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On the Łojasiewicz exponent

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Abstract. Let $\mathbb{K}$ be an algebraically closed field and let $X \subset \mathbb{K}^l$ be an n-dimensional affine variety of degree D. We give a sharp estimation of the degree of the set of non-properness for generically-finite separable and dominant mapping $f = (f_1, \ldots, f_n)\colon X \to \mathbb{K}^n$. We show that such a mapping must be finite, provided it has a sufficiently large geometric degree. Moreover, we estimate the Łojasiewicz exponent at infinity of a polynomial mapping $f\colon X \to \mathbb{K}^m$ with a finite number of zeroes.

Key words: polynomials, Łojasiewicz exponent, affine variety.

1. Introduction

Let $\mathbb{K}$ be an algebraically closed field and let $X \subset \mathbb{K}^l$ be an affine n-dimensional variety over $\mathbb{K}$. Let $f\colon X \to \mathbb{K}^n$ be a generically-finite dominant polynomial mapping. We say that f *is finite at a point* $y \in \mathbb{K}^n$, if there exists a Zariski open neighborhood U of y such that the mapping $\mathrm{res}_{f^{-1}(U)} f\colon f^{-1}(U) \to U$ is finite.

The set S_f of points at which the mapping f is not finite, plays a fundamental role in the study of generically-finite morphisms of affine varieties (see [3], [4]). We say that the set S_f is *the set of non-properness* of the mapping f. In the first part of this paper we study the set S_f. Assume that X is of degree D, and $f = (f_1, \ldots, f_n)\colon X \to \mathbb{K}^n$ is a generically-finite separable and dominant mapping. We show that the set S_f is a hypersurface and

$$\deg S_f \le \frac{D(\prod_{i=1}^n \deg f_i) - \mu(f)}{\min_{1\le i\le n} \deg f_i},$$

where $\mu(f)$ is the geometric degree of f. We show also that this estimation is sharp. Moreover, we prove that such a mapping must be finite provided it has a sufficiently large geometric degree.

Now assume that $f = (f_1, \ldots, f_m)\colon X \to \mathbb{K}^m$ is a polynomial mapping

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with a finite set (possibly empty) of zeroes. Assume that $\deg f_i = d_i$ and $d_1 \geq d_2 \geq \cdots \geq d_m$. Let $|\cdot|_v$ be any non-trivial absolute value on the field $\mathbb{K}$ and for $a = (a_1, \ldots, a_m) \in \mathbb{K}^m$ define $\|a\|_v = \max_{i=1}^m |a_i|_v$. Recall that the Łojasiewicz exponent $e(f)$ of the mapping f at infinity is the number:

$$e(f) = \sup\{a\colon \text{there is a constant } C > 0\colon \\ \|f(x)\|_v > C\|x\|_v^a, \text{ for } x \in X \text{ and } \|x\|_v \gg 0\}.$$

For $X = \mathbb{C}^n$ with the Euclidean norm, the estimation of the Łojasiewicz exponent has been done in [1], [6], [2]. For a sequence $d_1 \geq d_2 \geq \cdots \geq d_m > 0$, put $N(d_1, \ldots, d_m; n) = \prod_{i=1}^m d_i$ for $m \leq n$, $N(d_1, \ldots, d_m; n) = (\prod_{i=1}^{n-1} d_i) d_m$ for $m > n > 1$ and $N(d_1, \ldots, d_m; n) = d_m$ for $m > n = 1$. They have proved that if $f = (f_1, \ldots, f_m)$ has only finitely many zeros on $\mathbb{C}^n$, then:

$$e(f) \geq d_m - N(d_1, \ldots, d_m; n) + \sum_{f(a)=0} \mu_a(f),$$

where $\mu_a(f)$ stands for the local multiplicity of the mapping f at a point $a \in \mathbb{C}^n$ (see Definition 5.3). We generalize this result (using a quite different method) on every affine variety $X \subset \mathbb{C}^l$ and every non-trivial absolute value $|\cdot|_v$. We show that, if $X \subset \mathbb{C}^l$ is an affine n-dimensional variety of degree D and $f = (f_1, \ldots, f_m)$ has only finitely many zeros on X, then we have

$$e(f) \geq d_m - DN(d_1, \ldots, d_m; n) + \sum_{f(a)=0} \mu_a(f),$$

and this estimation is sharp. Here $\mu_a(f)$ stands for the local multiplicity of the mapping $f|_X$ at a point a (see Definition 5.3). In particular our result (Theorem 5.6) generalizes Proposition 1.10 from [6] and Theorem 7.3 from [2]. Moreover, in the general case (of arbitrary field $\mathbb{K}$) we prove (Theorem 5.2) that:

$$e(f) \geq d_m - DN(d_1, \ldots, d_m; n) + \nu,$$

where ν is the number of zeroes of f. We use here the methods from our recent paper [5]. We include proofs of most results which we use and thus our exposition is self-contained.

2. Terminology

We assume that $\mathbb{K}$ is an algebraically closed field. If $X \subset \mathbb{K}^l$ is an affine variety of codimension k, then by deg X we mean the number of common points of X and sufficiently general linear subspace M of dimension k. In particular if $X = \mathbb{K}^l$, then deg $X = 1$.

If $X \subset \mathbb{K}^l$ is an affine variety and $g \in \mathbb{K}[X]$ is a regular function, then we put

$$\deg g = \min\{\deg G\colon G \in \mathbb{K}[x_1, \ldots, x_l] \quad \text{and} \quad G|_X = g\}.$$

If $f\colon X \to Y$ is a polynomial generically-finite mapping of affine varieties, then we define the geometric degree of f, denoted $\mu(f)$, to be the number $[\mathbb{K}(X)\colon f^*\mathbb{K}(Y)]$. If the mapping f is separable, then it is well-known that the $\mu(f)$ is equal to the number of points in a generic fiber of f.

3. Perron Theorem

We start with the following important Generalized Perron Theorem (see [5] and [10]).

Theorem 3.1 (Generalized Perron Theorem) *Let $\mathbb{L}$ be a field and let $X \subset \mathbb{L}^k$ be an affine variety of dimension n and of degree D. Assume that $Q_1, \ldots, Q_{n+1} \in \mathbb{L}[X]$ are non-constant regular functions with* $\deg Q_i = d_i$. *If the mapping $Q = (Q_1, \ldots, Q_{n+1})\colon X \to \mathbb{L}^{n+1}$ is generically finite, then there exists a non-zero polynomial $W(T_1, \ldots, T_{n+1}) \in \mathbb{L}[T_1, \ldots, T_{n+1}]$ such that*

a) $W(Q_1, \ldots, Q_{n+1}) = 0$ *on X,*
b) $\deg W(T_1^{d_1}, T_2^{d_2}, \ldots, {T_{n+1}}^{d_{n+1}}) \leq D \prod_{j=1}^{n+1} d_j$.

Proof. We sketch the proof. Without loss of generality we can assume that the field $\mathbb{L}$ is algebraically closed. Let $\tilde{X} = \{(x, w) \in X \times \mathbb{L}^{n+1}\colon Q_i(x) = w_i^{d_i} + w_i\}$ (if $d_i = 1$ we take $Q_i(x) = w_i$). Let W be an irreducible polynomial such that $W(Q_1, \ldots, Q_{n+1}) = 0$ and take $P(T_1, \ldots, T_{n+1}) = W(T_1^{d_1} + T_1, \ldots, T_{n+1}^{d_{n+1}} + T_{n+1})$. Let $Y = \{w \in \mathbb{L}^{n+1}\colon P(w) = 0\}$.

Since the polynomial W is reduced it is not difficult to check that the polynomial P is also reduced. In particular we have deg $Y = \deg P$. The sets $\tilde{X}$, Y are affine sets of pure dimension n. Now consider the mapping

$$\pi\colon \tilde{X} \ni (x, w) \to w \in Y.$$

It is easy to see that π is a dominant generically finite mapping. Consequently

$$\deg \pi \deg Y \leq \deg \tilde{X}.$$

By the Bezout Theorem we have $\deg \tilde{X} \leq D \prod_{j=1}^{n+1} d_j$. This finishes the proof. □

4. The set S_f for polynomial mappings

Let $\mathbb{K}$ be an algebraically closed field and let X, Y be affine varieties over $\mathbb{K}$. Recall the following (see [3], [4]):

Definition 4.1 Let $f\colon X \to Y$ be a generically-finite dominant polynomial mapping of affine varieties. We say that f *is finite at a point* $y \in Y$, if there exists a Zariski open neighborhood U of y such that the mapping $\mathrm{res}_{f^{-1}(U)} f\colon f^{-1}(U) \to U$ is finite.

It is well-known that the set S_f of points at which the mapping f is not finite, is either empty or it is a hypersurface (see [3], [4] and Theorem 4.2). We say that the set S_f is *the set of non-properness* of the mapping f.

Let $X \subset \mathbb{K}^l$ be an affine variety of dimension n. In this section we give a sharp estimation of the degree of the hypersurface S_f for a generically-finite separable and dominant polynomial mapping $f = (f_1, \ldots, f_n)\colon X \to \mathbb{K}^n$. As a corollary, we show that if the geometric degree $\mu(f)$ of the mapping f is sufficiently large (relatively to the degree of X and the degrees of polynomials f_j), then the mapping f must be finite.

First we recall our result about the set S_f (see [3], [4]). Let X be an affine n-dimensional variety and let $f\colon X \to \mathbb{K}^n$ be a dominant, generically-finite polynomial mapping. We have:

Theorem 4.2 *Let $f\colon X \to \mathbb{K}^n$ be a dominant generically finite polynomial map and let $k(f_1, \ldots, f_n) \subset \mathbb{K}(X)$ be the induced field extension. Let $\mathbb{K}[X] = \mathbb{K}[g_1, \ldots, g_r]$ and*

$$t^{n_i} + \sum_{k=1}^{n_i} a_k^i(f) t^{n_i - k} = 0,$$

where the $a^i_k \in \mathbb{K}(f_1, \ldots, f_n)$ are rational functions, be the minimal equation of g_i over $\mathbb{K}(f_1, \ldots, f_n)$. Let S denote the union of poles of all functions a^i_k. Then f is finite at a point y if and only if $y \in \mathbb{K}^n \setminus S$.

Proof. $\Rightarrow$ It is enough to prove that the mapping

$$f\colon X \setminus f^{-1}(S) \to \mathbb{K}^n \setminus S$$

is finite. If S is the empty set, then $\mathbb{K}[f_1, \ldots, f_n] \subset \mathbb{K}[g_1, \ldots, g_r]$ is an integral extension and the mapping f is finite. Otherwise S, and hence $f^{-1}(S)$ is a hypersurface. Let $S = \{x\colon A(x) = 0\}$ for some polynomial A. Let $V = X \setminus f^{-1}(S)$ and let $W = \mathbb{K}^n \setminus S$. Then V, W are affine varieties and $\mathbb{K}[V] = \mathbb{K}[g_1, \ldots, g_r][(A(f)^{-1}]$, $\mathbb{K}[W] = \mathbb{K}[x_1, \ldots, x_n][A^{-1}]$. Hence $f_*\mathbb{K}[W] = \mathbb{K}[f_1, \ldots, f_n][(A(f)^{-1}]$. Since all functions a^i_k are regular in W we conclude that elements g_i are integral over $f_*\mathbb{K}[W]$. Of course a polynomial $A(f)^{-1}$ is also integral, and we get the integral extension $f_*\mathbb{K}[W] \subset \mathbb{K}[V]$.

$\Leftarrow$ The following lemma is well-known:

Lemma 4.3 *Let A, B be integral domains, $B = A[g_1, \ldots, g_n]$ such that the quotient field B_0 (of B) is finite field extension of the quotient field A_0 (of A). Assume that A is a normal ring.*

The ring B is a finite A-module if and only if the following condition holds: if $P_i \in A_0[t]$ is the minimal monic polynomial of g_i over A_0, then $P_i \in A[t]$, $i = 1, \ldots, n$.

Now let f be finite over $x \in \mathbb{K}^n$. It means that there is a affine neighborhood U of y such that the mapping $\mathrm{res}_{f^{-1}(U)} f\colon f^{-1}(U) \to U$ is finite. Of course, we can assume that $U = \mathbb{K}^n \setminus \{x\colon A(x) = 0\}$, where A is a polynomial. By the assumption, the ring $\mathbb{K}[f^{-1}(U)] = \mathbb{K}[g_1, \ldots, g_r][(A(f)^{-1}]$ is integral over the ring $f_*\mathbb{K}[U] = \mathbb{K}[f_1, \ldots, f_n][(A(f)^{-1}]$. By Lemma 4.3 we have, that the coefficients $a^i_k(f)$ of polynomials

$$t^{n_i} + \sum_{k=1}^{n_i} a^i_k(f) t^{n_i - k}, \quad i = 1, \ldots, m$$

belong to the ring $\mathbb{K}[f_1, \ldots, f_n][(A(f))^{-1}]$. Hence

$$a^i_k \in \mathbb{K}[x_1, \ldots, x_n][(A(x))^{-1}]$$

and consequently they are regular in U. Thus $U \subset \mathbb{K}^n \setminus S$. $\square$

Corollary 4.4 *Let X be an affine n-dimensional variety and let $f = (f_1, \dots, f_n)\colon X \to \mathbb{K}^n$ be a generically finite mapping. Let S be a set of non-properness of the mapping f. Then for every polynomial $G \in \mathbb{K}[X]$, if $W_G(T_1, \dots, T_n, t) = \sum_{i=0}^{s} a_i(T)t^{s-i} \in \mathbb{K}[T, t]$ is irreducible and*

$$W_G(f_1, \dots, f_n, G) = 0,$$

then

$$\{T\colon a_0(T) = 0\} \subset S.$$

Proof. Observe that the mapping $f\colon X \setminus f^{-1}(S) \to \mathbb{K}^n \setminus S$, is finite. Moreover, $\mathbb{K}[\mathbb{K}^n \setminus S] = \mathbb{K}[x_1, \dots, x_n]_h$, where h is the reduced equation of S. Now Corollary 4.4 follows directly from Lemma 4.3 □

Corollary 4.5 *Let $X \subset \mathbb{K}^l$ be an affine $n-$dimensional variety and let $f\colon X \to \mathbb{K}^n$ be a dominant generically finite and separable polynomial mapping. Assume that $p \in \mathbb{K}[x_1, \dots, x_l]$ and that*

$$P_p(f, p) := \sum_{k=0}^{\mu} a_k(f)p^{\mu-k} = 0,$$

(where $a_k(T) \in \mathbb{K}[T_1, \dots, T_n]$) is the minimal irreducible equation of p over $\mathbb{K}[f_1, \dots, f_n]$. Then there is a linear form p, such that

a) $\mu = \mu(f)$,

b) $S_f = \{T \in \mathbb{K}^n\colon a_0(T) = 0\}$.

Proof. For $t \in \mathbb{K}$ let $\alpha_t = \sum_{i=0}^{l} t^i x_i$. Let S_t denote the set of poles of coefficients of the minimal equation of α_t over $\mathbb{K}(f_1, \dots, f_n)$. By Theorem 4.2 we have $S_t \subset S_f$. Since the hypersurface S_f has only finite number of irreducible components, we have (by the Dirichlet box principle) that there is an infinite subset $T \subset \mathbb{K}$, such that if $t, t' \in T$ then $S_t = S_{t'}$.

Since the extension $\mathbb{K}(f) \subset \mathbb{K}(X)$ is separable, we have that there are a finite number of fields between $\mathbb{K}(f)$ and $\mathbb{K}(X)$. In particular, we can assume that for $t, t' \in T$ we have $\mathbb{K}(f)(\alpha_t) = \mathbb{K}(f)(\alpha_{t'})$.

Take $t_1, \dots, t_l \in T$, where $t_i \neq t_j$ for $i \neq j$. It is easy to check that linear forms α_{t_i}, $i = 1, \dots, l$ generates the algebra $\mathbb{K}[X]$. By Theorem 4.2, we have $S_f = \bigcup_{j=1}^{l} S_{t_j}$. Since $S_{t_i} = S_{t_j}$ we have in fact that $S_f = S_{t_i}$. Moreover, since $\mathbb{K}(f)(\alpha_t) = \mathbb{K}(f)(\alpha_{t'})$ for $t \in T$, we obtain that $\mathbb{K}(f)(\alpha_t) = \mathbb{K}(X)$. In particular if we take $p = \alpha_t$ (where $t \in T$), we have $\mu = \mu(f)$ and $S_f = \{T \in \mathbb{K}^n\colon a_0(T) = 0\}$. □

The following result gives a (sharp) estimation of the degree of the set S_f:

Theorem 4.6 *Let $X \subset \mathbb{K}^l$ be an affine n-dimensional variety of degree D and let $f = (f_1, \ldots, f_n)\colon X \to \mathbb{K}^n$ be a generically finite dominant and separable mapping. Then the set S_f of non-properness of the mapping f is a hypersurface* (*or the empty set*) *and*

$$\deg S_f \leq \frac{D(\prod_{i=1}^n \deg f_i) - \mu(f)}{\min_{1\leq i\leq n} \deg f_i},$$

where $\mu(f)$ is the geometric degree of f.

Proof. Let p be a linear form as in Corollary 4.5. Consider the mapping

$$\Phi\colon X \ni x \to (f_1(x), \ldots, f_n(x), p(x)) \in \mathbb{K}^{n+1}.$$

Let $\deg f_i = d_i$. By the Generalized Perron Theorem (Theorem 3.1) there exists a non-zero polynomial $W(T_1, \ldots, T_n, Y) \in \mathbb{K}[T_1, \ldots, T_n, Y]$ such that

a) $W(f_1, \ldots, f_n, p) = 0$ on X,
b) $\deg W(T_1^{d_1}, T_2^{d_2}, \ldots, T_n^{d_n}, Y) \leq D \prod_{j=1}^n d_j$.

Let

$$P_p = \sum_{k=0}^{\mu} a_k(f) p^{\mu-k} = 0,$$

where the $a_k \in \mathbb{K}[f_1, \ldots, f_n]$, be the minimal irreducible equation of p over $\mathbb{K}[f_1, \ldots, f_n]$. By the minimality of P_p we have

$$P_p(T, Y) | W(T, Y),$$

in particular $\deg P_p(T_1^{d_1}, T_2^{d_2}, \ldots, {T_n}^{d_n}, Y) \leq D \prod_{j=1}^n d_j$. Thus

$$\deg a_0(T_1^{d_1}, T_2^{d_2}, \ldots, {T_n}^{d_n}) Y^{\mu(f)} \leq D \prod_{j=1}^n d_j.$$

This means that

$$\deg a_0(T_1^{d_1}, T_2^{d_2}, \ldots, {T_n}^{d_n}) \leq D \prod_{j=1}^n d_j - \mu(f).$$

Clearly $\deg a_0(T_1^{d_1}, T_2^{d_2}, \ldots, {T_n}^{d_n}) \geq (\min_{1\leq i\leq n} d_i)(\deg a_0)$. Finally by Corollary 4.5 we have

$$\deg S_f \leq \frac{D(\prod_{i=1}^n \deg f_i) - \mu(f)}{\min_{1\leq i\leq n} \deg f_i}.$$

□

Corollary 4.7 *Let $X \subset \mathbb{K}^l$ be an affine n-dimensional variety of degree D and let $f = (f_1, \ldots, f_n)\colon X \to \mathbb{K}^n$ be a generically finite, separable and dominant mapping. If*

$$\mu(f) > D\Big(\prod_{i=1}^n \deg f_i\Big) - \min_{1\leq i\leq n} \deg f_i,$$

then the mapping f is finite.

Example 4.8 Let $n \geq 2$ and set $\Gamma_n = \{(x_1, \ldots, x_n) \in \mathbb{K}^n\colon \prod_{i=1}^n x_i = 1\}$. Take $f\colon \Gamma_n \ni (x_1, \ldots, x_n) \to (x_1, \ldots, x_{n-1}) \in \mathbb{K}^{n-1}$. It is easy to see that

$$\deg \Gamma_n = n \quad \text{and} \quad S_f = \bigcup_{i=1}^{n-1} \{x\colon x_i = 0\}.$$

Thus deg $S_f = n - 1$. On the other hand deg $f_i = 1$ and $\mu(f) = 1$. Hence

$$D\Big(\prod_{i=1}^n \deg f_i\Big) - \mu(f) = n - 1 = \deg S_f.$$

This means that our estimation is sharp.

5. On the Łojasiewicz exponent

We begin with the folowing important fact (see also [5]):

Theorem 5.1 *Let $\mathbb{K}$ be an algebraically closed field and take $m \leq n$. Let $f_1, \ldots, f_m \in \mathbb{K}[x_1, \ldots, x_l]$ be polynomials with* $\deg f_i = d_i$ *and let $X \subset \mathbb{K}^l$ be an affine algebraic n-dimensional variety of degree D. Assume that the set $V(f_1, \ldots, f_m) \cap X$ is finite. If we take a sufficiently general system of coordinates $(x_1, \ldots, x_l)$, then there exist polynomials $g_{ij} \in \mathbb{K}[x_1, \ldots, x_l]$ and non-zero polynomials $\phi_i(x_i) \in \mathbb{K}[x_i]$, such that*

a) $\deg g_{ij} f_j \leq DN(d_1, \ldots, d_m; n)$,

b) $\phi_i(x_i) = \sum_{j=1}^m g_{ij} f_j$ *for every $i = 1, \ldots, l$ (on X).*

Proof. Let $V(f_1, \dots, f_m) = \{a_1, \dots, a_r\}$. The mapping

$$\Phi\colon X \times \mathbb{K} \ni (x, z) \to (x, f_1(x)z, \dots, f_m(x)z) \in \mathbb{K}^l \times \mathbb{K}^m$$

is a (non-closed) embedding outside the set $\{a_1, \dots, a_r\} \times \mathbb{K}$. Take $\Gamma = cl(\Phi(X \times \mathbb{K}))$. Let $s = l + m$ and $\pi\colon \Gamma \to \mathbb{K}^{n+1}$ be a generic projection of the form

$$\pi\colon X \ni (x_1, \dots, x_s) \to \Big(\sum_{j=1}^{s} a_{1j}x_j, \sum_{j=2}^{s} a_{2j}x_j, \dots, \sum_{j=n}^{s} a_{nj}x_j\Big) \in \mathbb{K}^n.$$

Hence π is a finite mapping. Define $\Psi := \pi \circ \Phi(x, z)$. We have

$$\begin{aligned}\Psi &= \Big(\sum_{j=1}^{m} \gamma_{1j} f_j z + l_1(x), \dots, \sum_{j=n}^{m} \gamma_{nj} f_j z + l_n(x), l_{n+1}(x)\Big)\\ &:= \big(\Psi_1(x, z), \dots, \Psi_{n+1}(x, z)\big),\end{aligned}$$

where $l_1, \dots, l_{n+1}$ are generic linear forms. In particular we can assume that l_{n+1} is a variable x_1 in a generic system of coordinates.

Apply the Generalized Perron Theorem to $\mathbb{L} = \mathbb{K}(z)$, polynomials $\Psi_1, \dots, \Psi_{n+1} \in \mathbb{L}[x]$ and to the variety X considered over $\mathbb{L}$. Thus there exists a non-zero polynomial $W(T_1, \dots, T_{n+1}) \in \mathbb{L}[T_1, \dots, T_{n+1}]$ such that $W(\Psi_1, \dots, \Psi_{n+1}) = 0$ on X, and

$$\deg W(T_1^{d_1}, T_2^{d_2}, \dots, {T_{n+1}}^{d_{n+1}}) \le D \prod_{j=1}^{n} d_j.$$

Since coefficients of a polynomial W are in $\mathbb{K}(z)$, we obtain that there is a non-zero polynomial $\tilde{W} \in \mathbb{K}[T_1, \dots, T_{n+1}, Y]$, such that

a) $\tilde{W}(\Psi_1(x, z), \dots, \Psi_{n+1}(x, z), z) = 0$,

b) $\deg_T \tilde{W}(T_1^{d_1}, T_2^{d_2}, \dots, {T_{n+1}}^{d_{n+1}}, Y) \le D \prod_{j=1}^{n} d_j$, where $\deg_T$ denotes the degree with respect to variables $T = (T_1, \dots, T_{n+1})$.

Note that the mapping $\Psi = (\Psi_1, \dots, \Psi_{n+1})\colon X \times \mathbb{K} \to \mathbb{K}^{n+1}$ is finite outside the set $\bigcup_{j=1}^{r} \{T \in \mathbb{K}^n \colon T_{n+1} = a_{j1}\}$, where a_{j1} is the first coordinate of a_j (recall that we consider a generic system of coordinates in which $x_1 = l_{n+1}$!). In particular the set of non-properness of the mapping Ψ is contained in the hypersurface $S = \{T \in \mathbb{K}^{n+1} \colon \prod_{j=1}^{r} (T_{n+1} - a_{j1}) = 0\}$.

Since the mapping $\Psi = (\Psi_1, \dots, \Psi_{n+1})\colon X \times \mathbb{K} \to \mathbb{K}^{n+1}$ is finite outside S, for every polynomial $G \in \mathbb{K}[x_1, \dots, x_n, z]$ there is a minimal polynomial $P_G(T, Y) \in \mathbb{K}[T_1, \dots, T_{n+1}][Y]$, such that $P_G(\Psi_1, \dots, \Psi_{n+1}, G) =$

$\sum_{i=0}^{s} b_i(\Psi_1, \ldots, \Psi_{n+1})G^{s-i} = 0$ and the coefficient b_0 satisfies $\{T\colon b_0(T) = 0\} \subset S$ (see Corollary 4.4).

Take here $G = z$. By the minimality of P_z, we have

$$P_z(T, Y)|\tilde{W}(T, Y),$$

in particular $\deg_T P_z(T_1^{d_1}, T_2^{d_2}, \ldots, {T_{n+1}}^{d_{n+1}}, z) \leq D\prod_{j=1}^{n} d_j$. Let N be the degree of P_z with respect to Y. Add all terms of the form $z^N Q(x)$ which occur in the expression $P_z(\Psi_1, \ldots, \Psi_{n+1}, z)$. It is easy to see that Q must be either equal to $b_0(x_1)$ or must be of a form $f_1^{s_1}\cdots f_n^{s_n}P(x)$, where $\sum s_i > 0$ and deg $f_1^{s_1}\cdots f_n^{s_n}P(x) \leq D\prod_{j=1}^{n} d_j$. Thus we have the equality $b_0(x_1) + \sum f_i g_i = 0$, where $\deg f_i g_i \leq D\prod_{j=1}^{n} d_j$. Take $\phi_1 = b_0$. By the construction the polynomial ϕ_1 has zeroes only in $a_{11}, \ldots, a_{r1}$.

Further, since the form l_{n+1} was generic, we can find n forms of this type which are linearly independent. Hence in a similar way as above we can construct polynomials $\phi_i(x_i)$, $i = 2, \ldots, l$ as in b). □

Now we are in a position to prove:

Theorem 5.2 *Let $\mathbb{K}$ be an algebraically closed field with a non-trivial absolute value $|-|_v\colon \mathbb{K} \to \mathbb{R}$. Let $X \subset \mathbb{K}^l$ be an affine n-dimensional variety of degree D. Assume that $f_1, \ldots, f_m \in \mathbb{K}[x_1, \ldots, x_l]$ have only finite number ν (possibly $\nu = 0$) of zeros on X. Let $d_i = \deg f_i$ (where $d_1 \geq d_2 \geq \cdots \geq d_m > 0$). Put $f = (f_1, \ldots, f_m)$. Then there is a constant $C > 0$, such that for $x \in X$*

$$\|f(x)\|_v \geq C\|x\|_v^{d_m - DN(d_1, \ldots, d_m; n) + \nu},$$

if $\|x\|_v$ is sufficiently large.

Proof. Take a general linear combination:

$$F_1 = f_1,\ F_i = \sum_{j=i}^{m} \gamma_{ij} f_j,\ i = 2, \ldots, n,\ (\text{or } F_1 = f_m \text{ for } n = 1).$$

Since the number of zeroes of $F_1, \ldots, F_n$ is finite and greater or equal to ν, we can assume that $m \leq n$. We can also assume that $f_i = F_i$. Now Theorem 5.2 is a consequence of the Elimination Theorem (Theorem 5.1). Indeed, we can assume that the system of coordinates is sufficiently generic and there exist polynomials $g_{ij} \in \mathbb{K}[x_1, \ldots, x_l]$ and polynomials $\phi_i(x_i) \in \mathbb{K}[x_i]$, such that

a) $\deg g_{ij} f_j \leq DN(d_1, \ldots, d_m; n)$,
b) $\phi_i(x_i) = \sum_{j=1}^{m} g_{ij} f_j$,
Observe that we have $\deg \phi_i \geq \nu$ (we can assume that all zeroes of $f_1, \ldots, f_n$ have all different coordinates!). Further, if $G(x_1, \ldots, x_l) \in \mathbb{K}[x_1, \ldots, x_l]$ is a polynomial of degree at most d, then

$$|G(x)|_v < C\|x\|_v^d$$

for large $\|x\|_v$. On the other hand if $\phi(t)$ is a polynomial of one variable t of degree at least ν, then $|\phi(t)|_v > C'|t|_v^{\nu}$ for large $|t|_v$. In particular from b) we get

$$A\|x\|_v^{DN(d_1, \ldots, d_m; n) - d_m}\|f(x)\| > B\|x\|_v^{\nu}$$

and consequently

$$\|f(x)\|_v \geq C\|x\|_v^{d_m - DN(d_1, \ldots, d_m; n) + \nu},$$

for $\|x\|_v$ sufficiently large. □

Now we can prove our main result. First we introduce the notion of a local multiplicity:

Definition 5.3 Let $X \subset \mathbb{C}^l$ be an affine variety and let $f: X \to \mathbb{C}^m$ be a polynomial mapping. Assume that the fiber $f^{-1}(0)$ is finite and non-empty and let $a \in f^{-1}(0)$. Let $Y = cl(f(X))$ and let $\mathbf{Y_0} = \bigcup_{j=1}^{l} \mathbf{Y_j}$ be a decomposition of the analytic germ of Y at 0 into irreducible components. We define the local multiplicity of the mapping f at the point a, denoted $\mu_a(f)$, to be the number of points in $U \cap (\bigcup_{j=1}^{l} f^{-1}(y_j))$, where U is a sufficiently small neighborhood of a (in the classical topology) and $y_j \in \mathbf{Y_j}$ are sufficiently general points.

Remark 5.4 If $m = \dim X$, then $\mu_a(f)$ is the standard multiplicity, see e.g., [9].

We need also the following:

Lemma 5.5 *Let $\Delta \subset \mathbb{C}^l$ be a polydisc and let $Y \subset \Delta$ be an analytic set of pure dimension n. Let $F_k: Y \to \mathbb{C}^n$, $k = 1, 2, \ldots$ be holomorphic mappings and assume that F_k converges to F almost uniformly on Y. If the fiber $F^{-1}(0)$ is finite and non-empty, then there exists a number k_0 and an open neighborhood U of 0 and an open neighborhood V of $F^{-1}(0)$ such that all* *mappings*

$$F_k\colon V\cap F_k^{-1}(U)\to U,\quad k\ge k_0,$$

are proper. Moreover, we can take V and U as small as we want.

Proof. Let V be a relatively compact neighborhood of $F^{-1}(0)$. Let U_i be a ball around 0 of radius $1/i$. Assume that Lemma does not hold. Then for every i we find an arbitrary large number n_i such that the mapping $F_{n_i}\colon V\cap F_{n_i}^{-1}(U_i)\to U_i$ is not proper. Since $\overline{V}$ is compact, this means that the set $(\overline{V}\setminus V)\cap F_{n_i}^{-1}(U_i)$ is not empty, e.g., it contains a point $x_i\in\overline{V}\setminus V$. Hence we have a sequence of points $x_i\in\overline{V}\setminus V$, such that $F_{n_i}(x_i)\in U_i$. Since the set $\overline{V}\setminus V$ is compact, we can assume that the sequence x_i, $i=1, 2, \ldots$ has a limit x_0 in $\overline{V}\setminus V$. Moreover, we can assume that $n_1<n_2<n_3<\cdots$ Now we have $0=\lim F_{n_i}(x_i)=F(x_0)$. Thus $x_0\in F^{-1}(0)$. Since $x_0\notin V$, it is a contradiction. □

Finally we have our main result:

Theorem 5.6 *Let $X\subset\mathbb{C}^l$ be an affine n-dimensional variety of degree D. Let $|-|_v\colon\mathbb{C}\to\mathbb{R}$ be a non-trivial absolute value. Assume that $f_1, \ldots, f_m\in\mathbb{C}[x_1, \ldots, x_l]$ have only finite set (possibly empty) of zeros on X. Let $d_i=\deg f_i$ (where $d_1\ge d_2\ge\cdots\ge d_m>0$) and put $f=(f_1, \ldots, f_m)$. Then there is a constant $C>0$, such that for $x\in X$*

$$\|f(x)\|_v\ge C\|x\|_v{}^{d_m-DN(d_1, \ldots, d_m;n)+\sum_{f(a)=0}\mu_a(f)},$$

if $\|x\|_v$ is sufficiently large.

Proof. Taking a general linear combination as before, it is not difficult to check that we can assume $m\le n$. If $m<n$, then $V(f_1, \ldots, f_m)=\emptyset$ and Theorem 5.6 follows directly from Theorem 5.2. Hence we can assume that $m=n$ and $V(f_1, \ldots, f_n)\ne\emptyset$.

Arguing as in the proof of Theorem 5.2, we see that it is enough to prove that $\deg\phi_i\ge\sum_{f(a)=0}\mu_a(f)$ (the notation is as in Theorem 5.2). Let $V(f_1, \ldots, f_n)=\{a_1, \ldots, a_r\}$ and let $a_i=(a_{i1}, \ldots, a_{il})$.

Consider polynomials ϕ_i as in Theorem 5.1. Take $\phi_1=\phi$. From the proof of Theorem 5.1 we have the equality:

$$\phi(x_1)z^s+\sum_{j=1}^{s}a_j\Big(\sum_{j=1}^{n}\gamma_{1j}f_jz+l_1(x), \ldots,$$

$$\sum_{j=n}^{n} \gamma_{nj} f_j z + l_n(x),\, x_1\Big) z^{s-j} = 0, \quad (5.1)$$

where $a_j \in \mathbb{K}[T_1, \ldots, T_{n+1}]$. In particular

$$\phi(x_1) = -\Big(\sum_{j=1}^{s} a_j(F_1(x)z + l_1(x), \ldots, F_n(x)z + l_n(x), x_1) z^{-j}\Big), \quad (5.2)$$

where $F_i = \sum_{j=i}^{n} \gamma_{ij} f_j$. Take $\overline{a}_j(T) = a_j(T_1, \ldots, T_{n+1}) - a_j(0, \ldots, 0, T_{n+1})$ and $\psi_(x_1, z) = \sum_{j=1}^{s} a_j(0, \ldots, 0, x_1) z^{s-j}$. We have

$$\phi(x_1) + \frac{\psi(x_1, z)}{z^s} = -\Big(\sum_{j=1}^{s} \overline{a}_j(F_1(x)z + l_1(x), \ldots,$$
$$F_n(x)z + l_n(x), x_1) z^{-j}\Big), \quad (5.3)$$

where $\overline{a}_j(0, \ldots, 0, x_1) \equiv 0$ and $\psi(x_1, z)/z^s$ tends to 0 almost uniformly, when $|z|$ tends to the infinity.

Let $z_i \to \infty$. From the proof of Theorem 5.1 it follows that we can modify the linear forms l_i by adding any constant c_i i.e., without any change we can consider $l_i + c_i$ as a new l_i. Take $c = (c_1, \ldots, c_n)$ in this way that c is a regular value of every mapping $F_{z_k} = z_k(F_1 + l_1/z_k, \ldots, F_n + l_n/z_k)$. Such a c does exist, because a countable union of hypersurfaces in $\mathbb{C}^n$ is a nowhere dense subset of $\mathbb{C}^n$. Now change l_i by $l_i + c_i$. Thus for every k we have that F_{z_k} have only smooth simple zeroes.

Let $\Delta(a_1, r)$ be a polydisc around the point a_1 such that the point a_1 is the unique zero of the mapping $F = (F_1, \ldots, F_n)$ in $\Delta(a_1, r)$. Take $F_k = F_{z_k}/z_k$ and use Lemma 5.5 to the sequence F_k and $Y = X \cap \Delta(a, r)$. Hence we have a neighborhood V of a_1 and a neighborhood U of 0, such that $F_k \colon V \cap F_k^{-1}(U) \to U$ are proper mappings. We can assume that V is so small that $\sharp(V \cap F^{-1}(c)) = \mu_{a_1}(F) = \mu_{a_1}(f)$ for generic small $c \in U$. In fact we can choose $c \in U$ such that the fiber $F^{-1}(c)$ consists of smooth points of X and $\sharp(V \cap F^{-1}(c)) = \mu_{a_1}(F)$ and $\sharp(V \cap F_k^{-1}(c)) = \sharp(V \cap F_k^{-1}(0))$ for large k (note that F_k has only smooth simple zeroes). Let $G \subset V \cap (X \setminus \mathrm{Sing}(X))$ be a small open neighborhood of $F^{-1}(c)$. By the Rouche Theorem for large k mappings $F_k - c$ and $F - c$ have the same number of zeroes in G. This means that F_k has at least $\mu_{a_1}(F) = \mu_{a_1}(f)$ different zeroes in V. Since $F_{z_k} = z_k F_k$, we have that F_{z_k} also have at least $\mu_{a_1}(f)$ different zeroes in

V.

Let $\pi\colon \mathbb{C}^m \ni (x_1, \dots, x_m) \to x_1 \in \mathbb{C}$ be a projection. Let S be a small disc around a_{11} which contains $\pi(V)$. Note that for $|z| \gg 0$, we have by the Rouche Theorem that the polynomial $\phi(x_1)$ has in S the same number of zeroes (equal to $\mu_{a_{11}}(\phi)$) as the polynomial $\phi(x_1) + \psi(x_1, z)/z^s$.

Since the coordinates are generic, we have by (5.3), that for a large k a polynomial $\phi(x_1) + \psi(x_1, z_k)/z_k^s$ has at least $\mu_{a_1}(f)$ different zeroes in S. Thus $\mu_{a_{11}}(\phi) \geq \mu_{a_1}(f)$. In the same way we have that ϕ has multiplicity $\mu_{a_{i1}}(\phi) \geq \mu_{a_i}(f)$ at every point a_{i1}. Thus $\deg \phi \geq \sum_{f(a)=0} \mu_a(f)$. Similarly $\deg \phi_i \geq \sum_{f(a)=0} \mu_a(f)$ for every i. □

Example 5.7 We show that Theorem 5.6 is sharp, i.e., for every $D, d_1, \dots, d_m$ (where $d_1 \geq d_2 \cdots \geq d_m > 0$), there exists an n-dimensional affine variety $X \subset \mathbb{K}^l$ of degree D and polynomials $f_i \in \mathbb{K}[X]$ of degrees $d_1, \dots, d_m$ such that $e((f_1, \dots, f_m)) = d_m - DN(d_1, \dots, d_m; n) + \sum_{f(a)=0} \mu_a(f)$. Moreover, we show this for mappings with non-empty fiber $f^{-1}(0)$ as well as for mappings with empty fiber $f^{-1}(0)$.

a) First we consider the case $f^{-1}(0) \neq \emptyset$. Then $m \geq n$. Take $X = \{x \in \mathbb{C}^{n+1} \colon \sum_{i=1}^n a_i x_i = x_{n+1}^D\}$, where $a_i \in \mathbb{C}$ are sufficiently general numbers. Let

$$f\colon X \ni (x_1, \dots, x_n, x_{n+1}) \to \Big(x_1^{d_1} - 1, \dots, {x_{n-1}}^{d_{n-1}} - 1, (x_n^{d_m} - 1)x_n^{d_n - d_m}, \dots, (x_n^{d_m} - 1)x_n^{d_{m-1} - d_m}, x_n^{d_m} - 1\Big) \in \mathbb{C}^m.$$

It is easy to see that $DN(d_1, \dots, d_m; n) = \sum_{f(a)=0} \mu_a(f)$ and consequently

$$e(f) = d_m = d_m - DN(d_1, \dots, d_m; n) + \sum_{f(a)=0} \mu_a(f).$$

(we left details to the reader).

b) Now we consider the case $f^{-1}(0) = \emptyset$. We modify Kollár's Example 2.3 from [6]. Take $X = \{x \in \mathbb{C}^{n+1} \colon x_n x_{n+1}^{D-1} = 1\}$. For $m \leq s \leq n$ set $f_s = x_1^{d_s}$. Further take $f_{n-1} = x_1 x_{n+1}^{d_{n-1}-1} - x_2^{d_{n-1}}$, $f_{n-2} = x_2 x_{n+1}^{d_2 - 1} - x_3^{d_2}, \dots, f_1 = x_{n-1} x_{n+1}^{d_1 - 1} - x_n^{d_1}$. Clearly $\deg f_i = d_i$. Put $f = (f_1, \dots, f_m)$. It is easy to check that $f^{-1}(0) = \emptyset$ and

$$e(f) = d_m - DN(d_1, \dots, d_m; n)$$

(we left details to the reader).

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