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On blocking semiovals with an 8-secant in projective planes of order 9

Nobuo Nakagawa and Chihiro Suetake

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Abstract. Let S be a blocking semioval in an arbitrary projective plane Π of order 9 which meets some line in 8 points. According to Dover in [2], $20 \leq |S| \leq 24$. In [8] one of the authors showed that if Π is desarguesian, then $22 \leq |S| \leq 24$. In this note all blocking semiovals with this property in all non-desarguesian projective planes of order 9 are completely determined. In any non-desarguesian plane Π it is shown that $21 \leq |S| \leq 24$ and for each $i \in \{21, 22, 23, 24\}$ there exist blocking semiovals of size i which meet some line in 8 points. Therefore, the Dover's bound is not sharp.

Key words: blocking semioval, projective plane, ternary function, finite field, collineation group.

1. Introduction

Let $\Pi = (\mathcal{P}, \mathcal{L})$ be a projective plane. A *blocking set* in Π is a set B of points such that for every line $l \in \mathcal{L}$, $l \cap B \neq \phi$ but l is not entirely contained in B. A *semioval* in Π is a set M of points such that for every point $P \in M$ there exists a unique line $l \in \mathcal{L}$ such that $l \cap M = \{P\}$. The idea of a semioval was introduced in [1] and [9]. A *blocking semioval* in Π is a set S of points which is both a semioval and a blocking set.

Let Π be a projective plane of order $q \geq 3$, and let S be a blocking semioval in Π. Dover [3] showed that if S has a $(q-k)$-secant, $1 \leq k < q-1$, then $|S| \geq \big((3k+4)/(k+2)\big)q - k$. We consider whether this bound is sharp or not, when $k = 1$. From the above Dover's result and Dover [2], it follows that if S has a $(q-1)$-secant, then $(7/3)q - 1 \leq |S| \leq 3q - 3$ and the upperbound is met if and only if S is a vertexless triangle. Hence we assume that $|S| < 3q - 3$ in the followings.

Let $\Pi = (\mathcal{P}, \mathcal{L})$ be a projective plane of order 9, and let S be a blocking semioval with 8-secant in Π which is not a vertexless triangle. Let x_i denote the number of lines of Π which meet S in exactly i points. Then $x_0 = 0$ and $x_1 = |S|$ by the definition of S. By Dover [2], $x_9 = 0$. Set $X(S) =$

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$(x_1, x_2, \ldots, x_8)$. If Π is a non-desarguesian plane, then Π is the Hughes plane, the nearfield plane or the dual nearfield plane by Lam, Kolesova and Thiel [6]. We show that if Π is a non-desarguesian plane, $21 \le |S| \le 23$ and for each $i \in \{21, 22, 23\}$ there exist blocking semiovals of size i with $x_8 \ne 0$ using a computer. Since the size of any blocking semioval with $x_8 \ne 0$ which is not a vertexless triangle in $PG(2, 9)$ is 22 or 23, this shows that the Dover's lower bound is not sharp. We remark that there are many $X(S)$'s in a non-desargueian plane Π as compared with $X(S)$'s in $PG(2, 9)$.

2. Blocking semiovals with $x_8 \ne 0$

Let $\Pi = (\mathcal{P}, \mathcal{L})$ be a projectve plane of order q. We coordinate Π using Kallaher's method [5, Chapter 2]. Choose four points U, V, W, I, no three of which are collinear. Let $Q = GF(q)$ as a set. Then, there exists a one-to-one correspondence α between Q and the points in $WI - (UV \cap WI)$ such that $0\alpha = W$, and $1\alpha = I$. Using the set Q, Π is coordinated as follows:

(1) To a point $P \in WI - (UV \cap WI)$ assign the coordinates $((b, b))$, where $b\alpha = P$.

(2) If $P \notin WI$, and $P \notin UV$, then assign to P the coordinates $((a, b))$, where $PV \cap WI = ((a, a))$, and $PU \cap WI = ((b, b))$.

(3) If $P \in UV$ and $P \ne V$, then assign to P the coordinate $((m))$, where $WP \cap IV = ((1, m))$.

(4) To V assign the coordinate $((\infty))$, where ∞ is a symbol not in Q.

Thus $\mathcal{P} = \{((x, y)) \mid x, y \in Q\} \cup \{((a)) \mid a \in Q \cup \{\infty\}\}$. The line l through the points $((m))$ and $((0, k))$ is assigned the coordinates $[[m, k]]$. The line g through the points $((\infty))$ and $((k, 0))$ is assigned the coordinates $[[\infty, k]]$. The line h through the points $((\infty))$ and $((0))$ is assigned the coordinate $[[\infty]]$. Thus $\mathcal{L} = \{[[m, k]] \mid m \in Q \cup \{\infty\}, k \in Q\} \cup \{[[\infty]]\}$.

A *ternary function* F is defined on Q as follows: If $a, m, k \in Q$, then $F(a, m, k)$ is the second coordinate of the point $((a, 0))V \cap ((m))((0, k))$. Thus $[[\infty]] = \{((x)) | x \in Q \cup \{\infty\}\}$, $[[\infty, k]] = \{((k, y)) \mid y \in Q\} \cup \{((\infty))\}$ and $[[m, k]] = \{((x, y)) \mid x \in Q, y = F(x, m, k)\} \cup \{((m))\}$ for $m, k \in Q$.

Now let Π be a projective plane of order 9 and let S be a blocking semioval in Π with $x_8 \ne 0$ which is not a vertexless triangle. Set $|S| = 17 + n$. Then, by the Dover's bound $20 \le |S| \le 23$ and $3 \le n \le 6$. Since $x_8 \ne 0$, we may assume that UV is the 8-secant of S. Then $S \supseteq \{((x)) | x \in Q^* = Q - \{0\}\}$. Since the remaining lines $[[\infty, a]] = \{((a, y)) \mid y \in Q\} \cup \{((\infty))\}$

through $((\infty))$ must also intersect S, there exists a mapping $f\colon Q \ni x \longmapsto f(x) \in Q$ such that $\{((x,\, f(x))) \mid x \in Q\} \subseteq S$. Thus

$$S = \{((x,\, f(x))) \mid x \in Q\} \cup \{((y)) \mid y \in Q^*\} \\ \cup \{((a_i,\, b_i)) \mid 1 \le i \le n\} \quad (*)$$

for some $a_i,\, b_i \in Q (1 \le i \le n)$, where $(a_i,\, b_i) \ne (a_j,\, b_j)$, $1 \le i \ne j \le n$ and $f(a_i) \ne b_i$, $1 \le i \le n$.

Theorem 2.1 ([8, Theorem 2.4]) *Let S be a point set of Π of size $17 + n$ satisfying the condition $(*)$. Then S is a blocking semioval if and only if the following hold.*

(1) *For any $a \in Q^*$, there exists a unique element $b \in Q$ such that $f(x) \ne F(x,\, a,\, b)$ for all $x \in Q$ and $F(a_i,\, a,\, b) \ne b_i$ for all $i \in \{1, 2, \ldots, n\}$.*
(2) *$b_1, b_2, \ldots, b_n$ are pairwise distinct.*
(3) *$Q \ni x \longmapsto f(x) \in Q - \{b_1, b_2, \ldots, b_n\}$ is a surjection.*
(4) *If $f(a_i) = f(x)$, then $x = a_i$.*
(5) *If $a \in Q$ $(a \ne a_1, a_2, \ldots, a_n)$, then there exists $x \in Q$ $(x \ne a)$ such that $f(a) = f(x)$.*

Let $l(n)$ be the number of distinct elements in $\{a_1, a_2, \ldots, a_n\}$.

Lemma 2.2 ([8, Lemma 2.6]) $9 \le l(n) + 2n$

Lemma 2.3 *If $l(n) < 9$, then $8 \ge l(n) + n$.*

Proof. By Theorem 2.1(4), if $f(a_i) = f(a_j)$, then $a_i = a_j$. By Theorem 2.1(3), (4) and (5), $Q = f(Q) \cup \{b_1, b_2, \ldots, b_n\} = \{f(x) \mid x \in Q,\, x \ne a_i \ (i = 1, 2, \ldots, n)\} \cup \{f(a_i) \mid i = 1, 2, \ldots, n\} \cup \{b_1, b_2, \ldots, b_n\}$, where the right-hand side of the equality is a disjoint union. This yields $9 \ge 1 + l(n) + n$. Thus we have the lemma. □

3. The Hughes plane

In this section, we completely determine blocking semiovals with $x_8 \ne 0$ in the Hughes plane of order 9 ([4]). On the field $Q = GF(9)$, define a new multiplication $\circ$ as follows:

$$x \circ y = \begin{cases} xy & \text{if} \quad y^4 = 1, \\ x^3 y & \text{if} \quad y^4 = -1, \\ 0 & \text{if} \quad y = 0. \end{cases}$$

The set Q with field addition $+$, forms a nearfield which is not a field. Then the Hughes plane $\Pi = (\mathcal{P}, \mathcal{L})$ of order 9 is defined as follows:

$$\mathcal{P} = \{[x, y, z] \mid x, y, z \in Q, \ (x, y, z) \neq (0, 0, 0)\},$$

where $[x, y, z] = \{(x \circ k, y \circ k, z \circ k) \mid k \in Q^*\}$.

$$\mathcal{L} = \{L_s A^m \mid s = 1 \quad \text{or} \quad s \in Q - GF(3), \ 0 \leq m \leq 12\},$$

where $A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and $L_s A^m = \{[(x, y, z)^T(A^m)] \mid x + s \circ y + z = 0\}$.

Here ${}^T(A^m)$ is the transpose matrix of A^m.
For $B \in GL(3, 3)$ $\widetilde{B}\colon \mathcal{P} \ni [x, y, z] \longmapsto [(x, y, z)^T B] \in \mathcal{P}$ is a collineation of Π. Set $G_1 = \{\widetilde{B} \mid B \in GL(3, 3)\}$. Then, $G_1 \cong PGL(3, 3)$.

Let $t \in Q$ such that $t^2 = 1 + t$. Then t is a generator of the multiplicative group of the field GF(9). Then, $\tau\colon Q \ni a + bt \longmapsto a - bt \in Q$ $(a, b \in GF(3))$ is an automorphism of the nearfield Q and τ induces a collineation $\widetilde{\tau}$ of Π, that is, $\widetilde{\tau}[x, y, z] = [\tau(x), \tau(y), \tau(z)]$. $\varphi\colon Q \ni a + bt \longmapsto (a + b) + bt \in Q$ $(a, b \in GF(3))$ is also an automorphism of Q. Let $\widetilde{\varphi}$ be the collineation of Π induced by φ. Set $G_2 = \langle \widetilde{\tau}, \widetilde{\varphi} \rangle$. Then, G_2 is isomorphic to the symmetric group of degree 3.

Theorem 3.1 ([10]) (1) $\operatorname{Aut}\Pi = G_1 G_2 = G_1 \times G_2$.
(2) $\operatorname{Aut}\Pi$ *has two orbits* $\mathcal{P}_0 = \{[x, y, z] \mid x, y, z \in GF(3), \ (x, y, z) \neq (0, 0, 0)\}$ *and* $\mathcal{P} - \mathcal{P}_0$ *on* $\mathcal{P}$.
(3) $(\operatorname{Aut}\Pi)_{[0,0,1]}$ *has four orbits* $\{[0, 0, 1]\}$, $\Omega_1 := \{[a, b, c] \mid a, b, c \in GF(3), (a, b) \neq (0, 0)\}$, $\Omega_2 := [t^5, 1, 0](\operatorname{Aut}\Pi)_{[0,0,1]}$, $\Omega_3 := [0, 1, t](\operatorname{Aut}\Pi)_{[0,0,1]}$ *on* $\mathcal{P}$.
(4) $(\operatorname{Aut}\Pi)_{[t^5,1,0]}$ *has four orbits* $\{[t^5, 1, 0]\}$, $\Lambda_1 := \{[t^6, 1, 0], [t^3, 1, 0]\}$, $\Lambda_2 := \{[t, 1, 0], [t^7, 1, 0], [t^2, 1, 0]\}$, $\Lambda_3 := [0, 1, t](\operatorname{Aut}\Pi)_{[t^5,1,0]}$ *on* $\mathcal{P} - \mathcal{P}_0$.

Let S be a blocking semioval in Π with $x_8 \neq 0$. Let U, V, W, I be four points of Π, no three of which are collinear, and let $S \supseteq UV - \{U, V\}$. From Theorem 3.1, we may consider the following six coordinatizations by ((,)), (( )) for the points and [[,]], [[ ]] for the lines in Π (see Section 2). Namely when $U = [0, 0, 1]$, there are three cases of $V \in \Omega_1$, $V \in \Omega_2$ or $V \in \Omega_3$, and when $U = [t^5, 1, 0]$, there are three cases of $V \in \Lambda_1$, $V \in \Lambda_2$ or $V \in \Lambda_3$ as follows.

Case 1: $U = [0,\,0,\,1] = ((0)),\ V = [0,\,1,\,0] = ((\infty)),\ W = [1,\,0,\,0] = ((0,\,0)),\ I = [1,\,1,\,1] = ((1,\,1))$.

Case 2: $U = [0,\,0,\,1] = ((0)),\ V = [t^5,\,1,\,0] = ((\infty)),\ W = [0,\,1,\,0] = ((0,\,0)),\ I = [1,\,1,\,1] = ((1,\,1))$.

Case 3: $U = [0,\,0,\,1] = ((0)),\ V = [0,\,1,\,t] = ((\infty)),\ W = [1,\,0,\,0] = ((0,\,0)),\ I = [1,\,1,\,1] = ((1,\,1))$.

Case 4: $U = [t^5,\,1,\,0] = ((0)),\ V = [t^6,\,1,\,0] = ((\infty)),\ W = [0,\,0,\,1] = ((0,\,0)),\ I = [1,\,1,\,1] = ((1,\,1))$.

Case 5: $U = [t^5,\,1,\,0] = ((0)),\ V = [t,\,1,\,0] = ((\infty)),\ W = [0,\,0,\,1] = ((0,\,0)),\ I = [1,\,1,\,1] = ((1,\,1))$.

Case 6: $U = [t^5,\,1,\,0] = ((0)),\ V = [0,\,1,\,t] = ((\infty)),\ W = [1,\,0,\,0] = ((0,\,0)),\ I = [1,\,1,\,1] = ((1,\,1))$.

First, we consider Case 6. We want to determine the ternary function $F\colon Q \times Q \times Q \longrightarrow Q$ corresponding to the coodinatization of the case. Since the line through the point $[1,\,0,\,0]$ and the point $[1,\,1,\,1]$ is $L_1A^7 = \{[x,\,1,\,1] \mid x \in Q\} \cup \{[1,\,0,\,0]\}$ and the line through the point $[0,\,1,\,t]$ and the point $[t^5,\,1,\,0]$ is $L_{t^5}A^9$, the coordinates $((x,\,x))$ for $x \in Q$ can be determined for example as follows:

$$((0,\,0)) = [1,\,0,\,0],\ ((t^3,\,t^3)) = [0,\,1,\,1] \quad \text{and} \quad ((x,\,x)) = [x,\,1,\,1] \quad \text{for} \quad x \in Q - \{0,\,t^3\}.$$

For the coordinates $((x,\,x))$, all coordinates $((x,\,y))$, $((z))$, $[[x,\,y]]$, $[[z]]$ and the ternary function F can be uniquely determined by a computer research as follows:

k ↓

$m \longrightarrow F(1,\,m,\,k)$

	0	1	t	t^2	t^3	-1	t^5	t^6	t^7
0	0	1	t	t^2	t^3	-1	t^5	t^6	t^7
1	1	t^7	0	t	t^5	t^6	t^3	-1	t^2
t	t	t^5	t^2	t^3	-1	0	1	t^7	t^6
t^2	t^2	0	t^5	1	t^6	t^3	t^7	t	-1
t^3	t^3	t^6	t^7	0	t	t^5	-1	t^2	1
-1	-1	t^3	t^6	t^5	t^7	t^2	t	1	0
t^5	t^5	t	-1	t^7	t^2	1	t^6	0	t^3
t^6	t^6	t^2	t^3	-1	1	t^7	0	t^5	t
t^7	t^7	-1	1	t^6	0	t	t^2	t^3	t^5

k
$\downarrow$
$m \longrightarrow F(t,\, m,\, k)$

	0	1	t	t^2	t^3	-1	t^5	t^6	t^7
0	0	1	t	t^2	t^3	-1	t^5	t^6	t^7
1	t	0	t^3	t^6	-1	t^2	t^7	1	t^5
t	1	t^2	t^6	0	t^5	t^7	-1	t^3	t
t^2	t^3	t	t^7	t^5	1	t^6	t^2	-1	0
t^3	t^2	t^3	t^5	-1	0	1	t	t^7	t^6
-1	t^7	t^5	-1	t	t^6	0	t^3	t^2	1
t^5	t^6	t^7	t^2	1	t	t^3	0	t^5	-1
t^6	t^5	-1	1	t^3	t^7	t	t^6	0	t^2
t^7	-1	t^6	0	t^7	t^2	t^5	1	t	t^3

k
$\downarrow$
$m \longrightarrow F(t^2,\, m,\, k)$

	0	1	t	t^2	t^3	-1	t^5	t^6	t^7
0	0	1	t	t^2	t^3	-1	t^5	t^6	t^7
1	t^2	t^6	1	t^3	t^7	t	-1	t^5	0
t	t^7	0	-1	t^5	1	t^2	t^6	t	t^3
t^2	t^5	-1	t^6	t^7	t^2	1	t^3	0	t
t^3	t^6	t^7	t^2	t	-1	t^3	0	1	t^5
-1	t^3	t	t^5	1	0	t^7	t^2	-1	t^6
t^5	t	t^5	t^7	-1	t^6	0	1	t^3	t^2
t^6	-1	t^3	0	t^6	t	t^5	t^7	t^2	1
t^7	1	t^2	t^3	0	t^5	t^6	t	t^7	-1

k
$\downarrow$
$m \longrightarrow F(t^3,\, m,\, k)$

	0	1	t	t^2	t^3	-1	t^5	t^6	t^7
0	0	1	t	t^2	t^3	-1	t^5	t^6	t^7
1	t^3	t^5	-1	1	0	t^7	t^2	t	t^6
t	-1	t^3	t^5	t^6	t^2	t	t^7	1	0
t^2	t^7	t^6	1	t	t^5	0	-1	t^2	t^3
t^3	t	t^2	0	t^5	t^7	t^6	1	t^3	-1
-1	t^5	-1	t^3	t^7	t	1	t^6	0	t^2
t^5	t^2	0	t^6	t^3	-1	t^5	t	t^7	1
t^6	1	t	t^7	0	t^6	t^2	t^3	-1	t^5
t^7	t^6	t^7	t^2	-1	1	t^3	0	t^5	t

$$k$$
$$\downarrow$$
$m \longrightarrow F(-1, m, k)$

	0	1	t	t^2	t^3	-1	t^5	t^6	t^7
0	0	1	t	t^2	t^3	-1	t^5	t^6	t^7
1	-1	t	t^5	0	t^2	t^3	t^6	t^7	1
t	t^6	t^7	t^3	-1	t	1	0	t^2	t^5
t^2	t	t^2	-1	t^3	0	t^7	1	t^5	t^6
t^3	t^5	-1	t^6	t^7	1	t	t^3	0	t^2
-1	t^2	0	1	t^6	-1	t^5	t^7	t	t^3
t^5	1	t^3	0	t^5	t^7	t^6	t^2	-1	t
t^6	t^7	t^6	t^2	1	t^5	0	t	t^3	-1
t^7	t^3	t^5	t^7	t	t^6	t^2	-1	1	0

$$k$$
$$\downarrow$$
$m \longrightarrow F(t^5, m, k)$

	0	1	t	t^2	t^3	-1	t^5	t^6	t^7
0	0	1	t	t^2	t^3	-1	t^5	t^6	t^7
1	t^5	-1	t^2	t^7	t^6	1	t	0	t^3
t	t^3	t	0	1	t^7	t^6	t^2	t^5	-1
t^2	t^6	t^7	t^3	-1	t	t^5	0	1	t^2
t^3	t^7	0	1	t^3	t^5	t^2	t^6	-1	t
-1	1	t^6	t^7	0	t^2	t	-1	t^3	t^5
t^5	-1	t^2	t^5	t^6	1	t^7	t^3	t	0
t^6	t^2	t^5	-1	t	0	t^3	1	t^7	t^6
t^7	t	t^3	t^6	t^5	-1	0	t^7	t^2	1

$$k$$
$$\downarrow$$
$m \longrightarrow F(t^6, m, k)$

	0	1	t	t^2	t^3	-1	t^5	t^6	t^7
0	0	1	t	t^2	t^3	-1	t^5	t^6	t^7
1	t^6	t^3	t^7	-1	1	t^5	0	t^2	t
t	t^5	-1	1	t^7	t^6	t^3	t	0	t^2
t^2	-1	t^5	t^2	0	t^7	t	t^6	t^3	1
t^3	1	t	-1	t^6	t^2	0	t^7	t^5	t^3
-1	t	t^2	0	t^3	t^5	t^6	1	t^7	-1
t^5	t^7	t^6	t^3	t	0	t^2	-1	1	t^5
t^6	t^3	t^7	t^6	t^5	-1	1	t^2	t	0
t^7	t^2	0	t^5	1	t	t^7	t^3	-1	t^6

k
$\downarrow$
$m \longrightarrow F(t^7, m, k)$

	0	1	t	t^2	t^3	-1	t^5	t^6	t^7
0	0	1	t	t^2	t^3	-1	t^5	t^6	t^7
1	t^7	t^2	t^6	t^5	t	0	1	t^3	-1
t	t^2	t^6	t^7	t	0	t^5	t^3	-1	1
t^2	1	t^3	0	t^6	-1	t^2	t	t^7	t^5
t^3	-1	t^5	t^3	1	t^6	t^7	t^2	t	0
-1	t^6	t^7	t^2	-1	1	t^3	0	t^5	t
t^5	t^3	-1	1	0	t^5	t	t^7	t^2	t^6
t^6	t	0	t^5	t^7	t^2	t^6	-1	1	t^3
t^7	t^5	t	-1	t^3	t^7	1	t^6	0	t^2

Then, S is described by $(*)$ of Section 2. The elements $a_1, \ldots, a_n, b_1, \ldots, b_n$ and the mapping f from Q to Q must satisfy the conditions (1), …, (5) of Theorem 2.1. Let $l(n)$ be the number of distinct elements in $\{a_1, \ldots, a_n\}$ as in Lemma 2.2 and Lemma 2.3. Let $Q = \{a_1, \ldots, a_n, a_{n+1}, \ldots, a_{9+n-l(n)}\} = \{b_1, \ldots, b_n, b_{n+1}, \ldots, b_9\}$.

Suppose that $|S| = 20$. Then, $n = 3$ and $l(3) = 3$ by Lemma 2.2. Therefore, by (2), …, (5) of Theorem 2.1, $Q = \{a_1, \ldots, a_9\} = \{b_1, \ldots, b_9\}$, and we may assume that $f(a_1) = b_4$, $f(a_2) = b_5$, $f(a_3) = b_6$, $f(a_4) = f(a_5) = b_7$, $f(a_6) = f(a_7) = b_8$, $f(a_8) = f(a_9) = b_9$. But there is no $(a_1, \ldots, a_9, b_1, \ldots, b_9)$ satisfying the condition (1) of Theorem 2.1 using a computer.

Suppose that $|S| = 21$. Then, $n = 4$ and $l(4) = 1$, 2, 3 or 4. Assume that $l(4) = 3$. Then, $Q = \{a_1 = a_2, a_3, a_4, \ldots, a_{10}\}$, and we may assume that $f(a_1) = b_5$, $f(a_3) = b_6$, $f(a_4) = b_7$. There are the following two cases. The first case is $f(a_5) = f(a_6) = f(a_7) = f(a_8) = b_8$, $f(a_9) = f(a_{10}) = b_9$. Then, we get (2) or (4) in Appendix as $X(S)$ and S, where each S is an example. The second case is $f(a_5) = f(a_6) = f(a_7) = b_8$, $f(a_8) = f(a_9) = f(a_{10}) = b_9$. Then, we get (1) in Appendix as $X(S)$ and S, or (2), (4) in Appendix as $X(S)$. By a similar argument, when $l(4) = 4$, we get (5) in Appendix. For the other cases, we can not get new $X(S)$'s.

Suppose that $|S| = 22$. Then, $n = 5$ and $l(5) = 1$, 2 or 3 by Lemma 2.3. When $l(5) = 1$, (7), (8), (10) or (12) in Appendix holds. When $l(5) = 2$, (9), (11) or (13) in Appendix holds except $X(S)$ obtained already. When $l(5) = 3$, new $X(S)$'s do not hold.

Suppose that $|S| = 23$. Then, $n = 6$ and $l(6) = 1$ or 2 by Lemma 2.3. When $l(6) = 1$, (15), (16) or (17) in Appendix holds. When $l(6) = 2$, new

$X(S)$'s do not hold.

Case 1 yields (8), (11), (15), (16) or (17) in Appendix as $X(S)$.

Case 2 yields (2), (3), (6), (8), (10), (11), (12), (13), (15), (16) or (17) in Appendix as $X(S)$, where for example S of (3) or (6) in Appendix holds for $X(S)$ of (3) or (6) in Appendix, respectively.

Case 3 yields (7), (8), (10), (11), (12), (13), (15), (16) or (17) in Appendix as $X(S)$.

Case 4 yields (2), (8), (11), (15), (16) or (17) in Appendix as $X(S)$.

Case 5 yields (8), (10), (11), (13), (14), (15), (16) or (17) in Appendix as $S(X)$, where for example S in (14) in Appendix holds for $X(S)$ of (14) in Appendix. Thus we have the following theorem.

Theorem 3.2 *Let S be a blocking semioval in the Hughes plane of order 9 with $x_8 \neq 0$ and $|S| \neq 24$. The following hold*:

(1) $|S| = 21$, 22 *or* 23.

(2) *If* $|S| = 21$, *then*

$$\begin{aligned} X(S) = {} & (21, 43, 16, 6, 1, 3, 0, 1),\ (21, 44, 14, 6, 3, 2, 0, 1), \\ & (21, 44, 16, 0, 9, 0, 0, 1),\ (21, 45, 11, 9, 2, 2, 0, 1), \\ & (21, 45, 12, 6, 5, 1, 0, 1) \quad \textit{or} \quad (21, 46, 8, 12, 1, 2, 0, 1). \end{aligned}$$

(3) *If* $|S| = 22$, *then*

$$\begin{aligned} X(S) = {} & (22, 33, 23, 6, 5, 1, 0, 1),\ (22, 34, 20, 9, 4, 1, 0, 1), \\ & (22, 34, 21, 6, 7, 0, 0, 1),\ (22, 35, 17, 12, 3, 1, 0, 1), \\ & (22, 35, 18, 9, 6, 0, 0, 1),\ (22, 36, 14, 15, 2, 1, 0, 1), \\ & (22, 36, 15, 12, 5, 0, 0, 1) \quad \textit{or} \quad (22, 37, 12, 15, 4, 0, 0, 1). \end{aligned}$$

(4) *If* $|S| = 23$, *then*

$$\begin{aligned} X(S) = {} & (23, 21, 32, 12, 0, 1, 1, 1),\ (23, 23, 27, 15, 1, 0, 1, 1) \\ & \textit{or} \quad (23, 24, 24, 18, 0, 0, 1, 1). \end{aligned}$$

4. The nearfield plane

In this section, we completely determine blocking semiovals with $x_8 \neq 0$ in the nearfield plane of order 9. Let $Q = GF(9)$ with the new multiplication $\circ$ and the field addition $+$ be the nearfield of order 9 defined in Section 3.

Then, the nearfield plane $\Pi = (\mathcal{P}, \mathcal{L})$ of order 9 is defined as follows:

$$\mathcal{P} = \{(x, y) \mid x, y \in Q\} \cup \{(a) \mid a \in Q \cup \{\infty\}\},$$
$$\mathcal{L} = \{[m, k] \mid m \in Q \cup \{\infty\},\ k \in Q\} \cup \{[\infty]\},$$

where $[m, k] = \{(x, x \circ m + k) \mid x \in Q\} \cup \{(m)\}$ for $m \in Q$.

Theorem 4.1 ([8], Section 8) *Let H be the full collineation group of Π.*
(1) *H acts transitively on $[\infty]$.*
(2) *$H_{(\infty)} = H_{(0)}$ and $H_{(\infty)}$ acts tansitively on $[\infty] - \{(0), (\infty)\}$.*
(3) *H acts 2-transitively on $\{(x, y) \mid x, y \in Q\}$.*
(4) *The translation group of Π acts transitively on $\{(x, y) \mid x, y \in Q\}$.*

Let S be a blocking semioval with $x_8 \neq 0$. Let U, V, W, I be four points of Π, no three of which are collinear, and let $S \supseteq UV - \{U, V\}$. From Theorem 4.1, we may consider the following four coordinatizations by ((,)), (( )) for the points and [[,]], [[ ]] for the lines in Π (see Section 2). Namely we will take $[\infty]$ as the 8-secant in the last two cases of the following.

Case 1: $U = (0, 0) = ((0)), V = (0, 1) = ((\infty)), W = (1, 0) = ((0, 0)),$
$I = (1, 1) = ((1, 1)).$

Case 2: $U = (0, 0) = ((0)), V = (\infty) = ((\infty)), W = (0) = ((0, 0)),$
$I = (1, 1) = ((1, 1)).$

Case 3: $U = (1) = ((0)), V = (\infty) = ((\infty)), W = (0, 0) = ((0, 0)),$
$I = (1, 0) = ((1, 1)).$

Case 4: $U = (0) = ((0)), V = (\infty) = ((\infty)), W = (0, 0) = ((0, 0)),$
$I = (1, 1) = ((1, 1)).$

Then, S is described by $(*)$ of Section 2. The elements $a_1, \dots, a_n, b_1, \dots, b_n$ and the mapping f must satisfy the conditions (1), ..., (5) in Theorem 2.1. By a similar argument as in Section 3, we have the following.

Case 1 yields (2), (8), (12), (10), (7), (13), (11), (15), (16) or (17) in Appendix as $X(S)$ and S is for example

$$\{(t^7, 1), (t, 1), (1, t), (t^6, t^6), (t^2, 1), (t^6, t^2), (-1, t^3), (-1, 0), (t^6, 0), (t^2), (1, 0), (t^3, t^5), (t^5, t^7), (\infty)\} \cup \{(0, x) \mid x \in Q - \{0, 1\}\},$$

$$\{(1,1),(t^5,1),(t^7,1),(-1,1),(t,1),(t^3,1),\\ (-1,t^6),(1,t^2),(t^3,t^5),(t^7,t),(t^6),(t^6,-1),\\ (t^3,0),(1,0),(\infty)\}\cup\{(0,x)\mid x\in Q-\{0,1\}\},$$

$$\{(t^5,t^5),(t^3,-1),(t^6,t^2),(t^2,t^3),(t,t^7),(t^7,0),\\ (t^3,1),(-1,t^3),(t^7,-1),(1,t^7),(t,-1),(t^2,t^7),\\ (t^2,-1),(t,t^3),(\infty)\}\cup\{(0,x)\mid x\in Q-\{0,1\}\},$$

$$\{(1,t),(-1,t^6),(t^3,-1),(t^6,t^2),(t^2,t^3),(t^7),\\ (t^7,t^5),(-1,t^2),(t^5,t^3),(1,t^6),(1,1),(t^3,t^3),\\ (t^6,0),(1,0),(\infty)\}\cup\{(0,x)\mid x\in Q-\{0,1\}\},$$

$$\{(t^5,t^5),(t^3,-1),(t^6,t^2),(t^2,t^3),(t,t^7),(t^7,0),\\ (t^5,1),(t),(-1,t^5),(t^3,t^5),(t^7,t),(t^5,t^7),\\ (t^5,t^6),(t^2,t^5),(\infty)\}\cup\{(0,x)\mid x\in Q-\{0,1\}\},$$

$$\{(1,1),(t^6,1),(t^7,1),(t^3,1),(t^6,t^2),(0),\\ (t^2,t^3),(t^7,t^5),(-1,t^2),(t^5,t^3),(t^3,t),(t^3,t^2),\\ (t^6,t^3),(-1,t^5),(\infty)\}\cup\{(0,x)\mid x\in Q-\{0,1\}\},$$

$$\{(1,t),(-1,t^6),(t^3,-1),(t^2,t^3),(-1,1),(t,t^7),\\ (0),(-1,t^3),(t^6,t),(t,t^2),(t^7,-1),(t,t),\\ (t^3,t^3),(-1,-1),(\infty)\}\cup\{(0,x)\mid x\in Q-\{0,1\}\},$$

$$\{(1,1),(t^5,1),(t^6,1),(t^7,1),(-1,1),(t,1),\\ (t^2,1),(t^7),(-1,t^3),(t^6,t),(t,t^2),(t^5,t^6),\\ (t^7,-1),(t^2),(1,0),(\infty)\}\cup\{(0,x)\mid x\in Q-\{0,1\}\},$$

$$\{(1,1),(t^5,1),(t^6,1),(t^7,1),(-1,1),(t^2,1),\\ (t^3,1),(t^2,t^3),(t^5,-1),(t^5),(t^3,t^6),(1,t^5),\\ (t^3,0),(t^2),(1,0),(\infty)\}\cup\{(0,x)\mid x\in Q-\{0,1\}\}\quad\text{or}$$

$$\{(1,1),(t^5,1),(t^6,1),(t^7,1),(-1,1),(t,1),\\ (t^3,1),(t,t^7),(t^7,t^5),(-1,t^2),(t^5,t^3),(t^6,0),\\ (t^3,0),(t^2),(1,0),(\infty)\}\cup\{(0,x)\mid x\in Q-\{0,1\}\},\quad\text{respectively}$$

Case 2 yields (6), (3), (2), (8), (10), (7), (11), (9), (15), (16) or (17) in Appendix as $X(S)$, where S is for example

$$\{(1,\,1),\,(t,\,t^7),\,(t^6,\,t),\,(t^5),\,(1,\,-1),\,(t,\,t^3),\,(t^6,\,t^5),\,(t),\,(t^2,\,0),\,(t^3,\,0),\,(-1,\,0),\,(t^5,\,0),\,(t^7,\,0)\}\cup\{(0,\,x)|x\in Q^*\},$$

$$\{(t^5),\,(t^6),\,(t^7),\,(-1),\,(0),\,(1,\,1),\,(-1,\,-1),\,(t,\,t^3),\,(t^5,\,t^7),\,(t^2,\,t),\,(t^6,\,t^5),\,(t^3,\,t^2),\,(t^7,\,t^6)\}\cup\{(0,\,x)|x\in Q^*\}\quad\text{or}$$

$$\{(1,\,1),\,(1,\,t^5),\,(1,\,t^6),\,(t,\,t^2),\,(t,\,t^6),\,(1,\,-1),\,(t,\,0),\,(t^3,\,t^5),\,(-1,\,t^6),\,(t^7,\,t),\,(t^2),\,(t^2,\,t^7),\,(t^5,\,1),\,(t^6,\,t^3)\}\cup\{(0,\,x)|x\in Q^*\}$$

for $X(S)$ of (6), (3) or (9) in Appendix, respectively.

Case 3 yields (8), (15) or (17) in Appendix as $X(S)$.

Case 4 also yields (8), (15) or (17) in Appendix as $X(S)$. Thus we have the following theorem.

Theorem 4.2 *Let S be a blocking semioval in the nearfield plane of order 9 with $x_8 \neq 0$ and $|S| \neq 24$. The following hold*:

(1) $|S| = 21$, 22 *or* 23.

(2) *If* $|S| = 21$, *then*

$$X(S) = (21,\,44,\,16,\,0,\,9,\,0,\,0,\,1)\quad or\quad (21,\,46,\,8,\,12,\,1,\,2,\,0,\,1).$$

(3) *If* $|S| = 22$, *then*

$$X(S) = (22,\,33,\,23,\,6,\,5,\,1,\,0,\,1),\ (22,\,34,\,20,\,9,\,4,\,1,\,0,\,1),\ (22,\,34,\,21,\,6,\,7,\,0,\,0,\,1),\ (22,\,35,\,17,\,12,\,3,\,1,\,0,\,1),\ (22,\,35,\,18,\,9,\,6,\,0,\,0,\,1),\ (22,\,36,\,14,\,15,\,2,\,1,\,0,\,1)\ or\ (22,\,36,\,15,\,12,\,5,\,0,\,0,\,1).$$

(4) *If* $|S| = 23$, *then*

$$X(S) = (23,\,21,\,32,\,12,\,0,\,1,\,1,\,1),\ (23,\,23,\,27,\,15,\,1,\,0,\,1,\,1)\ or\ (23,24,\,24,\,18,\,0,\,0,\,1,\,1).$$

5. The dual nearfield plane

In this section, we completely determine blocking semiovals with $x_8 \neq 0$ in the dual nearfield plane of order 9. Let $Q = GF(9)$ with the new multiplication $\circ$ and the field addition $+$ be the nearfield plane of order 9 defined in Section 3. Let $\Pi = (\mathcal{P}, \mathcal{L})$ be the nearfield plane defined using Q in Section 4. Let $t \in GF(9)$ such that $t^2 = 1 + t$. Then $GF(9)^* = \langle t \rangle$. Set $G = (\operatorname{Aut}\Pi)_{(0,0)}$, and let T be the translation group of Π. Then, $T = \{t(a, b) | a, b \in Q\}$, where $t(a, b)\colon (x, y) \longmapsto (x + a, x + b)$, $\operatorname{Aut}\Pi = GT$ and T is a normal subgroup of $\operatorname{Aut}\Pi$.

Theorem 5.1 ([8], Section 8)
(1) $\operatorname{Aut}\Pi$ *has two orbits* $\{[\infty]\}$, $\mathcal{L} - \{[\infty]\}$ *on* $\mathcal{L}$.
(2) $(\operatorname{Aut}\Pi)_{[\infty, 0]}$ *is transitive on* $\{l \in \mathcal{L} \mid l \ni (\infty), l \neq [\infty, 0], [\infty]\}$.
(3) $(\operatorname{Aut}\Pi)_{[\infty, 0]}$ *has two orbits* $\Gamma_1 := \{[m, k] \mid m \in Q^*, k \in Q\}$ *and* $\Gamma_2 := \{[0, k] \mid k \in Q\}$ *on* $\{l \in \mathcal{L} \mid l \not\ni (\infty)\}$.

Proof. Since G is transitive on $[\infty]$, (1) holds.

Since $G((0), [\infty, 0]) = \{\varphi \in G \mid \varphi$ is a perspectivity with the center (0) and the axis $[\infty, 0]\} = \{(x, y) \longmapsto (x \circ a, y) \mid a \in Q^*\}$, (2) holds.

$(\operatorname{Aut}\Pi)_{[\infty, 0]}$ fixes (∞) and (0). Since Π is a translation plane, $\{[0, k] \mid k \in Q\}$ is an orbit of $(\operatorname{Aut}\Pi)_{[\infty, 0]}$. Since $G((\infty), [0, 0]) = \{(x, y) \longmapsto (x, y \circ a) \mid a \in Q^*\}$ and Π is a translation plane, $\{[m, k] \mid m \in Q^*, k \in Q\}$ is an orbit of $(\operatorname{Aut}\Pi)_{[\infty, 0]}$. Thus, (3) holds. □

Let S be a blocking semioval in the dual plane Π^d of the plane Π with $x_8 \neq 0$ and $|S| \neq 24$. Let U, V, W, I be four points of Π^d, no three of which are collinear, and let $S \supseteq UV - \{U, V\}$. From Theorem 5.1, we may consider the following four coodinatizations by $((\ ,\ ))$, $((\ ))$ for the points and $[[\ ,\ ]]$, $[[\ ]]$ for the lines in Π^d (see Section 2), namely four cases of $V = [\infty]$, $V = [\infty, 1]$, $V \in \Gamma_1$ or $V \in \Gamma_2$.

Case 1: $U = [\infty, 0] = ((0))$, $V = [\infty] = ((\infty))$, $W = [0, 0] = ((0, 0))$, $I = [1, 1] = ((1, 1))$.

Case 2: $U = [\infty, 0] = ((0))$, $V = [\infty, 1] = ((\infty))$, $W = [0, 0] = ((0, 0))$, $I = [1, 1] = ((1, 1))$.

Case 3: $U = [\infty, 0] = ((0))$, $V = [0, 0] = ((\infty))$, $W = [\infty] = ((0, 0))$, $I = [1, 1] = ((1, 1))$.

Case 4: $U = [\infty, 0] = ((0))$, $V = [1, 0] = ((\infty))$, $W = [\infty] = ((0, 0))$, $I = [-1, 1] = ((1, 1))$.

Then, S is described by $(*)$ of Section 2. The elements $a_1, \ldots, a_n, b_1, \ldots, b_n$ and the mapping f must satisfy the conditions (1), ..., (5) in Theorem 2.1. By a similar argument as in Section 3, we have the following.

Case 1 yields (3), (8), (15) or (17) in Appendix as $X(S)$ and S is for example

$$\{[1, 1], [t, t^2], [t^2, t^3], [t^3, t], [1, -1], [t, t^6], [t^2, t^7], [t^3, t^5], [-1, 0], [t^5, 0], [t^6, 0], [t^7, 0], [0, 0]\} \cup \{[\infty, x] \mid x \in Q^*\},$$

$$\{[0, 1], [0, t], [0, t^2], [0, -1], [0, t^6], [0, 0], [1, t^3], [t, t^3], [t^2, t^3], [-1, t^3], [t^3, t^7], [t^5, t^7], [t^6, t^5], [t^7, t^5]\} \cup \{[\infty, x] \mid x \in Q^*\},$$

$$\{[0, 1], [0, t], [0, t^2], [0, t^3], [0, -1], [0, t^5], [0, 0], [1, t^6], [t, t^6], [t^2, t^6], [-1, t^6], [t^5, t^6], [t^6, t^6], [t^3, t^7], [t^7, t^7]\} \cup \{[\infty, x] \mid x \in Q^*\} \quad \text{or}$$

$$\{[0, 1], [0, t], [0, t^2], [0, t^3], [0, -1], [0, t^5], [0, 0], [1, t^6], [t, t^6], [-1, t^6], [t^5, t^6], [t^2, t^7], [t^3, t^7], [t^6, t^7], [t^7, t^7]\} \cup \{[\infty, x] \mid x \in Q^*\}, \quad \text{respectively.}$$

Case 2 yields (8), (15), (16) or (17) in Appendix as $X(S)$, where S is for example

$$\{[-1, 1], [t^5, t], [t^7, t^3], [1, -1], [t^2, t^6], [t^3, t^7], [0, 0], [t^6, t^2], [t^7, t^2], [t^2, t^2], [t^5, t^2], [-1, t^2], [t^6, t^5], [t^3, t^5], [1, t^5], [\infty]\} \cup \{[\infty, x] \mid x \in Q - \{0, 1\}\}$$

for $X(S)$ of (16) in Appendix.

Case 3 yields (6), (8), (10), (11), (15), (16) or (17) in Appendix as $X(S)$, where S is for example

$$\{[0, t], [0, t^2], [0, t^3], [0, -1], [0, t^6], [1, 1], [-1, 1], [t^6, t^7], [t^2, t^7], [t^7, t^5], [t^3, t^5], [\infty, t^7],$$

$$[\infty,\ t^3]\} \cup \{[x,\ 0] \mid x \in Q^*\},$$

$$\{[1,\ 1],\ [t,\ t],\ [t^2,\ t^2],\ [t^3,\ t^3],\ [t^5,\ t^5],\ [\infty,\ -1],\ [t^6,\ t^7],\ [t,\ t^7],\ [t^2,\ t^7],\ [t^5,\ t^6],\ [1,\ t^6],\ [0,\ t^6],\ [1,\ -1],\ [t,\ -1]\} \cup \{[x,\ 0] \mid x \in Q^*\} \quad \text{or}$$

$$\{[1,\ 1],\ [t,\ t],\ [t^6,\ t^6],\ [t^2,\ t^3],\ [-1,\ t^5],\ [\infty,\ -1],\ [t^6,\ t^7],\ [1,\ t^2],\ [t,\ t^2],\ [t^6,\ t^2],\ [t^3,\ t^2],\ [t^6,\ -1],\ [t,\ -1],\ [0,\ -1]\} \cup \{[x,\ 0] \mid x \in Q^*\}$$

for $X(S)$ of (6), (10) or (11) in Appendix, respectively.

Case 4 yields (8), (7), (13), (11), (15), (16) or (17) in Appendix as $X(S)$, where S is for example

$$\{[-1,\ 1],\ [t^7,\ t^2],\ [t^6,\ t^3],\ [0,\ -1],\ [t^3,\ t^5],\ [\infty,\ -1],\ [t^3,\ t^7],\ [t^7,\ t^7],\ [0,\ t^7],\ [t^2,\ t^6],\ [t^3,\ t^6],\ [1,\ t^6],\ [t^3,\ t],\ [t^6,\ t]\} \cup \{[x,\ 0] \mid x \in Q - \{1\}\} \quad \text{or}$$

$$\{[t^2,\ t],\ [t^7,\ t^2],\ [0,\ -1],\ [t^3,\ t^5],\ [t^7,\ 1],\ [t^5,\ t^6],\ [t^3,\ t^7],\ [\infty,\ t^6],\ [\infty,\ t^7],\ [\infty,\ t],\ [\infty,\ t^3],\ [\infty],\ [t,\ t^3],\ [t^7,\ t^3]\} \cup \{[x,\ 0] \mid x \in Q - \{1\}\}$$

for $X(S)$ of (7) or (13) in Appendix, respectively. Thus we have the following theorem.

Theorem 5.2 *Let S be a blocking semioval in the dual nearfield plane of order 9 with $x_8 \neq 8$ and $|S| \neq 24$. The following hold:*
(1) $|S| = 21$, 22 *or* 23.
(2) *If* $|S| = 21$*, then*

$$X(S) = (21,\ 44,\ 16,\ 0,\ 9,\ 0,\ 0,\ 1) \quad \text{or} \quad (21,\ 46,\ 8,\ 12,\ 1,\ 2,\ 0,\ 1).$$

(3) *If* $|S| = 22$*, then*

$$X(S) = (22,\ 33,\ 23,\ 6,\ 5,\ 1,\ 0,\ 1),\ (22,\ 34,\ 20,\ 9,\ 4,\ 1,\ 0,\ 1),\ (22,\ 35,\ 17,\ 12,\ 3,\ 1,\ 0,\ 1),\ (22,\ 35,\ 18,\ 9,\ 6,\ 0,\ 0,\ 1) \ \text{or}\ (22, 36,\ 15,\ 12,\ 5,\ 0,\ 0,\ 1).$$

(4) *If* $|S| = 23$, *then*

$$X(S) = (23,\ 21,\ 32,\ 12,\ 0,\ 1,\ 1,\ 1),\ (23,\ 23,\ 27,\ 15,\ 1,\ 0,\ 1,\ 1) \quad or \quad (23,\ 24,\ 24,\ 18,\ 0,\ 0,\ 1,\ 1).$$

Appendix

(1) $X(S) = (21,\ 43,\ 16,\ 6,\ 1,\ 3,\ 0,\ 1)$

$$S = \{[t^2, 1, 1],\ [t^5, 1, t^3],\ [1, 0, t^6],\ [t^2, 1, t^2],\ [t^3, 1, -1],\ [t^6, 1, t^6],\ [-1, 1, 0],\ [1, 0, t^3],\ [0, 1, t^6],\ [-1, 1, 1],\ [t, 1, 1],\ [1, 1, t^7],\ [t^7, 1, t],\ [t^3, 1, 1],\ [t^7, 1, t^6],\ [t^2, 1, t^3],\ [t^6, 1, -1],\ [1, 0, 1],\ [1, 1, t^2],\ [-1, 1, t^7],\ [t, 1, t^5]\}$$

(2) $X(S) = (21,\ 44,\ 14,\ 6,\ 3,\ 2,\ 0,\ 1)$

$$S = \{[t^6, 1, t^2],\ [t^3, 1, -1],\ [t^6, 1, t^6],\ [1, 1, t^5],\ [1, 1, 0],\ [t^3, 1, t^2],\ [t^2, 1, t^2],\ [0, 1, t^7],\ [t, 1, t^2],\ [1, 1, t^7],\ [t^3, 1, t],\ [t^5, 1, t^2],\ [t^5, 1, t^7],\ [t^3, 1, 1],\ [t^7, 1, t^6],\ [t^2, 1, t^3],\ [t^6, 1, -1],\ [1, 0, 1],\ [1, 1, t^2],\ [-1, 1, t^7],\ [t, 1, t^5]\}$$

(3) $X(S) = (21,\ 44,\ 16,\ 0,\ 9,\ 0,\ 0,\ 1)$

$$S = \{[t, 1, t^2],\ [t^2, 1, t^6],\ [-1, 1, t^5],\ [1, 0, t^6],\ [t^7, 1, -1],\ [t^6, 1, 1],\ [t^6, 1, t^2],\ [1, 1, -1], [1, 1, 0],\ [0, 1, t^2],\ [0, 1, t],\ [t^3, 1, t^7],\ [t^3, 1, t^2],\ [t^5, 1, 1],\ [t^5, 1, t],\ [t^5, 1, t^2],\ [t^5, 1, t^3],\ [t^5, 1, -1],\ [t^5, 1, t^5],\ [t^5, 1, t^6],\ [t^5, 1, t^7]\}$$

(4) $X(S) = (21,\ 45,\ 11,\ 9,\ 2,\ 2,\ 0,\ 1)$

$$S = \{[0, 1, t^7],\ [0, 1, t^3],\ [t^7, 1, t^3],\ [t, 1, 1],\ [0, 1, 0],\ [t^5, 1, t^2],\ [t^6, 1, t^5],\ [t^3, 1, -1],\ [1, 1, t^6],\ [-1, 1, t^3],\ [1, 0, -1],\ [t^3, 1, t^7],\ [t^2, 1, t],\ [t^3, 1, 1],\ [t^7, 1, t^6],\ [t^2, 1, t^3],\ [t^6, 1, -1],\ [1, 0, 1],\ [1, 1, t^2],\ [-1, 1, t^7],\ [t, 1, t^5]\}$$

(5) $X(S) = (21,\ 45,\ 12,\ 6,\ 5,\ 1,\ 0,\ 1)$

$$S = \{[1, 1, 1],\ [t, 1, 1],\ [t^5, 1, t^3],\ [0, 1, 1],\ [t^7, 1, t^3],\ [t^6, 1, t^5],\ [t^3, 1, -1],\ [0, 1, t^3],\ [t^2, 1, 0],\ [t^7, 1, 0],$$

$[-1, 1, 0], [t, 1, 0], [1, 0, 0], [t^3, 1, 1], [t^7, 1, t^6], [t^2, 1, t^3],$
$[t^6, 1, -1], [1, 0, 1], [1, 1, t^2], [-1, 1, t^7], [t, 1, t^5]\}$

(6) $X(S) = (21, 46, 8, 12, 1, 2, 0, 1)$

$$S = \{[1, 1, 0], [t^3, 1, 0], [-1, 1, 0], [1, 0, 0], [t^6, 1, 0], [t, 1, 0], [t, 1, t^6], [t^7, 1, t^2], [t^7, 1, t^6], [0, 1, -1], [0, 1, 1], [t^2, 1, 1], [t^2, 1, -1], [t^5, 1, 1], [t^5, 1, t], [t^5, 1, t^2], [t^5, 1, t^3], [t^5, 1, -1], [t^5, 1, t^5], [t^5, 1, t^6], [t^5, 1, t^7]\}$$

(7) $X(S) = (22, 33, 23, 6, 5, 1, 0, 1)$

$$S = \{[0, 1, t^7], [0, 1, t^2], [0, 1, t^6], [0, 0, 1], [0, 1, t^5], [0, 1, t^3], [t^6, 1, 0], [t^7, 1, 0], [1, 0, t], [1, 1, t^7], [t^7, 1, t], [1, 1, t^3], [t^7, 1, t^5], [1, 1, 0], [t^3, 1, 1], [t^7, 1, t^6], [t^2, 1, t^3], [t^6, 1, -1], [1, 0, 1], [1, 1, t^2], [-1, 1, t^7], [t, 1, t^5]\}$$

(8) $X(S) = (22, 34, 20, 9, 4, 1, 0, 1)$

$$S = \{[-1, 1, t^6], [t, 1, -1], [t^5, 1, t^2], [t^2, 1, t^7], [t^3, 1, t^5], [t^6, 1, 0], [-1, 1, -1], [t^2, 1, 1], [0, 1, t^2], [t^3, 1, t^6], [t, 1, t^2], [t^7, 1, -1], [t^3, 1, t^2], [1, 1, t], [t^3, 1, 1], [t^7, 1, t^6], [t^2, 1, t^3], [t^6, 1, -1], [1, 0, 1], [1, 1, t^2], [-1, 1, t^7], [t, 1, t^5]\}$$

(9) $X(S) = (22, 34, 21, 6, 7, 0, 0, 1)$

$$S = \{[t^6, 1, t^2], [t^2, 1, 1], [t^5, 1, t^3], [t, 1, t^6], [t^3, 1, t^7], [1, 1, 0], [t, 1, t^3], [t^7, 1, t^2], [0, 1, t^5], [-1, 1, t^3], [t, 1, t], [1, 1, 1], [t, 1, t^2], [t^6, 1, t^3], [t^3, 1, 1], [t^7, 1, t^6], [t^2, 1, t^3], [t^6, 1, -1], [1, 0, 1], [1, 1, t^2], [-1, 1, t^7], [t, 1, t^5]\}$$

(10) $X(S) = (22, 35, 17, 12, 3, 1, 0, 1)$

$$S = \{[-1, 1, t^6], [1, 0, t^3], [t^5, 1, t^2], [t^2, 1, t^7], [t^3, 1, t^5], [t^6, 1, 0], [-1, 1, -1], [t^2, 1, 1], [0, 1, t^2], [t^3, 1, t^6], [t^7, 1, t^5], [t^3, 1, t^7], [1, 0, t^6], [t^3, 1, t], [t^3, 1, 1], [t^7, 1, t^6], [t^2, 1, t^3], [t^6, 1, -1], [1, 0, 1], [1, 1, t^2], [-1, 1, t^7], [t, 1, t^5]\}$$

(11) $X(S) = (22, 35, 18, 9, 6, 0, 0, 1)$

$$S = \{[-1, 1, t^6],\ [t, 1, -1],\ [1, 0, t^3],\ [t^5, 1, t^2],\ [-1, 1, t^5], [t^6, 1, 0],\ [t^2, 1, 1],\ [0, 1, t^3],\ [t^6, 1, t^7],\ [1, 1, -1],\ [t^2, 1, t^2], [-1, 1, t],\ [t^7, 1, t^2],\ [1, 0, -1],\ [t^3, 1, 1],\ [t^7, 1, t^6], [t^2, 1, t^3],\ [t^6, 1, -1],\ [1, 0, 1],\ [1, 1, t^2],\ [-1, 1, t^7],\ [t, 1, t^5]\}$$

(12) $X(S) = (22, 36, 14, 15, 2, 1, 0, 1)$

$$S = \{[1, 0, t^5],\ [t^7, 1, t^7],\ [t^5, 1, t^3],\ [t^3, 1, -1],\ [t, 1, t^6], [1, 1, 0],\ [1, 1, 1],\ [0, 1, t^7],\ [t, 1, t^2],\ [t^7, 1, -1],\ [-1, 1, -1], [t^6, 1, t],\ [t^2, 1, t^5],\ [1, 1, t^7],\ [t^3, 1, 1],\ [t^7, 1, t^6],\ [t^2, 1, t^3], [t^6, 1, -1],\ [1, 0, 1],\ [1, 1, t^2],\ [-1, 1, t^7],\ [t, 1, t^5]\}$$

(13) $X(S) = (22, 36, 15, 12, 5, 0, 0, 1)$

$$S = \{[1, 1, 1],\ [-1, 1, t^6],\ [1, 0, t^3],\ [t^5, 1, t^2],\ [1, 0, t^5], [t^3, 1, t^5],\ [1, 1, 0],\ [t^7, 1, t^2],\ [-1, 1, t^3],\ [t^6, 1, 1],\ [1, 0, -1], [t, 1, t],\ [0, 1, t^2],\ [1, 0, t^7],\ [t^3, 1, 1],\ [t^7, 1, t^6],\ [t^2, 1, t^3], [t^6, 1, -1],\ [1, 0, 1],\ [1, 1, t^2],\ [-1, 1, t^7],\ [t, 1, t^5]\}$$

(14) $X(S) = (22, 37, 12, 15, 4, 0, 0, 1)$

$$S = \{[1, 1, 1],\ [t^6, 1, -1],\ [-1, 1, t],\ [t^7, 1, t^5],\ [-1, 1, -1], [t^7, 1, t^2],\ [t^5, 1, t^6],\ [t^7, 1, t^7],\ [t^3, 1, t^2],\ [-1, 1, 1], [t^2, 1, -1],\ [t^3, 1, t^3],\ [0, 1, -1],\ [t, 1, 1],\ [0, 1, 0],\ [1, 1, 0], [t^2, 1, 0],\ [t^3, 1, 0],\ [-1, 1, 0],\ [t^6, 1, 0],\ [t^7, 1, 0],\ [1, 0, 0]\}$$

(15) $X(S) = (23, 21, 32, 12, 0, 1, 1, 1)$

$$S = \{[1, 1, 1],\ [t^7, 1, t^3],\ [t, 1, -1],\ [t^5, 1, t^2],\ [t^2, 1, t^7], [t^3, 1, t^5],\ [t^6, 1, 0],\ [t, 1, 1],\ [t^6, 1, t^2],\ [0, 1, -1],\ [t^3, 1, t^3], [t^2, 1, t^5],\ [1, 0, t],\ [t^2, 1, -1],\ [1, 1, t],\ [t^3, 1, 1],\ [t^7, 1, t^6], [t^2, 1, t^3],\ [t^6, 1, -1],\ [1, 0, 1],\ [1, 1, t^2],\ [-1, 1, t^7],\ [t, 1, t^5]\}$$

(16) $X(S) = (23, 23, 27, 15, 1, 0, 1, 1)$

$$S = \{[1, 1, 1],\ [t, 1, -1],\ [1, 0, t^3],\ [t^5, 1, t^2],\ [t^2, 1, t^7], [t^3, 1, t^5],\ [t^6, 1, 0],\ [t, 1, 1],\ [t^6, 1, t^2],\ [0, 1, -1],\ [t^3, 1, t^3],$$

$$[t^2, 1, t^5],\ [1, 1, t^5],\ [t^3, 1, t^6],\ [t^6, 1, t],\ [t^3, 1, 1],\ [t^7, 1, t^6],$$
$$[t^2, 1, t^3],\ [t^6, 1, -1],\ [1, 0, 1],\ [1, 1, t^2],\ [-1, 1, t^7],\ [t, 1, t^5]\}$$

(17) $X(S) = (23, 24, 24, 18, 0, 0, 1, 1)$

$$S = \{[-1, 1, t^6],\ [t, 1, -1],\ [1, 0, t^3],\ [t^5, 1, t^2],\ [t^2, 1, t^7],$$
$$[t^3, 1, t^5],\ [t^6, 1, 0],\ [t^2, 1, t^6],\ [-1, 1, t^5],\ [0, 1, t^7],\ [t, 1, t^2],$$
$$[t, 1, t^7],\ [1, 1, t^5],\ [t^3, 1, t^6],\ [t^6, 1, t],\ [t^3, 1, 1],\ [t^7, 1, t^6],$$
$$[t^2, 1, t^3],\ [t^6, 1, -1],\ [1, 0, 1],\ [1, 1, t^2],\ [-1, 1, t^7],\ [t, 1, t^5]\}$$

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N. Nakagawa
Department of Mathematics
Faculty of Science and Technology
Kinki University
Higashi-Osaka, Osaka 577-8502, Japan
E-mail: nakagawa@math.kindai.ac.jp

C. Suetake
Department of Mathematics
Faculty of Engineering
Oita University
700, Dan-noharu, Oita
870-1192, Japan
E-mail: suetake@csis.oita-u.ac.jp