20}$$

Then the fundamental solution $E_1(y_n, z_n)$ of the initial problem (15), (16) as $F = 0$ is a Fourier integral operator with the phase function $\varphi(y, z_n, \eta') -$

$\langle z', \eta' \rangle$. From (18) and the continuity property in $H_s(\mathbf{R}^n)$ of Fourier integral operators with non-degenerate phase functions (see Theorem 1.9 in Chapter 10 of [8]) we have the following

Lemma 4.3 *If $V(\cdot, z_n) \in C([0,T]; H_{-\infty}(\mathbf{R}^n))$, then $(E(y_n, z_n) - E_1(y_n, z_n))$ $V(\cdot\,, z_n)$ belongs to $C([0,T]^2;\ H_\infty(\mathbf{R}^n))$, where $H_{-\infty}(\mathbf{R}^n) = \cup_{s\in\mathbf{R}} H_s(\mathbf{R}^n)$ and $H_\infty(\mathbf{R}^n) = \cap_{s\in\mathbf{R}} H_s(\mathbf{R}^n)$.*

Making use of the lemmas, we shall prove Proposition 4.1. From (18) we have

$$V(\cdot\,, 0) = E(0, z_n)V(\cdot\,, z_n) + \int_{z_n}^{0} E(0,\tau)F(\cdot\,,\tau)\,d\tau. \tag{21}$$

Let $\chi(z_n)$ be a $C_0^\infty([0, T_1])$ function, where $0 < T_1 < T$, such that $\chi_0 = \int_0^{T_1} \chi(z_n)\,dz_n$ is not zero. Multiply $\chi(z_n)$ to the both side of (21), and integrate with respect to z_n from 0 to T_1. Then we have the following

$$\chi_0 V(\cdot\,,0) = \int_0^{T_1} E(0,\tau)\chi(\tau)V(\cdot\,,\tau)\,d\tau - \int_0^{T_1} E(0,\tau)\chi_1(\tau)F(\cdot\,,\tau)\,d\tau, \tag{22}$$

where $\chi_1(\tau) = \int_\tau^{T_1} \chi(z_n)\,dz_n$. In the first term of the right hand side of (22) from Lemma 4.3 we may consider

$$(E_2(\chi V))(y') = \int_0^{T_1} E_1(0,\tau)\chi(\tau)V(\cdot\,,\tau)\,d\tau,$$

which is a Fourier integral operator on $\mathbf{R}^n \times (\mathbf{R}^n \times (0,T))$ with the phase function $\Phi(y', \eta', z_n, z') = \phi(y', 0, z_n, \eta') - \langle z', \eta' \rangle$ and the amplitude $a(y', \eta', z_n, z') = e_1(y', 0, z_n, \eta')$, where $e_1(y', y_n, z_n, \eta')$ is the amplitude of $E_1(y_n, z_n)$. It is well known (cf. [5]) that $WF(E_2(\chi V))$ is contained in the set $\{(y', \eta') \in T^*\mathbf{R}^n \setminus 0$: there exists $(z, \zeta) \in WF(\chi V)$ such that $(y', \eta', z, -\zeta) \in C_\Phi\}$, where

$$\begin{aligned} C_\Phi = \{&(y', (\operatorname{grad}_{x'} \varphi)(y', 0, z_n, \zeta'), (\operatorname{grad}_{\eta'} \varphi)(y', 0, z_n, \zeta'), z_n, -\zeta', \\ &\partial\varphi/\partial z_n(y', 0, z_n, \zeta'))\colon y',\, z' \in \mathbf{R}^n,\ 0 < z_n < T,\ \zeta' \neq 0\}. \end{aligned}$$

This relation is equivalent to the following;

$$\begin{aligned} WF(E_2(\chi V)) \subset \big\{&\big(y'(z', 0, z_n, \zeta'), \eta'(z', 0, z_n, \zeta')\big)\colon \\ &\big(z, \zeta', -\partial\varphi/\partial z_n(y'(z', 0, z_n, \zeta'), 0, z_n, \zeta')\big) \in WF(\chi V)\big\}, \end{aligned}$$

where $\big(y'(z',t,z_n,\zeta'),\eta'(z',t,z_n,\zeta')\big)$ is the solution of Hamilton equation:

$$dy'/dt=-\partial\lambda/\partial\eta',\quad d\eta'/dt=\partial\lambda/\partial y',\quad y'(z_n)=z',\quad \eta'(z_n)=\zeta'.$$

From the relation

$$\varphi(y',t,z_n,\zeta')=\langle y',\zeta'\rangle+\int_{z_n}^{t}\lambda\big(y',\tau,\eta'(z'(y',\tau,z_n,\zeta'),\tau,z_n,\zeta')\big)\,d\tau,$$

where $z'=z'(y',t,z_n,\zeta')$ is the inverse function of $y'=y'(z',t,z_n,\zeta')$, we see that $|\partial\varphi/\partial z_n(y'(z',0,z_n,\zeta'))+\lambda(z',z_n,\zeta')|\le O(1)|z_n|\,|\zeta'|$. By the assumption on V we have $\chi V=V_1+V_2$ such that $WF(V_1)\cap\gamma_+=\emptyset$ and $V_2\in H_s(\mathbf{R}^n\times(0,T))$. Here from the theorem on propagations of regularities in interior of the domain and the above argument we may assume that $WF(V_1)\cap\big\{\big(y'(0,z_n,0,\eta^{0\prime}),z_n,\eta'(0,z_n,0,\eta^{0\prime}),-\partial\varphi/\partial z_n(0,z_n,\eta^{0\prime})\big);\ z_n\in\operatorname{supp}\chi\big\}=\emptyset$. It implies that $(0,\eta^{0\prime})$ does not belong to $WF(E_2(V_1))$. Clearly we have $E_2(V_2)\in H_s(\mathbf{R}^n)$.

Let us consider the second term in the right hand side of (22). From Lemma 4.3 and the assumption on F we may consider

$$\int_0^{T_1}E_1(0,\tau)\chi_1(\tau)(1-\phi_1)F(\,\cdot\,,\tau)\,d\tau,$$

where the symbol of $\phi_1(y,D_{y'})$ is 1 near $(0,\eta^{0\prime})$. Let $\phi_2(y,D_{y'})$ be a pseudodifferential operator such that the symbol of $\phi_2(y',\eta')$ satisfies $\phi_2(y',\eta')(1-\phi_1(y,\eta'))=0$ for $y_n\in[0,T_1]$. Let $\{F_m(\,\cdot\,,\tau)\}\subset C([0,T]:H_s(\mathbf{R}^n))$ be a convergence sequence to $F(\,\cdot\,,\tau)$ in $L^2([0,T]:H_s(\mathbf{R}^n))$. Then a continuity property in $H_s(\mathbf{R}^n)$ of $E_1(0,\tau)$ we see that $\int_0^{T_1}E_1(0,\tau)\chi_1(\tau)(1-\phi_1)F_m(\,\cdot\,,\tau)\,d\tau$ converges to $\int_0^{T_1}E_1(0,\tau)\chi_1(\tau)(1-\phi_1)F(\,\cdot\,,\tau)\,d\tau$ in $H_s(\mathbf{R}^n)$. Making use of an approximation by Riemann sum, we have

$$\begin{aligned}\phi_2(y',D_{y'})&\int_0^{T_1}E_1(0,\tau)\chi_1(\tau)(1-\phi_1)F_m(\,\cdot\,,\tau)\,d\tau\\&=\int_0^{T_1}\phi_2E_1(0,\tau)\chi_1(\tau)(1-\phi_1)F_m(\,\cdot\,,\tau)\,d\tau,\end{aligned}$$

which converges to $\int_0^{T_1}\phi_2E_1(0,\tau)\chi_1(\tau)(1-\phi_1)F(\,\cdot\,,\tau)\,d\tau$. This means that

$$\phi_2(y',D_{y'})\int_0^{T_1}E_1(0,\tau)\chi_1(\tau)(1-\phi_1)F(\,\cdot\,,\tau)\,d\tau$$

$$= \int_0^{T_1} \phi_2 E_1(0,\tau)\chi_1(\tau)(1-\phi_1)F(\,\cdot\,,\tau)\,d\tau.$$

From the property on the wave front set of $E_1(0,\tau)$ for fixed τ and the assumption on the support of $\phi_2(y',\eta')$ and $(1-\phi_1(y,\eta'))$, it follows that $\phi_2 E_1(0,\tau)(1-\phi_1)$ has a C^∞ kernel, if τ is sufficiently small, that is, $\phi_2(y',D_{y'})\int_0^{T_1} E_1(0,\tau)\chi_1(\tau)(1-\phi_1)F(\,\cdot\,,\tau)\,d\tau \in C_0^\infty(\mathbf{R}^n)$. Thus we have the statement i) of Proposition 4.1. If we use the formula

$$V(\,\cdot\,,y_n) = E(y_n,0)V(\,\cdot\,,0) + \int_0^{y_n} E(y_n,\tau)F(\,\cdot\,,\tau)\,d\tau,$$

then we get the statement ii) by the similar argument used in one of proving the statement i) of Proposition 4.1. The proof of Proposition 4.1 is complete.

Next we shall consider the following initial value problem;

$$(D_{y_n} - H(y,D_{y'}))V = F \quad \text{in} \quad \mathbf{R}^n \times \{0 < y_n < T\}, \tag{23}$$

$$V = G \quad \text{on} \quad \mathbf{R}^n \times \{y_n = 0\}. \tag{24}$$

We have the following

Proposition 4.4 *We assume that $F \in H_{(s,r_1)}(\bar{\mathbf{R}}_+^{n+1})$ for some $s \geq 0$ and $r_1 \in \mathbf{R}$, and that F and G belong to H_s at $(0,\eta^{0\prime})$. Then V belongs to H_s at $(0,\eta^{0\prime})$.*

Proof. By the argument used in the proof of Proposition 4.1 it is not difficult to show that there exists a pseudodifferential operator $\phi_1(y,D_{y'})$ of order 0, which is elliptic at $(0,\eta^{0\prime})$, such that $\phi_1(y,D_{y'})V \in C([0,T];\ H_s(\mathbf{R}^n)) \cap H_{(0,s)}(\bar{\mathbf{R}}_+^{n+1})$. Let $\phi_2(y,D_{y'})$ be a pseudodifferential operator of order 0 such that $\phi_2(y,D_{y'})$ is elliptic at $(0,\eta^{0\prime})$ and $WF(\phi_2)$ is contained in the set of elliptic points of ϕ_1. Then $(D_{y_n} - H)(\phi_2 V) = \phi_2 F + (D_{y_n}\phi_2)V - [H,\phi_2]V \in H_{(0,s)}(\bar{\mathbf{R}}_+^{n+1})$. Repeatedly making use of Theorem 4.3.1 of [5], we see that there exists a pseudodifferential operator $\phi(y,D_{y'})$ of order 0, which is elliptic at $(0,\eta^{0\prime})$, such that $\phi(y,D_{y'})V \in H_{([s]+1,s-[s]-1)}(\bar{\mathbf{R}}_+^{n+1}) \subset H_s(\bar{\mathbf{R}}_+^{n+1})$. The proof is completed. $\square$

Let us consider the case iii) in Introduction. Let $\varphi(y)$ be a function belonging to $C_0^\infty(U_0)$ such that $\varphi = 1$ near 0. At point $\rho_0 = (0,\eta^{0\prime}) \in T^*(\{y_n = 0\}) \setminus 0$ we suppose the condition $r_{\lambda+2\mu}(0,\eta^{0\prime}) < 0$. Let Γ_0 and Γ_1 be a conic neighbourhood of ρ_0 such that $\Gamma_1 \subset \Gamma_0$, and $\phi(y,D_{y'})$ be a pseudodifferential operator of order 0 such that the support of the symbol

$\phi(y,\eta')$ of $\phi(y,D_{y'})$ is contained in $\Gamma_0\times(-2\epsilon_0,2\epsilon_0)$ and $\phi(y,\eta')=1$ in $\Gamma_1\times(-\epsilon_0,\epsilon_0)$ with a small positive number ϵ_0. By the argument used in Section 1.3 in [19] there exist an elliptic pseudodifferential operator $S(y,D_{y'})$ of order 0 and a pseudodifferential operator $K(y,D_{y'})$ of order -1 such that the symbol of $A_1(y,D_{y'})=(D_{y_n}-\tilde{M})(1+K)S^{-1}-(1+K)S^{-1}(D_{y_n}-M)$ belongs to $S^{-\infty}(\Gamma_0\times(-2\epsilon_0,2\epsilon_0))$. Here the form of $\tilde{M}(y,D_{y'})$ is as follows;

$$\begin{pmatrix}\tilde{H}_+ & 0\\ 0 & \tilde{H}_-\end{pmatrix} \quad \text{with} \quad \tilde{H}_\pm=\begin{pmatrix}H_\pm & 0\\ 0 & h_\pm\end{pmatrix},$$

where the principal symbol of $H_\pm$ has a form $\lambda_\pm E_{n-1}$ with the real symbol $\lambda_\pm(y,\eta')$ and $h_\pm(y,D_{y'})$ is a scalar pseudodifferential operator with the real symbol. Let $U={}^t({}^t(\Lambda\varphi u),{}^t(D_{y_n}\varphi u))$, where $\Lambda=(1+|D_{y'}|^2)^{1/2}$, and put $V=(1+K)S^{-1}\phi U$, and

$$\begin{aligned}F&=(1+K)S^{-1}\phi^t(0,{}^t(\varphi f)-{}^t([L,\varphi]u))\\ &\quad-(1+K)S^{-1}(D_{y_n}\phi-[M,\phi])U+A_1\phi U,\end{aligned}\tag{25}$$

where L is the differential operator in (5). Then V satisfies the following

$$(D_{y_n}-\tilde{M}(y,D_{y'}))V=F \quad \text{in} \quad \{y_n>0\}.\tag{26}$$

Let us consider the boundary operator. Put $\tilde{B}=(E_n,0)$ if B_0 is the Dirichlet condition and $\tilde{B}=(B_1\Lambda^{-1},B_2)$ if B_0 is the free boundary condition. Here the forms of B_1 and B_2 are denoted in (1.6) of [19]. For the boundary operator $B(y,D_{y'})V=\tilde{B}S(1+K)^{-1}\ V$ satisfies the following

$$B(y,D_{y'})V=G \quad \text{on} \quad \{y_n=0\}.\tag{27}$$

Here if B_0 is the free boundary condition, then for $A_2(y,D_{y'})=S(1+K)^{-1}$ $(1+K)S^{-1}-I$ we have

$$G=\phi\varphi g-\phi[B_0,\varphi]u+[\tilde{B},\phi]U+\tilde{B}A_2\phi U,\tag{28}$$

and if B_0 is the Dirichlet condition, then we have

$$G=\phi\Lambda\varphi g+\tilde{B}A_2\phi U.\tag{29}$$

If $r_{\lambda+2\mu}(0,\eta^{0\prime})<0$, then there exists the null bicharacteristic strip $\tilde{\gamma}_\rho^\epsilon$ of $(\eta_n-a)^2+r_\rho(y,\eta')$ passing through $(0,\eta^{0\prime},a(0,\eta^{0\prime})+\epsilon(-r_\rho(0,\eta^{0\prime}))^{1/2})$, where $\rho=\mu$ or $\lambda+2\mu$ and $\epsilon=+$ or $-$. We denote $\tilde{\gamma}_\rho^\epsilon\cap T^*(\mathbf{R}^n\times(0,\infty))$ by γ_ρ^ϵ. Then we have the following

Theorem 4.5 *Let* $(0,\eta^{0\prime})$ *satisfy* $r_{\lambda+2\mu}(0,\eta^{0\prime})<0$. *We assume that for* $s\geq 0$ *and* $r_1\in\mathbf{R}$ f *of* (5) *belongs to* $H^{\mathrm{loc}}_{(s,r_1)}(U_0\cap\bar{\mathbf{R}}^{n+1}_+)$, *and satisfies the following condition*; f *belongs to* H_s *on* $\gamma^+_\mu\cup\gamma^-_\mu\cup\gamma^+_{\lambda+2\mu}\cup\gamma^-_{\lambda+2\mu}\cup\{(0,\eta^{0\prime})\}$. *We also assume that* $g\in H^{\mathrm{loc}}_{s_2}(U_0\cap\mathbf{R}^n)$ *of* (5) *belongs to* H_r *at* $(0,\eta^{0\prime})$, *where* $r=s+1$, *if* B_0 *is the Dirichlet condition, and* $r=s$, *if* B_0 *is the free boundary condition. Then we have the following two statements*:
i) *We shall consider the Dirichlet condition or the free boundary condition with Lamé constants so that* $\lambda>4\mu$. *Then if a solution* $u\in H^{\mathrm{loc}}_{(2,s_2)}(U_0\cap\bar{\mathbf{R}}^{n+1}_+)$ *of* (5) *and* (6) *satisfies that* u *belongs to* H_{s+1} *on* $\gamma^\epsilon_\mu\cup\gamma^{\epsilon'}_{\lambda+2\mu}$, *where* ϵ *and* ϵ' *are* $+$ *or* $-$, *then* u *belongs to* H_{s+1} *on* $\gamma^{-\epsilon}_\mu\cup\gamma^{-\epsilon'}_{\lambda+2\mu}\cup\{(0,\eta^{0\prime})\}$.
ii) *We shall consider the free boundary condition under the assumption on Lamé constants such that* $\lambda\leq 4\mu$. *Then if a solution* u *belongs to* $H^{\mathrm{loc}}_{(2,s_2)}(U_0\cap\bar{\mathbf{R}}^{n+1}_+)$ *of* (5) *and* (6) *satisfies that* u *belongs to* H_{s+1} *on* $\gamma^\epsilon_\mu\cup\gamma^{\epsilon}_{\lambda+2\mu}$, *where* ϵ *is* $+$ *or* $-$, *then* $u\in H_{s+1}$ *on* $\gamma^{-\epsilon}_\mu\cup\gamma^{-\epsilon}_{\lambda+2\mu}\cup\{(0,\eta^{0\prime})\}$.

Proof. Let $(b^+_1,\ldots,b^+_{n-1},b^+,b^-_1,\ldots,b^-_{n-1},b^-)$ be the principal symbol of $B(y,D_{y'})$. Then in the case considered in i) the matrices $(b^+_1,\ldots,b^+_{n-1},b^\pm)$ and $(b^-_1,\ldots,b^-_{n-1},b^\pm)$ are nonsingular at $(0,\eta^{0\prime})$. Since the principal symbol of S is denoted in (1.8) of [19], these facts are easily proved for the Dirichlet condition. For the free boundary condition these facts are proved in Remark 1.10 of [19]. In the case considered in ii) the two matrices $(b^+_1,\ldots,b^+_{n-1},b^+)$ and $(b^-_1,\ldots,b^-_{n-1},b^-)$ are nonsingular at $(0,\eta^{0\prime})$. These facts are verified in the proof of Theorem 1.9 of [19]. For simplicity we assume that $u\in H_{s+1}$ on $\gamma^+_\mu\cup\gamma^+_{\lambda+2\mu}$. Then from Proposition 2.3 we have $V=(1+K)S^{-1}\phi U\in H_s$ on $\gamma^+_\mu\cup\gamma^+_{\lambda+2\mu}$. We may assume that there exists a pseudodifferential operator $\phi_1(y,D_{y'})$ of order 0, which is elliptic at $(0,\eta^{0\prime})$ such that $\phi_1(y,D_{y'})u\in H_{(m,r_1)}(\bar{\mathbf{R}}^{n+1}_+)$ for some $m\geq 2$ and $r_1\in\mathbf{R}$. Under this condition we shall prove that $\phi_2(y,D_{y'})u\in H_{(m_1+2,-2)}(\bar{\mathbf{R}}^{n+1}_+)$, where $m_1=\min(s,m-1)$ and ϕ_2 satisfies the same conditions to these of ϕ_1. From (25), (28) and (29) there exists a pseudodifferential operator $\phi_3(y,D_{y'})$ which satisfies the same conditions to these of $\phi_1(y,D_{y'})$ such that $\phi_3F\in H_{m_1}(\bar{\mathbf{R}}^{n+1}_+)$ and $\phi_3G_{|y_n=0}\in H_{m_1}(\mathbf{R}^n)$, where $m_1=\min(s,m-1)$. Since $V_1=(1-\chi)\tilde{M}(y,D_{y'})V$ belongs to $H_{(m_1,r_1)}(\bar{\mathbf{R}}^{n+1}_+)$ for some $r_1\in\mathbf{R}$, $\phi_3V_1\in H_{m_1}(\bar{\mathbf{R}}^{n+1}_+)$, where $\chi(y')\in C^\infty_0(\mathbf{R}^n)$ and $\chi=1$ near 0. Thus in (26) we may assume that the symbol $\tilde{M}(y,D_{y'})$ is zero in $|y'|\geq M_0$ for some positive M_0. From the assumption on $\gamma_\mu^+ \cup \gamma_{\lambda+2\mu}^+$ and Proposition 4.1 we also have that $\phi_4^t({}^tV_+, v_+)_{|y_n=0} \in H_{m_1}(\mathbf{R}^n)$ with the same type ϕ_4, where $V = {}^t({}^tV_+, v_+, {}^tV_-, v_-)$. Since $(b_1^-, \dots, b_{n-1}^-, b^-)$ is elliptic at $(0, \eta^{0\prime})$, we see that $\phi_4^t({}^tV_-, v_-)_{|y_n=0}$ also belongs to $H_{m_1}(\mathbf{R}^n)$. This fact and Proposition 4.4 imply that there exists a pseudodifferential operator $\phi_5(y, D_{y'})$ with the same properties to these of ϕ_1 such that $\phi_5 V \in H_{m_1}(\bar{\mathbf{R}}_+^{n+1})$. It follows that $\phi_6 u \in H_{m_1+1}(\bar{\mathbf{R}}_+^{n+1})$ for same type $\phi_6(y, D_{y'})$. From Theorem 4.3.1 of [5] and $L(\phi u) = \phi f + [L, \phi]u$ we see that $\phi_7 u \in H_{(m_1+2,-2)}(\bar{\mathbf{R}}_+^{n+1})$ for same type $\phi_7(y, D_{y'})$. Repeatedly making use of this argument, we can prove that $\phi_1 u \in H_{(s+1,-2)}(\bar{\mathbf{R}}_+^{n+1})$. From (26), (28) and (29) it follows that there exists ϕ_8 with the same conditions to ϕ_1 such that $\phi_8 F \in H_s(\bar{\mathbf{R}}_+^{n+1})$, $\phi_8 G \in H_s(\mathbf{R}^n)$. From Proposition 4.1 we see that u belongs to H_s at $(0, \eta^{0\prime})$. The property on $\gamma_\mu^- \cup \gamma_{\lambda+2\mu}^-$ is easily proved from Proposition 2.3 and Theorem 3.1. The proof is completed. $\square$

5. Elliptic cases

First we shall consider the case ii) in Introduction. Let $E_\pm(y, D_{y'})$ be a pseudodifferential operator of order 1 with the principal symbol $e_\pm(y, \eta')$ such that $\pm e_\pm(y, \eta')$ satisfies the conditions (1.6) and (1.7) of Chapter III in [18] and $e_\pm(y, \eta')$ is independent of y if $|y|$ is sufficiently large. We shall consider the problem (15) and (16) for $D_{y_n} - E_\pm(y, D_{y'})$. In order to show similar statements to these denoted in Proposition 4.1 and Proposition 4.4 we need the parametrix of the forward Cauchy problem for $D_{y_n} - E_+(y, D_{y'})$ and one of the backward Cauchy problem for $D_{y_n} - E_-(y, D_{y'})$. The constructions of parametrices are done in Section 1 of Chapter III in [18]. Making use of the representation formula (1.45) and the estimate (1.11) of Chapter III in [18], we can easily prove the corresponding propositions to Proposition 4.1 and Proposition 4.4. If $r_{\lambda+2\mu}(0, \eta^{0\prime}) > 0 > r_\mu(0, \eta^{0\prime})$, then the principal symbol $(b_1^+, \dots, b_{n-1}^+, b^+, b_1^-, \dots, b_{n-1}^-, b^-)$ of $B(y, D_{y'})$, which is the boundary operator of the problem (1.14) in [19], satisfies that the matrix $(b_1^\pm, \dots, b_{n-1}^\pm, b^+)$ is non singular at $(0, \eta^{0\prime})$ (see the proof of Theorem 1.8 in [19]). From these observations we have the following

Theorem 5.1 *Let $(0, \eta^{0\prime})$ be a point such that $r_{\lambda+2\mu}(0, \eta^{0\prime}) > 0 > r_\mu(0, \eta^{0\prime})$. We assume that for $s \geq 0$ and $r \in \mathbf{R}$ $f \in H_{(s,r)}^{\mathrm{loc}}(U_0 \cap \bar{\mathbf{R}}_+^{n+1})$ of* (5) *satisfies the following conditions; f belongs to H_s on $\gamma_\mu^+ \cup \gamma_\mu^- \cup \{(0, \eta^{0\prime})\}$. We also assume the condition on g appeared in the statement of* Theorem 4.5. *Then* *we have the following statement on a solution* $u \in H^{\mathrm{loc}}_{(2,s_2)}(U_0 \cap \bar{\mathbf{R}}^{n+1}_+)$ *of the boundary value problem* (5), (6): *If* u *belongs to* H_{s+1} *on* γ^{ϵ}_{μ}, *then* u *belongs to* H_{s+1} *on* $\gamma^{-\epsilon}_{\mu} \cup \{(0, \eta^{0\prime})\}$.

Next we shall consider the case i). If $(0, \eta^{0\prime})$ satisfies the conditions $r_\mu(0, \eta^{0\prime}) > 0$ and $\eta^0_0 = 0$, then the forms of the principal symbol E_+ and E_- appeared in (1.7) of [19] are not simple (see the middle part of p.125 in [19]). However the principal symbol of E_+ satisfies the condition (1.7) in Chapter III of [18], and the determinant $R(y', \eta')$ of the principal symbol $B_+(y', D_{y'}) = (b^+_1, \ldots, b^+_{n-1}, b^+)$, which is called the Lopatinski determinant, is non singular at $(0, \eta^{0\prime})$ (see Lemma 1.4 in [19]). So we have the following

Theorem 5.2 *Let* $(0, \eta^{0\prime})$ *be a point such that* $r_\mu(0, \eta^{0\prime}) > 0$ *and* $\eta^0_0 = 0$. *We assume that for some* $s \geq 0$ *and* $r \in \mathbf{R}$ $f \in H^{\mathrm{loc}}_{(s,r)}(U_0 \cap \bar{\mathbf{R}}^{n+1}_+)$ *of* (5) *satisfies the condition* f *belongs to* H_s *at* $(0, \eta')$, *and assume that* $g \in H^{\mathrm{loc}}_{s_2}(U_0 \cap \mathbf{R}^n)$ *of* (6) *satisfies the same conditions to these in* Theorem 4.5. *Then a solution* $u \in H^{\mathrm{loc}}_{(2,s_2)}(U_0 \cap \bar{\mathbf{R}}^{n+1}_+)$ *of the boundary value problem* (5) *and* (6) *belongs to* H_{s+1} *at* $(0, \eta^{0\prime})$.

We assume that $(0, \eta^{0\prime})$ satisfies the conditions $r_\mu(0, \eta^{0\prime}) > 0$ and $\eta^0_0 \neq 0$. If the boundary condition is the Dirichlet condition, then the Lopatinski determinant is not zero at $(0, \eta^{0\prime})$. Thus we have the following

Theorem 5.3 *Let* $(0, \eta^{0\prime})$ *be a point such that* $r_\mu(0, \eta^{0\prime}) > 0$ *and* $\eta^0_0 \neq 0$. *If the boundary condition is the Dirichlet condition, then we have the exactly same statement to one of* Theorem 5.2.

If we consider the free boundary condition, then the Lopatinski determinant $R(y', \eta')$ has real zeros, and at these points $(\partial R / \partial \eta_0)(y', \eta')$ are not zero (see Lemma 1.3 of [19]). Thus we need some global condition in order to avoid the singularities along the null bicharacteristic strip of $R(y', \eta')$, which is corresponding to the Raylight wave in seismology. We assume that f and g in (1) and (2) are 0 and that a solution $u(t, x)$ of (1) and (2) belongs to $H_2((T_1, T_2) \times (\omega \cap \Omega))$, where ω is an open neighbourhood of $\mathbf{R}^n \setminus \Omega$. Furthermore we suppose that for $t_0 \in (T_1, T_2)$ data $F = {}^t({}^t u(t_0, x), {}^t(\partial_t u)(t_0, x))$ have the property such that there exist data $\tilde{F}$ belonging to $\mathcal{D}(A^N)$ which are equal to F in $\omega \cap \Omega$, where A is the operator defined in (0.5) of [14] for an isotropic elastic equation.

Proposition 5.4 *Let $u(t,x)$ be a solution of* (1) *and* (2), *which satisfies the above conditions. In the reduced boundary value problem* (5) *and* (6) *we assume that $(\bar{y}_0, 0, \eta^{0\prime})$ satisfies the conditions $r_\mu(\bar{y}_0, 0, \eta^{0\prime}) > 0$ and $R(\bar{y}_0, 0, \eta^{0\prime}) = 0$, where y_0 is corresponding to t in* (1). *Then the solution of* (5) *and* (6), *which is reduced from $u(t,x)$, belongs to H_s at $(\bar{y}_0, 0, \eta^{0\prime})$, if $s \leq N + 1/2$.*

Proof. The first observation is as follows: Let $U(t)$ be the one parameter group defined in Theorem 1.12 of [14]. If $\tilde{F} \in \mathcal{D}(A^N)$, then the first component $u_0(t,x)$ of $U(t)\tilde{F}$ belongs to $H^{\mathrm{loc}}_{N+1}(\mathbf{R} \times \bar{\Omega})$. Then $v(t,x) = u(t,x) - u_0(t,x)$ satisfies (1) and (2) and the condition $({}^t v(t_0,x), {}^t(\partial_t v)(t_0,x)) = 0$ if $x \in \omega \cap \Omega$. From these conditions and Theorem 3.1 of [14] it implies that $v(t,x) = 0$ for $|t - t_0| < \epsilon_0$ and $x \in \omega' \setminus \Omega$, where ϵ_0 is sufficiently small and ω' is an open subset of ω such that $\bar{\omega}' \subset \omega$. This means that $u(t,x)_{|\partial\Omega}$ belongs to $H_{N+1/2}((t_0 - \epsilon_0, t_0 + \epsilon_0) \times \partial\Omega)$.
The second observation is as follows: Let us consider the boundary value problem (26) and (27), where $\tilde{M}(y, D_{y'})$ has the form $\begin{pmatrix} E_+ & 0 \\ 0 & E_- \end{pmatrix} (y, D_{y'})$ with $n \times n$ matrix E_+ and E_-. Then we have the boundary equation

$$B_+(y', D_{y'})V_+ = G_+, \tag{30}$$

where $V_{|y_n=0} = {}^t({}^tV_+, {}^tV_-)$ and for any $s \leq N+1/2$ $G_+ = (G - B_+V_-)_{|y_n=0} \in H_s$ at $\rho_0 = (\bar{y}_0, 0, \eta^{0\prime})$. We assume that the Lopatinski determinant $R(y', D_{y'})$ of $B_+(y', D_{y'})$ is zero at ρ_0. Then there exists the null bicharacteristic strip γ of $R(y', \eta')$ passing through ρ_0. From the condition $\partial R/\partial\eta_0(\rho_0) \neq 0$ γ is parameterized by y_0. Let $\gamma^\pm$ be $\{\gamma(y_0)\colon \pm y_0 > \bar{y}_0\}$. In (30) from the left hand side multiply a pseudodifferential operator $C(y', D_{y'})$ whose principal symbol is the cofactor matrix of the principal symbol of $B_+(y', D_{y'})$. Then we get the equation $\tilde{R}(y', D_{y'})V_+ = \tilde{g}_+$, where the principal symbol of $\tilde{R}(y', D_{y'})$ is $R(y, \eta')E$ and $\tilde{g}_+ \in H_s$ at ρ_0. From the argument in the proof of Theorem 2.1 in Chapter VI of [16], we have the following statement: If V_+ belongs to H_{s-1} on $\gamma_+(\gamma_-)$, then V_+ belongs to H_{s-1} on γ. From the first observation we may always assume that $V_{+|y_n=0}$ belongs to H_{s-1} on γ_+ or γ_-, where $s - 1 \leq N - 1/2$. Repeatedly making use of the argument in the second observation we can prove the desired property on u. The proof is completed. $\square$

6. Diffractive cases

In this section we shall consider a point $(0, \eta^{0\prime})$ such that $r_\rho(0, \eta^{0\prime}) = 0$ and $\{\eta_n - a, r_\rho\}(0, \eta^{0\prime}) < 0$, where $\{\ ,\ \}$ is the Poisson bracket and ρ is μ or $\lambda + 2\mu$. This point is called a diffractive point with respect to ρ. The construction of a parametrix to the boundary value problem (5) and (6) with the diffractive boundary is done in [20] and propagations of regularities near diffractive points in C^∞ category is studied in [19] and [20]. The proofs of theorems appeared in this section are deeply depend on the arguments and the results in [19] and [20]. The readers refer to these papers. First we shall consider the free boundary condition and a diffractive point with respect to μ. Let γ_μ be the null bicharacteristic strip of $(\eta_n - a)^2 + r_\mu(y, \eta')$ passing through $(0, \eta^{0\prime}, 0)$. From the definition of diffractive points there exist half rays $\gamma_\mu^\pm$ such that $\gamma_\mu = \gamma_\mu^+ \cup (0, \eta^{0\prime}, 0) \cup \gamma_\mu^-$ and $\gamma_\mu^\pm \subset T^*\{y_n > 0\}$. We have the following

Theorem 6.1 *Let $(0, \eta^{0\prime})$ be a diffractive point with respect to μ and B_0 in* (6) *be the free boundary condition. In* (5) *we assume that f belongs to $C^\infty([0, T_0) : \mathcal{D}'(\mathbf{R}^n)) \cap L^2([0, T_0] : H_s^{\mathrm{loc}}(U_0 \cap \{y_n = 0\}))$ for $s \geq 0$ and $T_0 > 0$, and satisfies the following condition: There exists a pseudodifferential operator $\phi(y, D_{y'})$ of order* 0*, which is elliptic at $(0, \eta^{0\prime})$ such that $\phi(y, D_{y'})f \in C^\infty(\mathbf{R}^n \times [0, T_0))$. In* (6) *we suppose that g belongs to H_s at $(0, \eta^{0\prime})$. If a solution u in* (5) *and* (6) *belongs to H_{s+1} on γ_μ^ϵ, where $\epsilon = \pm$, then $u(y', 0)$ belongs to $H_{s+2/3}$ at $(0, \eta^{0\prime})$ and $(D_{y_n}u)(y', 0)$ belongs to $H_{s-1/3}$ at $(0, \eta^{0\prime})$.*

Proof. For simplicity we assume $n = 3$. In the arbitrary dimension case the proof is done by the same method to one appeared here. We shall use the decomposition (1.17), (1.18), (1.19) and (1.20) appeared in Section 1.4 of [19]. Here $V = {}^t({}^tV_2, v_+, v_-)$, $F = {}^t({}^tf_2, f_+, f_-)$ and $G = {}^t({}^tg_2, g_+, g_-)$ are similarly defined to these appeared in (25) and (28). If γ_μ^ϵ is defined by $\{(y(t), \eta(t));\ t > 0\}$, then we define $-\gamma_\mu^\epsilon$ by $\{y(-t), -\eta(-t);\ t < 0\}$, which is a part of the null bicharacteristic strip of $(\eta_n - a)^2 + r_\mu$ passing through $(0, -\eta^{0\prime})$. Let $\psi(y', D_{y'})$ be a pseudodifferential operator of order 0 whose symbol has the support in a small conic neighbourhood of $(0, -\eta^{0\prime})$. Then for any $h \in H_{-s+1/3}(\mathbf{R}^3)$ we shall consider the distribution $F_{\epsilon'}(\psi h)$ defined in Proposition 4.2 of [20]. Here ϵ' is $+$ or $-$ and $F_{\epsilon'}(\psi h)$ satisfies the condition $WF(F_{\epsilon'}(\psi h)_{|y_n>0}) \cap (-\gamma_\mu^{-\epsilon}) = \emptyset$. Since in the proof of Proposition 4.2 in [20] $F_{\epsilon'}(\psi h)$ is defined from $G^{(1)}v^{(1)}$ which is denoted in (18) of [20], we shall state the properties of $G^{(1)}v^{(1)}$. That is a continuous linear operator from $\mathcal{D}'(U_0)$ to $C^\infty([0,T) : \mathcal{D}'(U_0))$, where U_0 is an open neighbourhood of $0 \in \mathbf{R}^3$. From Theorem 2.5.11' in [7] and Proposition 4.2 of [20] the wave front set of $G^{(1)}(v^{(1)})(\,\cdot\,, y_3^0)$ for fixed y_3^0 is contained in the set $\{(y',\eta') \in T^*(\mathbf{R}^3) \setminus 0;\ (y',\eta')$ is a projected point to $T^*\{y_3 = y_3^0\}$ of $\gamma_\mu^{\epsilon'}$, whose starting point belongs to $WF(v^{(1)})\}$. There exist small positive numbers c_1 and c_2 such that for any $y_3 \in [c_1, c_2]$ $G^{(1)}(y_3)$ is a composition operator of a Fourier integral operator on $\mathbf{R}^3$ with a non-degenerate phase function and a symbol belonging to $S_{1,0}^{-1/6}$ and an operator A with the form $\widehat{Av}(\eta') = A(\alpha|\eta'|^{2/3})^{-1}\chi(\eta')\hat{v}(\eta')$, where $\left|A(\alpha|\eta'|^{2/3})^{-1}\right| \le C(1+|\eta'|)^{1/6}$ and $\chi(\eta')$ is a cut function near $-\eta^{0\prime}$. Thus if $v^{(1)} \in H_{-s}(\mathbf{R}^3)$, then $G^{(1)}v^{(1)} \in C([c_1, c_2];\ H_{-s}(\mathbf{R}^3))$. Here we remark that we can take arbitrary small c_1 depending on the size of the support of $\chi(\eta')$ appeared in (18) of [20].

We define the extension of V_2^c of V_2 such that $V_2^c = V_2$, if $y_3 \ge 0$, and $V_2^c = 0$, if $y_3 < 0$. Similarly we also define f_2^c. Take a scalar function $\rho(y) \in C_0^\infty(U)$ which is identically 1 near 0, where U is a neighbourhood of $0 \in \mathbf{R}^4$, and put ${}^tE_+(h) = \tilde{F}_{\epsilon'}(\psi h)$, where the component of $\tilde{F}_{\epsilon'}(\psi h)$ is equal to the first four components of $F_{\epsilon'}(\psi h)$. Then we have

$$\begin{aligned}&\langle (D_{y_n} - M_2)(V_2^c), \rho^tE_+(h)\rangle \\ &\quad = \langle f_2^c, \rho^tE_+(h)\rangle - i\langle V_2(\,\cdot\,,0), \rho^tE_+(h)(\,\cdot\,,0)\rangle_\partial,\end{aligned} \tag{31}$$

where $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle_\partial$ are usual bilinear forms on $L^2(\mathbf{R}^4)$ and $L^2(\mathbf{R}^3)$, respectively. The form (31) is equivalent to

$$\begin{aligned}&\langle V_2(\,\cdot\,,0), \rho^tE_+(h)(\,\cdot\,,0)\rangle_\partial = -i\langle V_2^c, \rho(D_{y_3} + {}^tM_2)^tE_+(h)\rangle \\ &\quad - i\langle V_2^c, [D_{y_3} + {}^tM_2, \rho]^tE_+(h)\rangle - i\langle \rho\phi f_2^c, {}^tE_+(h)\rangle \\ &\quad - i\langle f_2^c, {}^t(1-\phi)\rho^tE_+(h)\rangle,\end{aligned} \tag{32}$$

where $\phi = \phi(y', D_{y'})$ is a pseudodifferential operator of order 0 such that the support of the symbol $\phi(y',\eta')$ of $\phi(y', D_{y'})$ is in a conic neighbourhood of $(0,\eta^{0\prime}) \in T^*\{y_3 = 0\}$ and $\phi(y',\eta') = 1$ in $\{(y',\eta') \in T^*\{y_n = 0\};\ (y',-\eta') \in \operatorname{supp}\psi\}$. From Lemma 4.1 of [20] there exists the pseudodifferential operator $B_2(y, D_{y'})$ whose form is a 4×2 matrix such that $V_{2|y_3=0} = B_2(y',0,D_{y'})^t(v_2,v_3)_{|y_3=0} + g_1$, where $V_2 = (v_1,\dots,v_4)$ and g_1 belongs to H_s at $(0,\eta^{0\prime})$. Then the left hand side of (32) is equal to

$$\langle {}^t(v_2, v_3), \rho^t B_2^t E_+(h)(y', 0)\rangle_\partial + \langle {}^t(v_2, v_3), [{}^t B_2, \rho]^t E_+(h)(y', 0)\rangle_\partial \\ + \langle g_1, \rho^t E_+(h)(y', 0)\rangle_\partial. \tag{33}$$

From Proposition 4.2 of [20] for any $h \in H_{1/3-s}(\mathbf{R}_3)$ we have a distribution ${}^tE_+(h)$ which satisfies the conditions that ${}^tE_+(h)$ is a linear operator and there exists a pseudodifferential operator $\psi_1(y', D_{y'})$ such that $\psi_1\rho({}^tB_2({}^tE_+(h))_{|y_n=0} - \psi h)$, $(1-\psi_1)\rho^t B_2({}^tE_+(h))_{|y_n=0}$ and $(1-\psi_1)\rho\psi h$ are smooth functions on $\mathbf{R}^3$. Here we may assume that $\psi(y', D_{y'})$ is elliptic at $(0, -\eta^{0\prime})$. From the argument sated in the back part of (52) in [20] and the fourth line to the last of p.368 in [20] we have to take

$$\hat{v}^{(1)}(\eta') = \begin{pmatrix} c_0 K^{-1} & 0 \\ 0 & 1 \end{pmatrix} \hat{v}^{(2)}(\eta')$$

in (18) of [20], where

$$(c_0K^{-1})(\eta') = (1+|\eta'|^2)^{1/6} A(\alpha|\eta'|^{2/3})/A'(\alpha|\eta'|^{2/3})$$

with $\alpha = \eta_0/|\eta'|$ and Airy function $A(s)$, which belongs to $S^{1/3}_{1/3,0}$. If $h \in H_{-s+1/3}(\mathbf{R}^3)$, then we take $v^{(2)}$ as an element of $H_{-s+1/3}(\mathbf{R}^3)$, which means that $v^{(1)}$ is in $H_{-s}(\mathbf{R}^3)$. It implies that ${}^tE_+(h)$ is a continuous linear operator from $H_{1/3-s}(\mathbf{R}^3)$ to $C([0,T] : H_{-s}(\mathbf{R}^3))$.

Let us consider the null bicharacteristic strip $\{(y(t,\rho), \eta(t,\rho))\}$ of $(\eta_n - a)^2 + r_\mu$ starting at ρ near the diffractive point ρ_0. If ρ_0 is a diffractive point, then from the Malgrange's preparation theorem we have $y_3(t,\rho) = A(t,\rho)\{t^2 + B(\rho)t + C(\rho)\}$, where $A(0,\rho_0) > 0$, $B(\rho_0) = 0$ and $C(\rho_0) = 0$. It implies that there exist a small constant c_1, c_2 and a small conic neighbourhood Γ_0 of ρ_0 such that $(dy_3/dt)(t,\rho) \neq 0$ for all $c_1 < t < c_2$, $\rho \in \Gamma_0$. Thus we can use the statement i) of Proposition 4.1 for $c_1 < t < c_2$. From the conditions on f and ${}^tE_+(h)$, the statement i) of Proposition 4.1, and the Banach's closed graph theorem we see that all terms of the right hand side of (32) are continuous linear functionals on $H_{1/3-s}(\mathbf{R}^3)$. From (33) we also see that $\langle {}^t(v_2, v_3), \rho\psi h\rangle_\partial$ is a linear functional on $H_{1/3-s}(\mathbf{R}^3)$. This fact and Lemma 4.1 in [20] imply that $u(y', 0)$ belongs to $H_{s+2/3}$ at $(0, \eta^{0\prime})$ and $(D_{y_n}u)(y', 0)$ belongs to $H_{s-1/3}$ at $(0, \eta^{0\prime})$. The proof of Theorem 6.1 is complete. □

Next we shall consider the Dirichlet boundary condition. In this case the Lopatinski determinant $R(x', \eta')$ defined in (46) of [20] is not zero at

$(0,\eta^{0\prime})$. Thus we do not need the special form of $\hat{v}_1(\eta')$. We have the following

Theorem 6.2 *Let $(0,\eta^{0\prime})$ be a diffractive point with respect to μ, and B_0 in* (6) *be the Dirichlet condition. In* (5) *we assume that f satisfies the same condition stated in* Theorem 6.1. *In* (6) *we suppose that $g \in H_{s+1}$ at $(0,\eta^{0\prime})$, where $s \geq 0$. If a solution u of* (5) *and* (6) *belongs to H_{s+1} on γ_μ^ϵ, where $\epsilon = \pm$, then $u(y',0)$ belongs to H_{s+1} at $(0,\eta^{0\prime})$ and $(D_{y_n}u)(y',0)$ belongs to H_s at $(0,\eta^{0\prime})$.*

In order to prove a propagation of regularities we need a lemma and a theorem. An operator $P(y,D_y)$ has the form $D_{y_n}^m + \sum_{j=0}^{m-1} p_j(y,D_{y'})D_{y_n}^j$, where $p_j(y,D_{y'})$ is of order $m-j$ and the symbol of $p_j(y,D_{y'})$ is independent of y, if $|y|$ is large. Then we have the following

Lemma 6.3 *Let s be a non-negative number. Then for any $f \in H_s(\bar{\mathbf{R}}_+^{n+1})$ there exists $w \in H_{s+m}(\bar{\mathbf{R}}_+^{n+1})$ such that the* 0 *extension of $f + P(y,D_y)w$ to $\{y_n < 0\}$ belongs to $H_r(\mathbf{R}^{n+1})$ with $r = \max([s], s-1/2)$, where $[s]$ is the Gauss symbol of s.*

Proof. From the form of $P(y,D_y)$ and Theorem 9.4 in p.41 of [10] there exists $w \in H_{s+m}(\bar{\mathbf{R}}_+^{n+1})$ such that $D_{y_n}^j(Pw+f)(y',0) = 0$ $(j = 0,\ldots,[s-1/2])$. Put $g = Pw + f$. Then

$$D_{y_n}^j(g^c) = (D_{y_n}^j g)^c \qquad j = 1,\ldots,[s],$$

where h^c is the 0 extension to $\{y_n < 0\}$ of h. One of equivalent conditions of $h \in H_s(\Omega)$ is the condition that $D_n^j h \in H_{s-[s],[s]-j}(\Omega)$ for $j = 0,\ldots,[s]$, where $\Omega = \mathbf{R}^{n+1}$ or $\bar{\mathbf{R}}_+^{n+1}$. If $0 \leq s-[s] \leq 1/2$, then we have that $D_n^j(g^c) \in H_{0,[s]-j}(\mathbf{R}^{n+1})$ for $j = 0,\ldots,[s]$. This means that $g^c \in H_{[s]}(\mathbf{R}^{n+1}) \subseteq H_{s-1/2}(\mathbf{R}^{n+1})$. If $1/2 < s-[s] < 1$, then we have that $\Lambda^{[s]-j}D_n^j g \in H_{s-[s]}(\mathbf{R}_{n+1}^+)$ for $j = 0,\ldots,[s]$, where $\Lambda^k = (1+|D_{y'}|^2)^{1/2}$. From Theorem 11.4 in p.60 of [10] it follows that $(\Lambda^{[s]-j}D_n^j g)^c = \Lambda^{[s]-j}D_n^j(g^c) \in H_{s-[s]-1/2}(\mathbf{R}^{n+1})$ for $j = 0,\ldots,[s]$. This means that g^c belongs to $H_{s-1/2}(\mathbf{R}^{n+1})$. The proof is completed. □

The following theorem is due to the proof of Theorem 2.1.4 of [6] and the statement in Lemma 8.33 of [1].

Theorem 6.4 *Let $Q(y, D_y)$ be a classical pseudodifferential operator of order m in $\mathbf{R}^{n+1}$ with the symbol $q(y, \eta) \sim \sum q_j(y, \eta)$ such that for some C each term $q_j(y, \eta)$ is a rational function of η_n when $C|\eta_n| > |\eta'| + 1$, where $\eta' = (\eta_0, \dots, \eta_{n-1})$. Then for any μ, ν, N, and a small number a in the domain $\{0 \leq y_n < a\}$ $D^{\mu}_{y_n}(Q(u \otimes \delta^{(\nu)}_n))$ is expressed by the following form, where $\delta^{(\nu)}_n$ is the ν-th derivation of Dirac function with respect to $y_n = 0$;*

$$P_{\mu,\nu,N}(y, D_{y'})u + (2\pi)^{-n} \int e^{i\langle y', \eta'\rangle} q_{\mu,\nu,N}(y, \eta')\hat{u}(\eta')\, d\eta'. \tag{34}$$

Here $P_{\mu,\nu,N}(y, D_{y'})$ is a pseudodifferential operator with a symbol $p_{\mu,\nu,N}(y, \eta') \in C([0, a); S^{m+\mu+\nu+1}_{1,0}(\mathbf{R}^n))$ and $q_{\mu,\nu,N}(y, \eta')$ satisfies the condition that for any multi index α and any compact set K in $\mathbf{R}^n$ there exists a constant $C_{K,N}$ such that $|(D^{\alpha}_y q_{\mu,\nu,N})(y, \eta')| \leq C_{N,K}(1 + |\eta'|)^{-N}$ for $(y, \eta') \in [0, a) \times K \times \mathbf{R}^n_{\eta'}$.

Our theorem on propagations of regularities near diffractive points with respect to μ is as follows:

Theorem 6.5 *Let $(0, \eta^{0\prime})$ be a diffractive point with respect to μ and f and the solution u in* (5) *and* (6) *satisfies the all assumptions stated in* Theorem 6.1. *Moreover we suppose that $f \in H^{\mathrm{loc}}_s(U_0 \cap \{y_n > 0\})$ with some neighbourhood U_0 of $0 \in \mathbf{R}^{n+1}$. Then we have the following two statements.*
i) *If the boundary condition is the free boundary condition and g belongs to H_s at $(0, \eta^{0\prime})$, then u belongs to $H_{s+1/6}$ on $\gamma^{-\epsilon}_{\mu} \cup \{(0, \eta^{0\prime})\}$.*
ii) *If we consider the Dirichlet condition and g belongs to H_{s+1} at $(0, \eta^{0\prime})$, then u belongs to $H_{s+1/2}$ on $\gamma^{-\epsilon}_{\mu} \cup \{(0, \eta^{0\prime})\}$.*

Proof. We only show the statement i). Making use of a cut function we may consider the boundary value problem (5) and (6) in $\mathbf{R}^{n+1}_+$ with $f \in H_s(\bar{\mathbf{R}}^{n+1}_+)$. We apply Lemma 6.3 to $f \in H_s(\bar{\mathbf{R}}^{n+1}_+)$ and $L(y, D_y)$. Denote the 0 extension of g by g^c. Then we have the following

$$\begin{aligned} L((u + w)^c) = (f + Lw)^c - A(u + w)(y', 0) \otimes \delta^{(1)}_{y_n} \\ - i(AD_{y_n} + B)(u + w)(y', 0) \otimes \delta_{y_n}, \end{aligned} \tag{35}$$

where $L(y, D_y) = A(y, D_{y'})D^2_{y_n} + B(y, D_{y'})D_{y_n} + C(y, D_{y'})$. From Lemma 6.3, Theorem 6.1 and usual computations the right hand side of (35) belongs to $H_{s-5/6}$ at $(0, \eta^{0\prime}, 0)$. Thus by Theorem 3.1 it follows that $(u + w)^c$ belongs to $H_{s+1/6}$ on γ_{μ}. It implies that there exists a pseudodifferential operator $p(y, D_y)$ of order 0, whose symbol is supported in a small conic neighbourhood of $(0, \eta^{0\prime}, 0) \in T^*(\mathbf{R}^{n+1}) \setminus 0$ and is elliptic at $(0, \eta^{0\prime}, 0)$, such that

$$p(y, D_y)(u+w)^c \in H_{s+1/6}(\mathbf{R}^{n+1}). \tag{36}$$

Since L is elliptic near $(0, (0, \dots, 0), \eta_n)$ with $\eta_n \neq 0$, there exists a properly supported classical pseudodifferential operator $Q(y, D_y)$ of order -2, which satisfies the condition in Theorem 6.4, such that $QL = I + R$, where $WF(R) \cap \{(y, \eta) \in T^*R_{n+1} \setminus 0;\ |\eta'| \leq \delta_0 |\eta_n|, |y_n| \leq \delta_0\} = \emptyset$. From (35) we have

$$\begin{aligned}(u+w)^c = &-R(u+w)^c + Q(f+Lw)^c - Q(A(u+w)(y',0) \otimes \delta_{y_n}^{(1)} \\ &+ i(AD_{y_n} + B)(u+w)(y',0) \otimes \delta_{y_n}).\end{aligned} \tag{37}$$

The second and third terms in the right hand side of (37) belong to $H_{s+1/6}$ from Lemma 6.3 and Theorem 6.4. Let us consider the operator $\phi(y, D_{y'})R$, which is a pseudodifferential operator on $\mathbf{R}^{n+1}$ from Proposition 2.3. Since the support of the symbol $\phi(y, \eta')$ of $\phi(y, D_{y'})$ is contained in $\Gamma_1 = \big\{(y, \eta');\ |y| < \delta_1,\ \big|\eta^{0\prime}/|\eta^0| - \eta'/|\eta'|\big| < \delta_1\big\}$ for small δ_1 and $L(y, D_y)$ is elliptic in $(WF(R) \cap \{(y, \eta); (y, \eta') \in \Gamma_1\}) \setminus \{(y, \eta);\ (y, \eta') \in \Gamma_1,\ |\eta_n| < \delta_2 |\eta'|\}$ for small δ_2 from (36), we see that $\phi(y, D_{y'})R(u+w)^c$ belongs to $H_{s+1/6}$. The proof is completed. $\square$

Next we shall consider a diffractive point $(0, \eta^{0\prime})$ with respect to $\lambda + 2\mu$. In this case the situation does not depend on boundary condition. Since $r_\mu(0, \eta^{0\prime}) < 0$, we have the null bicharacteristic strip γ_μ passing through $\big(0, \eta^{0\prime}, a(0, \eta^{0\prime}) \pm (-r_\mu(0, \eta^{0\prime}))^{1/2}\big)$. The half rays belonging to $T^*\{y_n > 0\}$ are denoted by γ_μ^+ and γ_μ^-. The following theorem is proved by the decomposition $(1,54)_\pm$, $(1.55)_\pm$ and $(1.56)_\pm$ in [19] and the argument used in the proof of Theorem 6.5.

Theorem 6.6 *Let $(0, \eta^{0\prime})$ be a diffractive point with respect to $\lambda + 2\mu$. The data f and g in* (5) *and* (6) *satisfy conditions mentioned in* Theorem 6.1 *and* Theorem 6.2. *Moreover we suppose that $f \in H_s^{\mathrm{loc}}(U_0 \cap \{y_n > 0\})$ for some neighbourhood U_0 of $0 \in \mathbf{R}^{n+1}$. If a solution u of* (5) *and* (6) *belongs to H_{s+1} on $\gamma_\mu^\epsilon \cup \gamma_{\lambda+2\mu}^{\epsilon'}$, where ϵ and ϵ' are $+$ or $-$, then u belong to $H_{s+1/2}$ on $\gamma_{\lambda+2\mu}^{-\epsilon'} \cup \{(0, \eta^{0\prime})\}$ and u also belongs to H_{s+1} on $\gamma_\mu^{-\epsilon'}$.*

In our proof the following condition is essential: there exists a pseudodifferential operator $\phi(y, D_{y'})$ of order 0, which is elliptic at $(0, \eta^{0\prime})$, such that $\phi(y, D_{y'})f \in C^\infty(\mathbf{R}^n \times [0, T_0])$, because we can not prove the L^2 continuity of the operators with the form (18) in [20]. However under some conditions on f we can eliminate the above essential condition. We have the following

Remark 6.7 In (1) and (2) we assume that $f = f(x)$ has the following property; there exists $(0, \tilde{f}) \in \mathcal{D}(A^N)$ such that $f = \tilde{f}$ in $U_0 \cap \Omega$, where A is the operator defined in (0.5) of [14] for an isotropic elastic equation and U_0 is a neighbourhood of $0 \in \mathbf{R}^n$. Then for $s \leq N + 1$ a solution u of (5) and (6) has the same properties stated in Theorem 6.5 and Theorem 6.6.

7. Preliminaries for analysis near glancing points

We shall say that $(0, \eta^{0\prime}) \in T^*\{y_n = 0\} \setminus 0$ is a glancing point with respect to ρ, if $r_\rho(0, 0, \eta^{0\prime}) = 0$ with the function r_ρ appeared in (10). After this section we devote to analyze propagations of regularities to solutions near these glancing points. In this section we shall study the boundary value problem for the operator $P(y, D_{y'})$ which is reduced from (5) and (6) near the considered glancing point. $P(y, D_{y'})$ has the following form:

$$(D_{y_n} - a)E_{2k} - \begin{pmatrix} 0 & \Lambda E_k \\ -R_\rho E_k & 0 \end{pmatrix} - \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{38}$$

where E_k is the $k \times k$ unit matrix, $a = a(y, D_{y'})$ is the differential operator of order 1 defined by the symbol (9), $\Lambda = (1 + |D_{y'}|^2)^{1/2}$, $R_\rho(y, D_{y'})$ is of order 1 and its principal symbol is $|\eta'|^{-1} r_\rho(y, \eta')$, and $A_{ij} = A_{ij}(y, D_{y'})$ is of order 0. The boundary condition $B(y, D_{y'})$ is a $k \times 2k$ matrix and their components are of order 0. We assume that the principal symbol of $B(y, D_{y'})$ is maximal rank at $(0, \eta^{0\prime})$ which is a glancing point with respect to ρ. Let us consider a solution $V(y) \in C([0, \infty) : \mathcal{D}'(\mathbf{R}^n))$ of the following problem

$$P(y, D_y)V = F \quad \text{in} \quad \{y_n > 0\} \tag{39}$$

$$B(y, D_{y'})V = G \quad \text{on} \quad \{y_n = 0\}, \tag{40}$$

where for $s \geq 0$ and some r_1 and r_2 $F \in H_{(s, r_1)}(\bar{\mathbf{R}}^{n+1}_+) \cap H_{(1, r_2)}(\bar{\mathbf{R}}^{n+1}_+)$ satisfies the condition that there exists a pseudodifferential operator $\phi_0(y, D_{y'})$, which is elliptic at $(0, \eta^{0\prime})$, such that $\phi_0(y, D_{y'})F \in H_s(\bar{\mathbf{R}}^{n+1}_+) \cap H_{(1, s-1)}(\bar{\mathbf{R}}^{n+1}_+)$.

On the boundary operator we assume the following condition: Since the principal symbol of B is maximal rank at $(0, \eta^{0\prime})$, we may assume that there exist ${}^t\tilde{V} = (\tilde{v}_1, \ldots, \tilde{v}_k)$, where $\tilde{v}_j$ is one of the components of V, and $D_j(y, D_{y'})$ of order 0 ($j = 1,\, 2$) with a $k \times k$ matrix form such that $BV = G$ is denoted by $V_1 = D_1(y, D_{y'})\tilde{V} + G_1$ and $V_2 = D_2(y, D_{y'})\tilde{V} + G_2$. Here $V = {}^t({}^tV_1, {}^tV_2)$ and G_j ($j = 1,\, 2$) satisfies the same conditions to these which will be imposed on G. Furthermore we assume that there exist $k \times k$ matrices $G_1^{\pm}(y, \eta')$ and $F_1^{\pm}(y, \eta')$, whose components are positively homogeneous of degree 0 with respect to η', such that $G_1^{\pm}(y, \eta')$ is positive definite in a conic neighbourhood Γ of $(0, \eta^{0\prime})$ and

$$\pm\{D_1^* G_1^{\pm} D_2 + D_2^* G_1^{\pm} D_1 + D_2^* F_1^{\pm} D_2 - D_1^* F_1^{\pm} D_1 (r_\rho |\eta'|^{-2}) E_k\} \geq 0 \tag{41}$$

in a conic neighbourhood of $(0, \eta^{0\prime})$, where $D_j(y, \eta')$ ($j = 1,\, 2$) is the principal symbol of $D_j(y, D_{y'})$. Moreover we suppose the following condition: There exist a conic neighbourhood Γ of $(0, \eta^{0\prime})$ and a sufficiently small constant $a_0 > 0$ such that

$$\mathrm{Det}(B_1 + iaB_2)(y', 0, \eta') \neq 0 \tag{42}$$

for all $(y', \eta') \in \Gamma$ and $0 < a < a_0$, where the principal symbol of B in (6) is denoted by $(B_1(y, \eta'), B_2(y, \eta'))$ with the $k \times k$ matrix B_j ($j = 1,\, 2$).

For a subset $\Gamma^{(1)} \subset T^*\{y_n > 0\} \backslash 0$ and a subset $\Gamma^{(0)} \subset T^*\{y_n = 0\} \backslash 0$ the definition that V belongs to H_s on $\Gamma^{(0)} \cup \Gamma^{(1)}$ is denoted in Definition 3.1 and Definition 4.1. We denote by $\exp\{tH_{R_\rho^0}\}(0, \eta^{0\prime})$ the bicharacteristic strip of $R_\rho^0 = |\eta'|^{-1} r_\rho(y', 0, \eta')$ through $(0, \eta^{0\prime})$. Put

$$\begin{aligned}\Gamma_\epsilon^{(0)} = \big\{&(y', \eta') \in T^*\{y_n = 0\} \setminus 0;\ r_\rho(y', 0, \eta') \leq 0,\\ &\big|(y', \eta'/|\eta'|) - (0, \eta^{0\prime}/|\eta^{0\prime}|)\big| \leq \epsilon^2\big\}\end{aligned}$$

and

$$\begin{aligned}\Gamma_\epsilon^{(1)} = \big\{&(y, \eta) \in T^*\{y_n > 0\} \setminus 0;\ (\eta_n - a(y, \eta'))^2 + r_\rho(y, \eta') = 0,\\ &0 < y_n \leq \epsilon^2,\ \big|(y', \eta'/|\eta'|) - (0, \eta^{0\prime}/|\eta^{0\prime}|)\big| \leq \epsilon^2\big\}\end{aligned}$$

We shall show the following key theorem on propagations of regularities:

Theorem 7.1 *Let $V \in C([0, \infty) : \mathcal{D}'(\mathbf{R}^n))$ satisfy the boundary value problem* (39) *and* (40)*, and $(0, \eta^{0\prime})$ be a glancing point with respect to ρ. Here F satisfies the conditions stated in the back part of* (40)*. We assume* *that there exist a positive constant a_1 and a neighbourhood of U_1 of $0 \in \mathbf{R}^n$ such that for any $y \in U_1 \times (0, a_1)$ V belongs to H_s at (y, η) with $|\eta_n| > \alpha|\eta'|$, where $\alpha > 0$ depends on y. We suppose* (41) *and* (42)*, and that G belongs to $H_{s+1/2}$ at $(0, \eta^{0\prime})$. Then there exist $\epsilon_0 > 0$ and $\delta > 0$, which are independent of V, such that if V belongs to H_s on $\Gamma^{(0)}_{\epsilon_1} \cup \Gamma^{(1)}_{\epsilon_1}$ for some ϵ_1 ($\epsilon_1 \leq \epsilon_0$), then at any point (y', η') of $\{\exp\{tH_{R^0_\rho}\}(0, \eta^{0\prime});\ |t| < \delta\epsilon_1\}$ V belongs to H_s.*

The first remark to prove Theorem 7.1 is that from Theorem 4.3.1 in [5] we may show that there exists a pseudodifferential operator $\phi(y, D_{y'})$, which is elliptic at $(y', 0, \eta')$ with $(y', \eta') \in \{\exp\{tH_{R^0_\rho}\}(0, \eta^{0\prime});\ |t| < \delta\epsilon_1\}$, such that $\phi(y, D_{y'})V \in H_{(0,s)}(\bar{\mathbf{R}}^{n+1}_+)$. The second remark is that we may assume $a(y, D_{y'}) = 0$. In order to show this fact we need the coordinate transform χ; $z' = \chi'(y', y_n)$, $z_n = y_n$ with $\chi'(y', 0) = y'$ defined in Remark 1.2 of [19]. It is not difficult to show that if $F \in C^\infty([0, \epsilon_0) : \mathcal{D}'(\mathbf{R}^n))$, then $F \circ \chi \in C^\infty([0, \epsilon_2) : \mathcal{D}'(\mathbf{R}^n))$ for some $\epsilon_2 > 0$. We have the following lemma, which is easily proved from Theorem 6.4 in Chapter 2 of [8], if we regard that $z' = \chi'(y', y_n)$ is a coordinate transform in $\mathbf{R}^n$ with the smooth parameter y_n.

Lemma 7.2 *We assume that F, G and V in* (39) *and* (40) *satisfy all conditions in* Theorem 7.1*. Then we have the following three statements.*
i) *$V \circ \chi$ satisfies $\tilde{P}(z, D_z)(V \circ \chi) = F \circ \chi$ in $\{z_n > 0\}$, where the form of $\tilde{P}(z, D_z)$ is the same to one of* (5) *with $a = 0$.*
ii) *There exists a pseudodifferential operator $\tilde{\phi}_0(z, D_{z'})$ of order 0, which is elliptic at $(0, \zeta^{0\prime})$ with $\zeta^{0\prime} = \eta^{0\prime}$, such that $\tilde{\phi}_0(z, D_{z'})(F \circ \chi) \in H_{s+1/2}(\bar{\mathbf{R}}^{n+1}_+)$.*
iii) *The condition that V belongs to H_s on $\Gamma^{(0)}_{\epsilon_1} \cup \Gamma^{(1)}_{\epsilon_1}$ for some ϵ_1 is equivalent to one that $V \circ \chi$ belongs to H_s on $\tilde{\Gamma}^{(0)}_{\tilde{\epsilon}_1} \cup \tilde{\Gamma}^{(1)}_{\tilde{\epsilon}_1}$ for some $\tilde{\epsilon}_1$, where $\tilde{\Gamma}^{(0)}_{\tilde{\epsilon}_1}$ and $\tilde{\Gamma}^{(1)}_{\tilde{\epsilon}_1}$ are similarly defined for (z, ζ) coordinate as $a = 0$.*

From now on we assume that $a(y, D_{y'}) = 0$. In order to prove Theorem 7.1 we need the following

Lemma 7.3 *We assume that F and G in* (39) *and* (40) *satisfy the stated conditions in* Theorem 7.1*, B satisfies the condition* (42)*, and that a solution V of the boundary value problem satisfies the all conditions stated in* Theorem 7.1 *as $\epsilon_1 = \epsilon$. If the support of the symbol of a properly supported pseudodifferential operator $A(y, D_{y'})$, which is of order s, is contained in $\Gamma_\epsilon = \{(y, \eta');\ 0 \leq y_n \leq \epsilon^2,\ \left|(y', \eta'/|\eta'|) - (0, \eta^{0\prime}/|\eta^{0\prime}|)\right| \leq \epsilon^2\}$, then $A(y, D_{y'})V$ belongs to $L^2(\bar{\mathbf{R}}^{n+1}_+)$.*

Proof. First we shall show that if $(\bar{y}', 0, \bar{\eta}') \in \Gamma_\epsilon$ and $r_\rho(\bar{y}', 0, \bar{\eta}') > 0$, then V belongs to H_s at $(\bar{y}', 0, \bar{\eta}')$. Let us check the Lopatinski determinant of the boundary value problem (39) and (40). The eigen values of the matrix $\begin{pmatrix} 0 & \alpha E_k \\ \beta E_k & 0 \end{pmatrix}$ are $\pm(\alpha\beta)^{1/2}$, where $\alpha\beta \neq 0$, and the eigen vectors of $\pm(\alpha\beta)^{1/2}$ are ${}^t({}^t e_j, \pm\beta^{-1}(\alpha\beta)^{1/2\,t} e_j)$ $(j = 1, \ldots, k)$, where $e_j \in \mathbf{R}^k$ such that the j-th component of e_j is 1 and the other components are 0. Thus if $r_\rho(\bar{y}', 0, \bar{\eta}') > 0$, the boundary value problem (39) and (40) is a parabolic type near $(\bar{y}', 0, \bar{\eta}')$. The Lopatinski matrix of this boundary value problem is $B_1 + i\Lambda^{-1}(r_\rho)^{1/2}B_2$, which is non-singular at $(\bar{y}', 0, \bar{\eta}')$ from the assumption (42). Therefore by the argument used in Section 5 in [19] we have the desired property on V near $(\bar{y}', 0, \bar{\eta}')$. Thus from the assumptions on V there exists a_1 such that $A(y, D_{y'})V$ belongs to $L^2(\mathbf{R}^n \times [0, a_1))$. Let $P_1(y, \eta)$ is the principal symbol of $P(y, D_y)$. Then from the form of P in (38) we have $\operatorname{Det} P_1(y, \eta) = \big((\eta_n)^2 + r_\rho\big)^k$. Thus if $\bar{y}_n > 0$ and $(\bar{\eta}_n)^2 + r_\rho(\bar{y}', \bar{y_n}, \bar{\eta}') \neq 0$, $(\bar{y}, \bar{\eta})$ is an elliptic point of P. Moreover if $\bar{\eta}' \neq 0$, then from Proposition 2.3 we can construct a microlocal parametric of P near $(\bar{y}, \bar{\eta})$, which is a pseudodifferential operator in $\mathbf{R}^{n+1}$. From Proposition 2.3 and the assumption of F we see that F belongs to H_s at the all points contained in $\{(y, \eta) \in T^*(\mathbf{R}^{n+1}) \setminus 0\colon y_n > 0,\ \eta' \neq 0,\ \phi_0(y, \eta') \neq 0\}$, where $\phi_0(y, \eta')$ is the principal symbol of $\phi_0(y, D_{y'})$ appeared in the assumption of F in the back part of (40). From these observations and the assumption of V making use of the partition of unity in the cotangential space $T^*(\mathbf{R}^n \times a_0/2, \infty)) \setminus 0$ and Proposition 2.1, we can show the desired property on $A(y, D_{y'})V$. The proof is complete. $\square$

Let us consider the points along the ray $\exp\{tH_{R_\rho^0}\}(0, \eta^{0\prime})$ for $0 \le t < \delta\epsilon_1$. According to the argument in Section 2 of [11] we introduce coordinate $(t, s) = (s_1, \ldots, s_{2n-1}, t)$ in a neighbourhood of $(0, \eta^{0\prime})$, where $|\eta^{0\prime}| = 1$, such that $(0, \eta^{0\prime})$ is the origin, and the Hamilton vector field $H_{R_\rho^0}$ is $\partial/\partial t$, where $R_\rho^0 = r_\rho(y', 0, \eta')|\eta'|^{-1}$. We denote the coordinate transform by $y' = y'(s, t)$, $\eta' = \eta'(s, t)$ and its inverse transform by $s = s(y', \eta')$, $t = t(y', \eta')$. Let H be the Heaviside function, and put $\chi(u) = H(1 - u)\exp(1/(u - 1))$, and $f(u) = AH(2u-1)\exp(2/(1-2u))$, where $A > 2\exp 2$. Let $\beta \in C^\infty(\mathbf{R})$ vanish on $(-\infty, -1)$, be strictly increasing on $(-1, -1/2)$, and be equal to 1 on $(-1/2, \infty)$. We define the following two functions

$$q_\epsilon(y_n, s, t) = \beta\Big(\frac{at}{\epsilon^2}\Big)\chi\Big(\frac{at}{\delta\epsilon} + \frac{a^2 s^2}{\epsilon^4} + f\Big(\frac{y_n}{\epsilon^2}\Big)\Big) \tag{43}$$

$$g_\epsilon(y_n, s, t) = -\frac{a}{\delta\epsilon}\beta\Big(\frac{at}{\epsilon^2}\Big)\chi'\Big(\frac{at}{\delta\epsilon} + \frac{a^2 s^2}{\epsilon^4} + f\Big(\frac{y_n}{\epsilon^2}\Big)\Big), \tag{44}$$

where δ is fixed independently of ϵ and we choose $\epsilon_0 \ll \delta$. Here we choose $a > 1/2$ such that if $|t| \leq a^{-1}\epsilon^2$ and $|s| \leq a^{-1}\epsilon^2$, then $\big|(y'(t,s), \eta'(t,s))/|(y'(t,s), \eta'(t,s))| - (0, \eta^{0\prime})\big| \leq \epsilon^2$. We also define $q_\epsilon(y, \eta') = q_\epsilon(y_n, s(y', \eta'), t(y', \eta'))$ and $g_\epsilon(y, \eta') = g_\epsilon(y_n, s(y', \eta'), t(y', \eta'))$. For $s \in \mathbf{R}$, $1 \leq \lambda \leq \infty$, we define

$$q_\epsilon^{s,\lambda}(y, \eta') = \int_0^\lambda q_\epsilon(y, \eta'/r) r^{s-1}\, dr \tag{45}$$

and define similarly $g_\epsilon^{s,\lambda}(y, \eta')$. Here $q_\epsilon^{s,\infty}$ and $g_\epsilon^{s,\infty}$ are positively homogeneous of degree s and their supports are same sets and are independent of s. The following properties on $q_\epsilon^{s,\lambda}$ and $g_\epsilon^{s,\lambda}$ are stated in Section 2 of [11].

Lemma 7.4 *The symbols $q_\epsilon^{s,\lambda}$ and $g_\epsilon^{s,\lambda}$ satisfy the following properties*:
i) $q_{\epsilon'}(y, \eta') > 0$ *on* $\operatorname{supp} q_\epsilon$ *and* $g_{\epsilon'}(y, \eta') > 0$ *on* $\operatorname{supp} g_\epsilon$ *if* $0 < \epsilon < \epsilon' \leq \epsilon_0$.
ii) *There exists* $a_\epsilon^{s,\lambda}(y, \eta') \in C^\infty(\mathbf{R}^{n+1} \times \mathbf{R}^n \setminus 0)$ *such that* $g_\epsilon^{s,\lambda}(y, \eta') = (a_\epsilon^{s,\lambda}(y, \eta'))^2$ *and it satisfies symbol estimates* $|\partial_y^\alpha \partial_{\eta'}^\beta a_\epsilon^{s,\lambda}(y, \eta')| \leq C_{\alpha,\beta} |\eta'|^{s-|\beta|}$, *where* $C_{\alpha,\beta}$ *does not depend on* $\lambda \in [1, \infty)$.

We also need the following properties on $q_\epsilon^{s,\lambda}$, which are easily derived from the definition of $q_\epsilon^{s,\lambda}$ and the relation $\chi(u) = -\chi'(u)(1-u)^2$.

Lemma 7.5 *We have the following two statements*:
i) $0 \leq q_\epsilon(y, \eta') \leq 4a^{-1}\delta\epsilon g_\epsilon(y, \eta')$.
ii) *The first derivatives of* $q_\epsilon^{s,\lambda}$ *satisfy that*

$$\frac{\partial q_\epsilon^{s,\lambda}}{\partial y_j}(y, \eta') = \alpha_\epsilon^{s,\lambda}(y, \eta') + \beta_\epsilon^{s,\lambda}(y, \eta'),$$

$$\frac{\partial q_\epsilon^{s,\lambda}}{\partial \eta_j}(y, \eta') = \gamma_\epsilon^{s-1,\lambda}(y, \eta') + \delta_\epsilon^{s-1,\lambda}(y, \eta'),$$

where $\{\alpha_\epsilon^{s,\lambda}\}_{\lambda\geq 1}$ *and* $\{\beta_\epsilon^{s,\lambda}\}_{\lambda\geq 1}$ *are bounded sets in* $S_{1,0}^s$, $\{\gamma_\epsilon^{s-1,\lambda}\}_{\lambda\geq 1}$ *and* $\{\delta_\epsilon^{s-1,\lambda}\}_{\lambda\geq 1}$ *are bounded sets in* $S_{1,0}^{s-1}$, $|\alpha_\epsilon^{s,\lambda}(y, \eta')| + |\eta'|\,|\gamma_\epsilon^{s,\lambda}(y, \eta')| \leq O(1)\delta\epsilon^{-1} g_\epsilon^{s,\lambda}(y, \eta')$, *and* $\operatorname{supp} \beta_\epsilon^{s,\lambda} \cup \operatorname{supp} \delta_\epsilon^{s-1,\lambda}$ *is contained in* Γ_ϵ *for* $1 \leq \lambda \leq \infty$, *where* Γ_ϵ *is appeared in* Lemma 7.3.

Theorem 7.1 will follow if we can show that for all fixed $\epsilon < \epsilon_1$ and $r = s$

$$\sup_{\lambda \geq 1} \left\| \Lambda^{-1/2} A_\epsilon^{r+1/2,\lambda}(y, D_{y'}) V \right\|_{L^2} < \infty, \tag{46}$$

where $\Lambda = (1 + |D_{y'}|^2)^{1/2}$ and $A_\epsilon^{r,\lambda}(y, D_{y'}) = a_\epsilon^{r,\lambda}(y, D_{y'}) E_k$ with the pseudodifferential operator $a_\epsilon^{r,\lambda}(y, D_{y'})$ whose symbol is $a_\epsilon^{r,\lambda}(y, \eta')$ stated in ii) of Lemma 7.4. Inductively we may assume that (46) for $r = s - 1/2$ holds, because when r is sufficiently negative, it holds. Under this assumption we have the following

Lemma 7.6 *Let $b_\epsilon^{2s-1,\lambda}(y, D_{y'})$ be a pseudodifferential operator of order $2s-1$ with the symbol $b_\epsilon^{2s-1,\lambda}(y, \eta')$ such that $\{b_\epsilon^{2s-1,\lambda}(y, \eta')\}_{\lambda \geq 1}$ is a bounded set in $S_{1,0}^{2s-1}$ and $\operatorname{supp} b_\epsilon^{2s-1,\lambda} \subset \operatorname{supp} g_\epsilon^{s,\infty}$. If we assume that for $V = E_k u$ (46) holds as $r = s - 1/2$, then we have*

$$|(b_\epsilon^{2s-1,\lambda}(y, D_{y'}) u, u)| \leq C_{\epsilon,s}, \tag{47}$$

where $C_{\epsilon,s}$ does not depend on $\lambda \in [1, \infty)$.

Proof. From i) of Lemma 7.4 if $\epsilon < \epsilon' < \epsilon_0$, then $a_{\epsilon'}^{s-1/2,\infty}(y, D_{y'})$ is elliptic on $\operatorname{supp} g_\epsilon^{s,\infty}$. Thus there exists $C_\epsilon^{0,\lambda}(y, D_{y'})$ of order 0 such that the symbols of $C_\epsilon^{0,\lambda}(y, \eta')$ $(\lambda \geq 1)$ form a bounded set in $S_{1,0}^0$ and $b_\epsilon^{2s-1,\lambda}(y, D_{y'}) = ((\Lambda^{-1/2} a_{\epsilon'}^{s,\infty})^* C_\epsilon^{0,\lambda} (\Lambda^{-1/2} a_{\epsilon'}^{s,\infty})) + d_\epsilon^\lambda(y, D_{y'})$, where the symbols of $d_\epsilon^\lambda(y, D_{y'})$ $(\lambda \geq 1)$ form a bounded set in $S^{-\infty}$. From (46) for $r = s - 1/2$ we complete the proof. $\square$

8. The proof of Theorem 7.1

In this section making use of lemmas in Section 7, we shall prove Theorem 7.1. Let $q_2(y, D_{y'})$ be a formally self-adjoint pseudodifferential operator such that the support of the symbol is contained in $\operatorname{supp} g_\epsilon^{s,\infty}$ and its principal symbol is $q_\epsilon^{2s+1,\lambda}(y, \eta')$. The essential property of $q_1 = \Lambda^{-1} q_2$ is as follows

Lemma 8.1 *Let $R_\rho(y, D_{y'})$ be a properly supported pseudodifferential operator of order 1 with the principal symbol $r_\rho(y, \eta')|\eta'|^{-1}$. If (46) holds for $V = E_k u$ as $r = s - 1/2$ and u satisfies the same assumptions on V in*

Theorem 7.1*, then there exist positive constants C and $C_{\epsilon,s}$ such that*

$$\operatorname{Re}(i[R_\rho, q_1]u, u) \geq C\|\Lambda^{-1/2}a_\epsilon^{s+1/2,\lambda}u\|_{L^2}^2 - C_{\epsilon,s}, \tag{48}$$

where C does not depend on u, ϵ, δ and λ, and $C_{\epsilon,s}$ is independent of λ.

Proof. From Lemma 7.6 and (46) for $r = s - 1/2$ we calculate the principal symbols of pseudodifferential operators appeared in the following equality:

$$[R_\rho, q_1] = \Lambda^{-1}[R_\rho^0, q_2] + \Lambda^{-1}[R_\rho - R_\rho^0, q_2] + [R_\rho, \Lambda^{-1}]q_2, \tag{49}$$

where $R_\rho^0 = R_{\rho|y_n=0}$. From Lemma 7.4 and Lemma 7.5 the second and third terms of (49) are denoted by $A_\epsilon^{2s,\lambda}(y, D_{y'}) + B_\epsilon^{2s,\lambda}(y, D_{y'})$ such that $\{A_\epsilon^{2s,\lambda}(y,\eta')\}_{\lambda\geq 1}$ and $\{B_\epsilon^{2s,\lambda}(y,\eta')\}_{\lambda\geq 1}$ are bounded sets in $S_{1,0}^{2s}$, $\operatorname{supp} A_\epsilon^{2s,\lambda} \subset \operatorname{supp} g_\epsilon^{s,\infty}$, $\operatorname{supp} B_\epsilon^{2s,\lambda} \subset \Gamma_\epsilon$, where Γ_ϵ is appeared in Lemma 7.3, and the absolute value of the principal symbol to $A_\epsilon^{2s,\lambda}$ is dominated by $O(1)\delta\epsilon|\eta'|^{-1}g_\epsilon^{2s+1,\epsilon}(y,\eta')$, where we use the fact that $r_\rho(y,\eta') - r_\rho(y', 0, \eta') = 0(\epsilon^2)$ on $\operatorname{supp} q_\epsilon^{2s+1,\infty}$. Let $\chi_\epsilon(y,\eta')$ be a symbol in $S_{1,0}^0$ such that $\chi_\epsilon = 1$ in $\operatorname{supp} g_\epsilon^{s,\infty}$ and $\operatorname{supp}\chi_\epsilon \subset \operatorname{supp} g_{\epsilon'}^{s,\infty}$ for some $\epsilon < \epsilon' < \epsilon_0$. Making use of the sharp Gårding inequality for $A_\epsilon^{2s,\lambda}(y, D_{y'}) + O(1)\delta\epsilon(\Lambda^{-1/2}a_\epsilon^{s+1/2,\lambda})^*$ $(\Lambda^{-1/2}a_\epsilon^{s+1/2,\lambda})$, Lemma 7.3, Lemma 7.5 and Lemma 7.6 we have

$$\begin{aligned}\operatorname{Re}(Nu, u) &\geq \operatorname{Re}(N\chi_\epsilon(y, D_{y'})u, \chi_\epsilon(y, D_{y'})u) - C_{\epsilon,s}\\ &\geq -O(1)\delta\epsilon\|\Lambda^{-1/2}a_\epsilon^{s+1/2,\lambda}u\|^2 - O(1)\|\chi_\epsilon u\|_{s-1/2}^2 - C'_{\epsilon,s}\\ &\geq -O(1)\delta\epsilon\|\Lambda^{-1/2}a_\epsilon^{s+1/2,\lambda}u\|^2 - C''_{\epsilon,s}\end{aligned} \tag{50}$$

where $N = i(\Lambda^{-1}[R_\rho - R_\rho^0, q_2] + [R_\rho, \Lambda^{-1}]q_2)$ and $C_{\epsilon,s}$, $C'_{\epsilon,s}$ and $C''_{\epsilon,s}$ are independent of λ. Since the principal symbol of R_ρ^0 is positively homogeneous of degree 1, for $a(y,\eta') = \tilde{a}(y_n, t(y',\eta'), s(y',\eta'))$ we can show that

$$H_{R_\rho^0}(a(y,\eta'/\lambda)) = \frac{\partial\tilde{a}}{\partial t}\big(y_n, t(y',\eta'/\lambda), s(y',\eta'/\lambda)\big).$$

It implies that the principal symbol of $i[R_\rho^0, q_2]$ have the following form; $|\eta'|^{-1}g_\epsilon^{2s+1,\lambda}(y,\eta') + C_\epsilon^{2s,\lambda}(y,\eta')$, where $\{C_\epsilon^{2s,\lambda}(y,\eta')\}_{\lambda\geq 1}$ is a bounded set in $S_{1,0}^{2s}$ and $\operatorname{supp} C_\epsilon^{2s,\lambda} \subset \Gamma_\epsilon$, where Γ_ϵ is in Lemma 7.3. Making use of $\chi_\epsilon(y, D_{y'})$ in (50) and the sharp Gårding inequality, we see that

$$\operatorname{Re}(i[R_\rho^0, q_2]u, u) \geq C_1\|\Lambda^{-1/2}a_\epsilon^{s+1/2,\lambda}u\|^2 - C_{\epsilon,s}, \tag{51}$$

where C_1 is positive and independent of u, ϵ, δ and λ, and $C_{\epsilon,s}$ is independent of λ. From (50) and (51) the proof is completed. □

Let $G_1(y, D_{y'})$ and $F_1(y, D_{y'})$ be properly supported pseudodifferential operators with the principal symbols $G_1^+(y, \eta')$ and $F_1^+(y, \eta')$ appeared in (41), respectively. For a pseudodifferential operator $\bar{R}_\rho(y, D_{y'}) = \Lambda^{-1} R_\rho(y, D_{y'})$ with $\Lambda^{-1} = (1 + |D_{y'}|^2)^{-1/2}$ we define the properly supported pseudodifferential operator $q(y, D_{y'}) = (1 - \bar{R}_\rho) q_1$. We also define properly supported pseudodifferential operators

$$Q = \begin{pmatrix} 0 & qE_k \\ q^* E_k & 0 \end{pmatrix}, \quad G_0 = \begin{pmatrix} G_1 & 0 \\ 0 & G_1 \end{pmatrix}, \quad F_0 = \begin{pmatrix} F_1 & 0 \\ 0 & F_1 \end{pmatrix}. \tag{52}$$

From the integration by parts we have the following

Lemma 8.2 *Put $Q_1 = F_0 Q \Lambda^{-1} D_{y_n}$. Then for $V \in C^\infty(\bar{\mathbf{R}}_+; \mathcal{D}'(\mathbf{R}^n))$ we have*

$$\begin{aligned} &- i\{(G_0 QPV, V) - (G_0 QV, PV) + (Q_1 PV, V) - (Q_1 V, PV)\} \\ &\quad = i(G_0(QM - M^*Q)V, V) + i((Q_1 M - M^* Q_1)V, V) \\ &\quad\quad - i([M^*, G_0]QV, V) + (G_0' QV, V) + (G_0 Q' V, V) + (Q_1' V, V) \\ &\quad\quad + (G_1 q V_2, V_1)_\partial + (G_1 q^* V_1, V_2)_\partial + (F_1, q\Lambda^{-1} D_{y_n} V_2, V_1)_\partial \\ &\quad\quad + (F_1^* \Lambda^{-1} D_{y_n} V_1, V_2)_\partial, \end{aligned} \tag{53}$$

where $V = {}^t({}^t V_1, {}^t V_2)$, $P(y, D_y)$ in (39) is denoted by $D_{y_n} - M(y, D_{y'})$, $(\ ,\ )$ and $(\ ,\ )_\partial$ are L^2 products on $\bar{\mathbf{R}}^{n+1}_+$ and $\mathbf{R}^n$, respectively, and the symbols of G_0', Q' and Q_1' are derivatives with respect to y_n of the symbols of G_0, Q and Q_1, respectively.

Let V be a solution of (39) and (40) and satisfy the conditions of Theorem 7.1. For this V we shall check the each term of the both hand sides of (53). In order to estimate the terms we need the following

Lemma 8.3 *Let $C_\epsilon^{2s,\lambda}(y, D_{y'})$ be a $2k \times 2k$ matrix whose all (i, j) components $C_{\epsilon,ij}^{2s,\lambda}(y, D_y')$ are pseudodifferential operators of order $2s$. For all components we assume the following: The symbol $C_{\epsilon,ij}^{2s,\lambda}(y, \eta')$ of $C_{\epsilon,ij}^{2s,\lambda}(y, D_{y'})$ satisfies that $\{C_{\epsilon,ij}^{2s,\lambda}(y,\eta')\}_{\lambda \geq 1}$ is a bounded set of $S_{1,0}^{2s}$, $\operatorname{supp} C_{\epsilon,ij}^{2s,\lambda} \subset \operatorname{supp} g_\epsilon^{s,\infty}$, and the principal symbol $\bar{C}_{\epsilon,ij}^{2s,\lambda}(y, \eta')$ of $C_{\epsilon,ij}^{2s,\lambda}(y, D_{y'})$ has the estimate*

$$|\bar{C}_{\epsilon,ij}^{2s,\lambda}(y, \eta')| \leq C\alpha |\eta'|^{-1} g_\epsilon^{2s+1,\lambda}, \tag{54}$$

where $C > 0$ does not depend on ϵ, δ, λ and α. Then we have the following estimate;

$$|(C_\epsilon^{2s,\lambda}(y, D_{y'})V, V)| \leq C_1\alpha\|\Lambda^{1/2} A_\epsilon^{s+1/2,\lambda} V\|^2 + C_{\epsilon,s}, \tag{55}$$

where $C_1 > 0$ does not depend on ϵ, δ, λ and α, and $C_{\epsilon,s} > 0$ does not depend on λ.

Proof. We may assume that the form of $C_\epsilon^{2s,\lambda}(y, D_{y'})$ is Hermitian, and that for fixed (i,j) the (k,ℓ) component of $C_\epsilon^{2s,\lambda}$ is 0 except (k,ℓ) is equal to (i,j) or (j,i). Then the eigen values of $C\alpha|\eta'|^{-1} g_\epsilon^{2s+1,\lambda} E_{2k} \pm \bar{C}_\epsilon^{2s,\lambda}(y,\eta')$ are $C\alpha|\eta'|^{-1} g_\epsilon^{2s+1,\lambda}$ and $C\alpha|\eta'|^{-1} g_\epsilon^{2s+1,\lambda} \pm |\bar{C}_{\epsilon,ij}^{2s,\lambda}(y,\eta')|$. We denote $C_\epsilon^{2s,\lambda}(y, D_{y'})$ by $\{C_\epsilon^{2s,\lambda} \pm C\alpha(\Lambda^{-1/2} A_\epsilon^{s+1/2,\lambda})^*(\Lambda^{-1/2} A_\epsilon^{s+1/2,\lambda})\} \mp C\alpha(\Lambda^{-1/2} A_\epsilon^{s+1/2,\lambda})^*(\Lambda^{-1/2} A_\epsilon^{s+1/2,\lambda})$. The estimate (55) follows from the sharp Gårding inequality for $C\alpha(\Lambda^{-1/2} A_\epsilon^{s+1/2,\lambda})^*(\Lambda^{-1/2} A_\epsilon^{s+1/2,\lambda}) \pm C_\epsilon^{2s,\lambda}$ and the estimate for $\chi_\epsilon(y, D_{y'})V$, where $\chi_\epsilon(y, D_{y'})$ is used in (50). The proof is complete. □

The following lemma is a key part to verify Theorem 7.1.

Lemma 8.4 *Under the assumptions in* Theorem 7.1 *we have the following estimate*

$$\mathrm{Re}(i(G_0(QM - M^*Q)V, V) \geq C\|\Lambda^{-1/2} A_\epsilon^{s+1/2,\lambda} V\|^2 - C_{\epsilon,s}, \tag{56}$$

where C is positive and independent of V, ϵ, δ and λ, and $C_{\epsilon,s}$ is independent of λ.

Proof. The principal symbol $G_1(y,\eta')$ of $G_1(y, D_{y'})$ is positive definite near $(0, \eta^{0\prime})$. However we may assume that $G_1(y,\eta')$ is globally positive definite. Thus there exists a positive definite matrix $G_2(y,\eta')$, which is smooth, such that $G_1 = (G_2)^2$ by the implicit function theorem (see the proof of Lemma 3.2.3 in [7]). Making use of Proposition 2.2.2 in [7], we have the elliptic properly supported pseudodifferential operator $\tilde{G}(y, D_{y'})$ such that

$$\tilde{G} = \begin{pmatrix} G_2(y, D_{y'}) & 0 \\ 0 & G_2(y, D_{y'}) \end{pmatrix}, \quad G_0(y, D_{y'}) = \tilde{G}^*\tilde{G} + R,$$

where $R \in L^{-\infty}$. In (52) we put $q = (1 - \bar{R}_\rho)q_1$, where $\bar{R}_\rho = \Lambda^{-1} R_\rho$. Then from $q_1^*\Lambda - \Lambda q_1 = 0$ we see that

$$QM-M^*Q=\begin{pmatrix} C_1E_k & 0 \\ 0 & C_2E_k \end{pmatrix}+A_\epsilon^\lambda, \tag{57}$$

where $A_\epsilon^\lambda(y,D_{y'})$ is of order $2s$ whose symbol $A_\epsilon^\lambda(y,\eta')$ have the support in $\operatorname{supp} g_\epsilon^{s,\infty}$, $\{A_\epsilon^\lambda(y,\eta')\}_{\lambda\geq 1}$ is a bounded set in $S^{2s}_{1,0}$, and the principal symbols of all components in $A_\epsilon^\lambda(y,D_{y'})$ are dominated by $O(1)\delta\epsilon|\eta'|^{-1}g_\epsilon^{2s+1,\lambda}(y,\eta')$ from i) of Lemma 7.5. Furthermore $C_1(y,D_{y'})$ is equal to

$$C_1=[R_\rho,q_1]+R_\rho[q_2,\Lambda^{-1}]+[q_2,R_\rho]\Lambda^{-1}R_\rho\Lambda^{-1}, \tag{58}$$

and $C_2(y,D_{y'})$ is equal to

$$C_2=[R_\rho,q_1]. \tag{59}$$

Since $|r_\rho(y,\eta')|\leq O(1)\epsilon^2|\eta'|^2$ in $\operatorname{supp} g_\epsilon^{s,\infty}$, where $O(1)$ does not depend on ϵ, δ and λ, from Lemma 7.5 all terms in the right hand sides of (58) except $[R_\rho,q_1]$ satisfy the same conditions to these of $A_\epsilon^\lambda(y,D_{y'})$. Therefore we have

$$(G_0(QM-M^*Q)V,V)=([R_\rho,q_1]\tilde{G}V,\tilde{G}v)+(N_1V,V), \tag{60}$$

where $N_1=\tilde{G}^*[\tilde{G},[R_\rho,q_1]]+G_0A_{1,\epsilon}^\lambda$ with $A_{1,\epsilon}^\lambda(y,D_{y'})$ which satisfies the same conditions on $A_\epsilon^\lambda(y,D_{y'})$ in (57). Making use of Lemma 7.6, Lemma 8.1, Lemma 8.3 and the elliptic estimate for $\tilde{G}$, we have (56). The proof is completed. □

Next we shall consider $N_2=i(Q_1M-M^*Q_1-[M^*,G_0]Q-iG_0'Q)$ in (53). From the definitions of Q_1 and M we see that N_2 is denoted by the following form;

$$\left\{\begin{pmatrix} iF_1[R_\rho,q_1] & 0 \\ 0 & iF_1[R_\rho,q_1] \end{pmatrix}+B_{1,\epsilon}^\lambda\right\}\Lambda^{-1}D_{y_n}+B_{2,\epsilon}^\lambda, \tag{61}$$

where $B_{j,\epsilon}^\lambda$ ($j=1,2$) is of order $2s$, the symbol $B_{j,\epsilon}^\lambda(y,\eta')$ of $B_{j,\epsilon}^\lambda$ satisfies that $\operatorname{supp} B_{j,\epsilon}^\lambda\subset\operatorname{supp} g_\epsilon^{s,\infty}$ and $\{B_{j,\epsilon}^\lambda\}_{\lambda\geq 1}$ is a bounded set in $S^{2s}_{1,0}$, and the absolute values of the principal symbols of all components of $B_{j,\epsilon}^\lambda(y,D_{y'})$ are dominated by $O(1)\delta\epsilon|\eta'|^{-1}g_\epsilon^{2s+1,\lambda}(y,\eta')$. From the proof of Lemma 8.1 we see that $iF_1[R_\rho,q_1]=(\Lambda^{-1/2}A_\epsilon^{s+1/2,\lambda})^*F_1(\Lambda^{-1/2}A_\epsilon^{s+1/2,\lambda})+b_\epsilon^{2s-1,\lambda}$, where $b_\epsilon^{2s-1,\lambda}(y,D_{y'})$ is of order $2s-1$, the support of the symbol $b_\epsilon^{2s-1,\lambda}(y,\eta')$ is contained in $\operatorname{supp} g_\epsilon^{s,\infty}$, and $\{b_\epsilon^{2s-1,\lambda}(y,\eta')\}_{\lambda\geq 1}$ is a bounded set in $S^{2s-1}_{1,0}$. From Lemma 7.6, Lemma 8.3 and the equality $\Lambda^{-1}D_{y_n}V=\Lambda^{-1}MV+\Lambda^{-1}F$

it follows that

$$|(N_2V,V)| \le C\|\Lambda^{-1/2}A_\epsilon^{s+1/2,\lambda}\Lambda^{-1}D_{y_n}V\|\,\|\Lambda^{-1/2}A_\epsilon^{s+1/2,\lambda}V\| + C\delta\epsilon\|\Lambda^{-1/2}A_\epsilon^{s+1/2,\lambda}V\|^2 + C_{\epsilon,s}, \tag{62}$$

where C is independent of ϵ, δ and λ, and $C_{\epsilon,s}$ is independent of λ. From (1.40), (1.41) and (1.42) in [19] we have

$$|(N_2V,V)| \le C\delta\|\Lambda^{-1/2}A_\epsilon^{s+1/2,\lambda}V\|^2 + C_{\epsilon,s}. \tag{63}$$

Let us check $(G_0Q'V,V)$ in (53), which is equal to

$$(G_1q'V_2,V_1) + (G_1(q^*)'V_1,V_2),$$

where $V = {}^t({}^tV_1, {}^tV_2)$. Making use of $V_2 = (\Lambda + A_{12})^{-1}\{D_{y_n}V_1 + A_{11}V_1 - F_1\}$, where $F = {}^t({}^tF_1, {}^tF_2)$ and $\partial q_\epsilon/\partial y_n = -(\delta/\epsilon)f'(y_n/\epsilon^2)g_\epsilon$, we have

$$\begin{aligned} &G_1q'(\Lambda + A_{12})^{-1}(D_{y_n} + A_{11}) \\ &\quad = -\big\{\delta\epsilon^{-1}f'(y_n/\epsilon^2)(\Lambda^{-1/2}A_\epsilon^{s+1/2,\lambda})^*G_1(\Lambda^{-1/2}A_\epsilon^{s+1/2,\lambda}) \\ &\qquad + C_{\epsilon,1}^{2s-1,\lambda}\big\}\Lambda^{-1}D_{y_n} + C_{\epsilon,2}^{2s-1,\lambda}, \end{aligned} \tag{64}$$

where $C_{\epsilon,j}^{2s-1,\lambda}(y, D_{y'})$ $(j=1,2)$ is of order $2s-1$ such that $\{C_{\epsilon,j}^{2s-1,\lambda}(y,\eta')\}_{\lambda\ge1}$ is a bounded set in $S_{1,0}^{2s-1}$ and $\operatorname{supp} C_{\epsilon,j}^{2s-1,\lambda} \subset \operatorname{supp} g_\epsilon^{s,\infty}$. A similar representation holds for $q'G_1^*(\Lambda + A_{12})^{-1}(D_{y_n} + A_{11})$. From (1.40), (1.41) and (1.42) in [19] we have

$$|(G_0Q'V,V)| \le O(1)\delta\|\Lambda^{-1/2}A_\epsilon^{s+1/2,\lambda}V\|^2 + C_{\epsilon,s}. \tag{65}$$

Since $(Q_1'V,V) = (F_0'Q\Lambda^{-1}D_{y_n}V,V) + (F_0Q'\Lambda^{-1}D_{y_n}V,V)$, from the same argument of deriving (63) and (65) we have

$$|(Q_1'V,V)| \le O(1)\delta\|\Lambda^{-1/2}A_\epsilon^{s+1/2,\lambda}V\|^2 + C_{\epsilon,s}. \tag{66}$$

Let us start the estimates of the boundary terms in (53). From the assumption on B stated in the forward part of (41), $V_j = D_j(y', D_{y'})\tilde V + H_j$ $(j = 1, 2)$, where H_j satisfies the same condition to one of G. Thus $(G_1qV_2,V_1)_\partial + (G_1q^*V_1,V_2)_\partial$ is equal to

$$((D_1^*G_1D_2 + D_2^*G_1D_1)q\tilde V, \tilde V)_\partial + (N_3\tilde V,\tilde V)_\partial + R_3, \tag{67}$$

where $N_3 = D_1^*G_1[q,D_2] + D_2^*G_1([q_2,\Lambda^{-1}](1-\bar R_\rho) - q + [\bar R_\rho, \Lambda^{-1}q_2])D_1 + D_2^*G_1[q,D_1]$, and $R_3 = (G_1qD_2\tilde V, H_1)_\partial + (G_1qH_2, D_1\tilde V)_\partial + (G_1qH_2,H_1)_\partial +$

$(G_1q^*H_1, D_2\tilde{V})_\partial + (G_1q^*D_1\tilde{V}, H_2)_\partial + (G_1q^*H_1, H_2)_\partial$. From Lemma 7.5 we can denote N_3 by $C_{\epsilon,1}^{2s-1,\lambda}(y', D_{y'}) + C_{\epsilon,2}^{2s-1,\lambda}(y', D_{y'})$, where $\{C_{\epsilon,j}^{2s-1,\lambda}\}_{\lambda\geq 1}$ $(j = 1, 2)$ is a bounded set in $S_{1,0}^{2s-1}$, $\operatorname{supp} C_{\epsilon,1}^{2s-1,\lambda} \subset \operatorname{supp} g_\epsilon^{s,\infty}$, $\operatorname{supp} C_{\epsilon,2}^{2s-1,\lambda} \subset \Gamma_\epsilon$, where Γ_ϵ is in Lemma 7.3, and the absolute value of the principal symbol to $C_{\epsilon,1}^{2s-1,\lambda}(y', D_{y'})$ is dominated by $O(1)\delta\epsilon^{-1}|\eta'|^{-2}g_\epsilon^{2s+1,\lambda}(0, y', \eta')$. From the sharp Gårding inequality it is not difficult to show that

$$|(C_{\epsilon,1}^{2s-1,\lambda}\tilde{V}, \tilde{V})_\partial| \leq O(1)\delta\epsilon^{-1}\|\Lambda^{-1}A_\epsilon^{s+1/2,\lambda}\tilde{V}\|_\partial^2 + C_{s,\epsilon}\|\Lambda^{s-1}\chi_\epsilon\tilde{V}\|_\partial^2, \tag{68}$$

where $\|\cdot\|_\partial$ is the L^2 norm on $\{y_n = 0\}$ and $\chi_\epsilon(y, D_{y'})$ is the pseudodifferential operator appeared in (50). We shall use the classical trace inequality $\|\Lambda^{-1/2}v\|_\partial^2 \leq 2\|v\| \cdot \|\Lambda^{-1}D_{y_n}v\|$. Since there exists $k\times 2k$ matrix $M_2(y, D_{y'})$ of order 1 such that $D_{y_n}\tilde{V} = M_2(y, D_{y'})V + \tilde{F}$, where $\tilde{F}$ is made from F, we see that

$$\|\Lambda^{s-1/2}\chi_\epsilon\tilde{V}\| + \|\Lambda^{-1}D_{y_n}\Lambda^{s-1/2}\chi_\epsilon\tilde{V}\| \leq C_{s,\epsilon}. \tag{69}$$

From the classical trace inequality and (1.41) in [12], we also see that

$$\|\Lambda^{-1}A_\epsilon^{s+1/2,\lambda}\tilde{V}\|_\partial^2 \leq O(1)\epsilon\|\Lambda^{-1/2}A_\epsilon^{s+1/2,\lambda}\tilde{V}\|^2 + C_{s,\epsilon}. \tag{70}$$

From the classical trace inequality and the property on the support of $C_{\epsilon,2}^{2s-1\lambda}(y, D_{y'})$ it follows that

$$|(C_{\epsilon,2}^{2s-1,\lambda}\tilde{V}, \tilde{V})_\partial| \leq C_{s,\epsilon}. \tag{71}$$

From (68), (69) and (70) we have

$$|(N_3\tilde{V}, \tilde{V})_\partial| \leq O(1)\delta\|\Lambda^{-1/2}A_\epsilon^{s+1/2,\lambda}V\|^2 + C_{s,\epsilon}, \tag{72}$$

where $C_{s,\epsilon}$ is independent of λ.

The required conditions on q_2 are the following; q_2 is formally self adjoint and the principal symbol of q_2 is $q_\epsilon^{2s+1,\lambda}(y, \eta')$. Here since the function $q_\epsilon^{2s+1,\lambda}$ is nonnegative, from the proof of Lemma 2.28 in [11] there exists a function $b_\epsilon^{s+1/2,\lambda}(y, \eta')$ such that $q_\epsilon^{2s+1,\lambda} = (b_\epsilon^{s+1/2,\lambda})^2$, $\{b_\epsilon^{s+1/2,\lambda}\}_{\lambda\geq 1}$ is a bounded set in $S_{1,0}^{s+1/2}$ and

$$b_\epsilon^{s+1/2,\lambda}(y, \eta') \leq O(1)(\delta\epsilon)^{1/2}a_\epsilon^{s+1/2,\lambda}(y, \eta'). \tag{73}$$

Therefore we define

$$q_2(y, D_{y'}) = (b_\epsilon^{s+1/2,\lambda}(y, D_{y'}))^*(b_\epsilon^{s+1/2,\lambda}(y, D_{y'})),$$

where $b_\epsilon^{s+1/2,\lambda}(y, D_{y'})$ is the pseudodifferential operator with the symbol $b_\epsilon^{s+1/2,\lambda}(y, \eta')$. Let us check the term R_3 in (67). We shall consider the term $(G_1 q^* D_1 \tilde{V}, H_2)_\partial$ in R_3, which is equal to $(\Lambda^{-1} b_\epsilon^{s+1/2,\lambda} \tilde{V}, D_1^*(1-\bar{R}_\rho) b_\epsilon^{s+1/2,\lambda} G_1^* H_2)_\partial + ([b_\epsilon^{s+1/2,\lambda}, \Lambda^{-1}(1-\bar{R}_\rho)^* D_1]\tilde{V}, b_\epsilon^{s+1/2,\lambda} G_1^* H_2)_\partial + ([\Lambda^{-1}, (1-\bar{R}_\rho)D_1] b_\epsilon^{s+1/2,\lambda}\tilde{V}, b_\epsilon^{s+1/2,\lambda} G_1^* H_2)_\partial$. The absolute value of this quantity is dominated by $\|\Lambda^{-1} b_\epsilon^{s+1/2,\lambda}\tilde{V}\|_\partial^2 + \|\Lambda^{s-3/2}\chi_\epsilon \tilde{V}\|_\partial^2 + C_{s,\epsilon}$, where χ_ϵ is the pseudodifferential operator in (50). It is not difficult to show that $\|\Lambda^{-1} b_\epsilon^{s+1/2,\lambda}\tilde{V}\|_\partial^2 \le O(1)\delta\epsilon \|\Lambda^{-1} a_\epsilon^{s+1/2,\lambda}\tilde{V}\|_\partial^2 + C_{s,\epsilon}\|\Lambda^{s-1}\chi_\epsilon \tilde{V}\|_\partial^2$. Thus from the similar argument we have that

$$|R_3| \le O(1)\delta \|\Lambda^{-1/2} A_\epsilon^{s+1/2,\lambda} V\|^2 + C_{s,\epsilon}. \tag{74}$$

Since $\Lambda^{-1} D_{y_n} V_2 = \bar{R}_\rho V_1 + A_{21} V_1 + A_{22} V_2$ and $\Lambda^{-1} D_{y_n} V_1 = V_2 + A_{11} V_1 + A_{12} V_2$, where $A_{ij}(y, D_{y'})$ is of order -1, we see that $(F_1 q \Lambda^{-1} D_{y_n} V_2, V_1)_\partial + (F_1 q^* \Lambda^{-1} D_{y_n} V_1, V_2)_\partial$ is equal to

$$\big((D_2^* F_1 D_2 - D_1^* F_1 \bar{R}_\rho D_1) q \tilde{V}, \tilde{V}\big)_\partial + (N_4 \tilde{V}, \tilde{V})_\partial + R_4, \tag{75}$$

where N_4 is equal to $D_2^* F_1 [q^*, D_2] + D_2^* F_1 D_2 [q_2, \Lambda^{-1}] - D_1^* F_1 [q, \bar{R}_\rho^0] D_1 - D_1^* F_1 \bar{R}_\rho^0 [q, D_1] + D_1^* F_1 q (A_{21} D_1 + A_{22} D_2) + D_2^* F_1 q^* (A_{11} D_1 + A_{12} D_2)$. Here R_4 has the form $-(F_1 q \bar{R}_\rho D_1 \tilde{V}, H_1)_\partial + (F_1 q^* D_2 \tilde{V}, H_2)_\partial + \big(q(A_{-1}\tilde{V} + B_0 H_1 + C_0 H_2), D_0 \tilde{V} + E_0 H_1\big)_\partial + \big(q^*(\bar{A}_{-1}\tilde{V} + \bar{B}_0 H_1 + \bar{C}_0 H_2), \bar{D}_0 \tilde{V} + \bar{E}_0 H_2\big)_\partial$, where A_{-1} and $\bar{A}_{-1}$ are pseudodifferential operators of order -1, and B_0, C_0, D_0, E_0, $\bar{B}_0$, $\bar{C}_0$, $\bar{D}_0$ and $\bar{E}_0$ are pseudodifferential operators of order 0, which are independent of ϵ, δ and λ. From similar computations to derive (74) we have that

$$|R_4| \le O(1)\delta \|\Lambda^{-1/2} A_\epsilon^{s+1/2} V\|^2 + C_{s,\epsilon}. \tag{76}$$

N_4 has the same properties to these of N_3. Thus it follows that

$$|(N_4 \tilde{V}, \tilde{V})_\partial| \le O(1)\delta \|\Lambda^{-1/2} A_\epsilon^{s+1/2} V\|^2 + C_{s,\epsilon}. \tag{77}$$

Final part is the estimate of $(X q \tilde{V}, \tilde{V})_\partial$ which comes from (67) and (75), where $X = D_1^* G_1 D_1 + D_2^* G_1 D_1 + D_2^* F_1 D_2 - D_1^* F_1 \bar{R}_\rho D_1$ with $G_1 = G_1^+$ and

$F_1 = F_1^+$ in (41) and $q = (1-\bar{R}_\rho)\Lambda^{-1}q_2$. Put

$$\begin{aligned} N_5 = {} & X_1\Lambda^{-1/2}[\Lambda^{-1/2}, (b_\epsilon^{s+1/2,\lambda})^*]b_\epsilon^{s+1/2,\lambda} \\ & + [X_1, \Lambda^{-1/2}](b_\epsilon^{s+1/2,\lambda})^*\Lambda^{-1/2}b_\epsilon^{s+1/2,\lambda} \\ & + \Lambda^{-1/2}[X_1, (b_\epsilon^{s+1/2,\lambda})^*]\Lambda^{-1/2}b_\epsilon^{s+1/2,\lambda} \\ & + [\Lambda^{-1/2}, (b_\epsilon^{s+1/2,\lambda})^*]X_1\Lambda^{-1/2}b_\epsilon^{s+1/2,\lambda}, \end{aligned} \tag{78}$$

where $X_1 = X(1-\bar{R}_\rho)$. Then the absolute value of the principal symbol of N_5 is dominated by $O(1)\delta\epsilon^{-1}|\eta'|^{-2}g_\epsilon^{2s+1,\lambda}(y,\eta')$, and $(Xq\tilde{V},\tilde{V})_\partial = (X_1\Lambda^{-1/2}b_\epsilon^{s+1/2,\lambda}\tilde{V}, \Lambda^{-1/2}b_\epsilon^{s+1/2,\lambda}\tilde{V})_\partial + (N_5\tilde{V},\tilde{V})_\partial$. From the sharp Gårding inequality we have that

$$|(N_5\tilde{V},\tilde{V})_\partial \le O(1)\delta\epsilon^{-1}\|\Lambda^{-1}A_\epsilon^{s+1/2,\lambda}\tilde{V}\|_\partial^2 + C_{s,\epsilon}\|\Lambda^{s-1}\chi_\epsilon\tilde{V}\|_\partial^2,$$

where χ_ϵ is in (50). From (70) we see that

$$|(N_5\tilde{V},\tilde{V})_\partial| \le O(1)\delta\|\Lambda^{-1/2}A_\epsilon^{s+1/2}V\|^2 + C_{s,\epsilon}. \tag{79}$$

From the similar computation we have

$$\begin{aligned} & \mathrm{Re}(X_1\Lambda^{-1/2}b_\epsilon^{s+1/2}\tilde{V}, \Lambda^{-1/2}b_\epsilon^{s+1/2}\tilde{V})_\partial \\ & \quad \ge -O(1)\delta\|\Lambda^{-1/2}A_\epsilon^{s+1/2}V\|^2 - C_{s,\epsilon}, \end{aligned}$$

which implies that

$$\mathrm{Re}(X_1\Lambda^{-1}q_2\tilde{V},\tilde{V})_\partial \ge -O(1)\delta\|\Lambda^{-1/2}A_\epsilon^{s+1/2}V\|^2 - C_{s,\epsilon}, \tag{80}$$

where $C_{s,\epsilon}$ does not depend on λ.

Let us remark on the left hand side of (53). From the definition of $q_2(y, D_{y'})$ appeared in the back part of (73), Lemma 7.5 and Lemma 8.3 we see that the left hand side of (53) is dominated by

$$\begin{aligned} & \|\Lambda^{-1/2}b_\epsilon^{s+1/2,\lambda}F\|^2 + \|\Lambda^{-1/2}b_\epsilon^{s+1/2,\lambda}\Lambda^{-1}D_{y_n}F\|^2 \\ & \quad + O(1)\delta\epsilon\|\Lambda^{-1/2}A_\epsilon^{s+1/2,\lambda}V\|^2 + C_{\epsilon,s} \end{aligned}$$

Finally from (53), (58), (63), (65), (66), (77), (79) and (80), we have

$$\sup_\lambda \|\Lambda^{-1/2}A_\epsilon^{s+1/2,\lambda}V\|^2 \le C_{s,\epsilon}, \tag{81}$$

which is equal to (46) for $r = s$. The proof of Theorem 7.1 is complete.

The proof of Theorem 7.1 also implies the following

Theorem 8.5 *Let B, V, F and G satisfy all conditions in* Theorem 7.1. *We suppose that $G_1^{\pm}(y,\eta')$ and $F_1^{\pm}(y,\eta')$ in* (41) *satisfy the following strong condition than one of* (41): *$\pm X_{\pm}$ has a form $\begin{pmatrix} X_{\pm}^0 & 0 \\ 0 & X_{\pm}^1 \end{pmatrix}$, where $X_{\pm}$ is the principal symbol of $D_1^* G_1^{\pm} D_2 + D_2^* G_1^{\pm} D_1 + D_2^* F_1^{\pm} D_2 - D_1^* F_1^{\pm} D_1 \bar{R}_\rho$, and the $\ell \times \ell$ matrix $X_{\pm}^0(y,\eta')$ is positive definite near $(0,\eta^{0\prime})$ and the $(k-\ell) \times k-\ell)$ matrix $X_{\pm}^1(y,\eta')$ is non negative definite near $(0,\eta^{0\prime})$. Then $\tilde{V}_0 = (\tilde{V}_1,\dots,\tilde{V}_\ell)$ have the following regularity, where $\tilde{V}_j$ is the j-th component of $\tilde{V}$; For any point $(\bar{y}',\bar{\eta}')$ of $\{\exp\{tH_{R_\rho^0}\}(0,\eta^{0\prime});\ |t| \le \delta\epsilon_1\}$ $\tilde{V}_{0|y_n=0}$ belongs to H_s at $(\bar{y}',\bar{\eta}')$.*

Proof. Let $X^0(y,D_{y'})$ and $X^1(y,D_{y'})$ be pseudodifferential operators with the principal symbols $X_+^0(y,\eta')$ and $X_+^1(y,\eta')$, respectively. Put $\bar{X}_0 = X^0(1-\bar{R}_\rho)E_\ell$ and $\bar{X}_1 = X^1(1-\bar{R}_\rho)E_{k-\ell}$. Then for X_1 in (78) $(X_1\Lambda^{-1}q_2\tilde{V},\ \tilde{V})_\partial = (\bar{X}_0\Lambda^{-1/2}b_\epsilon^{s+1/2,\lambda}\tilde{V}_0,\ \Lambda^{-1/2}b_\epsilon^{s+1/2,\lambda}\tilde{V}_0)_\partial + (\bar{X}_1\Lambda^{-1/2}b_\epsilon^{s+1/2,\lambda}\tilde{V}_1,\ \Lambda^{-1/2}b_\epsilon^{s+1/2,\lambda}\tilde{V}_1)_\partial + ((X_1 - X_2)\Lambda^{-1/2}b_\epsilon^{s+1/2,\lambda}\tilde{V},\ \Lambda^{-1/2}b_\epsilon^{s+1/2,\lambda}\tilde{V})_\partial + (N_5\tilde{V},\tilde{V})_\partial$, where $\tilde{V}_1 = (\tilde{V}_{\ell+1},\dots,\tilde{V}_k)$, $X_2 = \begin{pmatrix} \bar{X}_0 & 0 \\ 0 & \bar{X}_1 \end{pmatrix}$ and N_5 is defined in (78). From the Gårding inequality we see that there exists a positive constant C such that

$$\begin{aligned} \mathrm{Re}(X_1\Lambda^{-1}q_2\tilde{V},\tilde{V})_\partial &\ge C\|\Lambda^{-1/2}b_\epsilon^{s+1/2,\lambda}\tilde{V}_0\|_\partial^2 \\ &\quad - O(1)\delta\|\Lambda^{-1/2}A_\epsilon^{s+1/2,\lambda}\tilde{V}\|^2 - C_{s,\epsilon}. \end{aligned} \tag{82}$$

Combining derived inequalities in the proof of Theorem 7.1 with (82), we can show that

$$\sup_\lambda\{\|\Lambda^{-1/2}A_\epsilon^{s+1/2,\lambda}V\|^2 + \|\Lambda^{-1/2}b_\epsilon^{s+1/2,\lambda}\tilde{V}_0\|_\partial^2\} < C_{s,\lambda}.$$

The proof is completed. □

We also need the following theorem under the week assumption that G belongs to H_s at $(0,\eta^{0\prime})$.

Theorem 8.6 *Let B, V and F satisfy all conditions in* Theorem 7.1. *We assume the week condition on G of* (40) *such that G belongs to H_s at $(0,\eta^{0\prime})$. We suppose that $\pm X_{\pm}$ of* Theorem 8.5 *is positive definite near $(0,\eta^{0\prime})$. Then at any point $(\bar{y}',\bar{\eta}')$ belonging to $\{\exp\{tH_{R_\rho^0}\}(0,\eta^{0\prime});\ |t| \le \delta\epsilon_1\}$ $V_{|y_n=0}$ belongs to H_s at $(\bar{y}',\bar{\eta}')$.*

Proof. Since X_1 in (82) is positive definite, we see that

$$\mathrm{Re}(X_1\Lambda^{-1}q_2\tilde{V},\tilde{V})_\partial \geq O(1)\|\Lambda^{-1/2}b_\epsilon^{s+1/2,\lambda}\tilde{V}\|_\partial^2 \\ - C\delta\|\Lambda^{-1/2}A_\epsilon^{s+1/2,\lambda}\tilde{V}\|^2 - C_{s,\epsilon}, \tag{83}$$

where C is positive and independent to ϵ, δ and λ. Let us check the terms R_3 in (67) and R_4 in (75) under the week assumption on G. From the form of R_3 we easily derive that for any $\epsilon' > 0$ $|R_3| \leq \epsilon'\|\Lambda^{-1/2}b_\epsilon^{s+1/2,\lambda}\tilde{V}\|_\partial^2 + C_{\epsilon',\epsilon,s}$, where $C_{\epsilon',\epsilon,s}$ is independent of λ. We also have that for any ϵ' $|R_4| \leq \epsilon'\|\Lambda^{-1/2}b_\epsilon^{s+1/2,\lambda}\tilde{V}\|_\partial^2 + O(1)\|\Lambda^{-1}b_\epsilon^{s+1/2,\lambda}\tilde{V}\|_\partial^2 + C_{\epsilon',\epsilon,s}$, where $C_{\epsilon',\epsilon,s}$ is independent of λ. From (83) we wee that

$$\sup_\lambda\{\|\Lambda^{-1/2}A_\epsilon^{s+1/2,\lambda}V\|^2+\|\Lambda^{-1/2}b_\epsilon^{s+1/2,\lambda}\tilde{V}\|_\partial^2\} < C_{s,\lambda}.$$

From the definition of $\tilde{V}$ stated in the forward part of (67) the proof is completed. □

9. Propagation of singularities at glancing points

First we shall show a similar theorem to Theorem 7.1 for the boundary value problem (5) and (6). The most complicate case is that B_0 is the free boundary condition and $(0,\eta^{0\prime})$ is a glancing point with respect to μ, that is, $r_\mu(0,\eta^{0\prime}) = 0$. First we shall consider this case. From the argument in Section 1.4 of [19] there exist pseudodifferential operator $S(y,D_{y'})$ and $K(y,D_{y'})$, which are of order 0, and -1, respectively, such that they satisfy the following condition (see (1.18), (1.19) and (1.20) in [19]): Let $\varphi(y)$ be a function belonging to $C_0^\infty(U_0)$ with U_0 in (5) such that $\varphi = 1$ near $0 \in \mathbf{R}^{n+1}$, and $\phi(y,D_{y'})$ be a pseudodifferential operator of order 0 such that the support of the symbol $\phi(y,\eta')$ of $\phi(y,D_{y'})$ is contained in a conic neighbourhood of $(0,0,\eta^{0\prime})$ and $\phi(y,\eta') = 1$ near $(0,\eta^{0\prime})$. Then

$$V = (1+K)S^{-1}\phi^t({}^t(\Lambda\varphi u), {}^t(D_{y_n}\varphi u)) \tag{84}$$

satisfies the following problems

$$(D_{y_n} - M_2)\bar{V} = F_2 \quad \text{in} \quad y_n > 0 \tag{85}$$
$$B_2(y,D_{y'})\bar{V} = G_2 \quad \text{on} \quad y_n = 0,$$
$$(D_{y_n} - e_+)V_+ = F_+ \quad \text{in} \quad y_n > 0 \tag{86}$$
$$V_+ = G_3 + B_3\bar{V} \quad \text{on} \quad y_n = 0$$
$$(D_{y_n} - e_-)V_- = F_- \quad \text{in} \quad y_n > 0. \tag{87}$$

Here $\Lambda = (1 + |D_{y'}|^2)^{1/2}$, $V = {}^t({}^t\bar{V}, V_+, V_-)$ and $F = {}^t({}^tF_2, F_+, F_-)$ are defined by (25), and G_2 and G_3 are defined from G denoted in (28) and they satisfy the same condition to one imposed on g. The form $D_{y_n} - M_2$ in (85) is same to one in (38) as $k = n - 1$ and $\rho = \mu$, and the principal symbol of $e_\pm(y, D_{y'})$ is $a(y, \eta') \pm i(r_{\lambda+2\mu}(y, \eta))^{1/2}$. First we shall check the condition appeared in Theorem 7.1.

Lemma 9.1 *Let an extensible distribution u to U_0 be a solution of the boundary value problem* (5) *and* (6). *Here we assume that $f \in H^{\mathrm{loc}}_{s-1}(U_0 \cap \{y_n > 0\})$ $(s \geq 0)$. Then there exists a_0 such that for any $y \in U_0 \cap \{0 < y_n < a_0\}$ V belongs to H_s at $(y, \eta) \in T^*(\mathbf{R}^{n+1}) \setminus 0$ with $|\eta_n| > \alpha|\eta'|$, where α depends on y. Moreover if f belongs to $H^{\mathrm{loc}}_{s-1}(U_0 \cap \{y_n > 0\}) \cap H_{(s,r_1)}(\bar{\mathbf{R}}^{n+1}_+) \cap H_{(1,r_2)}(\bar{\mathbf{R}}^{n+1}_+)$ for $s \geq 0$, r_1 and r_2, and if f also belongs to $H_s(\bar{\mathbf{R}}^{n+1}_+) \cap H_{(1,s-1)}(\bar{\mathbf{R}}^{n+1}_+)$ at $(0, \eta^{0\prime})$, then F defined in* (25) *satisfies the all conditions stated in the back part of* (40).

Proof. From (8) all points of $\{(y, 0, \eta_n) \in T^*(U_0) \setminus 0\}$ are elliptic points of $L(y, D_y)$. Thus for any point $(\bar{y}, \bar{\eta})$ with the conditions $\bar{y}_n > 0$ and $|\bar{\eta}_n| > \alpha|\bar{\eta}'|$ for some $\alpha > 0$ there exists a function $\varphi_1 \in C_0^\infty(U_0 \cap \{y_n > 0\})$ and $\chi_1(y, D_y)$ of order 0 such that $\chi_1(y, D_y)\varphi_1 u \in H_{s+1}(\mathbf{R}^{n+1})$. Here we may assume that $\varphi_1 = 1$ on some neighbourhood U_1 of $\bar{y}$ and that the symbol of $\chi_1(y, \eta)$ of $\chi(y, D_y)$ is equal to 1 in $\{(y, \eta);\ y \in U_1, |\eta_n| > (\alpha/2)|\eta'|\}$. We have that $u = u_1 + u_2 + u_3$, where $u_1 = \chi_1\varphi_1 u$, $u_2 = (1 - \chi_1)\varphi_1 u$ and $u_3 = (1 - \varphi_1)u$. For $\tilde{V}_j = (1 + K)S^{-1}\phi^t({}^t(\Lambda\varphi u_j), {}^t(D_{y_n}\varphi u_j))$ $(j = 1, 2, 3)$ we have $\tilde{V}_1 \in H_{s+1}(\bar{\mathbf{R}}^{n+1}_+)$. From Theorem 4.3.1 in [5] we see that u_3 belongs to $H^{\mathrm{loc}}_{s+1,r_1}(\bar{\mathbf{R}}^{n+1}_+)$ for some r_1. It implies that for $\varphi_2(y) \in C_0^\infty(U_1)$ $\varphi_2\tilde{V}_3 \in H_{s+1}(\bar{\mathbf{R}}^{n+1}_+)$. Let $\chi_2(y, D_y)$ be a pseudodifferential operator of order zero such that the support of the symbol of $\chi_2(y, D_y)$ is contained in $\{(y, \eta);\ \chi_1(y, \eta) = 1\}$ and $(\bar{y}, \bar{\eta})$ is an elliptic point of $\chi_2(y, D_y)$. Then from Proposition 2.3 $\chi_2\varphi_2(1+K)S^{-1}\phi^t(\Lambda, D_{y_n})\varphi(1-\chi_1)\varphi_1$ is a pseudodifferential operator of order $-\infty$. It follows that $\chi_2\varphi_2\tilde{V}_3$ belongs to $C^\infty(\mathbf{R}^{n+1})$. We see that $\chi_2\varphi_2 V$ belongs to H_{s+1} at $(\bar{y}, \bar{\eta})$. The last statement on F is clearly derived form the form (40). The proof is complete. $\square$

We shall prove the following

Theorem 9.2 *Let $(0, \eta^{0\prime})$ be a glancing point with respect to μ, and an extensible distribution u to U_0 be a solution of* (5) *and* (6). *We suppose that B_0 is the free boundary condition, and that f satisfies the all conditions* *imposed in* Lemma 9.1 *and g belongs to $H_{s+1/2}(\mathbf{R}^n)$ at $(0,\eta')$. If there exist ϵ_0 and $\delta > 0$, which are independent of u, such that if u belongs to H_{s+1} on $\Gamma_{\epsilon_1}^{(0)}$ and $\Gamma_{\epsilon_1}^{(1)}$ as $\rho = \mu$ for some $0 < \epsilon_1 \le \epsilon_0$, then at any point $(\bar{y'},\bar{\eta'})$ belonging to $\{\exp\{tH_{R_\mu^0}\}(0,\eta^{0\prime});\ |t| < \delta\epsilon_1\}$ u belongs to H_{s+1}.*

Proof. From the late half on p.131 of [19] the principal symbol $(B_{21},B_{22})(y,\eta')$ of $B_2(y,D_{y'})$ and the principal symbol $B_3(y,\eta')$ of $B_3(y,D_{y'})$ have the following forms:

$$B_2=(e_1,0,\dots,0,b_n',e_2,\dots,e_{n-1}),\quad B_3=(0,\dots,0,b_n,0,\dots,0), \tag{88}$$

where $e_j \in R_{n-1}$ is the unit vector whose j-th component is 1 and i-th component is 0 except $i \neq j$, b_n' is the n-th component of B_2, and b_n is n-th component of B_3. We check the condition (42) for B_2. From Lemma 1.20 in [19] the condition (42) holds for B_2 of (88). From Lemma 9.1 and the assumption of u V_2 satisfies the condition stated in Theorem 7.1 on $\Gamma_{\epsilon_1}^{(0)}$ and $\Gamma_{\epsilon_1}^{(0)}$ as $\rho = \mu$ for some $0 < \epsilon_1 < \epsilon_0$. Therefore let us apply Theorem 7.1 to the problem (85). Since the form of $B_2(y,D_{y'})$ is denoted by $(e_1,b_2',\dots,b_n',e_2,\dots,e_{n-1})$, for $\tilde V = {}^t(\bar V_n,\bar V_2,\dots,\bar V_{n-1})$, where $\bar V_j$ is the j-th component of $\bar V$ we have $V_1 = D_1\tilde V + {}^t(g_{21},0,\dots,0)$ and $V_2 = D_2\tilde V + {}^t(0,{}^tg_{22})$, where $\bar V = {}^t({}^tV_1,{}^tV_2)$, $G_2 = {}^t({}^tg_{12},{}^tg_{22})$ with $g_{22}(y) \in \mathbf{R}^{n-2}$, and the form of $D_1(y,D_{y'})$ and $D_2(y,D_{y'})$ are denoted in (1.45) of [19]. Put

$$G_1^{\pm}(y,D_{y'})=\begin{pmatrix}1 & b^*\\ b & E_{n-2}\end{pmatrix},\quad F_1^{\pm}(y,D_{y'})=\begin{pmatrix}\pm a & 0\\ 0 & \mp\bar R_\mu^0E_{n-2}\end{pmatrix}, \tag{89}$$

where $a>0$ and the principal symbol $b(y,\eta')$ in $G_1^{\pm}(y,D_{y'})$ is ${}^t(b_{2n},\dots,b_{n-1n})$ if $b_n' = {}^t(b_{1n},\dots,b_{n-1n})$ in (88). From Lemma 1.20 in [19] the principal symbol of $G_1^{\pm}$ is positive definite near $(0,\eta^{0\prime})$. Then the principal symbol of $X^{\pm} = D_1^*G_1^{\pm}D_2 + D_2^*G_1^{\pm}D_2 + D_2^*F_1^{\pm}D_1 - D_1^*F_1^{\pm}\bar R_\mu^0D_1$ is equal to

$$\pm\begin{pmatrix}a' & 0\\ 0 & (r_\mu|\eta'|^{-2})^2\end{pmatrix}, \tag{90}$$

where $a' = a(1-|b_{1n}|^2r_\mu|\eta'|^{-1}) + 2\operatorname{Re} b_{1n}(|b|^2-1)$. If we take sufficiently large a, then the principal symbol of $\pm X^{\pm}(y,D_{y'})$ satisfies the condition (41). From Theorem 8.5 it implies that there exists a pseudodifferential operator $\phi(y,D_{y'})$, which is elliptic at $(0,\eta^{0\prime})$, such that $\phi(y,D_{y'})\bar V_{n|y_n=0} \in H_s(\mathbf{R}^n)$. From Proposition 2.4 and its proof we see that $\phi(y,D_{y'})\bar V_{j|y_n=0} \in H_{s-1/2}(\mathbf{R}^n)$ $(j\neq n)$. From (88) it follows that $\phi(y,D_{y'})(G_3 + B_3\bar V)_{|y_n=0} \in$

$H_s(\mathbf{R}^n)$. Now we are able to use the argument of Section 5 in [19] to the problem (86), and can show that $\phi(y, D_{y'})V_{\pm|y_n=0} \in H_s(\mathbf{R}^n)$. The proof is completed. □

Next we assume that B_0 is the free boundary condition and $(0, \eta^{0\prime})$ is a glancing point with respect to $\lambda + 2\mu$. Then $r_\mu(0, \eta^{0\prime})$ is negative, and there exists the null bicharacteristic strip $\tilde{\gamma}_\mu^\pm$ through $\big(0, \eta^{0\prime}, a(0, \eta^{0\prime}) \pm (-r_\mu(0, \eta^{0\prime}))^{1/2}\big)$. Put $\gamma_\mu^\pm = \tilde{\gamma}_\mu^\pm \cap T^*\{y_n > 0\}$. We have the following

Theorem 9.3 *Let* $(0, \eta^{0\prime})$ *be a gliding point with respect to* $\lambda + 2\mu$ *and an extensible distribution* u *to* U_0 *be a solution of* (5) *and* (6), *where* B_0 *is the free boundary,* f *satisfies the all conditions stated in* Lemma 9.1. *We suppose that* g *belongs to* H_s *at* $(0, \eta^{0\prime})$ *and* u *belongs to* H_{s+1} *on* γ_μ^α, *where* α *is* $+$ *or* $-$. *Then there exist* $\epsilon_0 > 0$ *and* $\delta > 0$ *such that if* u *satisfies the condition stated in* Theorem 9.2 *on* $\Gamma_{\epsilon_1}^{(0)}$ *and* $\Gamma_{\epsilon_1}^{(1)}$ *as* $\rho = \lambda + 2\mu$ *for some* $0 < \epsilon_1 \le \epsilon_0$, *then* u *belongs to* H_{s+1} *on* $\gamma_\mu^{-\alpha} \cup \{\exp\{tH_{R^0_{\lambda+2\mu}}\}(0, \eta^{0\prime});\ |t| < \delta\epsilon_1\}$.

Proof. From the argument in Section 1.7 of [19] the boundary value problem (5) and (6) is reduced to the following problems (see $(1.54)_\pm$, $(1.55)_\pm$ and $(1.56)_\pm$ in [19]):

$$\begin{aligned}
&(D_{y_n} - H_+)V_+ = F_+ && \text{in} \quad y_n > 0, && (91)\\
&V_+ = g_1 - B_-V_- - cv_2 && \text{on} \quad y_n = 0 && \\
&(D_{y_n} - M_2)\bar{V} = F_2 && \text{in} \quad y_n > 0, && (92)\\
&B_2\bar{V} = g_2 - b_-V_- && \text{on} \quad y_n = 0, && \\
&(D_{y_n} - H_-)V_- = F_- && \text{on} \quad y_n > 0, && (93)
\end{aligned}$$

where $\bar{V} = {}^t(v_1, v_2)$, F_+, F_- and F_2 satisfy the same conditions to these of f, g_1 and g_2 satisfy the same conditions to these of g, the principal symbol of $H_\pm(y, D_{y'})$ is $a(y, \eta') + (\mp r_\mu(y, \eta'))^{1/2})E_{n-1}$, $D_{y_n} - M_2$ has the same form to (38) as $k = 1$ and $\rho = \lambda + 2\mu$, and $B_2(y, D_{y'}) = (1, d)$. Here we remark that $d(y, D_{y'})$ has the real principal symbol, which means that the condition (42) holds for B_2. If we assume that u belongs to H_{s+1} on γ_μ^-, then from (93) and Proposition 4.4 we see that $V_{-|y_n=0}$ belongs to H_s at $(0, \eta^{0\prime})$. Now we apply Theorem 8.6 to the problem (92). We can conclude that $(g_1 - B_-V_- - cv_2)_{|y_n=0}$ belongs to H_s at any point belonging to $\{\exp\{tH_{R^0_{\lambda+2\mu}}\}(0, \eta^{0\prime});\ |t| < \delta\epsilon_1\}$. From Proposition 4.4 we complete the proof. □

Now we shall consider the Dirichlet boundary condition. Let $(0, \eta^{0\prime})$ be a gliding point with respect to μ. Then from (1.16) of [19] the principal symbol of the boundary operator $B(y, D_{y'})$ in (1.17) of [19] has the form $(b_1, \ldots, b_{2n-2}, b_+, b_-) = \big((a\tilde{d}_1 - |\eta'|^2 e_n)|\eta'|^{-2}, d_2|\eta'|^{-1}, \ldots, d_{n-1}|\eta'|^{-1}, \tilde{d}_1|\eta'|^{-1}, 0, \ldots, 0, (\bar{\eta} + \alpha^+_{\lambda+2\mu} G)|\eta'|^{-1}\big)$. Here $\tilde{d}_1 = {}^t({}^t\eta', \eta' \cdot \operatorname{grad} g)$, $d_j = {}^t({}^t d'_j, d'_j \cdot \operatorname{grad} g)$, where $d'_2(\eta'), \ldots, d'_{n-1}(\eta')$ form a base of the orthogonal space of η' in $\mathbf{R}^n$ and are of degree 1, $\bar{\eta} = {}^t(\eta_0, \ldots, \eta_{n-1}, 0)$, $\alpha^{\pm}_{\lambda+2\mu} = a \pm i(r_{\lambda+2\mu})^{1/2}$ and $G = {}^t(-\operatorname{grad} g, 1)$. Since $\operatorname{grad} g(0) = 0$, $\det(b_1, \ldots, b_{n-2}, b_+)$ is not zero. Therefore multiply a pseudodifferential operator whose principal symbol is the inverse matrix of $(b_1, \ldots, b_{n-2}, b_+)$ from the left side, we may assume that boundary operator $B_2(y, D_{y'})$ in (1.18) of [19] is $(E_{n-1}, B_{22}(y, D_{y'}))$. Then if $D_1 = -B_{22}$, $D_2 = E_{n-1}$, $G_1^{\pm} = E_{n-1}$ and $F_1^{\pm} = \pm a E_{n-1}$, where a is sufficiently large, then the condition of Theorem 8.6 holds. Thus we can apply Theorem 8.6 to the problem (92). If $(0, \eta^{0\prime})$ is a glancing point with respect to $\lambda + 2\mu$, then from (1.8) of [19], s_{2n-1} and s_{2n}, which are defined at the first part of Section 1.7 of [19], the principal symbol of the boundary operator $B(y, D_{y'})$ in (1.52) of [19] have the form $(b_1, \ldots, b_{2n-2}, b_+, b_-)$, where $det(b_1, \ldots, b_{n-1}, b_{\pm})$ is not zero. Thus the situation is the same to one of the case that $(0, \eta^{0\prime})$ is a glancing point with respect to μ. We have the following

Theorem 9.4 *Let an extensible distribution u to U_0 be a solution of* (5) *and* (6), *where B_0 is the Dirichlet boundary condition and f satisfies the all conditions stated in* Lemma 9.1. *We suppose that g belongs to H_{s+1} at $(0, \eta^{0\prime})$, and that if $(0, \eta^{0\prime})$ is a glancing point with respect to $\lambda + 2\mu$, u belongs to H_{s+1} on γ^{α}_{μ}, where α is $+$ or $-$. Then all properties on u stated in* Theorem 9.2 *and* Theorem 9.3 *hold.*

Following [11] we shall define a generalized bicharacteristic for $p_\rho(y, \eta) = (\eta_n - a)^2 + r_\rho$.

Definition 9.1 a) A subset $\Sigma_{b,\rho}$ of $(T^*(\mathbf{R}^n \times (0, \infty)) \setminus 0) \cup (T^*\mathbf{R}^n \setminus 0)$ is the following set; If $(y, \eta) \in T^*(\mathbf{R}^n \times (0, \infty)) \setminus 0)$, then $p_\rho(y, \eta) = 0$ and if $(y', \eta') \in T^*\mathbf{R}^n \setminus 0$, then there exists η_n such that $p_\rho(y', 0, \eta', \eta_n) = 0$. Next we define various conic subsets of $\Sigma_{b,\rho}$. Define $\Sigma^0_{b,\rho} = \Sigma_{b,\rho} \cap (T^*(\mathbf{R}^n \times (0, \infty)) \setminus 0)$, $\Sigma^1_{b,\rho} = \{(y', \eta') \in T^*(\mathbf{R}^n) \setminus 0;\ r_\rho(y', 0, \eta') < 0\} \cap \Sigma_{b,\rho}$, $\Sigma^{2,\pm}_{b,\rho} = \{(y', \eta') \in T^*\mathbf{R}^n \setminus 0;\ r_\rho(y', 0, \eta') = 0,\ \mp(H^2_{p_\rho} y_n)(y', 0, \eta') > 0\} \cap \Sigma_{b,\rho}$, $\Sigma^{(3)}_{b,\rho} = \{(y', \eta') \in T^*\mathbf{R}^n \setminus 0;\ r_\rho(y', 0, \eta') = 0,\ (H^2_{p_\rho} y_n)(y', 0, \eta') = 0\} \cap \Sigma_{b,\rho}$ and $\Sigma^{\infty}_{b,\rho} =$

$\{(y', \eta') \in T^*\mathbf{R}^n \setminus 0;\ r_\rho(y', 0, \eta') = 0,\ (H^k_{p_\rho} y_n)(y', 0, \eta') = 0$ for all $k\} \cap \Sigma_{b,\rho}$.
b) Using the notations in a), we shall define a generalized bicharacteristic. Let Γ_0 be a small conic neighbourhood of $(0, \eta^{0\prime})$ and Σ_0 be $(\Gamma_0 \times [0, a) \times \mathbf{R}) \cup \Sigma_{b,\rho}$. A generalized bicharacteristic is a curve $\gamma_\rho\colon I \to \Sigma_0$, where I is interval, such that i) If $\gamma_\rho(t_0) \in \Sigma^0_{b,\rho}$, $t_0 \in I$, then $\gamma_\rho(t)$ is differential at t_0 and $\gamma_\rho(t_0) = H_{p_\rho}(\gamma_\rho(t_0))$, ii) If $\gamma_\rho(t_0) \in \Sigma^1_{b,\rho} \cup \Sigma^{2,-}_{b,\rho}$, $t_0 \in I$, then $\gamma_\rho(t) \in \Sigma^0_{b,\rho}$ for $0 \neq |t - t_0|$ small, iii) If $\gamma_\rho(t_0) \in \Sigma^{2,+}_{b,\rho} \cup \Sigma^{(3)}_{b,\rho}$, $t_0 \in I$, then $(y(t), \eta'(t))$ is differentiable at $t = t_0$ with the derivatives $(dy_n/dt)(t_0) = 0$, $d(y'(t), \eta'(t))/dt_{|t=t_0} = H_{r^0_\rho}(y'(t_0), \eta'(t_0))$, where $r^0_\rho(y', \eta') = r_\rho(y', 0, \eta')|\eta'|^{-1}$.

In [11] the following fact is proved: for any $(0, \eta^{0\prime}) \in \Sigma_{b,\rho}$ there exists a generalized bicharacteristic through this point, and that is unique if $(0, \eta^{0\prime}) \in \Sigma_{b,\rho} \setminus \Sigma^\infty_{b,\rho}$. In [15] an example is given for which there exist two generalized bicharacteristics through the same point, that belongs to $\Sigma^\infty_{b,\rho}$. Let $\gamma_\rho(t)$ be any generalized bicharacteristic through $(0, \eta')$. We set $\gamma^\pm_\rho = \cup\{\gamma_\rho(t);\ \pm t > 0\}$, where the union is taken over all generalized bicharacteristics satisfying the above conditions.

Using these notations we shall show theorems on propagations of regularities near glancing points. First we assume the condition to avoid diffractive points near $(0, \eta')$.

Condition A: There exists a conic neighbourhood Γ_0 of $(0, \eta^{0\prime})$ such that $\Gamma_0 \cap \Sigma^{(2),-}_{b,\rho} = \emptyset$.

In this case in the boundary value problem (5) and (6) we suppose that u is an extensible distribution to U_0 and f satisfies the all conditions stated in Lemma 9.1. The boundary datum g satisfies the following condition: First we suppose that $(0, \eta^{0\prime})$ is a glancing point with respect to μ. If B_0 is the free boundary condition, then $g \in H_{s+1/2}$ at $(0, \eta^{0\prime})$, and if B_0 is the Dirichlet condition, then $g \in H_{s+1}$ at $(0, \eta^{0\prime})$. Second we suppose that $(0, \eta^{0\prime})$ is a glancing point with respect to $\lambda + 2\mu$. If B_0 is the free boundary condition, then $g \in H_s$ at $(0, \eta^{0\prime})$, and if B_0 is the Dirichlet condition, then $g \in H_{s+1}$ at $(0, \eta^{0\prime})$. Then we have the following

Theorem 9.5 *We suppose that $(0, \eta^{0\prime}) \in \Sigma^{(2)}_{b,\rho}$ satisfies the condition A and that u, f and g satisfy the above conditions. Then we have the following two statements.*

i) *If* $\rho = \mu$, *and a solution* u *of* (5) *and* (6) *belongs to* H_{s+1} *on* $\gamma_\mu^\epsilon \cap \Gamma_0$, *where* ϵ *is* $+$ *or* $-$, *then* u *belongs to* H_{s+1} *on* $(\gamma_\mu^{-\epsilon} \cap \Gamma_0) \cup \{(0, \eta^{0\prime})\}$.
ii) *If* $\rho = \lambda + 2\mu$, *and a solution* u *of* (5) *and* (6) *belongs to* H_{s+1} *on* $(\gamma_\mu^\epsilon \cup \gamma_{\lambda+2\mu}^\epsilon) \cap \Gamma_0$, *where* ϵ *is* $+$ *or* $-$, *then* u *belongs to* H_{s+1} *on* $\big((\gamma_\mu^{-\epsilon} \cup \gamma_{\lambda+2\mu}^{-\epsilon}) \cap \Gamma_0\big) \cup \{(0, \eta^{0\prime})\}$.

From Theorem 3.1, Theorem 4.5, Theorem 5.1, Theorem 5.2, Theorem 9.2 Theorem 9.3 and Theorem 9.4 we can easily prove Theorem 9.5, if we use the argument to verify Theorem 2.44 of [12], which is denoted in the middle part of p.153 in [12].

In (a) in Definition 9.1 of this section if $(0, \eta^{0\prime}) \in \Sigma_{b,\rho}^{2,+}$, then the condition A holds. Thus Theorem 9.5 is one of generalizations of Theorem 4.15 in [1] in the Sobolev space. However near points belonging to $\Sigma_{b,\rho}^{2k+1} \cap \Sigma_{b,\rho}^{2k,-}$ the situations are quiet different. Here making use of functions $r_{\rho,0} = r_\rho(y', 0, \eta')$ and $r_{\rho,1} = (\partial r_0/\partial y_n)(y', 0, \eta')$, we put $\Sigma_{b,\rho}^{k,\pm} = \{(y', \eta') \in T^*\mathbf{R}^n \setminus 0;\ r_{\rho,0}(y', \eta') = 0,\ (H_{r_{\rho,0}}^j r_{\rho,1})(y', \eta') = 0,\ 0 \le j < k-2,\ \pm(H_{r_{\rho,0}}^{k-2} r_{\rho,1})(y', \eta') > 0\}$, where $H_{r_{\rho,0}}^j$ is the Hamilton vector field of $r_{\rho,0}$, and $\Sigma_{b,\rho}^k = \Sigma_{b,\rho}^{k,+} \cup \Sigma_{b,\rho}^{k,-}$ (see Lemma 3.4 of [11]). We have the following lemma

Lemma 9.6 *If* $(0, \eta^{0\prime})$ *belongs to* $\Sigma_{b,\rho}^{2k+1} \cup \Sigma_{b,\rho}^{2k,-}$, *then there exists a sequence* $\{\rho_m\}$ *such that* $\rho_m \in \Sigma_{b,\rho}^{2,-}$ *and* $\{\rho_m\}$ *converges to* $(0, \eta^{0\prime})$ *as* m *goes to* ∞.

Proof. After a non-homogeneous canonical change of coordinates in $T^*\mathbf{R}^n \setminus 0$ near $\rho_0 = (0, \eta^{0\prime})$, we may assume that $\rho_0 = (0, 0)$ and $r_{\rho,0}(y', \eta') = \eta_0$. Then the Hamiltonian vector field of $r_{\rho,0}$ becomes to $\partial/\partial y_0$. If $\rho_0 \in \Sigma_{b,\rho}^p$, then from the Malgrange's preparation theorem we have that near $(0, 0)$

$$r_{\rho,1}(y', \eta') = (y_0^p + a_1(y'', \eta') y_0^{p-1} + \cdots + a_p(y'', \eta')) A(y', \eta'),$$

where $y'' = (y_1, \ldots, y_{n-1})$, $A(0,0) \neq 0$, and $a_j(0,0) = 0$ $(j = 1, \ldots, k)$. Here if $\rho_0 \in \Sigma_{b,\rho}^{2k,-}$, then $A(0,0) < 0$. Thus if $\rho_0 \in \Sigma_{b,\rho}^{2k+1} \cup \Sigma_{b,\rho}^{2k,-}$, we can take $y_0^{(m)} \neq 0$ such that $\{y_0^{(m)}\}$ converges to 0 and $(\operatorname{sgn} A)(y_0^{(m)})^p < 0$, where p is $2k$ or $2k+1$. Since $a_j(0,0) = 0$, we can take $(y''^{(m)}, \eta''^{(m)})$ such that $|(a_1(y''^{(m)}, 0, \eta''^{(m)})(y_0^{(m)})^{p-1} + \cdots + a_p(y''^{(m)}, 0, \eta''^{(m)}))A(y^{(m)\prime}, 0, \eta''^{(m)})| < |(y_0^{(m)})^p A(y^{(m)\prime}, 0, \eta''^{(m)})|/2$ and $\{(y'^{(m)}, \eta''^{(m)})\}$ converges to $(0,0)$, which means that $(y'^{(m)}, 0, \eta''^{(m)})$ belongs to $\sum_{b,\rho}^{2,-}$. The proof is complete. $\square$

From Theorem 6.5 and Theorem 6.6 there exists a possibility that solutions of (5) and (6) lose a regularity at points belonging to $\sum_{b,\rho}^{2,-}$. Thus we suppose the following:

Condition B. There exists a conic neighbourhood of $\Gamma_0 \subset T^*\mathbf{R}^n \setminus 0$ of $(0, \eta^{0\prime})$ and $a > 0$ such that there is no generalized bicharacteristic connecting $\rho_1 \in \Sigma_{b,\rho}^{2,-}$ with $\rho_2 \in \Sigma_{b,\rho}^{(2)}$, which is contained in $\bar{\Gamma}_0 = \Gamma_0 \times [0, a) \times \mathbf{R}$.

Under this assumption we have the following

Theorem 9.7 *We suppose that $\rho_0 = (0, \eta^{0\prime})$ satisfies condition B, and that an extensible distribution u is a solution of* (5) *and* (6)*, where g satisfies the conditions in* Theorem 9.5. *In* (5) *f satisfies the condition that for some $T_0 > 0$ $f \in H_s^{\mathrm{loc}}(U_0 \cap \{y_n > 0\}) \cap C^\infty([0, T_0) : \mathcal{D}'(U_0 \cap \{y_n = 0\})$ and there exists a pseudodifferential operator $\phi_0(y, D_{y'})$, which is elliptic at $(0, \eta^{0\prime})$, such that $\phi_0(y, D_{y'})f \in C^\infty(\mathbf{R}^n \times [0, T_0))$. Then we have the following two statements*:

i) *If ρ_0 is a glancing point with respect to μ and $u \in H_{s+1}$ on $\gamma_\mu^\epsilon \cap \bar{\Gamma}_0$, where $\gamma_\mu^\epsilon(0) = \rho_0$, then u belongs to $H_{s+\alpha}$ at ρ_0, which means that for some δ_0 u belongs to $H_{s+\alpha}$ at all points in $\{\gamma_\mu^{-\epsilon};\ 0 < -\epsilon t < \delta_0\} \cap \bar{\Gamma}_0$. Moreover if for $\delta_1 > \delta_0$ $\{\gamma_\mu^{-\epsilon};\ \delta_0 < -\epsilon t < \delta_1\} \subset (\Sigma_{b,\rho}^{(0)} \cup \Sigma_{b,\rho}^1 \cup \Sigma_{b,\rho}^{2,+}) \cap \bar{\Gamma}_0$, then at all points in $\{\gamma_\mu^{-\epsilon};\ 0 < -\epsilon t < \delta_1\}$ u belongs to $H_{s+\alpha}$. Here ϵ is $+$ or $-$, and $\alpha = 1/6$, if B_0 is the free boundary condition, and $\alpha = 1/2$, if B_0 is the Dirichlet condition.*

ii) *If ρ_0 is a glancing point with respect to $\lambda + 2\mu$ and $u \in H_{s+1}$ on $(\gamma_\mu^\epsilon \cup \gamma_{\lambda+2\mu}^\epsilon) \cap \bar{\Gamma}_0$, where $\gamma_{\lambda+2\mu}^\epsilon(0) = \rho_0$ and $\gamma_\mu^\epsilon(0) = \big(0, \eta^{0\prime}, a(0, \eta^{0\prime}) + \epsilon(-r_\mu(0, \eta^{0\prime})^{1/2})\big)$ with $\rho_0 = (0, \eta^{0\prime}, 0)$, then u belongs to $H_{s+1/2}$ on $(\gamma_\mu^{-\epsilon} \cap \bar{\Gamma}_0) \cup \{\rho_0\}$. If u belongs to H_{s+1} on $\{\gamma_{\lambda+2\mu}^{-\epsilon};\ 0 < -\epsilon t < \delta_0\}$, then u belongs to $H_{s+1/2}$ on $\{\gamma_{\lambda+2\mu}^{-\epsilon};\ 0 < -\epsilon t < \delta_1\}$, where $\{\gamma_{\lambda+2\mu}^{-\epsilon};\ \delta_0 < -\epsilon t < \delta_1\}$ is contained in $(\Sigma_{b,\rho}^{(0)} \cup \Sigma_{b,\rho}^1 \cup \Sigma_{b,\rho}^{2,+}) \cap \bar{\Gamma}_0$.*

Proof. From the theorems proved in Section 3, 4, 5, 6 and 8, the argument to verify Theorem 2.44 of [12], which is stated in the middle part of p.153 in [12] we can get the following: In the case of the statement i) if u does not belong to $H_{s+\alpha}$ at ρ_0, then we can construct a generalized bicharacteristic γ_μ such that $\gamma_\mu(0) = \rho_0$ and u does not belong to $H_{s+\alpha}$ at all points in $\{\gamma_\mu(t);\ |t| < \delta\}$. In the case of the statement ii) if u belongs to H_{s+1} on $\gamma_\mu^\epsilon \cap \bar{\Gamma}_0$ and u does not belong to $H_{s+1/2}$ at ρ_0, then we can construct a generalized bicharacteristic $\gamma^{\epsilon}_{\lambda+2\mu}$ such that $\gamma^{\epsilon}_{\lambda+2\mu}(0)=\rho_0$ and u does not belong to $H_{s+1/2}$ at all points in $\{\gamma^{\epsilon}_{\lambda+2\mu}(t);\ 0<\epsilon<\delta\}$. Thus if we use Theorem 3.1 for $\Sigma^0_{b,\rho}$, Theorem 4.5 for $\Sigma^1_{b,\rho}$, and Theorem 6.5 and Theorem 6.6 for $\Sigma^{2,-}_{b,\rho}$, we have the desired properties. The proof is completed. □

We remark on the condition A and condition B. Let us come back to the original problem (1) and (2). Then null bicharacteristic strips of $\xi_0^2-\rho(\xi_1^2+\cdots+\xi_n^2)$ are lines. Thus we can state the following simple condition on $\partial\Omega$ of satisfying the condition A or condition B. We assume that $0\in\partial\Omega$. We say that Ω is locally convex near 0, if there exists a neighbourhood U_0 of 0 such that for all $x\in U_0\cap\partial\Omega\ H_x\cap U_0\subset\bar{\Omega}$, where H_x is the tangential plane of $\partial\Omega$ through x. If for all $x\in U_0\cap\partial\Omega\ H_x\cap U_0\subset\bar{\Omega}^c$, then we say that Ω is locally concave near 0. Then we have the following

Lemma 9.8 *If Ω is locally convex near 0, then the condition B holds for all $(0,\eta^{0\prime})\in\Sigma^{(2)}_{b,\rho}$. If Ω is locally concave near 0, then the condition A holds for all $(0,\eta^{0\prime})\in\Sigma^{(2)}_{b,\rho}$.*

Finally we remark that Remark 6.7 stated in the last part of Section 6 is valid for Theorem 9.5 and Theorem 9.7.

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