Hokkaido Mathematical Journal Vol. 35 (2006) p. 321–364

Navier-Stokes equations in a rotating frame in $\mathbb{R}^3$ with initial data nondecreasing at infinity

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(Received August 27, 2004; Revised May 2, 2005)

Abstract. Three-dimensional rotating Navier-Stokes equations are considered with a constant Coriolis parameter Ω and initial data nondecreasing at infinity. In contrast to the non-rotating case ($\Omega = 0$), it is shown for the problem with rotation ($\Omega \neq 0$) that Green's function corresponding to the linear problem (Stokes + Coriolis combined operator) does not belong to $L^1(\mathbb{R}^3)$. Moreover, the corresponding integral operator is unbounded in the space $L^\infty_\sigma(\mathbb{R}^3)$ of solenoidal vector fields in $\mathbb{R}^3$ and the linear (Stokes+Coriolis) combined operator does not generate a semigroup in $L^\infty_\sigma(\mathbb{R}^3)$. Local in time unique solvability of the rotating Navier-Stokes equations is proven for initial velocity fields in the space $L^\infty_{\sigma,a}(\mathbb{R}^3)$ which consists of L^∞ solenoidal vector fields satisfying vertical averaging property such that their baroclinic component belongs to a homogeneous Besov space $\dot{B}^0_{\infty,1}$ which is smaller than L^∞ but still contains various periodic and almost periodic functions. This restriction of initial data to $L^\infty_{\sigma,a}(\mathbb{R}^3)$ which is a subspace of $L^\infty_\sigma(\mathbb{R}^3)$ is essential for the combined linear operator (Stokes + Coriolis) to generate a semigroup. Using the rotation transformation, we also obtain local in time solvability of the classical 3D Navier-Stokes equations in $\mathbb{R}^3$ with initial velocity and vorticity of the form $\mathbf{V}(0) = \tilde{\mathbf{V}}_0(y) + (\Omega/2)e_3 \times y$, $\operatorname{curl}\mathbf{V}(0) = \operatorname{curl}\tilde{\mathbf{V}}_0(y) + \Omega e_3$ where $\tilde{\mathbf{V}}_0(y) \in L^\infty_{\sigma,a}(\mathbb{R}^3)$.

Key words: rotating Navier-Stokes equations, nondecreasing initial data, homogeneous Besov spaces, Riesz operators.

1. Introduction

In this paper we study initial value problem for the three-dimensional rotating Navier-Stokes equations in $\mathbb{R}^3$ with initial data nondecreasing at infinity:

$$\partial_t\mathbf{U} + (\mathbf{U}\cdot\nabla)\mathbf{U} + \Omega e_3 \times \mathbf{U} + \nu \operatorname{curl}^2 \mathbf{U} = -\nabla p,\ \nabla\cdot\mathbf{U} = 0, \quad (1.1)$$

$$\mathbf{U}(t, x)|_{t=0} = \mathbf{U}_0(x) \quad (1.2)$$

where $x = (x_1, x_2, x_3)$, $\mathbf{U}(t, x) = (U_1, U_2, U_3)$ is the velocity field and p is the pressure. In Eqs. (1.1) e_3 denotes the vertical unit vector and Ω is a constant Coriolis parameter; the term $\Omega e_3 \times \mathbf{U}$ restricted to divergence free vector fields is called the Coriolis operator. The initial velocity field

2000 Mathematics Subject Classification : Primary: 76D05, Secondary: 76U05.

$\mathbf{U}_0(x)$ depends on three variables x_1, x_2 and x_3. We consider initial data in spaces of solenoidal vector fields $L^\infty_\sigma(\mathbb{R}^3)$ nondecreasing at infinity ($L^\infty(\mathbb{R}^3)$ restricted to the divergence free subspace). The consideration of solutions not decaying at infinity is essential in the development of rigorous mathematical theory of 3D rotating turbulence (homogeneous statistical solutions [10]). In this paper we prove local in time unique solvability of the rotating Navier-Stokes equations in $\mathbb{R}^3$ under the condition that the initial velocity $\mathbf{U}_0 \in L^\infty_{\sigma,a}(\mathbb{R}^3)$, which is a subspace of $L^\infty_\sigma(\mathbb{R}^3)$ having vertical averaging property. We take initial data in the space

$$L^\infty_{\sigma,a}(\mathbb{R}^3) = \{u \in L^\infty(\mathbb{R}^3) \colon u - \overline{u} \in \dot{B}^0_{\infty,1}\}$$

where $\dot{B}^0_{\infty,1}$ is a Besov space which contains various periodic and almost periodic functions (see Appendix B). Here $\overline{u}$ denotes the vertical average of u. We use $\dot{B}^0_{\infty,1}$ since the Riesz operator is bounded in $\dot{B}^0_{\infty,1}$ but not in L^∞. The space $L^\infty_{\sigma,a}(\mathbb{R}^3)$ is a subspace of $L^\infty_\sigma(\mathbb{R}^3)$ which consists of bounded vector fields satisfying *vertical averaging property*. It is shown that the linear combined operator (Stokes+Coriolis) generates a bounded semigroup on $L^\infty_{\sigma,a}(\mathbb{R}^3)$ for each $\Omega \in \mathbb{R}$.

The above initial value problem (1.1)-(1.2) for the 3D rotating Navier-Stokes Equations is equivalent, via rotation transformation with respect to the vertical axis e_3, to the initial value problem for the classical (non-rotating) 3D Navier-Stokes Equations with initial data of the type $\mathbf{V}(0) = \tilde{\mathbf{V}}_0(y) + (\Omega/2)e_3 \times y$:

$$\partial_t \mathbf{V} + (\mathbf{V}\cdot\nabla)\mathbf{V} + \nu \operatorname{curl}^2 \mathbf{V} = -\nabla q, \quad \nabla\cdot\mathbf{V} = 0, \tag{1.3}$$

$$\mathbf{V}(t,\, y)|_{t=0} = \mathbf{V}(0) = \tilde{\mathbf{V}}_0(y) + \frac{\Omega}{2} e_3 \times y \tag{1.4}$$

where $y = (y_1,\, y_2,\, y_3)$, $\mathbf{V}(t,\, y) = (V_1,\, V_2,\, V_3)$ is the velocity field and q is the pressure. Since $\operatorname{curl}\big((\Omega/2)e_3 \times y\big) = \Omega e_3$, the vorticity vector at initial time $t = 0$ is $\operatorname{curl}\mathbf{V}(0,\, y) = \operatorname{curl}\tilde{\mathbf{V}}_0(y) + \Omega e_3$. This connection between initial value problems for the 3D Navier-Stokes Equations is made precise in the last section of the paper. Using the rotation transformation, our results for initial value problem (1.1)-(1.2) imply local in time solvability of the Navier-Stokes equations (1.3)-(1.4) in $\mathbb{R}^3$ under the condition that the initial velocity is of the form $\mathbf{V}(0) = \tilde{\mathbf{V}}_0(y) + (\Omega/2)e_3 \times y$ with $\tilde{\mathbf{V}}_0(y) \in L^\infty_{\sigma,a}(\mathbb{R}^3)$.

Let $\mathbf{J}$ be the matrix such that $\mathbf{Ja} = e_3 \times \mathbf{a}$ for any vector field $\mathbf{a}$. Then

$$\mathbf{J} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.5}$$

We define the Stokes operator $\mathbf{A}$:

$$\mathbf{AU} = \nu \operatorname{curl}^2 \mathbf{U} = -\nu\Delta\mathbf{U} \tag{1.6}$$

on divergence free vector fields. Let $\mathbf{P}$ be the projection operator on divergence free fields. We recall that the operator $\mathbf{P}$ is related to the Riesz operators:

$$\mathbf{P} = \{P_{ij}\}_{i,\,j=1,\,2,\,3}, \quad P_{ij} = \delta_{ij} + R_i R_j; \tag{1.7}$$

where $\delta_{i,\,j}$ is Kronecker's delta and R_j are the *scalar Riesz operators* defined by

$$R_j = \frac{\partial}{\partial x_j}(-\Delta)^{-1/2} \quad \text{for } j = 1,\, 2,\, 3; \tag{1.8}$$

the symbol $\sigma(R_j)$ of R_j equals $i\xi_j/|\xi|$, where $i = \sqrt{-1}$ (see e.g. [29]).

We transform (1.1)-(1.2) into the abstract differential equation for $\mathbf{U}$

$$\mathbf{U}_t + \mathbf{A}(\Omega)\mathbf{U} + \mathbf{P}(\mathbf{U}\cdot\nabla)\mathbf{U} = 0, \tag{1.9}$$

where

$$\mathbf{A}(\Omega)\mathbf{U} = \mathbf{AU} + \Omega\mathbf{SU} \quad \text{and} \quad \mathbf{S} = \mathbf{PJP} \tag{1.10}$$

and we have used $\mathbf{PJU} = \mathbf{PJPU}$ on solenoidal vector fields. The main difficulty that we face in our studies of local solvability for Eqs. (1.1)-(1.2), (1.3)-(1.4) is that the Coriolis term is an unbounded operator in $L^\infty_\sigma(\mathbb{R}^3)$. We find that it is necessary to restrict initial data on a subspace of $L^\infty_\sigma(\mathbb{R}^3)$ on which the combined operator (Stokes+Coriolis) generates a semigroup.

It is important to note that mathematical techniques for Eqs. (1.1)-(1.2) with initial data on compact manifolds (bounded domains and periodic lattices in $\mathbb{R}^3$) and for initial data in $L^p(\mathbb{R}^3)$, $1 < p < +\infty$ spaces of functions that decay at infinity are very different from those for initial data non-decaying at infinity in $\mathbb{R}^3$. In the former case, the Coriolis operator is a bounded zero order pseudo-differential operator with a *skew-symmetric* matrix symbol. Then local in time solvability for fixed Ω immediately follows by repeating classical arguments on local solvability of the 3D Navier-Stokes equations. Uniform in Ω solvability does not always hold for bounded domains and it requires careful consideration in each case. We note that for initial data on periodic lattices and in bounded cylindrical domains in $\mathbb{R}^3$ the time interval $[0, T]$ for existence of strong solutions is uniform in Ω. Moreover, regularization of solutions occurs for large Ω. Global regularity for large Ω of solutions of the three-dimensional Navier-Stokes equations (1.1)-(1.2), (1.3)-(1.4) with initial data $\mathbf{U}_0(x)$ on arbitrary periodic lattices and in bounded cylindrical domains in $\mathbb{R}^3$ was proven in [2], [3] and [21] without any conditional assumptions on the properties of solutions at later times. The method of proving global regularity for large fixed Ω is based on the analysis of fast singular oscillating limits (singular limit $\Omega \to +\infty$), nonlinear averaging and cancellation of oscillations in the nonlinear interactions for the vorticity field. It uses harmonic analysis tools of lemmas on restricted convolutions and Littlewood-Paley dyadic decomposition to prove global regularity of the limit resonant three-dimensional Navier-Stokes equations which holds without any restriction on the size of initial data and strong convergence theorems for large Ω.

The mathematical theory of the Navier-Stokes equations in $\mathbb{R}^n$ ($n = 2, 3$) with initial data in spaces of functions non-decaying at infinity is more difficult than those on bounded domains or with periodic boundary conditions and it was developed only recently although there are earlier works to construct mild solutions for L^∞ initial data [6], [8]. Since energy is infinite for the corresponding solutions, classical energy methods for estimating norms of solutions or Galerkin approximation procedures cannot be used and new techniques are required. For example, Giga, Inui and Matsui [12] showed the time-local existence of strong solutions to the Navier-Stokes equations with non-decaying initial data in $L^\infty_\sigma(\mathbb{R}^n)$, $n = 2, 3$. Moreover, they proved the uniqueness under the same conditions. There are several related works for L^∞ initial data [7], [20]. We do not intend to exhaust references on this topic. Giga, Matsui and Sawada [14] proved the global in time solvability of the 2D Navier-Stokes equations with initial velocity in $L^\infty_\sigma(\mathbb{R}^2)$ without smallness nor integrability condition on initial velocity.

Although there are several earlier works on the solvability of the Navier-Stokes equations with initial data in Besov type spaces, it requires decay at space infinity. The space $\dot{B}^0_{\infty,1}$ was first used to solve the Boussinesq equations by Sawada and Taniuchi [27] (see Taniuchi[30] for recent improvement). Hieber-Sawada [17] and Sawada [26] constructed a unique local solution for the Navier-Stokes equations (1.3) with initial data $Mx+v_0$ where M is a trace free matrix and $v_0 \in \dot{B}^0_{\infty,1}$. This includes (1.4). However, their existence time estimate is weaker than our estimate (4.4). This is because they transformed (RNS) to the following integral equation;

$$u(t) = \exp(t\Delta)u_0 - \int_0^t \exp\big((t-s)\Delta\big) \times \mathbf{P}\{\mathrm{div}(u \otimes u)(s) + \Omega e_3 \times u(s)\}ds \quad \text{for } t > 0$$

and regarded the Coriolis term as a perturbation. In this paper, we transformed (RNS) into (I) (see Section 4) to estimate the Coriolis term in the form $\exp(-\Omega \mathbf{S} t)$ as the leading term with the heat operator $\exp(t\Delta)$. Then, the behavior as $\Omega \to \infty$ of the operator $\exp(-\Omega \mathbf{S} t)$ can be reduced to that of the operator of the form $\exp(tR_3)$ as $t \to \infty$.

In [17] and [26], they assume that all components of the initial data belongs to $\dot{B}^0_{\infty,1}$. We assume that only baroclinic component of the initial data belongs to $\dot{B}^0_{\infty,1}$ to our space for initial data $L^\infty_{\sigma,a}$ (see Section 3). This is another difference between our results and theirs. Although we restrict initial data v_0 in $L^\infty_{\sigma,a}$, as noticed in Remark 4.1 (iii) we may take an arbitrary element of $\dot{B}^0_{\infty,1}$ provided that it is divergence free.

Unfortunately, the existence time of our solution is not uniform in the Coriolis parameter $\Omega \in \mathbb{R}$ (see (4.4)) since we can not get uniform estimate for the Coriolis solution operator $\exp(-\Omega t\mathbf{S})$ for $t > 0$ in Ω. We are able to prove that its operator norm in the space BMO or the Besov space $\dot{B}_{\infty,q}$ $(1 \le q \le \infty)$ is dominated by $C(1+\Omega t)^\alpha$, where the constant α is 4 or $(3/2+\delta)$, respectively (see Proposition 2.2, Proposition 3.1). Here, BMO is the space of functions of bounded mean oscillation (see e.g. [29]) and $\delta > 0$ is an arbitrary constant. We are skeptical the uniform boundedness of the operator norm of $\exp(-\Omega \mathbf{S} t)$ in t, $\Omega \in \mathbb{R}$ but we do not have a counterexample. The local existence with an existence time uniform in Ω is recently proved by authors [13] by choosing a different space, the space of Fourier images of finite Radon measures which still contains many nondecaying functions such as almost periodic functions. For the Euler equations in bounded cylindrical domains Nicolaenko, Bardos, Golse and the third author [22] proved local existence whose existence time is uniform in Ω.

The plan of the paper is as follows. In Section 2 we consider the linearized (i.e., Stokes+Coriolis) problem and calculate the symbol of the solution operator. In Section 3 we give definition for initial data by splitting it to 2D3C (2 dimensional 3 components) part and other baroclinic part (3 dimensional, 3 components with zero vertical average). The characterization is natural from an observation of the symbol calculus in Section 2. In Section 4 we give main theorems and the proofs. In Section 5 we restate the main theorems for the equations (1.3)-(1.4) in the rotating frame. In Appendix A we calculate the kernel of the the linearized solution operator whose symbol is given in Section 2. It turns out the operator is not bounded in L^∞. In Appendix B and C we show key estimates for the Coriolis solution operator. These estimates are crucial for the proof of our theorems. In Appendix D we show bilinear estimate which is used in nonlinear estimate in Section 4. We also give fractional power estimate for readers' convenience.

2. Linear problem and calculation of symbols of pseudo-differential operators

In this section we solve linear problem using Fourier transform and calculate symbols of the corresponding pseudo-differential operators in $\mathbb{R}^3$. We consider the linear problem (Stokes+Coriolis):

$$\begin{aligned} &\partial_t \mathbf{\Phi} - \nu\Delta\mathbf{\Phi} + \Omega e_3 \times \mathbf{\Phi} = -\nabla\pi, \quad \nabla\cdot\mathbf{\Phi} = 0, \\ &\mathbf{\Phi}(t,x)|_{t=0} = \mathbf{\Phi}_0(x). \end{aligned} \tag{2.1}$$

After applying projection $\mathbf{P}$ on divergence free vector fields, the above equation (2.1) can be written in operator form as follows

$$\mathbf{\Phi}_t + \mathbf{A}\mathbf{\Phi} + \Omega\mathbf{S}\mathbf{\Phi} = 0, \quad \mathbf{\Phi}(t)|_{t=0} = \mathbf{\Phi}_0. \tag{2.2}$$

We introduce Fourier integrals:

$$\begin{aligned} &F\mathbf{u}(\xi) = \hat{\mathbf{u}}(\xi) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} e^{-i\xi\cdot x}\mathbf{u}(x)dx, \\ &F^{-1}\mathbf{v}(x) = \check{\mathbf{v}}(x) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} e^{ix\cdot\xi}\mathbf{v}(\xi)d\xi. \end{aligned} \tag{2.3}$$

Clearly, $\xi\cdot\hat{\mathbf{u}}(\xi) = 0$ if $\mathbf{u}$ is divergence free. Recall that the operators $\mathbf{P}$ and curl in Fourier representation have symbols $\sigma(\mathbf{P})$ and $\sigma(\text{curl})$:

$$\sigma(\mathbf{P}) = \mathbf{I} - \frac{1}{|\xi|^2}\begin{pmatrix} \xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_2\xi_1 & \xi_2^2 & \xi_2\xi_3 \\ \xi_3\xi_1 & \xi_3\xi_2 & \xi_3^2 \end{pmatrix},$$

$$\sigma(\mathrm{curl}) = i\begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}. \quad (2.4)$$

Here $\mathbf{I}$ is the 3×3 identity matrix. In what follows, we shall freely denote singular integral operator, say R_j in (1.8), by its symbol, say $i\xi_j/|\xi|$ for simplicity.

We also define the *vector Riesz operator* $\mathbf{R}$ by introducing its symbol:

$$\sigma(\mathbf{R}) \equiv \mathbf{R}(\xi) = \begin{pmatrix} 0 & -\xi_3/|\xi| & \xi_2/|\xi| \\ \xi_3/|\xi| & 0 & -\xi_1/|\xi| \\ -\xi_2/|\xi| & \xi_1/|\xi| & 0 \end{pmatrix}. \quad (2.5)$$

We note that the symbol $\mathbf{R}(\xi)$ is a 3×3 skew-symmetric matrix. The vector Riesz operator $\mathbf{R}$ acting in the space of divergence free vector fields has the property:

$$\mathbf{R}^2 = -\mathbf{I}. \quad (2.6)$$

In fact, since $\mathbf{R}(\xi)\mathbf{v} = (1/|\xi|)\xi \times \mathbf{v}$, we calculate for any solenoidal vector field $\mathbf{v}$

$$\begin{aligned}\mathbf{R}(\xi)^2\mathbf{v} &= \mathbf{R}(\xi)\Big(\frac{1}{|\xi|}\xi \times \mathbf{v}\Big) = \frac{1}{|\xi|^2}\xi \times (\xi \times \mathbf{v}) \\ &= \frac{1}{|\xi|^2}\big((\xi \cdot \mathbf{v})\xi - (\xi \cdot \xi)\mathbf{v}\big) = -\frac{1}{|\xi|^2}|\xi|^2\mathbf{v} = -\mathbf{v}.\end{aligned}$$

Here, we used divergence free condition $(\xi \cdot v) = 0$. Because the scalar Riesz operators R_j satisfy $\sum_{j=1}^3 R_j^2 = -1$, it seems natural to call the operator $\mathbf{R}$ the vector Riesz operator. We now calculate 3×3 matrix symbol $\mathbf{S}(\xi)$ of the zero order pseudo-differential operator $\mathbf{S}$:

$$\sigma(\mathbf{S}) \equiv \mathbf{S}(\xi) = \mathbf{P}(\xi)\mathbf{J}\mathbf{P}(\xi). \quad (2.7)$$

We make an important observation that the operator $\mathbf{S} = \mathbf{PJP}$ is related to the Riesz operators and the curl operator. One can easily show by direct matrix multiplication that

$$\mathbf{S}(\xi) \equiv \mathbf{P}(\xi)\mathbf{J}\mathbf{P}(\xi) = \Big(\frac{\xi_3}{|\xi|}\Big)\mathbf{R}(\xi). \quad (2.8)$$

It implies that the symbol of the operator $\mathbf{S}$ commutes with the symbols of the operator *curl* and the Stokes operator $\mathbf{A}$. The symbol $\mathbf{S}(\xi)$ of the operator $\mathbf{S}$ is a homogeneous function of degree zero and it is expressed in terms of the scalar Riesz operators R_j for $j = 1, 2, 3$ (cf. (1.8)). Eqs. (2.5) and (2.8) imply

$$\mathbf{S} = R_3 \begin{pmatrix} 0 & R_3 & -R_2 \\ -R_3 & 0 & R_1 \\ R_2 & -R_1 & 0 \end{pmatrix}. \tag{2.9}$$

We recall that the Riesz operators R_j are bounded operators in $L^p(\mathbb{R}^3)$ for $1 < p < \infty$ and $BMO(\mathbb{R}^3)$. However, the Riesz operators are not bounded in $L^\infty(\mathbb{R}^3)$. We also note that the Riesz operators R_j are bounded from $L^\infty(\mathbb{R}^3)$ to $BMO(\mathbb{R}^3)$.

Since Riesz operators are bounded in $BMO(\mathbb{R}^3)$ and $L^p(\mathbb{R}^3)$ $(1 < p < +\infty)$, we have

Proposition 2.1 (1) $\mathbf{S}\colon BMO(\mathbb{R}^3) \to BMO(\mathbb{R}^3)$ *is a bounded operator.*
(2) $\mathbf{S}\colon L^p(\mathbb{R}^3) \to L^p(\mathbb{R}^3)$, $1 < p < +\infty$, *is a bounded operator.*
(3) *The symbol* $\mathbf{S}(\xi)\colon \mathbb{R}^3 \to \mathbb{R}^3$ *of the operator* $\mathbf{S}$ *is a* 3×3 *matrix with the following properties*:

$$\text{(a)} \quad \big(\mathbf{S}(\xi)\big)^* = -\mathbf{S}(\xi) \quad (\textit{skew-symmetric matrix}),$$

$$\text{(b)} \quad \big(\mathbf{S}(\xi)\big)^2 = -\frac{\xi_3^2}{|\xi|^2}\mathbf{I} = \Big(\frac{i\xi_3}{|\xi|}\Big)\Big(\frac{i\xi_3}{|\xi|}\Big)\mathbf{I} \quad \textit{i.e. } \mathbf{S}^2 = R_3^2\mathbf{I} \tag{2.10}$$

where $i\xi_3/|\xi|$ *is the symbol of the Riesz operator* R_3.
(4) $|\mathbf{S}(\xi)\mathbf{v}| = |\mathbf{v}|$ *on the linear subspace of* $\mathbb{R}^3$ *with the property* $\xi \cdot \mathbf{v} = 0$ (*subspace of solenoidal vector fields*). *Here* $|\mathbf{v}|$ *denotes length of the vector* $\mathbf{v} \in \mathbb{R}^3$.

Remark 2.1 The operator $\mathbf{S}$ is *not* a bounded operator in $L^\infty_\sigma(\mathbb{R}^3)$, however, $\mathbf{S}\colon L^\infty_\sigma(\mathbb{R}^3) \to BMO(\mathbb{R}^3)$.

Eq. (2.10) is useful in calculating the operator $\exp(\mathbf{S})$ directly using infinite series:

$$\exp(\mathbf{S}) = \sum_{j=0}^{+\infty} \frac{1}{j!}\mathbf{S}^j. \tag{2.11}$$

Then we can solve linear Stokes+Coriolis problem (2.1), (2.2) in $BMO(\mathbb{R}^3)$ and in $L^p(\mathbb{R}^3)$, $1 < p < +\infty$. Since the operators commute, the solution of (2.2) is given by

$$\begin{aligned}\mathbf{\Phi}(t) &= \exp((-\mathbf{A} - \Omega\mathbf{S})t)\mathbf{\Phi}_0 \\ &= \exp(\nu t\Delta)\exp(-\Omega t\mathbf{S})\mathbf{\Phi}_0 \quad \text{for } t > 0.\end{aligned} \tag{2.12}$$

Of course, in Eqs. (2.12), $\exp(\nu t\Delta)$ is the usual semigroup generated by the heat kernel. Since $\mathbf{S}$ is a bounded operator in $BMO(\mathbb{R}^3)$ and $L^p(\mathbb{R}^3)$, $1 < p < +\infty$, the operator $\exp(\Omega\mathbf{S}t)$ is also a bounded operator in these spaces for each $\Omega \in \mathbb{R}$ and $t \in \mathbb{R}$. It is defined by convergent series:

$$\exp(\Omega\mathbf{S}t) = \sum_{j=0}^{+\infty} \frac{1}{j!}(\Omega t)^j \mathbf{S}^j. \tag{2.13}$$

We can solve linear Stokes+Coriolis problem (2.1) using Fourier transform in $\mathbb{R}^3$. After applying Fourier transform and projecting on divergence free subspace, we obtain

$$\begin{aligned}&\partial_t\mathbf{\Phi}(t,\xi) + \nu|\xi|^2\mathbf{\Phi}(t,\xi) + \Omega\mathbf{S}(\xi)\mathbf{\Phi}(t,\xi) = 0, \\ &\mathbf{\Phi}(t,\xi)|_{t=0} = \mathbf{\Phi}_0(\xi).\end{aligned} \tag{2.14}$$

Direct calculation using infinite series (2.13) and the property (2.10) of $\mathbf{S}$ imply that

$$\exp(\Omega\mathbf{S}(\xi)t) = \cos\Big(\frac{\xi_3}{|\xi|}\Omega t\Big)\mathbf{I} + \sin\Big(\frac{\xi_3}{|\xi|}\Omega t\Big)\mathbf{R}(\xi), \tag{2.15}$$

where $\mathbf{R}(\xi)$ is defined in (2.5).

Then the solution of (2.14) is given by

$$\mathbf{\Phi}(t,\xi) = e^{-\nu|\xi|^2 t}\bigg(\cos\Big(\frac{\xi_3}{|\xi|}\Omega t\Big)\mathbf{I} - \sin\Big(\frac{\xi_3}{|\xi|}\Omega t\Big)\mathbf{R}(\xi)\bigg)\mathbf{\Phi}_0(\xi). \tag{2.16}$$

In physical space the solution is given by convolution of inverse Fourier transform of $e^{-\nu|\xi|^2 t}\cos\big((\xi_3/|\xi|)\Omega t\big)$ and $e^{-\nu|\xi|^2 t}\sin\big((\xi_3/|\xi|)\Omega t\big)\mathbf{R}(\xi)$ with $\mathbf{\Phi}_0(x)$.

Thus, the symbol of the vector pseudo-differential operator $\exp\big(-\mathbf{A}(\Omega)t\big)$ corresponding to the linear problem (Stokes Operator+$\Omega\mathbf{S}$) is given by

$$\sigma\big(\exp(-\mathbf{A}(\Omega)t)\big) = e^{-\nu|\xi|^2 t}\cos\Big(\frac{\xi_3}{|\xi|}\Omega t\Big)\mathbf{I}$$

$$- e^{-\nu|\xi|^2 t} \sin\Big(\frac{\xi_3}{|\xi|}\Omega t\Big)\mathbf{R}(\xi), \tag{2.17}$$

where $\mathbf{R}$ is the vector Riesz operator with the 3×3 matrix symbol $\mathbf{R}(\xi)$ defined above; $\mathbf{I}$ is the 3×3 identity matrix. From the calculations outlined in Appendix A it follows that

$$\begin{aligned} &F^{-1}\bigg(e^{-\nu|\xi|^2 t}\cos\Big(\frac{\xi_3}{|\xi|}\Omega t\Big)\bigg), \\ &\quad F^{-1}\bigg(e^{-\nu|\xi|^2 t}\sin\Big(\frac{\xi_3}{|\xi|}\Omega t\Big)\mathbf{R}(\xi)\bigg) \in L^q(\mathbb{R}^3), \quad 1 < q < +\infty. \end{aligned} \tag{2.18}$$

The symbol $\sigma\big(\exp(-\mathbf{A}(\Omega)t)\big)$ is discontinuous at $\xi = 0$ since the functions $e^{-\nu|\xi|^2 t}\sin\big((\xi_3/|\xi|)\Omega t\big)(\xi_j/|\xi|)$, $j = 1, 2$ are discontinuous at $\xi = 0$. Therefore, the integral kernel given by Fourier transform of the symbol cannot belong to $L^1(\mathbb{R}^3)$. More detailed consideration of the Fourier transform given in the Appendix A shows that it behaves as $|x|^{-3}$ for large $|x|$ and that it is not a bounded operator in $L^\infty_\sigma(\mathbb{R}^3)$.

We state a boundedness of the operator $\exp(-\mathbf{A}(\Omega)t)$ in $BMO(\mathbb{R}^3)$ which will be needed to estimate the nonlinear term in Lemma 4.3. The boundedness follows from Proposition B.1, which shall be shown in Appendix B. In what follows we shall denote by C various constants. In particular, $C = C(*, \dots, *)$ denotes constants depending only on the quantities in the parenthesis.

Proposition 2.2 (Estimate for the Coriolis solution operator - BMO version) *There exists a constant $C > 0$ independent of Ω and t such that*

$$\|\exp(-\Omega t\mathbf{S})\|_{BMO\to BMO} \le C(1 + (\Omega t)^4) \quad \textit{for} \quad t > 0. \tag{2.19}$$

Remark 2.2 (i) The operator $\exp\big(-\mathbf{A}(\Omega)t\big)$ has a sharp estimate as an operator from $\dot{B}^0_{\infty,q}$ to itself (see Proposition 3.1). Here $\dot{B}^0_{\infty,q}(1 \le q \le \infty)$ is a homogeneous Besov space whose definition will be given in Appendix B.

(ii) The same estimate for the Coriolis solution operator as (2.19) hold as $L^p \to L^p$ for $1 < p < \infty$. Actually, we have

$$\|\exp(-\Omega t\mathbf{S})\|_{L^p\to L^p} \le C(1 + (\Omega t)^4), \tag{2.20}$$

for $t > 0$. In fact, the proof of (2.20) parallels the proof of (2.19) using the Mikhlin multiplier theorem in L^p $(1 < p < \infty)$ spaces (cf. Lemma B.2).

The norm estimate is uniform in Ω and t for $p = 2$ (cf. Lemma C.2).

3. Stokes-Coriolis semigroup and splitting of initial data having vertical averaging property

Before defining the space for initial data we note that in the equality (2.8) there is ξ_3 (i.e., $\partial/\partial x_3$ in all components). This implies that

$$\mathbf{S}f = 0 \quad \text{hence} \quad \exp(\Omega t \mathbf{S})f = f \quad \text{for } t > 0 \tag{3.1}$$

if the vector f is a 2D3C vector field (vector field with 3 components where each component depends only on 2 variables x_1 and x_2). Hence it is natural to take 2D3C vector field from 3D3C vector field. We introduce vertical averaging property as one of ways to take 2D3C flow.

Definition 3.1 (vertical averaging) Let $\mathbf{U} \in L^\infty_\sigma(\mathbb{R}^3)$. We say that $\mathbf{U}$ admits vertical averaging if

$$\lim_{L\to+\infty} \frac{1}{2L}\int_{-L}^{L} \mathbf{U}(x_1, x_2, x_3)dx_3 \equiv \overline{\mathbf{U}}(x_1, x_2)$$

exists almost everywhere. The vector field $\overline{\mathbf{U}}(x_1, x_2)$ is called *vertical average* of $\mathbf{U}(x_1, x_2, x_3)$.

Remark 3.1 (i) Clearly, all periodic and almost periodic functions (or vector fields) admit vertical averaging.
(ii) The vector field $\overline{\mathbf{U}}(x_1, x_2) = (\overline{U}_1(x_1, x_2), \overline{U}_2(x_1, x_2), \overline{U}_3(x_1, x_2))$ has zero horizontal divergence:

$$\nabla \cdot \overline{\mathbf{U}} = \partial_{x_1}\overline{U}_1 + \partial_{x_2}\overline{U}_2 = 0. \tag{3.2}$$

(iii) Supposing $\mathbf{U} \in L^p_\sigma(\mathbb{R}^3)$ for $1 < p < \infty$, the vertical average always exists; moreover, $\overline{\mathbf{U}} \equiv 0$.
(iv) If $\mathbf{U} \in L^\infty(\mathbb{R}^3)$ admits vertical averaging (at (x_1, x_2)), then we have uniform convergence property, i.e.,

$$\lim_{L\to\infty} \sup_{|r|\le M} \frac{1}{2L}\int_{-L}^{L} \mathbf{U}(x_1, x_2, x_3 + r)dx_3 = \overline{\mathbf{U}}(x_1, x_2)$$

for each $M > 0$. Indeed, we may assume that $\overline{\mathbf{U}}(x_1, x_2) = 0$ by considering

$\mathbf{U} - \overline{\mathbf{U}}$ instead of $\mathbf{U}$. We suppress the dependence of (x_1, x_2). Since

$$\int_{-L}^{L} \mathbf{U}(x_3 + r)dx_3 = \left(\int_{-L-r}^{L+r} - \int_{-L-r}^{-L+r} \right) \mathbf{U}(x_3)dx_3,$$

we observe that

$$\left| \frac{1}{2L} \int_{-L}^{L} \mathbf{U}(x_3 + r)dx_3 \right| \\ \leq \frac{L+r}{L} \frac{1}{2(L+r)} \left| \int_{-L-r}^{L+r} \mathbf{U}(x_3)dx_3 \right| + \|\mathbf{U}\|_\infty \frac{2r}{2L}.$$

We take supremum in $r \in [-M, M]$ and send L to ∞ to get the desired result.

Eq. (3.2) follows if we apply vertical averaging operation to the 3D divergence free equation $\nabla \cdot \mathbf{U} = \partial_{x_1} U_1 + \partial_{x_2} U_2 + \partial_{x_3} U_3 = 0$ and notice that

$$\lim_{L \to +\infty} \frac{1}{2L} \int_{-L}^{L} \frac{\partial U_3}{\partial x_3} dx_3 \\ = \lim_{L \to +\infty} \frac{1}{2L} (U_3(x_1, x_2, L) - U_3(x_1, x_2, -L)) = 0, \quad (3.3)$$

since $U_3 \in L^\infty(\mathbb{R}^3)$.

The operation of vertical averaging defined above is called 'barotropic projection' and the vector field $\overline{\mathbf{U}}(x_1, x_2)$ is called 'barotropic component' of $\mathbf{U}(x_1, x_2, x_3)$. Then the 'baroclinic component' $\mathbf{U}^\perp(x_1, x_2, x_3)$ is defined as

$$\mathbf{U}^\perp(x_1, x_2, x_3) = \mathbf{U}(x_1, x_2, x_3) - \overline{\mathbf{U}}(x_1, x_2). \quad (3.4)$$

Now we define the space for initial data $\mathbf{U}_0$.

Definition 3.2 (Space for initial data) We define a subspace of L^∞_σ of the form

$$L^\infty_{\sigma, a}(\mathbb{R}^3) = \{\mathbf{U} \in L^\infty_\sigma(\mathbb{R}^3);\ \mathbf{U} \text{ admits vertical averaging} \\ \text{and } \mathbf{U}^\perp \in \dot{B}^0_{\infty, 1}\}.$$

The space $L^\infty_{\sigma, a}(\mathbb{R}^3)$ is a Banach space with the norm

$$\|\mathbf{U}\|_{L^\infty_{\sigma, a}} = \|\overline{\mathbf{U}}\|_{L^\infty(\mathbb{R}^2;\mathbb{R}^3)} + \|\mathbf{U}^\perp\|_{\dot{B}^0_{\infty, 1}}.$$

Indeed, let $\{\mathbf{U}_j\}$ be a Cauchy sequence of $L^\infty_{\sigma,a}$. Since $\|f\|_\infty \le C\|f\|_{\dot{B}^0_{\infty,1}}$, $\mathbf{U}_j$ converges to some $\mathbf{U} \in L^\infty_\sigma$ uniformly in $\mathbb{R}^3$. Since $\overline{\mathbf{U}_j}$ exists, so does $\overline{\mathbf{U}}$. Since $\|\overline{f}\|_\infty \le C\|f\|_\infty$, we conclude that $\overline{\mathbf{U}_j} \to \overline{\mathbf{U}}$ uniformly in $\mathbb{R}^2$. Since $\{\mathbf{U}_j^\perp\}$ is a Cauchy sequence in $\dot{B}^0_{\infty,1}$, there is a limit $\mathbf{v} \in \dot{B}^0_{\infty,1}$. However, $\mathbf{U}_j \to \mathbf{U}$, $\overline{\mathbf{U}_j} \to \overline{\mathbf{U}}$, so $\mathbf{v}$ must be equal to $\mathbf{U}^\perp$.

Remark 3.2 The space $L^\infty_{\sigma,a}$ has a topological direct sum decomposition of the form

$$L^\infty_{\sigma,a} = \mathcal{W} \oplus \mathcal{B}^0$$

with

$$\mathcal{W} = \{\mathbf{U} \in L^\infty_\sigma;\ \partial U_i/\partial x_3 \equiv 0 \quad \text{in distributional sense } \mathbb{R}^3 \quad \text{for } i = 1, 2, 3\},$$
$$\mathcal{B}^0 = \{\mathbf{U} \in \dot{B}^0_{\infty,1} \cap L^\infty_\sigma;\ \overline{\mathbf{U}}(x_1, x_2) \equiv 0 \quad \text{a.e. } (x_1, x_2) \in \mathbb{R}^2\}.$$

Indeed, for $\mathbf{U} \in L^\infty_{\sigma,a}$ we observe that $\overline{\mathbf{U}} \in \mathcal{W}$ and $\mathbf{U}^\perp \in \mathcal{B}^0$. Moreover, $\mathcal{W} \cap \mathcal{B}^0 = \{0\}$. The closedness of $\mathcal{W}$ and $\mathcal{B}^0$ can be proved using Definition 3.2.

The advantage of the Besov space $\dot{B}^0_{\infty,1}$ is that the Riesz operators and, consequently, the operator $\exp(-\mathbf{A}(\Omega)t)$ are bounded operators in this space. Also, this space contains all locally Lipschitz periodic functions with zero mean value and all almost periodic functions of the form

$$\sum_{j=1}^\infty \alpha_j e^{\sqrt{-1}\lambda_j \cdot x} \quad \text{with } \{\alpha_j\}_{j=1}^\infty \in l^1,\ \{\lambda_j\} \subset \mathbb{R}^3 \setminus \{0\}.$$

We consider boundedness of the Coriolis solution operator. Let $\mathbf{U} \in L^\infty_{\sigma,a}(\mathbb{R}^3)$. Then $\mathbf{U}$ admits vertical averaging and we have the following representation (splitting)

$$\mathbf{U} = \overline{\mathbf{U}} + \mathbf{U}^\perp, \tag{3.5}$$

where $\overline{\mathbf{U}}(x_1, x_2)$ is a 2D3C vector field such that $\overline{U}_j(x_1, x_2) \in L^\infty(\mathbb{R}^2)$ for $j = 1, 2, 3$. Hence we have

$$\exp\bigl(-\mathbf{A}(\Omega)t\bigr)\mathbf{U} = \exp(\nu t\Delta)\overline{\mathbf{U}} + \exp\bigl(-\mathbf{A}(\Omega)t\bigr)\mathbf{U}^\perp \tag{3.6}$$

since (3.1) implies

$$\exp(-\Omega t\mathbf{S})\overline{\mathbf{U}} = \overline{\mathbf{U}}. \tag{3.7}$$

Then the first term in RHS of (3.6) can be estimated in L^∞ by $\|\overline{\mathbf{U}}\|_{L^\infty}$ because of $\|G_{\nu t}\|_{L^1} = 1$. However, when $\Omega \neq 0$, the second term still contains the Coriolis solution operator $\exp(-\Omega t\mathbf{S})$ in $\exp(-\mathbf{A}(\Omega)t)$. Moreover, the derivative estimate for the heat kernel which shall appear as Lemma 4.1 does not apply the terms (3.6) since there is no derivative in them. In order to estimate the second term of RHS in (3.6) we need the following boundedness in $\dot{B}^0_{\infty,1}$. The space $\dot{B}^0_{\infty,1}$ is smaller than L^∞. We claim the boundedness in $\dot{B}^0_{\infty,q}$ with $1 \leq q \leq \infty$.

Proposition 3.1 (Estimate for the Coriolis solution operator - Besov version) *Let $1 \leq q \leq \infty$. For each $\delta > 0$ there exists a constant $C = C(\delta) > 0$ independent of q, Ω, t and f such that*

$$\|\exp(-\Omega t\mathbf{S})\|_{\dot{B}^0_{\infty,q} \to \dot{B}^0_{\infty,q}} \leq C(1 + \Omega t)^{3/2+\delta} \quad \textit{for} \quad t > 0. \tag{3.8}$$

The above proposition is an immediate consequence of Theorem C.1 whose proof is postponed until Appendix C. The estimate (3.8) in Besov spaces is applicable for both linear ($q = 1$) and nonlinear ($q = \infty$) estimates (see Lemma 4.2 and Lemma 4.4, respectively) by virtue of the embedding $\dot{B}_{\infty,1} \hookrightarrow L^\infty \hookrightarrow \dot{B}^0_{\infty,\infty}$, while Proposition 2.2 (BMO-version) is useful only for nonlinear estimate due to $L^\infty \hookrightarrow BMO$.

In the remainder of this section we shall prove that $\exp(-\mathbf{A}(\Omega)t)$ is a bounded semigroup in $L^\infty_{\sigma,a}$ for each $\Omega \in \mathbb{R}$. Since we have Proposition 3.1 together with (3.6) and (3.7), it suffices to prove

Proposition 3.2 *The operator $\exp(-\mathbf{A}(\Omega)t)$ maps from $L^\infty_{\sigma,a}$ to itself for all $t > 0$.*

Proof. It suffices to show that $\exp\bigl(-\mathbf{A}(\Omega)t\bigr)\mathbf{U} = \exp(-\Omega t\mathbf{S})\exp(\nu t\Delta)\mathbf{U} \in \mathcal{B}^0$ if $\mathbf{U} \in \mathcal{B}^0$. We first prove that $\exp(\nu t\Delta)\mathbf{U} \in \mathcal{B}^0$ if $\mathbf{U} \in \mathcal{B}^0$. Since

$$\begin{aligned}
&\bigl(\exp(\nu t\Delta)\mathbf{U}\bigr)(x) \\
&= \int_{\mathbb{R}^2}\left(\int_{-\infty}^{\infty}\mathbf{U}(x_1 - y_1,\, x_2 - y_2,\, x_3 - y_3)G_{\nu t}(y_3)dy_3\right) \\
&\qquad\qquad\qquad\qquad \times G_{\nu t}(y_1,\, y_2)dy_1dy_2
\end{aligned}$$

with the Gauss kernel $g_{\nu t}$, it suffices to prove that

$$\lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L}\int_{-\infty}^{\infty}\mathbf{U}(x_1 - y_1,\, x_2 - y_2,\, x_3 - y_3)G_{\nu t}(y_3)dy_3dx_3 = 0$$

for a.e. $(x_1-y_1,\, x_2-y_2)$. This follows from the uniform convergence property Remark 3.1(iv), since the Gauss kernel $G_{\nu t}(y_3)$ is integrable for large y_3. We thus proved that $\overline{\exp(\nu t\Delta)\mathbf{U}} = 0$. The divergence free property is clear, so we conclude that $\exp(\nu t\Delta)\mathbf{U} \in \mathcal{B}^0$ if $\mathbf{U} \in \mathcal{B}^0$. The proof will be complete if we prove

$$\exp(-\Omega t\mathbf{S})\mathbf{U} \in \mathcal{B}^0 \quad \text{if } \mathbf{U} \in \mathcal{B}^0.$$

We give the proof of this fact in Lemma B.4. □

4. Local existence and uniqueness

In this section we prove time-local existence and uniqueness for (1.1)-(1.2). The differential equations are formally transformed into the integral equation of the form:

$$\text{(I)} \qquad \mathbf{U}(t) = \exp\bigl(-\mathbf{A}(\Omega)t\bigr)\mathbf{U}_0 - N(\mathbf{U},\, t;\Omega) \quad \text{for } t > 0.$$

Here the nonlinear term $N(\mathbf{U},\, t;\Omega) = N(\mathbf{U},\, \mathbf{U},\, t;\Omega)$ is defined by

$$N(\mathbf{U},\, \mathbf{V},\, t;\Omega) = \int_0^t \exp\bigl(-\mathbf{A}(\Omega)(t-s)\bigr)\mathbf{P}\,\mathrm{div}(\mathbf{U}\otimes\mathbf{V})(s)ds.$$

We call a solution of the integral equation (I) a mild solution of the rotational Navier-Stokes equations. Since $\mathbf{PU} = \mathbf{U}$ for divergence free vector field and $\mathbf{P}\Delta = \Delta\mathbf{P}$, we have

$$\mathbf{A}(\Omega) = -\mathbf{P}\Delta + \Omega\mathbf{PJ} = -\Delta + \Omega\mathbf{PJP} = -\Delta + \Omega\mathbf{S}.$$

Note that

$$\exp\bigl(-\mathbf{A}(\Omega)t\bigr) = e^{t\Delta}\exp(-\Omega t\mathbf{S}),$$

where $e^{t\Delta}$ is the solution operator of the heat equation (in what follows we put $\nu = 1$ for simplicity). For an interval $I \subset [-\infty,\, \infty]$ and a Banach space X let $C(I;X)$ denote the space of all continuous functions with valued in X. The space $C_w(I;X)$ denotes the space of all X-valued star weakly continuous functions.

The goal of this section is to prove the following theorems.

Theorem 4.1 (Existence and uniqueness of mild solution $\mathbf{U}$) *Suppose that* $\mathbf{U}_0 \in L^\infty_{\sigma,a}(\mathbb{R}^3)$. *Then*

(1) *There exist* $T_0 = T_0(\Omega) > 0$ *and a unique solution* $\mathbf{U} = \mathbf{U}(t)$ *of* (I) *such that*

$$\mathbf{U} \in C([\eta, T_0]; L^\infty_\sigma) \cap C_w([0, T_0]; L^\infty_\sigma) \tag{4.1}$$

for any $\eta > 0$.
(2) *The solution* $\mathbf{U}$ *satisfies*

$$\sup_{t \in (0, T_0)} \|t^{1/2} \nabla \mathbf{U}\|_{L^\infty_\sigma} < \infty \quad \text{and} \quad \nabla \mathbf{U} \in C([\eta, T_0]; L^\infty_\sigma) \tag{4.2}$$

for any $\eta > 0$.

Theorem 4.2 (Existence of classical solution $\mathbf{U}$) *Suppose that* $\mathbf{U}_0 \in L^\infty_{\sigma, a}(\mathbb{R}^3)$. *Let* $\mathbf{U} = \mathbf{U}(t)$ *be a solution of* (I) *satisfying* (4.1) *and* (4.2). *If we set*

$$\nabla p(t) = \nabla \sum_{j, k=1}^{3} R_j R_k \mathbf{U}^j \mathbf{U}^k(t) - \Omega \begin{pmatrix} R_1(R_2\mathbf{U}^1 - R_1\mathbf{U}^2) \\ R_2(R_2\mathbf{U}^1 - R_1\mathbf{U}^2) \\ R_3(R_2\mathbf{U}^1 - R_1\mathbf{U}^2) \end{pmatrix} \quad \text{for } t > 0, \tag{4.3}$$

then the pair $(\mathbf{U}, \nabla p)$ *is a classical solution of* (1.1)-(1.2).

Such a solution (satisfying (4.1)-(4.3)) is unique. In fact a stronger version is available.

Theorem 4.3 (Uniqueness of classical solution $\mathbf{U}$) *Suppose that* $\mathbf{U}_0 \in L^\infty_{\sigma, a}(\mathbb{R}^3)$. *Let*

$$\mathbf{U} \in L^\infty\big((0, T) \times \mathbb{R}^3\big), \quad p \in L^1_{\text{loc}}([0, T); BMO)$$

be a solution of (1.1)-(1.2) *in a distributional sense for some* $T > 0$. *Then the pair* $(\mathbf{U}, \nabla p)$ *is unique. Furthermore, the relation* (4.3) *holds.*

Remark 4.1 (i) For a lower estimate for $T_0 > 0$ we get

$$T_0(1 + \Omega T_0)^{6+4\delta} \geq \frac{C}{\|\mathbf{U}_0\|^2_{L^\infty_{\sigma, a}}}, \tag{4.4}$$

where $\delta > 0$ can be taken arbitrarily, and $C = C(\delta) > 0$ is a constant independent of ν, Ω, T_0, and $\|\mathbf{U}_0\|_{L_{\sigma, a}}$.
(ii) For regularity we can get the same results as in [12]. The remark except (i) after Theorem 1 in [12] holds for our equation (I).

(iii) From the proof given below it is rather clear that one can take initial data in $\mathcal{W} + \dot{B}^0_{\infty,1}$, which is larger than $L^\infty_{\sigma,a}$. In particular, this class includes $\dot{B}^0_{\infty,1} \cap L^\infty_\sigma$ for which local existence is discussed in [26].
(iv) If in addition we assume that $\overline{\mathbf{U}_0} \in BUC$ so that $\mathbf{U}_0 \in BUC$, then by construction our solution $\mathbf{U} \in C([0, T_0]; BUC)$; here, BUC denotes the space of all bounded uniformly continuous functions in $\mathbb{R}^3$. Indeed, since $\dot{B}^0_{\infty,1} \subset BUC$ (see e.g. Example 2.3(iv) in [26]), $\mathbf{U}_0 \in BUC$. Since

$$e^{t\Delta}\overline{\mathbf{U}_0} \in C([0, \infty); BUC) \quad \text{(see Proposition A.1.1 in [12])}$$

and

$$\exp(-\Omega t\mathbf{S})\mathbf{U}_0^\perp \in C([0, \infty); \dot{B}^0_{\infty,1}),$$

it is easy to see that $\mathbf{U}_j \in C([0, \infty); BUC)$. Thus its uniform limit $\mathbf{U}$ belongs to $C([0, T_0]; BUC)$.

We note that Theorem 4.2 follows from Theorem 4.1 as observed in [12], where the case $\Omega = 0$ is discussed. We also note that the uniqueness (Theorem 4.3) can be proved along the line of [15], [19], where the case $\Omega = 0$ is discussed. We won't repeat the proofs. The proof of Theorem 4.1 is based on a standard iteration method, and is similar to that of [12]. We have already prepared two estimates for $\exp(-\Omega t\mathbf{S})$ in BMO and Besov spaces (Proposition 2.2 and Proposition 3.1). We further estimate its spatial derivatives.

Lemma 4.1 (Estimate for derivative of the heat operator) *There exists a constant* $C > 0$ (*depending only on space dimensions*) *that satisfies*

$$(1) \quad \|\nabla e^{t\Delta} f\|_{L^\infty} \le C t^{-1/2} \|f\|_{BMO} \qquad \text{for } t > 0 \quad \text{and } f \in BMO, \tag{4.5}$$

$$(2) \quad \|\nabla e^{t\Delta} f\|_{\dot{B}^0_{\infty,1}} \le C t^{-1/2} \|f\|_{\dot{B}^0_{\infty,\infty}} \qquad \text{for } t > 0 \quad \text{and } f \in \dot{B}^0_{\infty,\infty}. \tag{4.6}$$

Remark 4.2 (i) Because of $\|\cdot\|_{BMO} \le \|\cdot\|_{L^\infty}$ it follows from (4.5) that

$$\|\nabla e^{t\Delta} f\|_{L^\infty} \le C t^{-1/2} \|f\|_{L^\infty} \quad \text{for } t > 0 \quad \text{and } f \in L^\infty. \tag{4.7}$$

Since $\|\cdot\|_{L^\infty} \leq \|\cdot\|_{\dot{B}^0_{\infty,1}}$ and $\|\cdot\|_{\dot{B}^0_{\infty,\infty}} \leq \|\cdot\|_{L^\infty}$, the above estimate (4.7) is derived from (4.6), too.
(ii) In the case of $\Omega = 0$ (non-rotating case), the estimate (4.5) without using Besov spaces yields the boundedness of the nonlinear term in L^∞, that is, we get

$$\begin{aligned}\|e^{t\Delta}\mathbf{P}\nabla\cdot(\mathbf{U}\otimes\mathbf{U})\|_{L^\infty} &= \|\nabla\cdot e^{t\Delta}\mathbf{P}(\mathbf{U}\otimes\mathbf{U})\|_{L^\infty}\\ &\leq \|\nabla\cdot e^{t\Delta}\|_{BMO\to L^\infty}\|\mathbf{P}(\mathbf{U}\otimes\mathbf{U})\|_{BMO}\\ &\leq Ct^{-1/2}\|\mathbf{P}(\mathbf{U}\otimes\mathbf{U})\|_{BMO}\\ &\leq Ct^{-1/2}\|\mathbf{U}\otimes\mathbf{U}\|_{BMO}\\ &\leq Ct^{-1/2}\|\mathbf{U}\otimes\mathbf{U}\|_{L^\infty} \leq Ct^{-1/2}\|\mathbf{U}\|^2_{L^\infty}.\end{aligned}$$

Here we used the boundedness of the operator $\mathbf{P}$. The above estimate follows from the Besov estimate (4.6), too if we use the boundedness of the operator $\mathbf{P}$ in the Besov spaces.

Proof. Since the estimate (2) shall be proved in Appendix D, here we show only (1). In [9, Lemma 2.1] Carpio obtained for the Gauss kernel $G_t = G_t(x)$ that

$$\|\nabla G_t\|_{\mathcal{H}^1} \leq Ct^{-1/2}, \quad t > 0.$$

Here, $\mathcal{H}^1$ denotes the Hardy space. Since the dual space of the space $\mathcal{H}^1$ is BMO, we have

$$\|\nabla e^{t\Delta} f\|_{L^\infty} \leq \|\nabla G_t\|_{\mathcal{H}^1}\|f\|_{BMO} \leq Ct^{-1/2}\|f\|_{BMO}.$$

Lemma 4.1 has been proved. □

Using the above lemma and Proposition 3.1, the linear term is estimated as follows.

Lemma 4.2 (Estimate for the linear term) *For each $\delta > 0$ there exists a constant C (independent of Ω, t, f) that satisfies*

$$\big\|\exp\big(-\mathbf{A}(\Omega)t\big)f\big\|_{L^\infty} \leq C(1+\Omega t)^{(3/2)+\delta}\|f\|_{L^\infty_{\sigma,a}}, \quad t > 0, \quad \textit{and}$$

$$\big\|\nabla\exp\big(-\mathbf{A}(\Omega)t\big)f\big\|_{L^\infty} \leq Ct^{-1/2}(1+\Omega t)^{(3/2)+\delta}\|f\|_{L^\infty_{\sigma,a}}, \quad t > 0$$

for all $f = (f_i)_{1\leq i\leq 3} \in L^\infty_{\sigma,a}$.

Proof. By (3.6), Proposition 3.1 and $\|\cdot\|_{L^\infty} \le \|\cdot\|_{\dot{B}^0_{\infty,1}}$ we get for any $\delta > 0$ that

$$\begin{aligned}
\|\exp(-\mathbf{A}(\Omega)t)f\|_{L^\infty} &= \|e^{t\Delta}\overline{f} + e^{t\Delta}\exp(-t\Omega\mathbf{S})f^\perp\|_{L^\infty} \\
&\le \|e^{t\Delta}\overline{f}\|_{L^\infty} + \|e^{t\Delta}\exp(-t\Omega\mathbf{S})f^\perp\|_{L^\infty} \\
&\le \|\overline{f}\|_{L^\infty} + \|\exp(-t\Omega\mathbf{S})f^\perp\|_{L^\infty} \\
&\le \|\overline{f}\|_{L^\infty} + \|\exp(-t\Omega\mathbf{S})f^\perp\|_{\dot{B}^0_{\infty,1}} \\
&\le \|\overline{f}\|_{L^\infty} + C(1+\Omega t)^{(3/2)+\delta}\|f^\perp\|_{\dot{B}^0_{\infty,1}} \\
&\le C(1+\Omega t)^{(3/2)+\delta}\|f\|_{L^\infty_{\sigma,a}}.
\end{aligned}$$

Similarly Lemma 4.1 implies that

$$\begin{aligned}
&\|\nabla\exp(-\mathbf{A}(\Omega)t)f\|_{L^\infty} \\
&= \|\nabla e^{t\Delta}\overline{f} + \nabla e^{t\Delta}\exp(-t\Omega\mathbf{S})f^\perp\|_{L^\infty} \\
&\le \|\nabla e^{t\Delta}\overline{f}\|_{L^\infty} + \|\nabla e^{t\Delta}\exp(-t\Omega\mathbf{S})f^\perp\|_{L^\infty} \\
&\le Ct^{-1/2}\|\overline{f}\|_{BMO} + Ct^{-1/2}\|\exp(-t\Omega\mathbf{S})f^\perp\|_{\dot{B}^0_{\infty,1}} \\
&\le Ct^{-1/2}(\|\overline{f}\|_{BMO} + (1+\Omega t)^{(3/2)+\delta}\|f^\perp\|_{\dot{B}^0_{\infty,1}}) \\
&\le Ct^{-1/2}(1+\Omega t)^{(3/2)+\delta}(\|\overline{f}\|_{L^\infty} + \|f^\perp\|_{\dot{B}^0_{\infty,1}}) \\
&\le Ct^{-1/2}(1+\Omega t)^{(3/2)+\delta}\|f\|_{L^\infty_{\sigma,a}}.
\end{aligned}$$

We have proved Lemma 4.2. □

Next we prepare estimates for the nonlinear term.

Lemma 4.3 (Estimate for the nonlinear term - BMO version) *There exists a constant $C > 0$ independent of Ω, t, F and f such that*

$$\|\exp(-\mathbf{A}(\Omega)t)\mathbf{P}\operatorname{div}F\|_{L^\infty} \le Ct^{-1/2}(1+(\Omega t)^4)\|F\|_{BMO}, \quad t > 0,$$

and

$$\|\nabla\exp(-\mathbf{A}(\Omega)t)\mathbf{P}f\|_{L^\infty} \le Ct^{-1/2}(1+(\Omega t)^4)\|f\|_{BMO} \quad t > 0$$

for all $F = (F_{i,j})_{1\le i,j\le 3} \in BMO$, with $\operatorname{div}F \in BMO$ and for all $f = (f_i)_{1\le i\le 3} \in BMO$.

Proof. It is easy to see that

$$\mathbf{P}\,\mathrm{div}\,F = \mathrm{div}\,F + \mathrm{div}(\mathbf{P} - I)F^t,$$

where F^t is transposed matrix of F. We rewrite

$$\exp\bigl(-\mathbf{A}(\Omega)t\bigr)\mathbf{P}\,\mathrm{div}\,F = e^{t\Delta}\exp(-\Omega t\mathbf{S})\,\mathrm{div}\{F + (\mathbf{P} - I)F^t\}.$$

Since the symbol of the operator $e^{t\Delta}\exp(-\Omega t\mathbf{S})\,\mathrm{div}$ is represented by

$$\begin{aligned}
&\exp(-t|\xi|^2)\Bigl\{\cos\Bigl(\frac{\xi_3}{|\xi|}\Omega t\Bigr)\mathbf{I} - \mathbf{R}(\xi)\sin\Bigl(\frac{\xi_3}{|\xi|}\Omega t\Bigr)\Bigr\}i\xi_k \\
&= i\xi_k\exp(-t|\xi|^2)\cos\Bigl(\frac{\xi_3}{|\xi|}\Omega t\Bigr)\mathbf{I} \\
&\quad - i\xi_k\mathbf{R}(\xi)\exp(-t|\xi|^2)\sin\Bigl(\frac{\xi_3}{|\xi|}\Omega t\Bigr), \qquad (4.8)
\end{aligned}$$

one sees from Proposition 2.2 that

$$\begin{aligned}
&\|e^{t\Delta}\exp(-\Omega t\mathbf{S})\,\mathrm{div}\,\|_{BMO\to L^\infty} \qquad (4.9)\\
&\le \|\nabla e^{t\Delta}\|_{BMO\to L^\infty}\|\cos(-iR_3\Omega t)\|_{BMO\to BMO} \\
&\quad + \|\,\mathrm{curl}\,e^{t\Delta}\|_{BMO\to L^\infty}\|\sin(-iR_3\Omega t)\|_{BMO\to BMO}\|\mathbf{R}\|_{BMO\to BMO} \\
&\le Ct^{-1/2}\bigl(1+(\Omega t)^4\bigr), \qquad (4.10)
\end{aligned}$$

where $C > 0$ is independent of Ω and t. Thus, by Lemma 4.1, Proposition 2.2 and boundedness of the operator $\mathbf{P}$ in BMO we have

$$\begin{aligned}
&\|\exp\bigl(-\mathbf{A}(\Omega)t\bigr)\mathbf{P}\,\mathrm{div}\,F\|_{L^\infty} \\
&\le Ct^{-1/2}\bigl(1 + (\Omega t)^4\bigr)\|F + (\mathbf{P} - I)F^t\|_{BMO} \\
&\le Ct^{-1/2}\bigl(1 + (\Omega t)^4\bigr)\bigl(\|F\|_{BMO} + \|(\mathbf{P} - I)F^t\|_{BMO}\bigr) \\
&\le Ct^{-1/2}\bigl(1 + (\Omega t)^4\bigr)\|F\|_{BMO}.
\end{aligned}$$

Similarly, we get by (4.9)

$$\bigl\|\nabla\exp\bigl(-\mathbf{A}(\Omega)t\bigr)\mathbf{P}f\bigr\|_{L^\infty} \le Ct^{-1/2}\bigl(1 + (\Omega t)^4\bigr)\|f\|_{BMO}$$

because the symbol of the operator $\nabla\exp(-\mathbf{A}(\Omega)t)$ is the essentially same as that of $e^{t\Delta}E(-\Omega t)\,\mathrm{div}$. We have proved Lemma 4.3. □

Lemma 4.4 (Estimate for the nonlinear term - Besov version) *For each $\delta > 0$ there exists a constant $C = C(\delta) > 0$ independent of Ω, t, F and f such that*

$$\|\exp(-\mathbf{A}(\Omega)t)\mathbf{P}\,\mathrm{div}\,F\|_{\dot{B}^0_{\infty,1}} \le Ct^{-1/2}(1+\Omega t)^{(3/2)+\delta}\|F\|_{\dot{B}^0_{\infty,\infty}},\ t>0,$$

and

$$\|\nabla\exp(-\mathbf{A}(\Omega)t)\mathbf{P}f\|_{\dot{B}^0_{\infty,1}} \le Ct^{-1/2}(1+\Omega t)^{(3/2)+\delta}\|f\|_{\dot{B}^0_{\infty,\infty}} \quad t>0$$

for all $F = (F_{i,j})_{1\le i,j\le 3} \in \dot{B}^0_{\infty,\infty}$, with $\mathrm{div}\,F \in \dot{B}^0_{\infty,\infty}$ and for all $f = (f_i)_{1\le i\le 3} \in \dot{B}^0_{\infty,\infty}$.

Proof. For the operator $e^{t\Delta}E(-\Omega t)\,\mathrm{div}$ whose symbol has the form (4.8) we get from Proposition 3.1 and Lemma 4.1(2) that

$$\begin{aligned}
&\|e^{t\Delta}E(-\Omega t)\,\mathrm{div}\,\|_{\dot{B}^0_{\infty,\infty}\to\dot{B}^0_{\infty,1}}\\
&\le\|\nabla e^{t\Delta}\|_{\dot{B}^0_{\infty,\infty}\to\dot{B}^0_{\infty,1}}\|\cos(-iR_3\Omega t)\|_{\dot{B}^0_{\infty,\infty}\to\dot{B}^0_{\infty,\infty}}\\
&+\|\,\mathrm{curl}\,e^{t\Delta}\|_{\dot{B}^0_{\infty,\infty}\to\dot{B}^0_{\infty,1}}\|\sin(-iR_3\Omega t)\|_{\dot{B}^0_{\infty,\infty}\to\dot{B}^0_{\infty,\infty}}\|\mathbf{R}\|_{\dot{B}^0_{\infty,\infty}\to\dot{B}^0_{\infty,\infty}}\\
&\le Ct^{-1/2}(1+\Omega t)^{(3/2)+\delta},
\end{aligned}$$

where $C > 0$ is independent of Ω and t. We have proved Lemma 4.4. □

Proof of Theorem 4.1. Since we can employ both Lemma 4.3 (BMO estimate) and Lemma 4.4 (Besov estimate) for estimating the nonlinear term in L^∞, in this proof, read the power $\alpha = 4$ when $X = BMO$ or $\alpha = (3/2) + \delta$ with any $\delta > 0$ when $X = \dot{B}^0_{\infty,\infty}$, respectively. The proof parallels in both cases. We use the following successive iteration:

$$\begin{aligned}
&\mathbf{U}_1(t) = \exp(-\mathbf{A}(\Omega)t)\mathbf{U}_0,\\
&\qquad\mathbf{U}_{j+1}(t) = \exp(-\mathbf{A}(\Omega)t)\mathbf{U}_0 - N(\mathbf{U}_j, t;\Omega) \quad \text{for } j \ge 1.
\end{aligned}$$

For $j \ge 1$ and $T > 0$ we set

$$\begin{aligned}
&K_j = K_j(T) = \sup_{0<s<T}\|\mathbf{U}_j(s)\|_{L^\infty}\\
&\qquad\text{and}\quad K'_j = K'_j(T) = \sup_{0<s<T}\big(s^{1/2}\|\nabla\mathbf{U}_j(s)\|_{L^\infty}\big).
\end{aligned}$$

Put $K_0 = \|\mathbf{U}_0\|_{L^\infty_{\sigma,a}}$ and note that K_0 is independent of $T > 0$. It follows from Lemma 4.3 and $\|\cdot\|_X \leq \|\cdot\|_{L^\infty}$ that

$$\begin{aligned}
&\|N(\mathbf{U}_j, t; \Omega)\|_{L^\infty} \\
&\leq \int_0^t \|\exp(-\mathbf{A}(\Omega)(t-s))\mathbf{P}\,\mathrm{div}(\mathbf{U}_j \otimes \mathbf{U}_j)(s)\|_{L^\infty} ds \\
&\leq \int_0^t \|\exp(-\mathbf{A}(\Omega)(t-s))\mathbf{P}\,\mathrm{div}\,\|_{X\to L^\infty} \|(\mathbf{U}_j \otimes \mathbf{U}_j)(s)\|_X ds \\
&\leq \int_0^t C(t-s)^{-1/2}(1+\Omega(t-s))^\alpha \|(\mathbf{U}_j \otimes \mathbf{U}_j)(s)\|_X ds \\
&\leq Ct^{1/2}(1+\Omega t)^\alpha \sup_{0<s<t} \|(\mathbf{U}_j \otimes \mathbf{U}_j)(s)\|_X \\
&\leq Ct^{1/2}(1+\Omega t)^\alpha \sup_{0<s<t} \|(\mathbf{U}_j \otimes \mathbf{U}_j)(s)\|_{L^\infty} \\
&\leq Ct^{1/2}(1+\Omega t)^\alpha \sup_{0<s<t} (\|\mathbf{U}_j(s)\|_{L^\infty}^2) \\
&\leq Ct^{1/2}(1+\Omega t)^\alpha \big(\sup_{0<s<t} \|\mathbf{U}_j(s)\|_{L^\infty}\big)^2. \qquad (4.11)
\end{aligned}$$

Similarly we have from Lemma 4.3

$$\begin{aligned}
&\|\nabla N(\mathbf{U}_j, t; \Omega)\|_{L^\infty} \\
&\leq \int_0^t \|\nabla \exp(-\mathbf{A}(\Omega)(t-s))\mathbf{P}\,\mathrm{div}(\mathbf{U}_j \otimes \mathbf{U}_j)(s)\|_{L^\infty} ds \\
&\leq C\int_0^t (t-s)^{-1/2}(1+\Omega(t-s))^\alpha \|\mathrm{div}(\mathbf{U}_j \otimes \mathbf{U}_j)(s)\|_X ds \\
&\leq C\int_0^t (t-s)^{-1/2}(1+\Omega(t-s))^\alpha \|\mathrm{div}(\mathbf{U}_j \otimes \mathbf{U}_j)(s)\|_{L^\infty} ds \\
&\leq C\int_0^t (t-s)^{-1/2}(1+\Omega(t-s))^\alpha s^{-1/2}s^{1/2}\|\nabla \mathbf{U}_j(s)\|_{L^\infty}\|\mathbf{U}_j(s)\|_{L^\infty} ds \\
&\leq C(1+\Omega t)^\alpha \sup_{0<s<t}\big(s^{1/2}\|\nabla \mathbf{U}_j(s)\|_{L^\infty}\big) \sup_{0<s<t}\|\mathbf{U}_j(s)\|_{L^\infty}. \qquad (4.12)
\end{aligned}$$

By the above estimates and Lemma 4.2 there exist constants C_0, C_1, C_2 and C_3 independent of Ω and T such that

$$\begin{aligned}
K_{j+1}(T) &\leq C_0(1+\Omega T)^\beta K_0 + C_1 T^{1/2}(1+\Omega T)^\alpha (K_j(T))^2, \\
K'_{j+1}(T) &\leq C_2(1+\Omega T)^\beta K_0 + C_3 T^{1/2}(1+\Omega T)^\alpha K_j(T) K'_j(T)
\end{aligned}$$

for $j \geq 1$. Here, $\beta := (3/2) + \delta$ for any $\delta > 0$. Taking T_0 small so that

$T_0^{1/2}(1+\Omega T_0)^{\alpha+\beta} < 1/\big(4(C_0C_1+C_2C_3)K_0\big)$, we get

$$\sup_{j\geq 1} K_j(T) \leq 2C_0K_0(1+\Omega T)^{\beta} \text{ and}$$
$$\sup_{j\geq 1} K'_j(T) \leq 2C_2K_0(1+\Omega T)^{\beta} \quad \text{if } T \leq T_0. \tag{4.13}$$

Next we shall prove the convergence. For $j \geq 1$ and $0 < T < T_0$ put

$$L_j = L_j(T) = \sup_{0<s<T} \|\mathbf{U}_j(s) - \mathbf{U}_{j-1}(s)\|_{L^\infty},$$
$$L'_j = L'_j(T) = \sup_{0<s<T} \big(s^{1/2}\|\nabla\mathbf{U}_j(s) - \nabla\mathbf{U}_{j-1}(s)\|_{L^\infty}\big).$$

Since

$$\begin{aligned}&\mathbf{U}_{j+1}(t) - \mathbf{U}_j(t) = N(\mathbf{U}_j,\, \mathbf{U}_j,\, t;\Omega) - N(\mathbf{U}_j,\, \mathbf{U}_{j-1},\, t;\Omega)\\ &+ N(\mathbf{U}_j,\, \mathbf{U}_{j-1},\, t;\Omega) - N(\mathbf{U}_{j-1},\, \mathbf{U}_{j-1},\, t;\Omega),\end{aligned} \tag{4.14}$$

similarly as in (4.11) and (4.12), we get from (4.13) that

$$\begin{aligned}&\|\mathbf{U}_{j+1}(t) - \mathbf{U}_j(t)\|_{L^\infty}\\ &\leq \int_0^t C(t-s)^{-1/2}\big(1+\Omega(t-s)\big)^{\alpha}\\ &\quad \times \big(\|\mathbf{U}_j(s)\|_{L^\infty} + \|\mathbf{U}_{j-1}(s)\|_{L^\infty}\big)\|(\mathbf{U}_j - \mathbf{U}_{j-1})(s)\|_{L^\infty} ds\\ &\leq Ct^{1/2}(1+\Omega t)^{\alpha} \sup_{0<s<t}\big(\|\mathbf{U}_j(s)\|_{L^\infty} + \|\mathbf{U}_{j-1}(s)\|_{L^\infty}\big)\\ &\qquad\qquad\qquad \times \sup_{0<s<t} \|(\mathbf{U}_j - \mathbf{U}_{j-1})(s)\|_{L^\infty}\\ &\leq Ct^{1/2}(1+\Omega t)^{\alpha+\beta} \sup_{0<s<t} \|(\mathbf{U}_j - \mathbf{U}_{j-1})(s)\|_{L^\infty}\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}&\|\nabla\mathbf{U}_{j+1}(t) - \nabla\mathbf{U}_j(t)\|_{L^\infty}\\ &\leq C\int_0^t (t-s)^{-1/2}\big(1+\Omega(t-s)\big)^{\alpha}\\ &\qquad \times s^{-1/2}\{s^{1/2}\|\nabla\mathbf{U}_j(s)\|_{L^\infty}\|(\mathbf{U}_j - \mathbf{U}_{j-1})(s)\|_{L^\infty}\\ &\qquad\qquad + s^{1/2}\|\mathbf{U}_j(s)\|_{L^\infty}\|\nabla(\mathbf{U}_j - \mathbf{U}_{j-1})(s)\|_{L^\infty}\}ds\\ &\quad + C\int_0^t (t-s)^{-1/2}\big(1+\Omega(t-s)\big)^{\alpha}\\ &\qquad \times s^{-1/2}\{s^{1/2}\|\nabla\mathbf{U}_{j-1}(s)\|_{L^\infty}\|(\mathbf{U}_j - \mathbf{U}_{j-1})(s)\|_{L^\infty}\end{aligned}$$

$$+ s^{1/2}\|\mathbf{U}_{j-1}(s)\|_{L^\infty}\|\nabla(\mathbf{U}_j - \mathbf{U}_{j-1})(s)\|_{L^\infty}\}ds$$

$$\le C(1+\Omega t)^{\alpha+\beta}\{\sup_{0<s<t}\|(\mathbf{U}_j - \mathbf{U}_{j-1})(s)\|_{L^\infty} + \sup_{0<s<t}\|\nabla(\mathbf{U}_j - \mathbf{U}_{j-1})(s)\|_{L^\infty}\} \tag{4.16}$$

for $t < T_0$. Hence there exist C_4, $C_5 > 0$ independent of Ω and T such that

$$L_{j+1}(T) \le C_4 K_0 T^{1/2}(1+\Omega T)^{\alpha+\beta} L_j(T),$$
$$L'_{j+1}(T) \le C_5 K_0 T^{1/2}(1+\Omega T)^{\alpha+\beta}\big(L_j(T) + L'_j(T)\big)$$

for $j \ge 1$ and $T < T_0$. Taking T_1 small so that $T_1^{1/2}(1+\Omega T_1)^{\alpha+\beta} < 1/\big(2(C_4+C_5)K_0\big)$, it is easy to see that

$$\sup_{j\ge1}\frac{L_{j+1}(T)}{L_j(T)} < \frac{1}{2} \text{ and } \sup_{j\ge1}\frac{L_{j+1}(T)+L'_{j+1}(T)}{L_j(T)+L'_j(T)} < \frac{1}{2} \quad \text{if } T \le T_1.$$

Thus, choosing $T < \min(T_0, T_1)$, the approximations $\{\mathbf{U}_j(t)\}_{j\ge1}$ and $\{t^{1/2}\nabla\mathbf{U}_j(t)\}_{j\ge1}$ are Cauchy sequences in $L^\infty\big((0,T)\times\mathbb{R}^3\big)$. Denote its limits by $\mathbf{U}(t)$ and $\mathbf{V}(t)$, respectively. Since $\mathbf{U}_j$ satisfies (4.1), so does $\mathbf{U}$. Similar calculation as in (4.15) and (4.16) yields that

$$N(\mathbf{U}_j, t; \Omega) \to N(\mathbf{U}, t; \Omega) \quad \text{in } L^\infty\big((0,T)\times\mathbb{R}^3\big) \quad \text{as } j\to\infty,$$
$$\nabla N(\mathbf{U}_j, t; \Omega) \to \nabla N(\mathbf{U}, t; \Omega) \quad \text{in } L^\infty\big((0,T)\times\mathbb{R}^3\big) \quad \text{as } j\to\infty,$$

which guarantees that $t^{1/2}\nabla\mathbf{U} = \mathbf{V}$ and that the limit $\mathbf{U}$ solves the integral equation (I). The properties (4.2) for $\mathbf{U}$ are also inherited from $\mathbf{U}_j$'s.

It remains to prove the uniqueness. We set $\mathbf{W} = \mathbf{U}_1 - \mathbf{U}_2$ and observe that

$$\mathbf{W}(t) = N(\mathbf{U}_1, \mathbf{U}_1, t; \Omega) - N(\mathbf{U}_2, \mathbf{U}_2, t; \Omega).$$

Then the same calculation as (4.14) and (4.15) gives us $\mathbf{W} \equiv 0$. □

5. Concluding remarks

The above results for the 3D rotating Navier-Stokes Equations can be formulated for solutions of the three-dimensional Navier-Stokes Equations with initial data of the form $\mathbf{V}(t, y)|_{t=0} = \mathbf{V}(0) = \tilde{\mathbf{V}}_0(y) + (\Omega/2)e_3 \times y$:

$$\partial_t\mathbf{V} + (\mathbf{V}\cdot\nabla)\mathbf{V} + \nu\,\mathrm{curl}^2\,\mathbf{V} = -\nabla q, \quad \nabla\cdot\mathbf{V} = 0, \tag{5.1}$$

$$\mathbf{V}(t, y)|_{t=0} = \mathbf{V}(0) = \tilde{\mathbf{V}}_0(y) + \frac{\Omega}{2}e_3 \times y \tag{5.2}$$

where $y = (y_1, y_2, y_3)$, $\mathbf{V}(t, y) = (V_1, V_2, V_3)$ is the velocity field and q is the pressure. In Eqs. (5.1) e_3 denotes the vertical unit vector and Ω is a constant parameter. The field $\tilde{\mathbf{V}}_0(y)$ depends on three variables y_1, y_2 and y_3. Since $\operatorname{curl}\big((\Omega/2)e_3 \times y\big) = \Omega e_3$, the vorticity vector at initial time $t = 0$ is

$$\operatorname{curl} \mathbf{V}(0, y) = \operatorname{curl} \tilde{\mathbf{V}}_0(y) + \Omega e_3. \tag{5.3}$$

In (5.2) we take $\tilde{\mathbf{V}}_0(y) \in L^\infty_{\sigma, a}(\mathbb{R}^3)$.

We now detail the canonical rotation transformation between the original vector field $\mathbf{V}(t, y)$ and the vector field $\mathbf{U}(t, x)$. Let $\mathbf{J}$ be the matrix such that $\mathbf{J}\mathbf{a} = e_3 \times \mathbf{a}$ for any vector field $\mathbf{a}$. Then

$$\mathbf{J} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{\Upsilon}(t) \equiv e^{\Omega \mathbf{J} t/2} = \begin{pmatrix} \cos(\Omega t/2) & -\sin(\Omega t/2) & 0 \\ \sin(\Omega t/2) & \cos(\Omega t/2) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.4}$$

For any fixed parameter Ω we introduce the following fundamental rotation transformation:

$$\mathbf{V}(t, y) = e^{+\Omega \mathbf{J} t/2} \mathbf{U}(t, e^{-\Omega \mathbf{J} t/2} y) + \frac{\Omega}{2} \mathbf{J} y, \quad x = e^{-\Omega \mathbf{J} t/2} y. \tag{5.5}$$

The transformation (5.5) is invertible:

$$\mathbf{U}(t, x) = e^{-\Omega \mathbf{J} t/2} \mathbf{V}(t, e^{+\Omega \mathbf{J} t/2} x) - \frac{\Omega}{2} \mathbf{J} x, \quad y = e^{+\Omega \mathbf{J} t/2} x. \tag{5.6}$$

The transformations (5.5)-(5.6) establish one-to-one correspondence between solenoidal vector fields $\mathbf{V}(t, y)$ and $\mathbf{U}(t, x)$. We note that $x = y$ for $t = 0$ and therefore $\tilde{\mathbf{V}}_0(y) = \tilde{\mathbf{V}}_0(x)$. Let $x = (x_h, x_3)$ where $x_h = (x_1, x_2, 0)$, $|x_h|^2 = x_1^2 + x_2^2$ and similarly for y.

The following identities hold for the vector fields $\mathbf{V}(t, y)$ and $\mathbf{U}(t, x)$ and pressure π:

1. $\nabla_y \cdot \mathbf{V}(t, y) = \nabla_x \cdot \mathbf{U}(t, x)$.
2. $\nabla_y \pi = \mathbf{\Upsilon}(t) \nabla_x \pi$.
3. $\operatorname{curl}_y \mathbf{V}(t, y) = \mathbf{\Upsilon}(t) \operatorname{curl}_x \mathbf{U}(t, x) + \Omega e_3$,

$$\operatorname{curl}_y^2 \mathbf{V}(t, y) = \mathbf{\Upsilon}(t) \operatorname{curl}_x^2 \mathbf{U}(t, x).$$

4. $(D/Dt)\mathbf{V}(t, y) = \mathbf{\Upsilon}(t)\big((D/Dt)\mathbf{U}(t, x) + \Omega\mathbf{J}\mathbf{U} - (\Omega^2/4)x_h\big)$ where D/Dt are the corresponding Lagrangian derivatives, $\mathbf{J}\mathbf{U} = e_3 \times \mathbf{U}$.

The above identities 1-4 imply that the transformation (5.5)-(5.6) is canonical for Eqs. (5.1)-(5.2). From the property 1 it follows that $\nabla_x \cdot \mathbf{U}(t, x) = 0$ since $\nabla_y \cdot \mathbf{V}(t, y) = 0$. Now using 2-4 and the fact that $\mathbf{\Upsilon}(t)$ is unitary, we can express each term in (5.1) in x and t variables to obtain the equations for $\mathbf{U}(t, x)$. Under the canonical rotation transformation (5.5)-(5.6) Eqs. (5.1)-(5.2) turn into Navier-Stokes system (5.7)-(5.8) with an additional Coriolis term $\Omega e_3 \times \mathbf{U}$ and modified initial data and pressure:

$$\partial_t \mathbf{U} + (\mathbf{U} \cdot \nabla_x)\mathbf{U} + \nu \operatorname{curl}_x^2 \mathbf{U} + \Omega \mathbf{e}_3 \times \mathbf{U} = -\nabla_x p, \qquad \nabla_x \cdot \mathbf{U} = 0, \tag{5.7}$$

$$\mathbf{U}(t, x)|_{t=0} = \mathbf{U}(0, x) = \tilde{\mathbf{V}}_0(x), \tag{5.8}$$

where $x = y$ at $t = 0$ and $x_h = (x_1, x_2)$. The systems Eqs. (5.1)-(5.2) and (5.7)-(5.8) are equivalent for every Ω and the pair of transformations (5.5)-(5.6) establishes one-to-one correspondence between their fully three-dimensional solutions.

We now state our theorem for the initial value problem (5.1)-(5.2).

Theorem 5.1 (Existence of classical solution $\mathbf{v}$) *Suppose* $\tilde{\mathbf{v}}_0 \in L^\infty_{\sigma, a}(\mathbb{R}^3)$. *Then there exists a classical solution* $(\mathbf{V}, \nabla q)$ *of* (5.1)-(5.2) *satisfying*

$$\nabla q(t) = \nabla \sum_{j,k=1}^{3} R_j R_k \mathbf{V}^j \mathbf{v}^k(t) - \Omega \begin{pmatrix} R_1(R_2\mathbf{V}^1 - R_1\mathbf{V}^2) \\ R_2(R_2\mathbf{V}^1 - R_1\mathbf{V}^2) \\ R_3(R_2\mathbf{V}^1 - R_1\mathbf{V}^2) \end{pmatrix} + \nabla \frac{\Omega^2 |y_h|^2}{8} \quad \text{for } t > 0.$$

Such a solution is unique provided that $\mathbf{v} - (\Omega/2)\mathbf{J}y \in L^\infty\big(\mathbb{R}^n \times (0, T)\big)$.

This follows from Theorem 4.2 and Theorem 4.3.

A. Appendix: Calculation of integral kernels

In this section we analyze inverse Fourier transform of $e^{-\nu|\xi|^2 t}\cos\big((\xi_3/|\xi|)\Omega t\big)$, which gives the integral kernel in the convolution operator with $\mathbf{\Phi}_0(x)$ (diagonal terms in (2.16)-(2.17)). The calculation of the inverse Fourier transform of $e^{-\nu|\xi|^2 t}\sin\big((\xi_3/|\xi|)\Omega t\big)(\xi_j/|\xi|)$ (off-diagonal terms in (2.16)-(2.17)) is similar. The integral kernel is obtained in the form

$$(2\pi)^{-3/2}F^{-1}\Big(e^{-\nu|\xi|^2 t}\cos\Big(\frac{\xi_3}{|\xi|}\Omega t\Big) - e^{-\nu|\xi|^2 t}\sin\Big(\frac{\xi_3}{|\xi|}\Omega t\Big)\frac{\xi_j}{|\xi|}\Big)$$

since $F^{-1}mFf = (2\pi)^{-3/2}(F^{-1}m) * f$ for a symbol m and a function f.

We have

$$F^{-1}\Big(e^{-\nu|\xi|^2 t}\cos\Big(\frac{\xi_3}{|\xi|}\Omega t\Big)\Big) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}\cos\Big(\frac{\xi_3}{|\xi|}\Omega t\Big)e^{-\nu|\xi|^2 t}e^{ix\cdot\xi}d\xi, \quad (A.1)$$

where $x = (x_1, x_2, x_3)$ and we denote $|x|^2 = x_1^2 + x_2^2 + x_3^2$, $|x'|^2 = x_1^2 + x_2^2$. Using spherical coordinates with center at 0 and azimuthal angle θ measured from the axis determined by the vector x, one has ($0 \le \theta \le \pi$, $0 \le \psi \le 2\pi$, $\rho = |\xi|$)

$$\frac{\xi_3}{|\xi|} = -\frac{|x'|}{|x|}\sin\theta\sin\psi + \frac{x_3}{|x|}\cos\theta. \quad (A.2)$$

Then

$$\begin{aligned}
&\int_{\mathbb{R}^3}\cos\Big(\frac{\xi_3}{|\xi|}\Omega t\Big)e^{-\nu|\xi|^2 t}e^{ix\cdot\xi}d\xi \\
&= \int_0^{+\infty}\int_0^{2\pi}\int_0^{\pi}\cos\Big(\frac{\Omega t|x'|}{|x|}\sin\theta\sin\psi\Big) \\
&\qquad \times\cos\Big(\frac{\Omega t x_3}{|x|}\cos\theta\Big)e^{-\nu\rho^2 t}e^{i|x|\rho\cos\theta}\rho^2\sin\theta d\rho d\psi d\theta \\
&= 2\pi\int_0^{+\infty}\int_0^{\pi}J_0\Big(\frac{\Omega t|x'|}{|x|}\sin\theta\Big)\cos\Big(\frac{\Omega t x_3}{|x|}\cos\theta\Big)e^{-\nu\rho^2 t} \\
&\qquad \times e^{i|x|\rho\cos\theta}\rho^2\sin\theta d\rho d\theta, \qquad (A.3)
\end{aligned}$$

where we have used the identity

$$\int_0^{2\pi}\cos\Big(\frac{\Omega t|x'|}{|x|}\sin\theta\sin\psi\Big)d\psi = 2\pi J_0\Big(\frac{\Omega t|x'|}{|x|}\sin\theta\Big). \quad (A.4)$$

Here, $J_n(z)$ denotes the Bessel function for $n = 0, 1, 2, \ldots$.

Let $\mu = \cos\theta$. Then we have from (A.3)

$$\begin{aligned}
&\int_{\mathbb{R}^3} \cos\Big(\frac{\xi_3}{|\xi|}\Omega t\Big) e^{-\nu|\xi|^2 t} e^{ix\cdot\xi} d\xi \\
&= 2\pi \int_0^{+\infty} \int_{-1}^1 J_0\Big(\frac{\Omega t|x'|}{|x|}\sqrt{1-\mu^2}\Big) \\
&\qquad\qquad \times \cos\Big(\frac{\Omega t x_3}{|x|}\mu\Big) e^{-\nu\rho^2 t} \cos(|x|\rho\mu)\rho^2 d\rho d\mu \qquad \text{(A.5)}
\end{aligned}$$

since the function $\sin(|x|\rho\mu)$ is odd in μ and other functions are even in μ.

Now we calculate the integral in (A.5) involving integration with respect to ρ. We have after somewhat lengthy but elementary calculations (which also involves shifting contour of integration in complex plane) or from the Table of Integrals in [16, page 529, 3.952]:

$$\begin{aligned}
&\int_0^{+\infty} e^{-\nu\rho^2 t} \cos(|x|\rho\mu)\rho^2 d\rho \\
&\qquad\qquad = \frac{\sqrt{\pi}}{4(\sqrt{\nu t})^3}\Big(1 - \frac{|x|^2\mu^2}{2\nu t}\Big) e^{-|x|^2\mu^2/(4\nu t)}. \qquad \text{(A.6)}
\end{aligned}$$

Substituting (A.6) into (A.5), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} \cos\Big(\frac{\xi_3}{|\xi|}\Omega t\Big) e^{-\nu|\xi|^2 t} e^{ix\cdot\xi} d\xi \\
&= 2\pi \frac{\sqrt{\pi}}{4(\sqrt{\nu t})^3} \int_{-1}^1 J_0\Big(\frac{\Omega t|x'|}{|x|}\sqrt{1-\mu^2}\Big) \\
&\qquad\qquad \times \cos\Big(\frac{\Omega t x_3}{|x|}\mu\Big)\Big(1 - \frac{|x|^2\mu^2}{2\nu t}\Big) e^{-|x|^2\mu^2/(4\nu t)} d\mu. \qquad \text{(A.7)}
\end{aligned}$$

For $\Omega = 0$ the above expression reduces to the heat kernel $G_{\nu t}(x) = \big(1/(4\pi\nu t)^{3/2}\big) e^{-|x|^2/(4\nu t)}$. In fact, since

$$J_0\Big(\frac{\Omega t|x'|}{|x|}\sqrt{1-\mu^2}\Big)\Big|_{\Omega=0} = 1, \quad \cos\Big(\frac{\Omega t x_3}{|x|}\mu\Big)\Big|_{\Omega=0} = 1$$

and

$$\int_{-1}^1 \Big(1 - \frac{|x|^2\mu^2}{2\nu t}\Big) e^{-|x|^2\mu^2/(4\nu t)} d\mu = 2e^{-|x|^2/(4\nu t)},$$

we get

$$2\pi\frac{\sqrt{\pi}}{4(\sqrt{\nu t})^3}\int_{-1}^{1}\Big(1-\frac{|x|^2\mu^2}{2\nu t}\Big)e^{-|x|^2\mu^2/(4\nu t)}d\mu$$
$$=2\pi\frac{\sqrt{\pi}}{4(\sqrt{\nu t})^3}2(4\pi\nu t)^{3/2}\frac{1}{(4\pi\nu t)^{3/2}}e^{-|x|^2/(4\nu t)}=(2\pi)^{(3/2)\cdot 2}G_{\nu t}(x).$$

Hence the kernel is given by

$$(2\pi)^{-3/2}F^{-1}\bigg(e^{-\nu|\xi|^2t}\cos\Big(\frac{\xi_3}{|\xi|}\Omega t\Big)\bigg)=G_{\nu t}(x)$$

if $\Omega=0$.

Let Ω and t be fixed. The asymptotics of the integral kernel in $|x|$ can be analyzed using (A.7). Clearly, it is bounded for $|x|\to 0$. Now we deduce the behaviour for large $|x|$. The main obstacle to a rapid decay of the kernel for large $|x|$ is that the term $e^{-|x|^2\mu^2/(4\nu t)}$ appears in combination with $|x|^2\mu^2$ and $e^{-|x|^2\mu^2/(4\nu t)}|_{\mu=0}=1$. The main contribution to the kernel asymptotics for large $|x|$ is given in the integral (A.7) by a small interval containing $\mu=0$. If we expand the expression $J_0\big((\Omega t|x'|/|x|)\sqrt{1-\mu^2}\big)\cos\big((\Omega t x_3/|x|)\mu\big)$ under integral in powers of μ (valid uniformly in $|x|$ since $|x_3|/|x|$, $|x'|/|x|\le 1$), then first we recover the term (heat kernel) $\times$ $J_0\big((\Omega t|x'|/|x|)\big)$ which clearly rapidly decays as $|x|\to+\infty$. Since the function under integral is even in μ, the next term will be of the form (function independent of μ) $\times$ $\mu^2\big(1-|x|^2\mu^2/(2\nu t)\big)e^{-|x|^2\mu^2/(4\nu t)}$. Its asymptotic behaviour for large $|x|$ is given by the integral:

$$\int_{-1}^{1}\mu^2\Big(1-\frac{|x|^2\mu^2}{2\nu t}\Big)e^{-|x|^2\mu^2/(4\nu t)}d\mu$$
$$=\frac{1}{|x|^3}\int_{-|x|}^{|x|}\eta^2\Big(1-\frac{\eta^2}{2\nu t}\Big)e^{-\eta^2/(4\nu t)}d\eta\sim\frac{C(\nu t)}{|x|^3}\quad\text{for large } |x|.$$

Therefore, the integral kernel behaves as $1/|x|^3$ for large $|x|$. In particular, the integral kernel does not belong to $L^1(\mathbb{R}^3)$. The corresponding integral operator cannot be viewed as a bounded operator in $L^\infty(\mathbb{R}^3)$ since a characteristic function of the outside of a large ball is always mapped to ∞ by this operator.

The above analysis and similar considerations for

$$F^{-1}\bigg(e^{-\nu|\xi|^2t}\sin\Big(\frac{\xi_3}{|\xi|}\Omega t\Big)\frac{\xi_j}{|\xi|}\bigg)$$

show that

$$F^{-1}\Big(e^{-\nu|\xi|^2 t}\cos\Big(\frac{\xi_3}{|\xi|}\Omega t\Big)\Big),$$
$$F^{-1}\Big(e^{-\nu|\xi|^2 t}\sin\Big(\frac{\xi_3}{|\xi|}\Omega t\Big)\mathbf{R}(\xi)\Big) \in L^q(\mathbb{R}^3), \quad 1 < q < +\infty.$$

It is clear without any calculations that $F^{-1}\big(e^{-\nu|\xi|^2 t}\sin\big((\xi_3/|\xi|)\Omega t\big)\mathbf{R}(\xi)\big)$ does not belong to $L^1(\mathbb{R}^3)$ since $e^{-\nu|\xi|^2 t}\sin\big((\xi_3/|\xi|)\Omega t\big)(\xi_j/|\xi|)$ ($j = 1, 2$) are discontinuous at $\xi = 0$.

B. Appendix: Estimate for the Coriolis solution operator

In this section we introduce the homogeneous Besov spaces $\dot{B}^s_{p,q} = \dot{B}^s_{p,q}(\mathbb{R}^n)$ and show boundedness of the Coriolis solution operator $\exp(-\Omega t\mathbf{S})$ in the Hardy space, BMO, and the Besov spaces.

All assertions in the Appendixes B, C and D except Lemma B.4 hold in the general space dimension $n = 1, 2, 3, \ldots$ although the Rotating Navier-Stokes equations is valid only when $n = 3$.

Before introducing the homogeneous Besov spaces, we prepare some notations. By $\mathcal{S}$ we denote the class of rapidly decreasing functions. The dual of $\mathcal{S}$, the space of tempered distributions is denoted by $\mathcal{S}'$. By $\mathcal{H}^1$ we denote the Hardy space. It is well known that the dual space of the Hardy space $\mathcal{H}^1$ is BMO, the space of functions of bounded mean oscillation. Let $\{\phi_j\}_{j=-\infty}^{\infty}$ be the Littlewood-Paley dyadic decomposition satisfying

$$\widehat{\phi_j}(\xi) = \widehat{\phi_0}(2^{-j}\xi) \in C_c^\infty(\mathbb{R}^n),$$
$$\operatorname{supp}\widehat{\phi_0} \subset \Big\{\frac{1}{2} < |\xi| < 2\Big\},$$
$$\sum_{j=-\infty}^{\infty}\widehat{\phi_j}(\xi) = 1 \quad (\xi \neq 0). \quad \text{(B.1)}$$

Definition B.1 (See, e.g. [5] page 146) The homogeneous Besov space $\dot{B}^s_{p,q} = \dot{B}^s_{p,q}(\mathbb{R}^n)$ is defined by

$$\dot{B}^s_{p,q} \equiv \big\{f \in \mathcal{Z}'; \|f; \dot{B}^s_{p,q}\| < \infty\big\}$$

for $s \in \mathbb{R}$ and $1 \le p,\, q \le \infty$, where

$$\|f; \dot{B}^s_{p,\,q}\| \equiv \begin{cases} \Big[\sum_{j=-\infty}^{\infty} 2^{jsq} \|\phi_j * f; L^p\|^q \Big]^{1/q} & \text{if} \quad q < \infty, \\ \sup_{-\infty \le j \le \infty} 2^{js} \|\phi_j * f; L^p\| \quad \text{if} \quad q = \infty. \end{cases}$$

Here $\mathcal{Z}'$ is the topological dual space of the space $\mathcal{Z}$, which is defined by $\mathcal{Z} \equiv \{ f \in \mathcal{S}; D^\alpha \hat{f}(0) = 0 \text{ for all multi-indices } \alpha = (\alpha_1,\, \dots,\, \alpha_n) \}$.

The above definition yields that all polynomials vanish in $\dot{B}^s_{p,\,q}$, however, it is well known that

$$\dot{B}^s_{p,q} \cong \left\{ f \in \mathcal{S}'; \|f; \dot{B}^s_{p,q}\| < \infty \quad \text{and} \quad f = \sum_{j=\infty}^{\infty} \phi_j * f \text{ in } \mathcal{S}' \right\} \tag{B.2}$$

if

$$s < n/p \quad \text{or} \quad (s = n/p \text{ and } q = 1). \tag{B.3}$$

For the indices $s \in \mathbb{R}$, $1 < p,\, q \le \infty$ satisfying the negation of (B.3);

$$s > n/p \quad \text{or} \quad (s = n/p \text{ and } p \ne 1) \tag{B.4}$$

we define the space $\dot{\mathcal{B}}^s_{p,\,q}$ by duality as follows.

$$\dot{\mathcal{B}}^s_{p,\,q} := (\dot{B}^{-s}_{p',\,q'})' \quad \text{for} \quad \frac{1}{p} + \frac{1}{p'} = 1,\; \frac{1}{q} + \frac{1}{q'} = 1.$$

In particular, we define $\dot{\mathcal{B}}^0_{\infty,\,\infty} := (\dot{B}^0_{1,\,1})'$. We do not define $\dot{\mathcal{B}}^s_{p,\,q}$ for the case $p = 1$ or $q = 1$ satisfying (B.4) since we do not use the spaces. In this paper by abuse of notation we denote $\dot{\mathcal{B}}^s_{p,\,q}$ simply by $\dot{B}^s_{p,\,q}$. For the details and examples one can consult e.g. [26], [27], [30].

The key lemma of this section is as follows.

Lemma B.1 (Boundedness of convolution-type operator) *Let $1 \le q \le \infty$. For $h \in \mathcal{S}'$ let $T = h*$ be a convolution-type operator defined on $\mathcal{S}$. Assume that T is regarded as a bounded operator $\mathcal{H}^1 \to \mathcal{H}^1$. Then, the operator T is bounded from $\dot{B}^0_{\infty,\,q}$ to itself. Its norm $\|T\|_{\dot{B}^0_{\infty,\,q} \to \dot{B}^0_{\infty,\,q}}$ is bounded by $C\|T\|_{\mathcal{H}^1 \to \mathcal{H}^1}$ with $C = C(n) > 0$ independent of q, h.*

Proof. We give the proof only when $q = 1$. The proof can be easily modified to the case $1 < q \leq \infty$. By the definition of the Besov norm we have

$$\|Tf; \dot{B}^0_{\infty,1}\| = \sum_{j=-\infty}^{\infty} \|\phi_j * Tf\|_\infty = \sum_{j=-\infty}^{\infty} \|\phi_j * h * f\|_\infty.$$

Since only three terms in the family $\{\operatorname{supp} \widehat{\phi_j}\}_j$ are nonzero for any fixed point $\xi \in \mathbb{R}^n$, we derive

$$\begin{aligned}\|Tf; \dot{B}^0_{\infty,1}\| &= \sum_{j,\,k\in\mathbb{Z},\,|j-k|\leq 2} \|\phi_j * h * \phi_k * f\|_\infty \\ &\leq \sum_{j,\,k\in\mathbb{Z},\,|j-k|\leq 2} \|\phi_j * h\|_1 \|\phi_k * f\|_\infty.\end{aligned}$$

The fact $\|\cdot\|_{L^1} \leq \|\cdot\|_{\mathcal{H}^1}$ and the assumption yield

$$\begin{aligned}\|Tf; \dot{B}^0_{\infty,1}\| &\leq \sum_{j,\,k\in\mathbb{Z},\,|j-k|\leq 2} \|\phi_j * h\|_{\mathcal{H}^1} \|\phi_k * f\|_\infty \\ &\leq C \sum_{j,\,k\in\mathbb{Z},\,|j-k|\leq 2} \|\phi_j\|_{\mathcal{H}^1} \|\phi_k * f\|_\infty.\end{aligned}$$

Here, $\|\phi_j\|_{\mathcal{H}^1} = \|\phi_j\|_{L^1} + \sum_{k=1}^n \|iR_k\phi_j\|_{L^1}$ is a constant independent of j since $\|\phi_j\|_{\mathcal{H}^1} = \|\phi_0\|_{\mathcal{H}^1}$. Indeed, we obtain that

$$\begin{aligned}\|\phi_j\|_1 &= \int_{\mathbb{R}^n} |\phi_j(x)|dx = \int_{\mathbb{R}^n} \big|\big(F^{-1}(F\phi_j)(\xi)\big)(x)\big|dx \\ &= \int_{\mathbb{R}^n} \big|\big(F^{-1}(F\phi_0)(2^{-j}\xi)\big)(x)\big|dx \\ &= 2^{jn} \int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} e^{i2^j x\xi}\big(F\phi_0(\xi)\big)(x)d\xi\right|dx \\ &= 2^{jn} \int_{\mathbb{R}^n} |\phi_0(2^j x)|dx = \int_{\mathbb{R}^n} |\phi_0(x)|dx = \|\phi_0\|_1\end{aligned}$$

and similarly

$$\begin{aligned}\|iR_k\phi_j\|_1 &= \int_{\mathbb{R}^n} |iR_k\phi_j(x)|dx = \int_{\mathbb{R}^n} \left|\left(F^{-1}\frac{i\xi_k}{|\xi|}(F\phi_j)(\xi)\right)(x)\right|dx \\ &= \int_{\mathbb{R}^n} \left|\left(F^{-1}\frac{i\xi_k}{|\xi|}(F\phi_0)(2^{-j}\xi)\right)(x)\right|dx\end{aligned}$$

$$= 2^{jn} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{i2^j x\xi} \frac{i\xi_k}{|\xi|} (F\phi_0(\xi))(x) d\xi \right| dx$$
$$= 2^{jn} \int_{\mathbb{R}^n} |iR_k \phi_0(2^j x)| dx = \int_{\mathbb{R}^n} |iR_k \phi_0(x)| dx = \|iR_k \phi_0\|_1$$

for all k with $1 \leq k \leq n$. Thus we conclude

$$\|Tf; \dot{B}^0_{\infty,1}\| \leq C\|\phi_0\|_{\mathcal{H}^1} \sum_{j,\,k\in\mathbb{Z},\,|j-k|\leq 2} \|\phi_k * f\|_\infty$$
$$\leq 3C\|\phi_0\|_{\mathcal{H}^1} \sum_{j=-\infty}^{\infty} \|\phi_j * f\|_\infty = C\|f; \dot{B}^0_{\infty,1}\|.$$

This establishes the result. □

Lemma B.2 (Theorem 7.30 in [11], [18], Mikhlin-type theorem in the Hardy space and BMO) *Suppose $k > n/2$. Let $m(\xi) \in C^k(\mathbb{R}^n \setminus \{0\})$ satisfy*

$$|D^\alpha m(\xi)| \leq C_\alpha |\xi|^{-|\alpha|} \quad (\xi \neq 0)$$
$$\text{for all } |\alpha| = \alpha_1 + \cdots + \alpha_n \leq k. \quad \text{(B.5)}$$

Then the operator defined by $T_m = F^{-1} m F$ is bounded from $\mathcal{H}^1$ to itself and from BMO to itself.

Lemma B.3 (Boundedness of resolvent operator) *Let $1 \leq q \leq \infty$. Consider the operator $\lambda - iR_j : \dot{B}^0_{\infty,q} \to \dot{B}^0_{\infty,q}$ for $j = 1, 2, 3$. Then,* $\mathrm{Spec}(iR_j) \subset \mathbb{R}$. *Here* $\mathrm{Spec}(K)$ *denotes the spectrum set of an operator K.*

Proof. Assume $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Since it is easy to see that $m(\xi) = 1/\big(\lambda + (\xi_j/|\xi|)\big)$ satisfies (B.5), Lemma B.2 guarantees that $(\lambda - iR_j)^{-1}$ exists and bounded from $\mathcal{H}^1$ to itself. So, it follows from Lemma B.1 that $(\lambda - iR_j)^{-1}$ exists and bounded from $\dot{B}^0_{\infty,q}$ to itself. Thus $\lambda \in \mathbb{C} \setminus \mathbb{R}$ belong to the resolvent set. □

Proposition B.1 (Estimate for exponential of the operator tR_j - $(n+1)$-power version) *Let X be $\mathcal{H}^1$, BMO or $\dot{B}^0_{\infty,q}$ for $1 \leq q \leq \infty$. Then we have*

$$\|\exp(tR_j)\|_X \leq C(1 + t^{n+1})\|f\|_X \quad \text{for } t > 0.$$

Remark B.1 In the case $X = \dot{B}^0_{\infty,q}$, the power $n+1$ of t in the above estimate can be improved to $\delta + n/2$ with any $\delta > 0$ (see Theorem C.1).

Proof. In the proof we omit j and write R for R_j. Consider the Yosida approximation R_λ ($\lambda \in \mathbb{N}$) for R defined by $R_\lambda := RJ_\lambda$ where $J_\lambda := \lambda(\lambda - R)^{-1}$. Since $R_\lambda = -\lambda + \lambda J_\lambda$, we see

$$\begin{aligned}\|\exp(tR_\lambda)\|_{X\to X} &= e^{-t\lambda}\|\exp(t\lambda J_\lambda)\|_{X\to X}\\ &\le e^{-t\lambda}\sum_{k=0}^{\infty}\frac{(t\lambda)^k}{k!}\|(J_\lambda)^k\|_{X\to X}.\end{aligned}$$

To estimate $\|(J_\lambda)^k\|_{X\to X}$ we get pointwise estimate for its symbol $m(\xi) := \big(\lambda^k/(\lambda - i\xi/|\xi|)^k\big)$ to derive

$$\lambda^{-k}\max_{|\alpha|\le n+1}\sup_{\xi\ne 0}|\xi|^{|\alpha|}|D^\alpha m(\xi)| \le \begin{cases} C\dfrac{1}{\lambda^k} & \text{if } \lambda \ge k,\\[2mm] C\dfrac{1}{\lambda^k}\Big(\dfrac{k}{\lambda}\Big)^{n+1} & \text{if } \lambda < k\end{cases}$$

for some numerical constant $C > 0$. Hence, applying Lemma B.2 as $k = n+1$, we see

$$\begin{aligned}&\|\exp(tR_\lambda)\|_{X\to X}\\ &\le Ce^{-t\lambda}\Bigg(\sum_{k=0}^{\lambda}\frac{(t\lambda)^k}{k!} + \sum_{k=\lambda+1}^{\infty}\frac{(t\lambda)^k}{k!}\Big(\frac{k}{\lambda}\Big)^{n+1}\Bigg)\\ &\le C + Ce^{-t\lambda}\frac{1}{\lambda^{n+1}}\sum_{k=\lambda+1}^{\infty}\frac{(t\lambda)^{k-n-1}}{(k-n-1)!}\\ &\qquad\qquad\times\frac{k^{n+1}}{k(k-1)(k-2)\cdots(k-n)}(t\lambda)^{n+1}\\ &\le C + Ct^{n+1}e^{-t\lambda}\sum_{m=\lambda-n}^{\infty}\frac{(t\lambda)^m}{m!}\\ &\le C(1+t^{n+1}).\end{aligned}$$

Then the estimate for $\exp(tR)$ in $\mathcal{H}^1$ and BMO follows from the convergence $\exp(tR_\lambda) \to \exp(tR)$ as $\lambda \to \infty$. The same estimate in the Besov spaces $\dot{B}^0_{\infty,q}$ follows from Lemma B.1 since $\exp(tR) = \big(F^{-1}\exp(ti(\xi/|\xi|))\big) * f$ is a convolution type operator. □

Lemma B.4 (Persistency of vertical averaging property) *Assume that* $n = 3$. *If* $\mathbf{U} \in \mathcal{B}^0$, *then* $E(-\Omega t)\mathbf{U} \in \mathcal{B}^0$, *where* $E(-\Omega t) = \exp(-t\Omega\mathbf{S})$.

Proof. It suffices to prove that $\overline{R_j f} = 0$ if $\overline{f} = 0$ for $f \in \dot{B}^0_{\infty,1}$, where R_j is a scalar Riesz operator and f is a scalar function. We approximate f by a finite sum $\sum \phi_j * f$.

We set $f_l = \sum_{|k| \le l} \phi_k * f$ for $l > 0$. By a similar argument to prove that $\overline{\exp(\nu t \Delta)\mathbf{U}} = 0$ for $\overline{\mathbf{U}} = 0$ in the proof of Proposition 3.2 we obtain that $\overline{R_j \phi_k * f} = 0$ if $\overline{f} = 0$, since $R_j \phi_k$ is a rapidly decreasing function. This implies that $\overline{R_j f_l} = 0$. Since $f_l \to f$ in $\dot{B}^0_{\infty,1}$ as $l \to \infty$, the Riesz operator R_j is bounded and the subspace of the zero vertical average is closed in $\dot{B}^0_{\infty,1}$, we conclude that $\overline{R_j f} = 0$. □

Remark B.2 The fact that Mikhlin's condition (B.5) implies that a bound for the operator $T_m = F^{-1} m F$ in $\dot{B}^0_{\infty,q}$ can be proved directly without using Lemma B.1; see e.g. Amann [1]. However, Lemma B.1 is not included in [1] and seems to be new.

C. Appendix: Improved estimate for the Coriolis solution operator in the homogeneous Besov space

In the previous appendix we already obtained an estimate for the Coriolis solution operator $\exp(-\Omega t \mathbf{S})$ with the power $n+1$ of Ωt. Although the estimate in Proposition B.1 is valid in 3 kinds of spaces, $\mathcal{H}^1$, BMO, and $\dot{B}^0_{\infty,q}$, we can get sharper estimate if we restrict function space to $\dot{B}^0_{\infty,q}$.

Theorem C.1 (Estimate for exponential of the operator tR_j - $\big((n/2)+\delta\big)$-power version) *Let $1 \le q \le \infty$. For each $\delta > 0$ there exists a constant $C = C(\delta, n) > 0$ independent of q, t, j, and f such that*

$$\|(\exp tR_j)f\|_{\dot{B}^0_{\infty,q}} \le C(1+t)^{(n/2)+\delta} \|f\|_{\dot{B}^0_{\infty,q}}$$

for all $t > 0$ and $f \in \dot{B}^0_{\infty,q}$.

For the proof the first step is to observe

Lemma C.1 *Let $1 \le q \le \infty$. Let ϕ_0 be the function of the Paley-Littlewood decomposition in the definition of the Besov norm. Then*

$$\|(\exp tR_j)f\|_{\dot{B}^0_{\infty,q}} \le 3\|(\exp tR_j)\phi_0\|_{L^1} \|f\|_{\dot{B}^0_{\infty,q}} \tag{C.1}$$

Proof. By definition

$$\begin{aligned}\|(\exp tR_j)f\|_{\dot{B}^0_{\infty,q}} &= \Big(\sum_{k=-\infty}^{\infty}\|(\exp tR_j)(\phi_k * f)\|_\infty^q\Big)^{1/q}\\ &= \Big(\sum_{h=-\infty}^{\infty}\sum_{|h-k|\le 1}\|(\exp tR_j)(\phi_k*\phi_h*f)\|_\infty^q\Big)^{1/q}\\ &\le \Big(\sum_{h=-\infty}^{\infty}\sum_{|h-k|\le 1}\|(\exp tR_j)\phi_k\|_{L^1}^q\|\phi_h*f\|_\infty^q\Big)^{1/q}\end{aligned}$$

Since $\|F^{-1}f_\lambda\|_{L^1} = \|F^{-1}f\|_{L^1}$ for $f_\lambda(\xi) = f(\lambda\xi)$, $\lambda > 0$ and $\xi_j/|\xi|$ is invariant under this scaling, we have

$$\begin{aligned}\|(\exp tR_j)\phi_k\|_{L^1} &= \Big\|F^{-1}\Big(\exp\Big(\frac{it\xi_j}{|\xi|}\Big)\hat{\phi}_0(2^{-k}\xi)\Big)\Big\|_{L^1}\\ &= \|(\exp tR_j)\phi_0\|_{L^1}.\end{aligned}$$

Hence one sees that

$$\begin{aligned}\|(\exp(tR_j)f\|_{\dot{B}^0_{\infty,q}} &\le 3\|\exp(tR_j)\phi_0\|_{L^1}\Big(\sum_{h=-\infty}^{\infty}\|\phi_h*f\|_\infty^q\Big)^{1/q}\\ &\le 3\|\exp(tR_j)\phi_0\|_{L^1}\|f\|_{\dot{B}^0_{\infty,q}}.\end{aligned}$$

The proof is now complete. □

We shall estimate $\|(\exp tR_j)\phi_0\|_{L^1}$ in RHS of (C.1) by using an weighted L^2 estimate.

Lemma C.2 *For each $\sigma \ge 0$ there exists a constant $K_\sigma = K(\sigma) > 0$ independent of $t > 0$ such that*

$$\|(1+|x|^2)^\sigma(\exp tR_j)\phi_0\|_{L^2} \le K_\sigma(1+t)^{2\sigma} \quad \textit{for all } t > 0.$$

Proof. We shall prove this Lemma only for $\sigma \in [0, 1]$ since we only use such a σ and the idea of the proof for large σ is the same. By the Parseval equality we have

$$\begin{aligned}&\|(1+|x|^2)(\exp tR_j)\phi_0\|_{L^2} = \Big\|(1-\Delta_\xi)\Big(\exp it\frac{\xi_j}{|\xi|}\Big)\hat{\phi}_0(\xi)\Big\|_{L^2}\\ &\le \|\hat{\phi}_0\|_{L^2} + \|\Delta_\xi\hat{\phi}_0\|_{L^2} + t^2\||\xi|^{-2}\hat{\phi}_0\|_{L^2} + 2t\||\xi|^{-1}\nabla_\xi\hat{\phi}_0\|_{L^2}.\end{aligned}$$

Since $\hat{\phi}_0 \in C_0^\infty$ and $0 \notin \operatorname{supp}\hat{\phi}_0$, we have the desired estimate for $\sigma = 1$.

The case $\sigma = 0$ directly follows from the Parseval equality. For $\sigma \in (0, 1)$ we use the Hölder inequality to get

$$\begin{aligned}\|(1+|x|^2)^\sigma h\|_{L^2} &\le \|(1+|x|^2)^\sigma h^\sigma\|_{L^{2/\sigma}} \|h^{1-\sigma}\|_{L^{2/(1-\sigma)}} \\ &\le \|(1+|x|^2)h\|_{L^2}^\sigma \|h\|_{L^2}^{1-\sigma}\end{aligned}$$

for any measurable function $h \geq 0$.

We thus interpolate the estimate for $\sigma = 0$ and $\sigma = 1$ to get the desired inequality for $\sigma \in (0, 1)$. □

Proof of Theorem C.1. By Lemma C.1 it suffices to prove that

$$\|(\exp tR_j)\phi_0\|_{L^1} \le C_\delta (1+t)^{(n/2)+\delta}, \quad t > 0$$

with C_δ independent of $t > 0$. By the Schwarz inequality we have

$$\|h\|_{L^1} \le \|(1+|x|^2)^{-\sigma}\|_{L^2} \|(1+|x|^2)^\sigma h\|_{L^2}.$$

We take $h = (\exp tR_j)\phi_0$ and apply Lemma C.2 with $\sigma = \delta/2 + n/4$ to get our desired estimate for $\sigma = \delta/2 + n/4$. The proof is now complete. □

D. Appendix: Estimate for fractional power of Laplacian of the heat kernel

In this appendix we give the proof of Lemma 4.1(2). For this we need an estimate for convolution in the Besov space $\dot{B}^0_{\infty,1}$. We shall also show estimates for derivative and fractional powers of the heat kernel.

Lemma D.1 *There exists a constant $C > 0$ independent of f and g such that*

$$\|f * g\|_{\dot{B}^0_{\infty,1}(\mathbb{R}^n)} \le C \|f\|_{\dot{B}^0_{1,1}(\mathbb{R}^n)} \|g\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^n)}$$

for $f \in \dot{B}^0_{1,1}(\mathbb{R}^n)$ and $g \in \dot{B}^0_{\infty,\infty}(\mathbb{R}^n)$.

Proof. By Young's inequality we have

$$\begin{aligned}\|f * g\|_{\dot{B}^0_{\infty,1}} &\le \sum_{j\in\mathbb{Z}} \|\phi_j * (f * g)\|_{L^\infty} \le \sum_{j,k\in\mathbb{Z}} \|\phi_j * (f * g) * \phi_k\|_{L^\infty} \\ &\le \sum_{j,k\in\mathbb{Z},\, |j-k|\le 2} \|\phi_j * f\|_{L^1} \|g * \phi_k\|_{L^\infty}\end{aligned}$$

$$\leq 3 \sup_{k\in\mathbb{Z}} \|g * \phi_k\|_{L^\infty} \sum_{j\in\mathbb{Z}} \|\phi_j * f\|_{L^1} \leq 3\|g\|_{\dot{B}^0_{\infty,\infty}} \|f\|_{\dot{B}^0_{1,1}}$$

□

Then we turn to the proof of Lemma 4.1(2).

Lemma D.2 *Let G_t be the heat kernel $(4\pi t)^{-n/2}\exp\big(-|x|^2/(4t)\big)$ for $t > 0$. Then*

(1) $\|\nabla G_t(x)\|_{\dot{B}^0_{1,1}(\mathbb{R}^n)} \leq Ct^{-1/2}$.

(2) (Lemma 4.1(2)) $\|\nabla e^{t\Delta} f\|_{\dot{B}^0_{\infty,1}(\mathbb{R}^n)} \leq Ct^{-1/2}\|f\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^n)}$ *for* $f \in \dot{B}^0_{\infty,\infty}(\mathbb{R}^n)$.

Proof. (1) Since $\phi_j(x) = 2^{jn}\phi_0(2^j x)$, we see

$$\begin{aligned}
\|\phi_j * \nabla G_t\|_1 &= \|\nabla(\phi_j) * G_t\|_1 \\
&= 2^j \left\| \int_{\mathbb{R}^n} |2^{jn}(\nabla\phi_0)(2^j y) G_t(x-y)| dy \right\|_1 \\
&\leq 2^j \|2^{jn}(\nabla\phi_0)(2^j\cdot)\|_1 \|G_t\|_1 \leq 2^j \|\nabla\phi_0\|_1 \|G_t\|_1. \quad \text{(D.1)}
\end{aligned}$$

On the other hand, we get by the mean value theorem and $\int \phi_0(z)dz = 0$

$$\begin{aligned}
(\phi_j * \nabla G_t)(x) &= \int_{\mathbb{R}^n} \phi_j(y)(\nabla G_t)(x-y)dy \\
&= \int_{\mathbb{R}^n} 2^{jn}\phi_0(2^j y)(\nabla G_t)(x-y)dy \\
&= \int_{\mathbb{R}^n} \phi_0(z)(\nabla G_t)(x - 2^{-j}z)dz \\
&= \int_{\mathbb{R}^n} \phi_0(z)\{(\nabla G_t)(x-2^{-j}z) - (\nabla G_t)(x)\}dz \\
&= \int_{\mathbb{R}^n} \phi_0(z) 2^{-j} z \left(\int_0^1 (\nabla^2 G_t)(x - \theta 2^{-j} z)d\theta \right) dz.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\phi_j * \nabla G_t\|_1 &\leq 2^{-j} \int_{\mathbb{R}^n} \left| \phi_0(z) z \int_0^1 (\nabla^2 G_t)(x - \theta 2^{-j} z) d\theta \right| dz \\
&\leq 2^{-j} \int_{\mathbb{R}^n} |\phi_0(z)||z| \big\|\nabla^2 G_t\big\|_1 dz \\
&\leq 2^{-j} \big\|\phi_0(z)|z|\big\|_1 \big\|\nabla^2 G_t\big\|_1. \quad \text{(D.2)}
\end{aligned}$$

We set $C_0 = \|\nabla\phi_0\|_1$, $C_1 = \|\phi_0(z)|z|\|_1$. Then the inequalities (D.1), (D.2) and $\|G_t\|_1 = 1$ yield

$$\|\phi_j * \nabla G_t\|_1 \leq \begin{cases} C_0 2^j, \\ C_2 2^{-j} t^{-1}. \end{cases}$$

Here, $C_2 = C_1\|\nabla^2 G_t\|_1 t$ is independent of t. Thus we get for any $N \in \mathbb{Z}$

$$\begin{aligned}
\|\nabla G_t(x)\|_{\dot{B}^0_{1,1}(\mathbb{R}^n)} &= \sum_{j=-\infty}^{\infty} \|\phi_j * \nabla G_t(x)\|_1 \\
&= \Big(\sum_{j=-\infty}^{N} + \sum_{j=N}^{\infty}\Big)\|\phi_j * \nabla G_t(x)\|_1 \\
&\leq C_0 \sum_{j=-\infty}^{N} 2^j + C_2 t^{-1} \sum_{j=N}^{\infty} 2^{-j} \\
&= C_0 2^{N+1} + C_2 2^{-N} t^{-1}.
\end{aligned}$$

Taking $N \in \mathbb{Z}$ such that $(C_2/C)t^{-1/2} \leq 2^N \leq (1/2C_0)t^{-1/2}$, we obtain the desired result.

(2) This is a direct consequence of (1) and Lemma D.1. □

The above estimates can be generalized to the fractional power of the Laplacian.

Lemma D.3 *Let* $0 < \alpha \leq 1$. *Then there exists a constant* $C_\alpha = C(\alpha) > 0$ *such that*

(1) $\|(-\Delta)^\alpha G_t\|_{\dot{B}^0_{1,1}(\mathbb{R}^n)} \leq C_\alpha t^{-\alpha}$ *for* $t > 0$,

(2) $\|(-\Delta)^\alpha \exp(t\Delta) f\|_{\dot{B}^0_{\infty,1}(\mathbb{R}^n)} \leq C_\alpha t^{-\alpha} \|f\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^n)}$ *for* $t > 0$, $f \in \dot{B}^0_{\infty,\infty}(\mathbb{R}^n)$.

Proof. Since the estimate (2) is a direct consequence of (1) and Lemma D.1, we show only (1). Setting $x = t^{1/2} z$, it is easy to see that

$$((-\Delta)^\alpha G_t)(x) = t^{-(n/2)-\alpha}((-\Delta)^\alpha G_1)(z) \quad \text{for } t > 0. \tag{D.3}$$

Hence, it is sufficient to show only the case $t = 1$. In fact, by scaling invariance $\|f(\lambda\cdot)\|_{\dot{B}^0_{1,1}(\mathbb{R}^n)} \approx \lambda^{-n}\|f\|_{\dot{B}^0_{1,1}(\mathbb{R}^n)}$ for $\lambda > 0$ we get

$$\|((-\Delta)^\alpha G_t)(x)\|_{\dot{B}^0_{1,1}} = t^{-(n/2)-\alpha}\|((-\Delta)^\alpha G_1)(t^{-1/2}x)\|_{\dot{B}^0_{1,1}}$$

$$\le Ct^{-(n/2)-\alpha}t^{n/2}\|(-\Delta)^\alpha G_1\|_{\dot{B}^0_{1,1}}$$
$$\le C_\alpha t^{-\alpha}.$$

For any fixed $j \in \mathbb{Z}$ one sees that

$$\phi_j * (-\Delta)^\alpha G_1 = \phi_j * (\mathcal{F}^{-1}(|\xi|^{2\alpha}\widehat{G_1})) = \mathcal{F}^{-1}(\widehat{\phi_j}|\xi|^{2\alpha}\widehat{G_1}).$$

Since $\widehat{\phi_j}(\xi) = \widehat{\phi_0}(2^{-j}\xi)$, we continue

$$\begin{aligned}\phi_j * (-\Delta)^\alpha G_1 &= \int e^{ix\cdot\xi}\widehat{\phi_0}(2^{-j}\xi)|\xi|^{2\alpha}\widehat{G_1}(\xi)d\xi \\ &= \int e^{ix\cdot 2^j\xi}\widehat{\phi_0}(\xi)|2^j\xi|^{2\alpha}\widehat{G_1}(2^j\xi)2^{jn}d\xi \\ &= 2^{jn+j2\alpha}\int e^{i2^jx\cdot\xi}|\xi|^{2\alpha}\widehat{\phi_0}(\xi)\widehat{G_1}(2^j\xi)d\xi \\ &= 2^{jn+j2\alpha}[\mathcal{F}^{-1}(|\xi|^{2\alpha}\widehat{\phi_0}(\xi)\widehat{G_1}(2^j\xi))](2^jx) \\ &= 2^{jn+j2\alpha}[\mathcal{F}^{-1}(|\xi|^{2\alpha}\widehat{\phi_0}(\xi)) * \mathcal{F}^{-1}(\widehat{G_1}(2^j\xi))](2^jx).\end{aligned}$$

It follows from

$$\mathcal{F}^{-1}(\widehat{G_1}(2^j\xi)) = \mathcal{F}^{-1}\left(\frac{1}{2^{jn}}\left[\mathcal{F}\left(G_1\left(\frac{x}{2^j}\right)\right)\right](\xi)\right) = \frac{1}{2^{jn}}G_1\left(\frac{x}{2^j}\right)$$

and $\mathcal{F}^{-1}(|\xi|^{2\alpha}\widehat{\phi_0}(\xi)) = (-\Delta)^\alpha\phi_0$ that

$$\phi_j * (-\Delta)^\alpha G_1 = 2^{j2\alpha}\left[(-\Delta)^\alpha\phi_0 * G_1\left(\frac{\cdot}{2^j}\right)\right](2^jx). \tag{D.4}$$

Hence Young's inequality yields

$$\begin{aligned}\|\phi_j * (-\Delta)^\alpha G_1\|_1 &= 2^{j2\alpha}\left\|\left[(-\Delta)^\alpha\phi_0 * G_1\left(\frac{\cdot}{2^j}\right)\right](2^jx)\right\|_1 \\ &= 2^{j2\alpha-jn}\left\|\left[(-\Delta)^\alpha\phi_0 * G_1\left(\frac{\cdot}{2^j}\right)\right](x)\right\|_1 \\ &\le 2^{j2\alpha-jn}\|(-\Delta)^\alpha\phi_0\|_1\left\|G_1\left(\frac{\cdot}{2^j}\right)\right\|_1 \\ &= 2^{j2\alpha}\|(-\Delta)^\alpha\phi_0\|_1\|G_1\|_1 \\ &\le C_\alpha 2^{j2\alpha}.\end{aligned} \tag{D.5}$$

Here we used $\|G_1\|_1 = 1$ and $\|(-\Delta)^\alpha\phi_0\|_1 = \|F^{-1}(|\xi|^{2\alpha}\widehat{\phi_0})\|_1 \le C_\alpha$ because

$|\xi|^{2\alpha}\widehat{\phi_0} \in \mathcal{S}$. We next shift $(-\Delta)^\alpha$ to $G_1(\,\cdot\,/2^j)$ in RHS of (D.4) to get

$$\begin{aligned}\phi_j * (-\Delta)^\alpha G_1 &= 2^{j2\alpha}\left[\phi_0 * (-\Delta)^\alpha G_1\Big(\frac{\cdot}{2^j}\Big)\right](2^j x)\\ &= 2^{j2\alpha}\left[\big((-\Delta)^{-\beta}\phi_0\big) * (-\Delta)^{\alpha+\beta} G_1\Big(\frac{\cdot}{2^j}\Big)\right](2^j x).\end{aligned}$$

Here we set $(-\Delta)^{\beta-\beta} = \mathbf{I}$ for $\beta > 0$. Then by Young's inequality for any fixed $j \in \mathbb{N}$ we have

$$\begin{aligned}&\|\phi_j * (-\Delta)^\alpha G_1\|_1\\ &= 2^{j2\alpha}\left\|\left[\big((-\Delta)^{-\beta}\phi_0\big) * (-\Delta)^{\alpha+\beta} G_1\Big(\frac{\cdot}{2^j}\Big)\right](2^j x)\right\|_1\\ &= 2^{j2\alpha-jn}\left\|\left[\big((-\Delta)^{-\beta}\phi_0\big) * (-\Delta)^{\alpha+\beta} G_1\Big(\frac{\cdot}{2^j}\Big)\right](x)\right\|_1\\ &\le 2^{j2\alpha-jn}\|(-\Delta)^{-\beta}\phi_0\|_1\left\|(-\Delta)^{\alpha+\beta}\left(G_1\Big(\frac{\cdot}{2^j}\Big)\right)\right\|_1.\end{aligned}$$

Noting that $(-\Delta)^\gamma(G_1(\,\cdot\,/a)) = a^{-2\gamma}((-\Delta)^\gamma G_1)(\,\cdot\,/a)$ for $a > 0$, $\gamma > 0$, and $\|(-\Delta)^{-\beta}\phi_0\|_1 = \|F^{-1}(|\xi|^{-2\beta}\widehat{\phi_0})\|_1 \le C_\beta$ for some $C_\beta > 0$ because $|\xi|^{-2\beta}\widehat{\phi_0} \in \mathcal{S}$ we continue

$$\begin{aligned}\|\phi_j * (-\Delta)^\alpha G_1\|_1 &\le C_\beta 2^{j2\alpha-jn}\left\|2^{-2(\alpha+\beta)j}\big((-\Delta)^{\alpha+\beta}G_1\big)\Big(\frac{x}{2^j}\Big)\right\|_1\\ &\le C_\beta 2^{j2\alpha-jn}2^{-2(\alpha+\beta)j}2^{jn}\left\|\big((-\Delta)^{\alpha+\beta}G_1\big)\Big(\frac{x}{2^j}\Big)\right\|_1\\ &= C_\beta 2^{-2\beta j}\big\|\big((-\Delta)^{\alpha+\beta}G_1\big)(x)\big\|_1.\end{aligned}$$

Since

$$\|(-\Delta)^\gamma G_1\|_1 \le C_\gamma \quad \text{for } \gamma > 0,$$

we get

$$\|\phi_j * (-\Delta)^\alpha G_1\|_1 \le C_{\alpha,\beta} 2^{-2\beta j}. \tag{D.6}$$

Fix $\beta > 0$ to get from (D.5) and (D.6) that

$$\|(-\Delta)^\alpha G_t(x)\|_{\dot{B}^0_{1,1}(\mathbb{R}^n)} \le \sum_{j\le 0} C_\alpha 2^{j2\alpha} + \sum_{j>0} C_\alpha 2^{-j2\beta} \le C_\alpha.$$

□

Acknowledgment This work is partly supported by the Grant-in-Aid fund of COE "Mathematics of Nonlinear Structure via Singularities". The work of the first author is partly supported by the Grant-in-Aid for Scientific Research, No. 14204011, the Japan Society of the Promotion of Science. Much of the work was done when the third author visited the Mathematics Department of Hokkaido University in January-February 2004. Its hospitality is gratefully acknowledged, as is support from the AFOSR contract FG9620-02-1-0026. The second author would like to thank Dr. Okihiro Sawada and Mr. Hironobu Sasaki for giving him instruction on Besov spaces. The authors are grateful to anonymous referee for valuable suggestions.

References

[1] Amann H., *Operator-Valued Fourier Multipliers, Vector-Valued Besov Spaces, and Applications*. Math. Nachr. **186** (1997), 5–56.

[2] Babin A., Mahalov A. and Nicolaenko B., *3D Navier-Stokes and Euler Equations with initial data characterized by uniformly large vorticity*. Indiana University Mathematics Journal **50** (2001), 1–35.

[3] Babin A., Mahalov A. and Nicolaenko B., *Global regularity of the 3D Rotating Navier-Stokes Equations for resonant domains*. Indiana University Mathematics Journal (3) **48** (1999), 1133–1176.

[4] Babin A., Mahalov A. and Nicolaenko B., *Regularity and integrability of the 3D Euler and Navier-Stokes Equations for uniformly rotating fluids*. Asympt. Anal. (2) **15** (1997), 103–150.

[5] Bergh J. and Löfström J., *Interpolation Spaces An Introduction*. Springer-Verlag, 1976.

[6] Cannon J.R. and Knightly G.H., *A note on the Cauchy problem for the Navier-Stokes equations*. SIAM J. Appl. Math. **18** (1970), 641–644.

[7] Cannone M., *Ondelettes, Praproduits et Navier-Stokes, Diderot Editeur*. Arts et Sciences Paris-New York-Amsterdam, 1995.

[8] Cannone M. and Meyer Y., *Littlewood-Paley decomposition and Navier-Stokes equations*. Methods and Applications of Analysis **2** (1995), 307–319.

[9] Carpio A., *Large-time behaviour in incompressible Navier-Stokes equations*. SIAM J. Math. Anal. **27** (1996), 449–475.

[10] Foias C., Manley O., Rosa R. and Temam R., *Navier-Stokes Equations and Turbulence*. Encyclopedia of Mathematics and Its Applications, vol. 83, Cambridge University Press, 2001.

[11] García-Cuerva J. and Rubio De Francia J.L., *Weighted Norm Inequalities and Related Topics*. North-Holland, 1985.

[12] Giga Y., Inui K. and Matsui S., *On the Cauchy problem for the Navier-Stokes equations with nondecaying initial data*. Quaderni di Matematika **4** (1999), 28–68.

[13] Giga Y., Inui K., Mahalov A. and Matsui S., *Uniform local solvability for the Navier-Stokes equations with the Coriolis force.* to appear in Methods and Applications of Analysis.

[14] Giga Y., Matsui S. and Sawada O., *Global existence of two-dimensional Navier-Stokes flow with nondecaying initial velocity.* J. Math. Fluid Mech. **3** (2001), 302–315.

[15] Giga Y., Inui K., Kato J. and Matsui S., *Remarks on the uniqueness of bounded solutions of the Navier-Stokes equations.* Nonlinear Analysis **47** (2001), 4151–4156.

[16] Gradshtein I.S. and Ryzhik I.M., *Table of Integrals, Series, and Products.* fifth edition, Academic Press, 1994.

[17] Hieber M. and Sawada O., *The Navier-Stokes equations in* $\mathbb{R}^n$ *with linearly growing initial data.* Arch. Rational Mech. Anal. (2) **175** (2005), 269–285.

[18] Hounie J., *A note on the Mihlin-Hormander multiplier theorem.* preprint, 2003.

[19] Kato J., *The uniqueness of nondecaying solutions for the Navier-Stokes equations.* Arch. Rational Mech. Anal. **169** (2003), 159–175.

[20] Koch H. and Tataru D., *Well-posedness for the Navier-Stokes equations.* Advances in Mathematics **157** (2001), 22–35.

[21] Mahalov A. and Nicolaenko B., *Global regularity of the 3D Navier-Stokes equations with weakly aligned large initial vorticity.* Russian Math. Surveys (2) (350) **58** (2003), 287–318.

[22] Mahalov A., Nicolaenko B., Bardos C. and Golse F., *Regularity of Euler Equations for a Class of Three-Dimensional Initial Data.* Special Volume on Trends in Partial Differential Equations of Mathematical Physics, Nonlinear Analysis Series, vol. 61, Birkhauser-Verlag, 2004, pp. 161–185.

[23] Mahalov A., *The instability of rotating fluid columns subjected to a weak external Coriolis force.* Physics of Fluids A (4) **5** (1993), 891–900.

[24] Mahalov A. and Zhou Y., *Analytical and phenomenological studies of rotating turbulence.* Phys. of Fluids (8) **8** (1996), 2138–2152.

[25] Poincaré H., *Sur la précession des corps déformables.* Bull. Astronomique **27** (1910), 321–356.

[26] Sawada O., *The Navier-Stokes flow with linearly growing initial velocity in the whole space.* Bol. Soc. Paran. Mat. **22** (2004), 75–96.

[27] Sawada O. and Taniuchi Y., *On Boussinesq flow with non-decaying initial data.* Funkcial. Ekvac. (2) **47** (2004), 225–250.

[28] Sobolev S.L., *Ob odnoi novoi zadache matematicheskoi fiziki.* Izvestiia Akademii Nauk SSSR, Ser. Matematicheskaia (1) **18** (1954), 3–50.

[29] Stein E.M., *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals.* Princeton University Press, 1993.

[30] Taniuchi Y., *Remarks on global solvability of 2-D Boussinsq equations with non-decaying initial data.* preprint.

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