Spacelike CMC 1 surfaces
with elliptic ends in de Sitter 3-Space

(Dedicated to Professor Takeshi Sasaki on the occasion of his sixtieth birthday)

Shoichi Fujimori

Abstract. We show that an Osserman-type inequality holds for spacelike surfaces of constant mean curvature 1 with singularities and with elliptic ends in de Sitter 3-space. An immersed end of a constant mean curvature 1 surface is an "elliptic end" if the monodromy representation at the end is diagonalizable with eigenvalues in the unit circle. We also give a necessary and sufficient condition for equality in the inequality to hold, and in the process of doing this we derive a condition for determining when elliptic ends are embedded.

Key words: de Sitter 3-space, spacelike CMC 1 surface, admissible singularities.

Introduction

It is known that there is a representation formula, using holomorphic null immersions into $SL(2, \mathbb{C})$, for spacelike constant mean curvature (CMC) 1 immersions in de Sitter 3-space $\mathbb{S}_1^3$ [AA]. Although this formula is very similar to a representation formula for CMC 1 immersions in hyperbolic 3-space $\mathbb{H}^3$ (the so-called Bryant representation formula [B, UY1]), and the global properties of CMC 1 immersions in $\mathbb{H}^3$ have been investigated [CHR, RUY1, RUY2, UY1, UY2, UY3, Yu], global properties and singularities of spacelike CMC 1 immersions in $\mathbb{S}_1^3$ are not yet well understood. One of the biggest reasons for this is that the only complete spacelike CMC 1 immersion in $\mathbb{S}_1^3$ is the flat totally umbilic immersion [Ak, R]. This situation is somewhat parallel to the relation between minimal immersions in Euclicean 3-space $\mathbb{R}^3$ and spacelike maximal immersions in Lorentz 3-space $\mathbb{R}_1^3$, the $\mathbb{R}^3$ case having been extensively studied while the $\mathbb{R}_1^3$ case still being not well understood. Recently, Umehara and Yamada defined spacelike maximal surfaces with certain kinds of "admissible" singularities, and named them "maxfaces" [UY4]. They then showed that maxfaces are rich objects with respect to global geometry. So in Section 1, we introduce

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admissible singularities of spacelike CMC 1 surfaces in $\mathbb{S}_1^3$ and name these surfaces with admissible singularities "CMC 1 faces".

For CMC 1 immersions in $\mathbb{H}^3$, the monodromy representation at each end is always diagonalizable with eigenvalues in $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, but this is not true for CMC 1 faces in general. However, when the monodromy representation at each end of a CMC 1 face is diagonalizable with eigenvalues in $\mathbb{S}^1$, we can directly apply many results for CMC 1 immersions in $\mathbb{H}^3$ to CMC 1 faces. So in Section 2, we give a definition of "elliptic ends" of CMC 1 faces. In Section 3, we give a necessary and sufficient condition for elliptic ends to be embedded.

The total curvature of complete minimal immersions in $\mathbb{R}^3$ of finite total curvature satisfies the Osserman inequality [O1, Theorem 3.2]. Furthermore, the condition for equality was given in [JM, Theorem 4]. This inequality is a stronger version of the Cohn-Vossen inequality. For a minimal immersion, the total curvature is equal to the degree of its Gauss map, multiplied by -4π. So the Osserman inequality can be viewed as an inequality about the degree of the Gauss map. In the case of a CMC 1 immersion in $\mathbb{H}^3$, the total curvature never satisfies equality of the Cohn-Vossen inequality [UY1, Theorem 4.3], and the Osserman inequality does not hold in general. However, using the hyperbolic Gauss map instead of the total curvature, an Osserman type inequality holds for CMC 1 immersions in $\mathbb{H}^3$ [UY2]. Also, Umehara and Yamada showed that the Osserman inequality holds for maxfaces [UY4]. In Section 4, we give the central result of this paper, that the Osserman inequality holds for complete CMC 1 faces of finite type with elliptic ends (Theorem 4.7). *We remark that the assumptions of finite type and ellipticity of the ends can actually be removed, because, in fact, any complete CMC 1 face must be of finite type. This deep result will be shown in the forthcoming paper [FRUYY].*

Lee and Yang were the first to construct a numerous collection of examples of CMC 1 faces [LY]. Furthermore, they constructed complete irreducible CMC 1 faces with three elliptic ends, by using hypergeometric functions [LY]. We will also give numerous examples here in Section 5, by using a method for transferring CMC 1 immersions in $\mathbb{H}^3$ to CMC 1 faces in $\mathbb{S}_1^3$. Applying this method to examples in [MU, RUY1, RUY2], we give many examples of complete reducible CMC 1 faces of finite type with elliptic ends.

We remark that the generic singularities of maxfaces in $\mathbb{R}_1^3$ and CMC 1

faces in $\mathbb{S}_1^3$ are investigated in [FSUY].

1. CMC 1 faces

Let $\mathbb{R}_1^4$ be the 4-dimensional Lorentz space with the Lorentz metric

$$\langle (x_0,\, x_1,\, x_2,\, x_3),\, (y_0,\, y_1,\, y_2,\, y_3) \rangle = -x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3.$$

Then de Sitter 3-space is

$$\mathbb{S}_1^3 = \mathbb{S}_1^3(1) = \{(x_0,\, x_1,\, x_2,\, x_3) \in \mathbb{R}_1^4 \mid -x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\},$$

with metric induced from $\mathbb{R}_1^4$. $\mathbb{S}_1^3$ is a simply-connected 3-dimensional Lorentzian manifold with constant sectional curvature 1. We can consider $\mathbb{R}_1^4$ to be the 2×2 self-adjoint matrices ($X^* = X$, where $X^* = {}^t\overline{X}$ and ${}^t X$ denotes the transpose of X), by the identification

$$\mathbb{R}_1^4 \ni X = (x_0,\, x_1,\, x_2,\, x_3) \leftrightarrow X = \sum_{k=0}^{3} x_k e_k = \begin{pmatrix} x_0 + x_3 & x_1 + i x_2 \\ x_1 - i x_2 & x_0 - x_3 \end{pmatrix},$$

where

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $\mathbb{S}_1^3$ is

$$\mathbb{S}_1^3 = \{X \mid X^* = X,\, \det X = -1\} = \{F e_3 F^* \mid F \in SL(2,\, \mathbb{C})\}$$

with the metric

$$\langle X,\, Y \rangle = -\frac{1}{2} \operatorname{trace} \left(X e_2 ({}^t Y) e_2 \right).$$

In particular, $\langle X,\, X \rangle = -\det X$. An immersion in $\mathbb{S}_1^3$ is called *spacelike* if the induced metric on the immersed surface is positive definite.

Aiyama and Akutagawa [AA] showed the following representation formula for simply-connected spacelike CMC 1 immersions.

Theorem 1.1 The representation of Aiyama-Akutagawa *Let D be a simply-connected domain in $\mathbb{C}$ with a base point $z_0 \in D$. Let $g\colon D \to (\mathbb{C} \cup \{\infty\}) \setminus \{z \in \mathbb{C} \mid |z| \le 1\}$ be a meromorphic function and ω a holomorphic 1-form on D such that*

$$d\hat{s}^2 = (1 + |g|^2)^2 \omega \bar{\omega} \tag{1.1}$$

is a Riemannian metric on D. Choose the holomorphic immersion $F = (F_{jk})\colon D \to SL(2, \mathbb{C})$ so that $F(z_0) = e_0$ and F satisfies

$$F^{-1}dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega. \tag{1.2}$$

Then $f\colon D \to \mathbb{S}^3_1$ defined by

$$f = Fe_3F^* \tag{1.3}$$

is a conformal spacelike CMC 1 immersion. The induced metric $ds^2 = f^*(ds^2_{\mathbb{S}^3_1})$ on D, the second fundamental form h, and the hyperbolic Gauss map G of f are given as follows:

$$ds^2 = (1 - |g|^2)^2 \omega\bar\omega, \ \ h = Q + \overline{Q} + ds^2, \ \ G = \frac{dF_{11}}{dF_{21}} = \frac{dF_{12}}{dF_{22}}, \quad (1.4)$$

where $Q = \omega dg$ is the Hopf differential of f.

Conversely, any simply-connected spacelike CMC 1 immersion can be represented this way.

Remark 1.2 We make the following remarks about Theorem 1.1:

(1) Following the terminology of Umehara and Yamada, g is called the *secondary* Gauss map. The pair (g, ω) is called the *Weierstrass data*.

(2) The oriented unit normal vector field N of f is given as

$$N = \frac{1}{|g|^2 - 1} F \begin{pmatrix} |g|^2 + 1 & 2g \\ 2\bar g & |g|^2 + 1 \end{pmatrix} F^*,$$

which is a future pointing timelike vector field (See [KY]).

(3) The hyperbolic Gauss map has the following geometric meaning: Let $\mathbb{S}^2_\infty$ be the future pointing ideal boundary of $\mathbb{S}^3_1$. Then $\mathbb{S}^2_\infty$ is identified with the Riemann sphere $\mathbb{C} \cup \{\infty\}$ in the standard way. Let γ_z be the geodesic ray starting at $f(z)$ in $\mathbb{S}^3_1$ with the initial velocity vector $N(z)$, the oriented unit normal vector of $f(D)$ at $f(z)$. Then $G(z)$ is the point $\mathbb{S}^2_\infty$ determined by the asymptotic class of γ_z. See [B, UY1, UY3].

(4) For the holomorphic immersion F satisfying Equation (1.2), $\hat f := FF^*\colon D \to \mathbb{H}^3$ is a conformal CMC 1 immersion with first fundamental form $\hat f^*(ds^2_{\mathbb{H}^3}) = d\hat s^2$ as in Equation (1.1), and with second fundamental form $\hat h = -\hat Q - \overline{\hat Q} + d\hat s^2$. The Hopf differential $\hat Q$ and the hyperbolic Gauss map $\hat G$ of $\hat f$ are the same as Q and G, since f and

$\hat{f}$ are both constructed from the same F, and F determines Q and G, by Equations (1.2) and (1.4).

(5)   By Equation (2.6) in [UY1], G and g and Q have the following relation:

$$2Q = S(g) - S(G),$$

where $S(g) = S_z(g)dz^2$ and

$$S_z(g) = \left(\frac{g''}{g'}\right)' - \frac{1}{2}\left(\frac{g''}{g'}\right)^2 \quad (' = d/dz)$$

is the Schwarzian derivative of g.

Our first task is to extend the above theorem to non-simply-connected CMC 1 surfaces with singularities, along the same lines as in [UY4]. To do this, we first define admissible singularities.

Definition 1.3   Let M be an oriented 2-manifold. A smooth (that is, C^∞) map $f\colon M \to \mathbb{S}_1^3$ is called a *CMC* 1 *map* if there exists an open dense subset $W \subset M$ such that $f|_W$ is a spacelike CMC 1 immersion. A point $p \in M$ is called a *singular point* of f if the induced metric ds^2 is degenerate at p.

Definition 1.4   Let $f\colon M \to \mathbb{S}_1^3$ be a CMC 1 map and $W \subset M$ an open dense subset such that $f|_W$ is a CMC 1 immersion. A singular point $p \in M \setminus W$ is called an *admissible* singular point if
(1)   there exists a C^1-differentiable function $\beta\colon U \cap W \to \mathbb{R}^+$, where U is a neighborhood of p, such that βds^2 extends to a C^1-differentiable Riemannian metric on U, and
(2) $df(p) \neq 0$, that is, df has rank 1 at p.
We call a CMC 1 map f a *CMC* 1 *face* if each singular point is admissible.

To extend Theorem 1.1 to CMC 1 faces that are not simply-connected, we prepare two propositions. First, we prove the following proposition:

Proposition 1.5   *Let M be an oriented* 2*-manifold and $f\colon M \to \mathbb{S}_1^3$ a CMC* 1 *face, where $W \subset M$ an open dense subset such that $f|_W$ is a CMC* 1 *immersion. Then there exists a unique complex structure J on M such that*
(1)   *$f|_W$ is conformal with respect to J, and*
(2)   *there exists an immersion $F\colon \widetilde{M} \to SL(2,\mathbb{C})$ which is holomorphic*

with respect to J such that

$$\det(dF) = 0 \quad and \quad f \circ \varrho = Fe_3 F^*,$$

where $\varrho \colon \widetilde{M} \to M$ is the universal cover of M.
This F is called a holomorphic null lift *of f.*

Remark 1.6 The holomorphic null lift F of f is unique up to right multiplication by a constant matrix in $SU(1, 1)$. See also Remark 1.11 below.

Proof of Proposition 1.5.   Since the induced metric ds^2 gives a Riemannian metric on W, it induces a complex structure J_0 on W. Let p be an admissible singular point of f and U a local simply-connected neighborhood of p. Then by definition, there exists a C^1-differentiable function $\beta \colon U \cap W \to \mathbb{R}^+$ such that βds^2 extends to a C^1-differentiable Riemannian metric on U. Then there exists a positively oriented orthonormal frame field $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ with respect to βds^2 which is C^1-differentiable on U. Using this, we can define a C^1-differentiable almost complex structure J on U such that

$$J(\boldsymbol{v}_1) = \boldsymbol{v}_2 \quad and \quad J(\boldsymbol{v}_2) = -\boldsymbol{v}_1. \tag{1.5}$$

Since ds^2 is conformal to βds^2 on $U \cap W$, J is compatible with J_0 on $U \cap W$. There exists a C^1-differentiable decomposition

$$\Gamma\big(T^*M^{\mathbb{C}} \otimes \mathfrak{sl}(2, \mathbb{C})\big)$$
$$= \Gamma\big(T^*M^{(1,0)} \otimes \mathfrak{sl}(2, \mathbb{C})\big) \oplus \Gamma\big(T^*M^{(0,1)} \otimes \mathfrak{sl}(2, \mathbb{C})\big) \tag{1.6}$$

with respect to J, where $\Gamma(E)$ denotes the sections of a vector bundle E on U. Since f is smooth, $df \cdot f^{-1}$ is a smooth $\mathfrak{sl}(2, \mathbb{C})$-valued 1-form. We can take the $(1, 0)$-part ζ of $df \cdot f^{-1}$ with respect to this decomposition. Then ζ is a C^1-differentiable $\mathfrak{sl}(2, \mathbb{C})$-valued 1-form which is holomorphic on $U \cap W$ with respect to the equivalent complex structures J_0 and J (which follows from the fact that $f|_W$ is a CMC 1 immersion, so the hyperbolic Gauss map G of f is holomorphic on W, which is equivalent to the holomorphicity of ζ with respect to J and J_0 on $U \cap W$). Hence $d\zeta \equiv 0$ on $U \cap W$. Moreover, since W is an open dense subset and ζ is C^1-differentiable on U, $d\zeta \equiv 0$ on U. Similarly, $\zeta \wedge \zeta \equiv 0$ on U. In particular,

$$d\zeta + \zeta \wedge \zeta = 0. \tag{1.7}$$

As U is simply-connected, the existence of a C^1-differentiable map $F_U =$

$(F_{jk})_{j,k=1,2} \colon U \to SL(2, \mathbb{C})$ such that $dF_U \cdot F_U^{-1} = \zeta$ is equivalent to the condition (1.7). Hence such an F exists.

Note that since f takes Hermitian matrix values, we have $df = \zeta f + (\zeta f)^*$. So $df(p) \neq 0$ (that is, p is an admissible singularity) implies $\zeta \neq 0$. Then at least one entry dF_{jk} of dF_U does not vanish at p. Using this F_{jk}, we define the function $z = F_{jk} \colon U \to \mathbb{C} = \mathbb{R}^2$. Then, z gives a coordinate system on U. Since $z = F_{jk}$ is a holomorphic function on $U \cap W$, it gives a complex analytic coordinate around p compatible with respect to that of $U \cap W$. The other entries of F_U are holomorphic functions with respect to z on $U \cap W$ and are C^1-differentials on U, so each entry of F_U is holomorphic with respect to z on U, by the Cauchy-Riemann equations. Since p is an arbitrary fixed admissible singularity, the complex structure of W extends across each singular point p.

This complex structure can be seen to be well-defined at singular points as follows: Let $p' \in M \setminus W$ be another singular point and U' a neighborhood of p' so that $U \cap U' \neq \emptyset$. Then by the same argument as above, there exists a C^1-differentiable almost complex structure J' on U' and C^1-differentiable map $F'_{U'} = (F'_{j'k'})_{j',k'=1,2} \colon U' \to SL(2, \mathbb{C})$ such that $dF'_{U'} \cdot F'^{-1}_{U'}$ is the $(1, 0)$-part of $df \cdot f^{-1}$ with respect to Equation (1.6). Define $z' = F'_{j'k'}$ so that $dF'_{j'k'} \neq 0$. Then by uniqueness of ordinary differential equations, $F_U = F'_{U'} A$ for some constant matrix A. So z and z' are linearly related, and hence they are holomorphically related. Also, because dz and dz' are nonzero, we have $dz/dz' \neq 0$ on $U \cap U'$.

For local coordinates z on M compatible with J, $\partial f \cdot f^{-1} := (f_z dz) \cdot f^{-1}$ (which is equal to ζ) is holomorphic on M and there exists a holomorphic map $F \colon \widetilde{M} \to SL(2, \mathbb{C})$ such that

$$dF \cdot F^{-1} = \partial f \cdot f^{-1}. \tag{1.8}$$

Since $\partial f \cdot f^{-1} \neq 0$, also $dF \neq 0$, and hence F is an immersion. Also, since f is conformal, $0 = \langle \partial f, \partial f \rangle = -\det(\partial f)$. Thus $\det(dF) = 0$.

Finally, we set $\hat{f} = Fe_3F^*$, defined on $\widetilde{M}$. We consider some simply-connected region $V \subset W$. By Theorem 1.1, there exists a holomorphic null lift $\hat{F}$ of f,

$$f = \hat{F}e_3\hat{F}^*, \tag{1.9}$$

defined on that same V. Then by Equations (1.8) and (1.9), we have

$$d\hat{F} \cdot \hat{F}^{-1} = dF \cdot F^{-1},$$

and hence $\hat{F} = FB$ for some constant $B \in SL(2, \mathbb{C})$. We are free to choose the solution F of Equation (1.8) so that $B = e_0$, that is, $\hat{F} = F$, so $f = \hat{f}$ on V. By the holomorphicity of F, $\hat{f}$ is real analytic on $\widetilde{M}$. Also, $f \circ \varrho$ is real analytic on $\widetilde{M}$, by Equation (1.8) and the holomorphicity of F. Therefore $f \circ \varrho = \hat{f}$ on $\widetilde{M}$, proving the proposition. $\qquad\square$

By Proposition 1.5, the 2-manifold M on which a CMC 1 face $f \colon M \to \mathbb{S}_1^3$ is defined always has a complex structure. So throughout this paper, we will treat M as a Riemann surface with a complex structure induced as in Proposition 1.5.

The next proposition is the converse to Proposition 1.5:

Proposition 1.7   *Let M be a Riemann surface and $F \colon M \to SL(2, \mathbb{C})$ a holomorphic null immersion. Assume the symmetric $(0, 2)$-tensor*

$$\det[d(Fe_3F^*)] \tag{1.10}$$

is not identically zero. Then $f = Fe_3F^* \colon M \to \mathbb{S}_1^3$ is a CMC 1 face, and $p \in M$ is a singular point of f if and only if $\det[d(Fe_3F^*)]_p = 0$. Moreover,

$$-\det[d(FF^*)] \quad \text{is positive definite} \tag{1.11}$$

on M.

Proof.   Since (1.10) is not identically zero, the set

$$W := \{p \in M \mid \det[d(Fe_3F^*)]_p \neq 0\}$$

is open and dense in M. Since $F^{-1}dF$ is a $\mathfrak{sl}(2, \mathbb{C})$-valued 1-form, there exist holomorphic 1-forms a_1, a_2 and a_3 such that

$$F^{-1}dF = \begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix}.$$

Since F is a null immersion, that is, $\operatorname{rank}(dF) = 1$, a_j $(j = 1, 2, 3)$ satisfy

$$a_1^2 + a_2a_3 = 0 \quad \text{and} \quad |a_1|^2 + |a_2|^2 + |a_3|^2 > 0. \tag{1.12}$$

Since

$$d(Fe_3F^*) = F\left(F^{-1}dFe_3 + (F^{-1}dFe_3)^*\right)F^*$$

$$= F \begin{pmatrix} a_1 + \overline{a_1} & -a_2 + \overline{a_3} \\ a_3 - \overline{a_2} & a_1 + \overline{a_1} \end{pmatrix} F^*,$$

we have

$$
\begin{aligned}
-\det[d(Fe_3 F^*)] &= -2|a_1|^2 + |a_2|^2 + |a_3|^2 \\
&= -2|a_2 a_3| + |a_2|^2 + |a_3|^2 \\
&= |a_2|^2 - 2|a_2||a_3| + |a_3|^2 = (|a_2| - |a_3|)^2 \geq 0.
\end{aligned}
$$

So $-\det[d(Fe_3 F^*)]$ is positive definite on W. Set $f = Fe_3 F^*$. Then $f|_W \colon W \to \mathbb{S}_1^3$ determines a conformal immersion with induced metric

$$ds^2 = f^* ds_{\mathbb{S}_1^3}^2 = \langle df,\, df \rangle = -\det[d(Fe_3 F^*)].$$

Furthermore, f is CMC 1 by Theorem 1.1. Also, by (1.12),

$$-\det[d(FF^*)] = 2|a_1|^2 + |a_2|^2 + |a_3|^2$$

is positive definite on M. Thus if we set

$$\beta = \frac{\det[d(FF^*)]}{\det[d(Fe_3 F^*)]}$$

on W, β is a positive function on W such that

$$\beta ds^2 = -\det[d(FF^*)]$$

extends to a Riemannian metric on M. Also,

$$\partial f \cdot f^{-1} = dF \cdot F^{-1} = F(F^{-1}dF)F^{-1} \neq 0,$$

and so $df \neq 0$. This completes the proof. $\qquad\square$

Remark 1.8 Even if F is a holomorphic null immersion, $f = Fe_3 F^*$ might not be a CMC 1 face. For example, for the holomorphic null immersion

$$F \colon \mathbb{C} \ni z \mapsto \begin{pmatrix} z+1 & -z \\ z & -z+1 \end{pmatrix} \in SL(2,\, \mathbb{C}),$$

$f = Fe_3 F^*$ degenerates everywhere on $\mathbb{C}$. Note that $\det[d(Fe_3 F^*)]$ is identically zero here.

Using Propositions 1.5 and 1.7, we can now extend the representation of Aiyama-Akutagawa for simply-connected CMC 1 immersions to the case of CMC 1 faces with possibly non-simply-connected domains.

Theorem 1.9 *Let M be a Riemann surface with a base point $z_0 \in M$. Let g be a meromorphic function and ω a holomorphic 1-form on the universal cover $\widetilde{M}$ such that $d\hat{s}^2$ in* Equation (1.1) *is a Riemannian metric on $\widetilde{M}$ and $|g|$ is not identically 1. Choose the holomorphic immersion $F = (F_{jk}) \colon \widetilde{M} \to SL(2, \mathbb{C})$ so that $F(z_0) = e_0$ and F satisfies* Equation (1.2)*. Then $f \colon \widetilde{M} \to \mathbb{S}_1^3$ defined by* Equation (1.3) *is a CMC 1 face that is conformal away from its singularities. The induced metric ds^2 on M, the second fundamental form h, and the hyperbolic Gauss map G of f are given as in* Equation (1.4)*. The singularities of the CMC 1 face occur at points where $|g| = 1$.*

Conversely, let M be a Riemann surface and $f \colon M \to \mathbb{S}_1^3$ a CMC 1 face. Then there exists a meromorphic function g (so that $|g|$ is not identically 1) and holomorphic 1-form ω on $\widetilde{M}$ such that $d\hat{s}^2$ is a Riemannian metric on $\widetilde{M}$, and such that Equation (1.3) *holds, where $F \colon \widetilde{M} \to SL(2, \mathbb{C})$ is an immersion which satisfies* Equation (1.2)*.*

Proof. First we prove the first paragraph of the theorem. Since

$$d(F e_3 F^*) = F(F^{-1} dF \cdot e_3 + (F^{-1} dF \cdot e_3)^*) F^*,$$

we have

$$-\det[d(F e_3 F^*)] = (1 - |g|^2)^2 \omega \bar{\omega}.$$

Also, since $d\hat{s}^2$ gives a Riemannian metric on $\widetilde{M}$, ω has a zero of order k if and only if g has a pole of order $k/2 \in \mathbb{N}$. Therefore $\det[d(F e_3 F^*)] = 0$ if and only if $|g| = 1$. Hence by Proposition 1.7, $f = F e_3 F^* \colon \widetilde{M} \to \mathbb{S}_1^3$ is a CMC 1 face, and $p \in \widetilde{M}$ is a singular point of f if and only if $|g(p)| = 1$, proving the first half of the theorem.

We now prove the second paragraph of the theorem. By Proposition 1.5, there exists a holomorphic null lift $F \colon \widetilde{M} \to SL(2, \mathbb{C})$ of the CMC 1 face f. Then by the same argument as in the proof of Proposition 1.7, we may set

$$F^{-1} dF = \begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix},$$

where a_j ($j = 1, 2, 3$) are holomorphic 1-forms such that (1.12) holds. By changing F into FB for some constant $B \in SU(1, 1)$, if necessary, we may assume that a_3 is not identically zero. We set

$$\omega := a_3, \quad g := \frac{a_1}{a_3}.$$

Then ω is a holomorphic 1-form and g is a meromorphic function. Since $a_2/a_3 = a_2 a_3/a_3^2 = -(a_1/a_3)^2 = -g^2$, we see that Equation (1.2) holds. Since $|a_2| - |a_3|$ is not identically zero (by the same argument as in the proof of Proposition 1.7), $|g|$ is not identically one. Also, since $g^2\omega = a_2$ is holomorphic, (1.11) implies $-\det[d(FF^*)] = (1+|g|^2)^2\omega\bar\omega = d\hat s^2$ is positive definite, so $d\hat s^2$ gives a Riemannian metric on $\widetilde{M}$, proving the converse part of the theorem. $\qquad\square$

Remark 1.10 Since the hyperbolic Gauss map G has the geometric meaning explained in (3) of Remark 1.2, G is single-valued on M itself, although F might not be. We also note that, by Equation (1.4), the Hopf differential Q is single-valued on M as well.

Remark 1.11 Let F be a holomorphic null lift of a CMC 1 face f with Weierstrass data (g, ω). For any constant matrix

$$B = \begin{pmatrix} \bar p & -q \\ -\bar q & p \end{pmatrix} \in SU(1, 1), \quad p\bar p - q\bar q = 1, \tag{1.13}$$

FB is also a holomorphic null lift of f. The Weierstrass data $(\hat g, \hat\omega)$ corresponding to $(FB)^{-1}d(FB)$ is given by

$$\hat g = \frac{pg + q}{\bar q g + \bar p} \quad \text{and} \quad \hat\omega = (\bar q g + \bar p)^2\omega. \tag{1.14}$$

Two Weierstrass data (g, ω) and $(\hat g, \hat\omega)$ are called *equivalent* if they satisfy Equation (1.14) for some B as in Equation (1.13). See Equation (1.6) in [UY1]. We shall call the equivalence class of the Weierstrass data (g, ω) the Weierstrass data associated to f. When we wish to emphasis that (g, ω) is determined by F, not FB for some B, we call (g, ω) the Weierstrass data associated to F.

On the other hand, the Hopf differential Q and the hyperbolic Gauss map G are independent of the choice of F, because

$$\hat\omega d\hat g = \omega dg \quad \text{and} \quad \frac{\bar p dF_{11} + \bar q dF_{12}}{\bar p dF_{21} + \bar q dF_{22}} = \frac{dF_{11}}{dF_{21}}, \quad \text{where} \quad F = (F_{jk}).$$

This can also be seen from (3) of Remark 1.2, which implies that G is determined just by f. Then $S(g) = S(\hat g)$ and (5) of Remark 1.2 imply Q is independent of the choice of F as well.

2.   CMC 1 faces with elliptic ends

It is known that the only complete spacelike CMC 1 immersion is a flat totally umbilic immersion [Ak, R] (see Example 5.1 below). In the case of non-immersed CMC 1 faces in $\mathbb{S}_1^3$, we now define the notions of completeness and finiteness of total curvature away from singularities, like in [KUY, UY4].

Definition 2.1   Let M be a Riemann surface and $f\colon M \to \mathbb{S}_1^3$ a CMC 1 face. Set $ds^2 = f^*(ds_{\mathbb{S}_1^3}^2)$. f is *complete* (resp. of *finite type*) if there exists a compact set C and a symmetric $(0, 2)$-tensor T on M such that T vanishes on $M \setminus C$ and $ds^2 + T$ is a complete (resp. finite total curvature) Riemannian metric.

Remark 2.2   For CMC 1 immersions in $\mathbb{S}_1^3$, the Gauss curvature K is non-negative. So the total curvature is the same as the total absolute curvature. However, for CMC 1 faces with singular points the total curvature is never finite, not even on neighborhoods of singular points, as can be seen from the form $K = 4dgd\bar{g}/(1 - |g|^2)^4\omega\bar{\omega}$ for the Gaussian curvature, see also [ER]. Hence the phrase "finite type" is more appropriate in Definition 2.1.

Remark 2.3   The universal covering of a complete (resp. finite type) CMC 1 face might not be complete (resp. finite type), because the singular set might not be compact on the universal cover.

Let $f\colon M \to \mathbb{S}_1^3$ be a complete CMC 1 face of finite type. Then $(M, ds^2 + T)$ is a complete Riemannian manifold of finite total curvature. So by [H, Theorem 13], M has finite topology, where we define a manifold to be of finite topology if it is diffeomorphic to a compact manifold with finitely many points removed. The *ends* of f correspond to the removed points of that Riemann surface.

Let $\varrho\colon \widetilde{M} \to M$ be the universal cover of M, and $F\colon \widetilde{M} \to SL(2, \mathbb{C})$ a holomorphic null lift of a CMC 1 face $f\colon M \to \mathbb{S}_1^3$. We fix a point $z_0 \in M$. Let $\gamma\colon [0, 1] \to M$ be a loop so that $\gamma(0) = \gamma(1) = z_0$. Then there exists a unique deck transformation τ of $\widetilde{M}$ associated to the homotopy class of γ. We define the monodromy representation Φ_γ of F as

$$F \circ \tau = F\Phi_\gamma.$$

Note that $\Phi_\gamma \in SU(1, 1)$ for any loop γ, since f is well-defined on M. So Φ_γ is conjugate to either

$$\mathcal{E} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \quad \text{or} \quad \mathcal{H} = \pm \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix}$$

$$\text{or} \quad \mathcal{P} = \pm \begin{pmatrix} 1+i & 1 \\ 1 & 1-i \end{pmatrix} \quad (2.1)$$

for $\theta \in [0, 2\pi)$, $s \in \mathbb{R} \setminus \{0\}$.

Definition 2.4 Let $f \colon M \to \mathbb{S}_1^3$ be a complete CMC 1 face of finite type with holomorphic null lift F. An end of f is called an *elliptic end* or *hyperbolic end* or *parabolic end* if its monodromy representation is conjugate to $\mathcal{E}$ or $\mathcal{H}$ or $\mathcal{P}$ in $SU(1, 1)$, respectively.

Remark 2.5 A matrix

$$X = \begin{pmatrix} p & q \\ \bar{q} & \bar{p} \end{pmatrix} \in SU(1, 1)$$

acts on the hyperbolic plane in the Poincaré model $\mathbb{H}^2 = (\{w \in \mathbb{C} \mid |w| < 1\}, ds_{\mathbb{H}^2}^2 = 4dwd\bar{w}/(1 - |w|^2)^2)$ as an isometry:

$$\mathbb{H}^2 \ni w \mapsto \frac{pw + q}{\bar{q}w + \bar{p}} \in \mathbb{H}^2.$$

X is called *elliptic* if this action has only one fixed point which is in $\mathbb{H}^2$. X is called *hyperbolic* if there exist two fixed points, both in the ideal boundary $\partial\mathbb{H}^2$. X is called *parabolic* if there exists only one fixed point which is in $\partial\mathbb{H}^2$. This is what motivates the terminology in Definition 2.4.

Since any matrix in $SU(2)$ is conjugate to $\mathcal{E}$ in $SU(2)$, CMC 1 immersions in $\mathbb{H}^3$ and CMC 1 faces with elliptic ends in $\mathbb{S}_1^3$ share many analogous properties. So in this paper we consider CMC 1 faces with only elliptic ends. We leave the study of hyperbolic ends and parabolic ends for another occasion.

Proposition 2.6 *Let V be a neighborhood of an end of f and $f|_V$ a spacelike CMC 1 immersion of finite total curvature which is complete at the end. Suppose the end is elliptic. Then there exists a holomorphic null lift $F \colon \widetilde{V} \to SL(2, \mathbb{C})$ of f with Weierstrass data (g, ω) associated to F such that*

$$d\hat{s}^2|_V = (1 + |g|^2)^2 \omega\bar{\omega}$$

is single-valued on V. Moreover, $d\hat{s}^2|_V$ has finite total curvature and is complete at the end.

Proof. Let $\gamma \colon [0, 1] \to V$ be a loop around the end and τ the deck transformation associated to γ. Take a holomorphic null lift $F_0 \colon \widetilde{V} \to SL(2, \mathbb{C})$ of f. Then by definition of an elliptic end, there exists a $\theta \in [0, 2\pi)$ such that

$$F_0 \circ \tau = F_0 P E_\theta P^{-1},$$

where $P \in SU(1, 1)$ and

$$E_\theta = \mathrm{diag}(e^{i\theta}, e^{-i\theta}) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

Defining the holomorphic null lift $F = F_0 P$ of f and defining (g, ω) to be the Weierstrass data associated to F, we have

$$F \circ \tau = F E_\theta, \quad g \circ \tau = e^{-2i\theta} g \quad \text{and} \quad \omega \circ \tau = e^{2i\theta} \omega.$$

Thus, $|g \circ \tau| = |g|$ and $|\omega \circ \tau| = |\omega|$. This implies $d\hat{s}^2|_V$ is single-valued on V. Let T be a $(0, 2)$-tensor as in Definition 2.1. Then by Equation (1.4), we have $ds^2 + T \leq d\hat{s}^2|_V$ on $V \setminus C$. Thus, if $ds^2 + T$ is complete, $d\hat{s}^2|_V$ is also complete. We denote the Gaussian curvature of the metric $d\hat{s}^2|_V$ by $K_{d\hat{s}^2|_V}$ (note that $K_{d\hat{s}^2|_V}$ is non-positive). Then we have

$$(-K_{d\hat{s}^2|_V}) d\hat{s}^2|_V = \frac{4 dg d\bar{g}}{(1 + |g|^2)^2} \leq \frac{4 dg d\bar{g}}{(1 - |g|^2)^2} = K ds^2$$

on $V \setminus C$. Thus, if $ds^2 + T$ is of finite total curvature, the total curvature of $d\hat{s}^2|_V$ is finite, proving the proposition. $\square$

Proposition 2.7   *Let $f : M \to \mathbb{S}_1^3$ be a complete CMC 1 face of finite type with elliptic ends. Then there exists a compact Riemann surface $\overline{M}$ and finite number of points $p_1, \ldots, p_n \in \overline{M}$ so that M is biholomorphic to $\overline{M} \setminus \{p_1, \ldots, p_n\}$. Moreover, the Hopf differential Q of f extends meromorphically to $\overline{M}$.*

Proof.   Since f is of finite type, M is finitely connected, by [H, Theorem 13]. Consequently, there exists a compact region $M_0 \subset M$, bounded by a finite number of regular Jordan curves $\gamma_1, \ldots, \gamma_n$, such that each component M_j of $M \setminus M_0$ can be conformally mapped onto the annulus $D_j = \{z \in \mathbb{C} \mid r_j < |z| < 1\}$, where γ_j corresponds to the set $\{|z| = 1\}$. Then by Proposition 2.6, there exists $d\hat{s}^2|_{M_j}$ which is single-valued on M_j and is of finite total curvature and is complete at the end, and so that $K_{d\hat{s}^2|_{M_j}}$ is non-

positive. Therefore by [L, Proposition III-16] or [O2, Theorem 9.1], $r_j = 0$, and hence each M_j is biholomorphic to the punctured disk $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$. We can, using the biholomorphism from M_j to D_j, replace M_j in M with D_j without affecting the conformal structure of M. Thus, without loss of generality, $M = \overline{M} \setminus \{p_1, \ldots, p_n\}$ for some compact Riemann surface $\overline{M}$ and a finite number of points $p_1, \ldots, p_n$ in $\overline{M}$, and each M_j becomes a punctured disk about p_j. Hence, by (4) of Remark 1.2, we can apply [B, Proposition 5] to $\hat{f}_j := \hat{f}|_{M_j}$ to see that $Q = \hat{Q}$ extends meromorphically to $M_j \cup \{p_j\}$, proving the proposition. $\qquad\square$

Let $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ and $\Delta^* = \Delta \setminus \{0\}$. Let $f \colon \Delta^* \to \mathbb{S}_1^3$ be a conformal spacelike CMC 1 immersion of finite total curvature which has a complete elliptic end at the origin. Then we can take the Weierstrass data associated to f in the following form:

$$\begin{cases} g = z^\mu \tilde{g}(z), & \mu > 0, & \tilde{g}(0) \neq 0, \\ \omega = w(z)dz = z^\nu \tilde{w}(z)dz, & \nu \leq -1, & \tilde{w}(0) \neq 0, \end{cases} \tag{2.2}$$

where $\tilde{g}$ and $\tilde{w}$ are holomorphic functions on Δ and $\mu + \nu \in \mathbb{Z}$. (See [UY1] and also [B, Proposition 4] for case that $\mu, \nu \in \mathbb{R}$. Then applying a transformation as in Equation (1.14) if necessary, we may assume $\mu > 0$. Completeness of the end then gives $\nu \leq -1$.)

Definition 2.8 The end $z = 0$ of $f \colon \Delta^* \to \mathbb{S}_1^3$ is called *regular* if the hyperbolic Gauss map G extends meromorphically across the end. Otherwise, the end is called *irregular*.

Since Q extends meromorphically to each end, we have the following proposition, by (4) and (5) of Remark 1.2:

Proposition 2.9 ([B, Proposition 6]) *An end* $f \colon \Delta^* \to \mathbb{S}_1^3$ *is regular if and only if the order at the end of the Hopf differential of f is at least* -2.

Remark 2.10 In [LY], Lee and Yang define *normal ends* and *abnormal ends*. Both normal and abnormal ends are biholomorphic to a punctured disk Δ^*, and the Hopf differential has a pole of order 2 at the origin. Normal ends are elliptic ends, and abnormal ends are hyperbolic ends. Moreover, the Lee-Yang catenoids with normal ends are complete in our sense. However, the Lee-Yang catenoids with abnormal ends include incomplete examples, because the singular set of these examples accumulates at the ends. (In fact, CMC 1 face with hyperbolic ends cannot be complete, which is shown

in [FRUYY]).

3.   Embeddedness of elliptic ends

In this section we give a criterion for embeddedness of elliptic ends, which is based on results in [UY1]. This criterion will be applied in the next section.

Let $f\colon \Delta^* \to \mathbb{S}_1^3$ be a conformal spacelike CMC 1 immersion of finite total curvature with a complete regular elliptic end at the origin. Let $\gamma\colon [0, 1] \to \Delta^*$ be a loop around the origin and τ the deck transformation of Δ^* associated to the homotopy class of γ. Then by the same argument as in the proof of Proposition 2.6, there exists the holomorphic null lift F of f such that $F \circ \tau = F E_\theta$ for some $\theta \in [0, 2\pi)$, where $E_\theta = \mathrm{diag}(e^{i\theta}, e^{-i\theta})$. Since $E_\theta \in SU(2)$, $\hat{f} := FF^*$ in $\mathbb{H}^3$ is single-valued on Δ^*.

Let (g, ω) be the Weierstrass data associated to F, defined as in (2.2). Then by (4) in Remark 1.2, $\hat{f}\colon \Delta^* \to \mathbb{H}^3$ is a conformal CMC 1 immersion with the induced metric $d\hat{s}^2 = (1 + |g|^2)^2 \omega\bar{\omega}$. Thus by the final sentence of Proposition 2.6, $\hat{f}$ has finite total curvature and is complete at the origin.

Since $\hat{f}$ has the same Hopf differential Q as f, f having a regular end immediately implies that $\hat{f}$ has a regular end.

Furthermore we show the following theorem:

Theorem 3.1   *Let $f\colon \Delta^* \to \mathbb{S}_1^3$ be a conformal spacelike CMC 1 immersion of finite total curvature with a complete regular elliptic end at the origin. Then there exists a holomorphic null lift $F\colon \widetilde{\Delta^*} \to SL(2, \mathbb{C})$ of f (that is, $f = F e_3 F^*$) such that $\hat{f} = FF^*$ is a conformal CMC 1 finite-total-curvature immersion from Δ^* into $\mathbb{H}^3$ with a complete regular end at the origin. Moreover, f has an embedded end if and only if $\hat{f}$ has an embedded end.*

Remark 3.2   The converse of the first part of Theorem 3.1 is also true, that is, the following holds: Let $\hat{f}\colon \Delta^* \to \mathbb{H}^3$ be a conformal CMC 1 immersion of finite total curvature with a complete regular end at the origin. Take a holomorphic null lift $F\colon \widetilde{\Delta^*} \to SL(2, \mathbb{C})$ of $\hat{f}$ (that is, $f = FF^*$) such that the associated Weierstrass data (g, ω) is written as in (2.2). Then $f = F e_3 F^*$ is a conformal spacelike CMC 1 finite-total-curvature immersion from Δ^* into $\mathbb{S}_1^3$ with a complete regular elliptic end at the origin. Moreover, $\hat{f}$ has an embedded end if and only if f has an embedded end. See Proposition 5.5 below.

We already know that such an F exists. So we must prove the equivalency of embeddedness between the ends of f and $\hat{f}$. To prove this we prepare three lemmas.

Lemma 3.3 [UY1, Lemma 5.3]   *There exists a* $\Lambda \in SL(2, \mathbb{C})$ *such that*

$$\Lambda F = \begin{pmatrix} z^{\lambda_1}a(z) & z^{\lambda_2}b(z) \\ z^{\lambda_1-m_1}c(z) & z^{\lambda_2-m_2}d(z) \end{pmatrix}, \tag{3.1}$$

where a, b, c, d are holomorphic functions on Δ *that do not vanish at the origin, and* $\lambda_j \in \mathbb{R}$ *and* $m_j \in \mathbb{N}$ *(j = 1, 2) are defined as follows:*
(1)   *If* $\mathrm{Ord}_0 Q = \mu + \nu - 1 = -2$, *then*

$$m_1 = m_2, \quad \lambda_1 = \frac{-\mu + m_j}{2} \quad and \quad \lambda_2 = \frac{\mu + m_j}{2}. \tag{3.2}$$

(2)   *If* $\mathrm{Ord}_0 Q = \mu + \nu - 1 \geq -1$, *then*

$$m_1 = -(\nu + 1), \ m_2 = 2\mu + \nu + 1, \ \lambda_1 = 0 \quad and \quad \lambda_2 = m_2. \tag{3.3}$$

Note that in either case we have $\nu < -1$ and

$$\lambda_1 < \lambda_2, \quad \lambda_1 - m_1 < \lambda_2 - m_2 \quad and \quad \lambda_1 - m_1 < 0. \tag{3.4}$$

Note also that in the second case we have $m_1 < m_2$.

Proof of Lemma 3.3. F satisfies Equation (1.2), which is precisely Equation (1.5) in [UY1]. So we can apply [UY1, Lemma 5.3], since that lemma is based on Equation (1.5) in [UY1]. This gives the result. $\square$

It follows that

$$\begin{aligned} \Lambda f \Lambda^* &= (\Lambda F)e_3(\Lambda F)^* \\ &= \begin{pmatrix} |z|^{2\lambda_1}|a|^2 - |z|^{2\lambda_2}|b|^2 & |z|^{2\lambda_1}\bar{z}^{-m_1}a\bar{c} - |z|^{2\lambda_2}\bar{z}^{-m_2}b\bar{d} \\ |z|^{2\lambda_1}z^{-m_1}\bar{a}c - |z|^{2\lambda_2}z^{-m_2}\bar{b}d & |z|^{2(\lambda_1-m_1)}|c|^2 - |z|^{2(\lambda_2-m_2)}|d|^2 \end{pmatrix}. \end{aligned}$$

Note that $\Lambda f \Lambda^*$ is congruent to $f = Fe_3F^*$, because $(\Lambda F)^{-1}d(\Lambda F) = F^{-1}dF$ determines both the first and second fundamental forms as in Equation (1.4).

To study the behavior of the elliptic end f, we present the elliptic end in a 3-ball model as follows: We set

$$\begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} = \Lambda f \Lambda^*.$$

Since

$$x_0 = \frac{1}{2}\,\mathrm{trace}(\Lambda f\Lambda^*)$$
$$= \frac{1}{2}\big(|z|^{2\lambda_1}|a|^2 - |z|^{2\lambda_2}|b|^2 + |z|^{2(\lambda_1-m_1)}|c|^2 - |z|^{2(\lambda_2-m_2)}|d|^2\big),$$

(3.4) implies that $\lim_{z\to 0} x_0(z) = \infty$. So we may assume that $x_0(z) > 1$ for any $z \in \Delta^*$. So we can define a bijective map

$$p\colon \{(x_0,\,x_1,\,x_2,\,x_3) \in \mathbb{S}_1^3 \mid x_0 > 1\} \to \mathbb{D}^3 := D_1^3 \setminus \overline{D_{1/\sqrt{2}}^3}$$

as

$$p(x_0,\,x_1,\,x_2,\,x_3) := \frac{1}{1+x_0}(x_1,\,x_2,\,x_3),$$

where D_r^3 denotes the open ball of radius r in $\mathbb{R}^3$ and $\overline{D_r^3} = D_r^3 \cup \partial D_r^3$ (See Fig. 1).
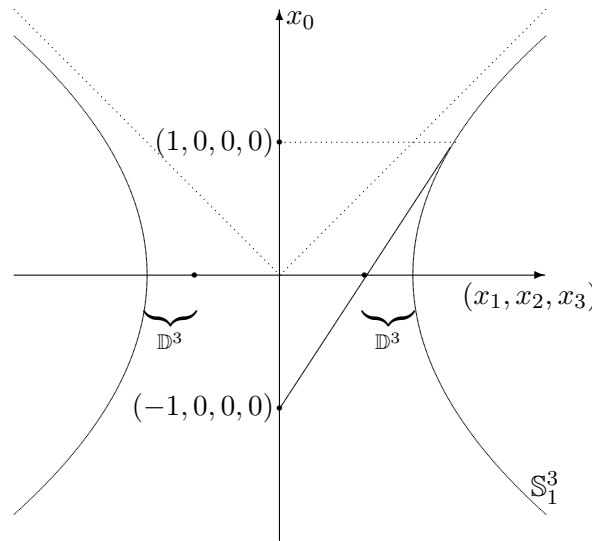

Fig. 1.   The projection p.

We set $(X_1,\,X_2,\,X_3) = p \circ (\Lambda f\Lambda^*)$. Then we have

$$X_1 + iX_2$$
$$= \frac{2a\bar{c}|z|^{2(\lambda_1-m_1)}z^{m_1} - 2b\bar{d}|z|^{2(\lambda_2-m_2)}z^{m_2}}{2 + |a|^2|z|^{2\lambda_1} + |c|^2|z|^{2(\lambda_1-m_1)} - |b|^2|z|^{2\lambda_2} - |d|^2|z|^{2(\lambda_2-m_2)}},\ (3.5)$$

$$X_3 =$$
$$\frac{|a|^2|z|^{2\lambda_1} - |c|^2|z|^{2(\lambda_1-m_1)} - |b|^2|z|^{2\lambda_2} + |d|^2|z|^{2(\lambda_2-m_2)}}{2 + |a|^2|z|^{2\lambda_1} + |c|^2|z|^{2(\lambda_1-m_1)} - |b|^2|z|^{2\lambda_2} - |d|^2|z|^{2(\lambda_2-m_2)}}. \quad (3.6)$$

We now define a function $U \colon \Delta^* \to \mathbb{C}$ that will be useful for proving Theorem 3.1:

$$U(z) = z^{-m_1}(X_1 + iX_2). \quad (3.7)$$

Then by making just a few sign changes to the argument in [UY1, Lemma 5.4], we have the following lemma:

Lemma 3.4 *The function U is a C^∞ function on Δ^* that extends continuously to Δ with $U(0) \neq 0$. Moreover,*

$$\lim_{z \to 0} z \frac{\partial U}{\partial z} = 0 \quad and \quad \lim_{z \to 0} z \frac{\partial U}{\partial \bar{z}} = 0. \quad (3.8)$$

Proof. If $\mathrm{Ord}_0\, Q = -2$, then Equation (3.7) is reduced to

$$U(z) = \frac{2a\bar{c} - 2b\bar{d}|z|^{2\mu}}{2|z|^{\mu+m_1} + |a|^2|z|^{2m_1} + |c|^2 - |b|^2|z|^{2(\mu+m_1)} - |d|^2|z|^{2\mu}}. \quad (3.9)$$

Then we have $U(0) = 2a(0)/c(0) \neq 0$, because $\mu > 0$ and $m_1 \geq 1$. Also, by a straightforward calculation, we see that Equation (3.8) holds.

If $\mathrm{Ord}_0\, Q \geq -1$, then Equation (3.7) is reduced to

$$U(z) = \frac{2a\bar{c} - 2b\bar{d}z^{m_2}\bar{z}^{m_1}}{(2 + |a|^2 - |d|^2)|z|^{2m_1} - |b|^2|z|^{2(m_1+m_2)} + |c|^2}. \quad (3.10)$$

Then we have $U(0) = 2a(0)/c(0) \neq 0$, because $m_1, m_2 \geq 1$. Also, since U is C^1-differentiable at the origin, we see that Equation (3.8) holds. $\qquad \square$

Also, we have the following lemma, analogous to Remark 5.5 in [UY1]:

Lemma 3.5 $p \circ (\Lambda f \Lambda^*) = (X_1,\, X_2,\, X_3)$ *converges to the single point* $(0,\, 0,\, -1) \in \partial D^3_1$. *Moreover,* $p \circ (\Lambda f \Lambda^*)$ *is tangent to* ∂D^3_1 *at the end* $z = 0$.

Proof. By (3.2)–(3.6), we see that

$$\lim_{z \to 0}(X_1,\, X_2,\, X_3) = (0,\, 0,\, -1).$$

Define a function $V \colon \Delta^* \to \mathbb{C}$ by $V := z^{-m_1}(X_3 + 1)$. Then from either

case of the proof of Lemma 3.4, we see that

$$\lim_{z \to 0} V = 0 \quad \text{and} \quad \lim_{z \to 0} z \frac{\partial V}{\partial z} = 0.$$

Therefore

$$0 = \lim_{z \to 0} z \frac{\partial V}{\partial z} = \lim_{z \to 0} \left(z^{-m_1+1} \frac{\partial X_3}{\partial z} \right) - m_1 V(0),$$

$$0 = \lim_{z \to 0} z \frac{\partial U}{\partial z} = \lim_{z \to 0} \left(z^{-m_1+1} \frac{\partial (X_1 + iX_2)}{\partial z} \right) - m_1 U(0)$$

imply that

$$\lim_{X_1 + iX_2 \to 0} \frac{\partial X_3}{\partial (X_1 + iX_2)} = \frac{V(0)}{U(0)} = 0,$$

proving the lemma.   □

Proof of Theorem 3.1.   [UY1, Theorem 5.2] shows that $\hat{f}$ has an embedded end if and only if $m_1 = 1$, so the theorem will be proven by showing that f also has an embedded end if and only if $m_1 = 1$. We now show this:

Since $U(0) \neq 0$, by taking a suitable branch we can define the function $u : \Delta \to \mathbb{C}$ by

$$u(z) = z \big(U(z) \big)^{1/m_1}.$$

Then u is a C^1 function such that

$$\frac{\partial u}{\partial z}(0) \neq 0 \quad \text{and} \quad \frac{\partial u}{\partial \bar{z}}(0) = 0. \tag{3.11}$$

Assume that $m_1 = 1$. By Equation (3.7), $X_1 + iX_2 = u$ holds. Then by Equation (3.11), $X_1 + iX_2$ is one-to-one on some neighborhood of the origin $z = 0$. Hence f has an embedded end.

Conversely, assume that f has an embedded end. Let $p_3 : \mathbb{D}^3 \to \Delta$ be the projection defined by $p_3(X_1, X_2, X_3) = X_1 + iX_2$. By Equation (3.7) we have

$$p_3 \circ p \circ (\Lambda f \Lambda^*) = u^{m_1}. \tag{3.12}$$

By Equations (3.11) and (3.12) the map $p_3 \circ p \circ (\Lambda f \Lambda^*)$ is an m_1-fold cover of $\Delta_\varepsilon^* = \{ z \in \mathbb{C} \mid 0 < |z| < \varepsilon \}$ for a sufficiently small $\varepsilon > 0$. Thus, by Lemma 3.5, m_1 must be 1, by the same topological arguments as at the end of the proof of Theorem 5.2 in [UY1].

Therefore, we have that f has an embedded end if and only if $m_1 = 1$.

$\square$

4. The Osserman-type inequality

Let $f\colon M = \overline{M} \setminus \{p_1, \ldots, p_n\} \to \mathbb{S}_1^3$ be a complete CMC 1 face of finite type with hyperbolic Gauss map G and Hopf differential Q.

Definition 4.1 We set

$$d\hat{s}^{\sharp 2} := (1 + |G|^2)^2 \frac{Q}{dG} \overline{\left(\frac{Q}{dG}\right)} \tag{4.1}$$

and call it the *lift-metric* of the CMC 1 face f. We also set

$$d\hat{\sigma}^{\sharp 2} := (-K_{d\hat{s}^{\sharp 2}}) d\hat{s}^{\sharp 2} = \frac{4 dG d\overline{G}}{(1 + |G|^2)^2}.$$

Remark 4.2 Since G and Q are defined on M, both $d\hat{s}^{\sharp 2}$ and $d\hat{\sigma}^{\sharp 2}$ are also defined on M.

We define the order of pseudometrics as in [UY3, Definition 2.1], that is:

Definition 4.3 A pseudometric $d\varsigma^2$ on $\overline{M}$ is of order m_j at p_j if $d\varsigma^2$ has a local expression

$$d\varsigma^2 = e^{2u_j} dz d\bar{z}$$

around p_j such that $u_j - m_j \log |z - z(p_j)|$ is continuous at p_j. We denote m_j by $\mathrm{Ord}_{p_j}(d\varsigma^2)$. In particular, if $d\varsigma^2$ gives a Riemannian metric around p_j, then $\mathrm{Ord}_{p_j}(d\varsigma^2) = 0$.

We now apply [UY2, Lemma 3] for regular ends in $\mathbb{H}^3$ to regular elliptic ends in $\mathbb{S}_1^3$, that is, we show the following proposition:

Proposition 4.4 *Let* $f\colon \Delta^* \to \mathbb{S}_1^3$ *be a conformal spacelike CMC 1 immersion of finite total curvature with a complete regular elliptic end at the origin $z = 0$. Then the following inequality holds*:

$$\mathrm{Ord}_0(d\hat{\sigma}^{\sharp 2}) - \mathrm{Ord}_0(Q) \geq 2. \tag{4.2}$$

Moreover, equality holds if and only if the end is embedded.

Proof. By Theorem 3.1, there exists a holomorphic null lift $F \colon \widetilde{\Delta^*} \to SL(2, \mathbb{C})$ of f such that $\hat{f} = FF^* \colon \Delta^* \to \mathbb{H}^3$ is a conformal CMC 1 immersion of finite total curvature with a complete regular end at the origin. Then by [UY2, Lemma 3], we have (4.2). Moreover, equality holds if and only if the end of $\hat{f}$ is embedded, by [UY2, Lemma 3]. This is equivalent to the end of f being embedded, by Theorem 3.1, proving the proposition.   □

The following lemma is a variant on known results in [Yu, KTUY]. In fact, [Yu] showed that $d\hat{s}^2$ is complete if and only if $d\hat{s}^{\sharp 2}$ is complete, see also [KTUY, Lemma 4.1].

Lemma 4.5   *Let $f \colon M \to \mathbb{S}_1^3$ be a CMC 1 face. Assume that each end of f is regular and elliptic. If f is complete and of finite type, then the lift-metric $d\hat{s}^{\sharp 2}$ is complete and of finite total curvature on M. In particular,*

$$\mathrm{Ord}_{p_j}(d\hat{s}^{\sharp 2}) \leq -2 \tag{4.3}$$

holds at each end p_j ($j = 1, \ldots, n$).

Proof. Since f is complete and of finite type, each end is complete and has finite total curvature. Then by (4.2) and the relation

$$\mathrm{Ord}_{p_j}(d\hat{s}^{\sharp 2}) + \mathrm{Ord}_{p_j}(d\hat{\sigma}^{\sharp 2}) = \mathrm{Ord}_{p_j}(Q) \tag{4.4}$$

(which follows from the Gauss equation $d\hat{\sigma}^{\sharp 2} d\hat{s}^{\sharp 2} = 4Q\overline{Q}$), we have (4.3) at each end p_j. Hence $d\hat{s}^{\sharp 2}$ is a complete metric. Also, again by (4.2), we have

$$\mathrm{Ord}_{p_j}(d\hat{\sigma}^{\sharp 2}) \geq 2 + \mathrm{Ord}_{p_j}(Q) \geq 0,$$

because p_j is regular (that is, $\mathrm{Ord}_{p_j}(Q) \geq -2$, by Proposition 2.9). This implies that the total curvature of $d\hat{s}^{\sharp 2}$ is finite.   □

Remark 4.6   Consider a CMC 1 face with regular elliptic ends. If it is complete and of finite type, then, by Lemma 4.5, the lift-metric is complete and of finite total curvature. But the converse is not true. See [FRUYY].

Theorem 4.7 (Osserman-type inequality)   *Let $f \colon M \to \mathbb{S}_1^3$ be a complete CMC 1 face of finite type with n elliptic ends and no other ends. Let G be its hyperbolic Gauss map. Then the following inequality holds:*

$$2\deg(G) \geq -\chi(M) + n, \tag{4.5}$$

where $\deg(G)$ *is the mapping degree of* G *(if* G *has essential singularities, then we define* $\deg(G) = \infty$*). Furthermore, equality holds if and only if each end is regular and embedded.*

Remark 4.8 As we remarked in the introduction, the completeness of a CMC 1 face f implies that f must be of finite type. See the forthcoming paper [FRUYY].

Proof of Theorem 4.7. Recall that we can set $M = \overline{M} \setminus \{p_1, \ldots, p_n\}$, where $\overline{M}$ is a compact Riemann surface and $p_1, \ldots, p_n$ is a set of points in $\overline{M}$, by Proposition 2.7. If f has irregular ends, then G has essential singularities at those ends. So $\deg(G) = \infty$ and then (4.5) automatically holds. Hence we may assume f has only regular ends. Using the Riemann-Hurwitz formula and the Gauss equation $d\hat{s}^{\sharp 2} d\hat{\sigma}^{\sharp 2} = 4Q\overline{Q}$, we have

$$
\begin{aligned}
2\deg(G) &= \chi(\overline{M}) + \sum_{p \in \overline{M}} \mathrm{Ord}_p \, d\hat{\sigma}^{\sharp 2} \\
&= \chi(\overline{M}) + \sum_{p \in \overline{M}} \left(\mathrm{Ord}_p \, Q - \mathrm{Ord}_p \, d\hat{s}^{\sharp 2} \right) \\
&= \chi(\overline{M}) + \sum_{p \in \overline{M}} \mathrm{Ord}_p \, Q - \sum_{p \in M} \mathrm{Ord}_p \, d\hat{s}^{\sharp 2} - \sum_{j=1}^{n} \mathrm{Ord}_{p_j} \, d\hat{s}^{\sharp 2} \\
&= -\chi(\overline{M}) - \sum_{j=1}^{n} \mathrm{Ord}_{p_j} \, d\hat{s}^{\sharp 2} \\
&\geq -\chi(\overline{M}) + 2n \quad \text{(because } d\hat{s}^{\sharp 2} \text{ is complete, by (4.3))} \\
&= -\chi(M) + n.
\end{aligned}
$$

Equality in (4.5) holds if and only if equality in (4.3) holds at each end, which is equivalent to equality in (4.2) holding at each end, by Equation (4.4). Thus by Proposition 4.4, we have the conclusion. $\qquad \square$

5. Examples

To visualize CMC 1 faces, we use the *hollow ball model* of $\mathbb{S}^3_1$, as in [KY, LY, Y]. For any point

$$
\begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} \leftrightarrow (x_0, x_1, x_2, x_3) \in \mathbb{S}^3_1,
$$

define

$$y_k = \frac{e^{\arctan x_0}}{\sqrt{1 + x_0^2}} x_k, \quad k = 1, 2, 3.$$

Then $e^{-\pi} < y_1^2 + y_2^2 + y_3^2 < e^{\pi}$. The identification $(x_0, x_1, x_2, x_3) \leftrightarrow (y_1, y_2, y_3)$ is then a bijection from $\mathbb{S}_1^3$ to the hollow ball

$$\mathscr{H} = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid e^{-\pi} < y_1^2 + y_2^2 + y_3^2 < e^{\pi}\}.$$

So $\mathbb{S}_1^3$ is identified with the hollow ball $\mathscr{H}$, and we show the graphics here using this identification to $\mathscr{H}$.

We shall first introduce four basic examples, using the same Weierstrass data as for CMC 1 immersions in $\mathbb{H}^3$.

Example 5.1 The CMC 1 face associated to horosphere in $\mathbb{H}^3$ is given by the Weierstrass data $(g, \omega) = (c_1, c_2 dz)$ with $c_1 \in \mathbb{C} \setminus \mathbb{S}^1$, $c_2 \in \mathbb{C} \setminus \{0\}$ on the Riemann surface $\mathbb{C}$. This CMC 1 face has no singularities. So this example is indeed a complete spacelike CMC 1 immersion. [Ak] and [R, Theorem 7] independently showed that the only complete spacelike CMC 1 immersion in $\mathbb{S}_1^3$ is a flat totally umbilic immersion.
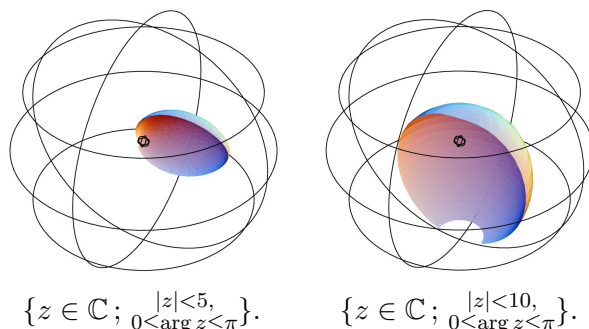

$$\{z \in \mathbb{C} ; \begin{smallmatrix} |z|<5, \\ 0 \leq \arg z \leq \pi \end{smallmatrix}\}. \qquad \{z \in \mathbb{C} ; \begin{smallmatrix} |z|<10, \\ 0 \leq \arg z \leq \pi \end{smallmatrix}\}.$$

Fig. 2. Pictures of Example 5.1. The left-hand side is drawn with $c_1 = 1.2$, $c_2 = 1$ and the right-hand side is drawn with $c_1 = 0$, $c_2 = 1$.

Example 5.2 The CMC 1 face associated to the Enneper cousin in $\mathbb{H}^3$ is given by the Weierstrass data $(g, \omega) = (z, cdz)$ with $c \in \mathbb{R} \setminus \{0\}$ on the Riemann surface $\mathbb{C}$. The induced metric $ds^2 = c^2(1-|z|^2)^2 dz d\bar{z}$ degenerates

where $|z| = 1$. Take a $p \in \mathbb{C}$ which satisfies $|p| = 1$. Define

$$\beta := \left(\frac{1 + |z|^2}{1 - |z|^2} \right)^2.$$

Then

$$\lim_{\substack{z \to p \\ z \in W}} \beta ds^2 = 4c^2 dz d\bar{z}.$$

So all singularities are admissible and hence this is a CMC 1 face. Moreover, it is easily seen that this CMC 1 face is complete and of finite type. Since this CMC 1 face is simply-connected, the end of this CMC 1 face is an elliptic end. Since $\mathrm{Ord}_\infty Q = -4 < -2$, the end of this CMC 1 face is irregular. Hence this CMC 1 face does not satisfy equality in the inequality (4.5).

$$\{z \in \mathbb{C} \,;\, |z| < 1.3\}. \qquad \{z \in \mathbb{C} \,;\, \substack{0.8 < |z| < 1.3 \\ \pi - 1 < \arg z < \pi + 1}\}.$$
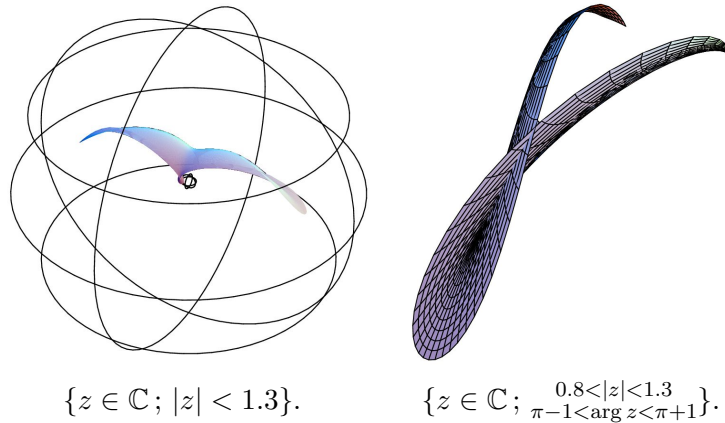
Fig. 3.   Pictures of Example 5.2, where $c = 1$.

Example 5.3   The CMC 1 face associated to the helicoid cousin in $\mathbb{H}^3$ is given by the Weierstrass data $(g, \omega) = (e^z, ice^{-z}dz)$ with $c \in \mathbb{R} \setminus \{0\}$ on the Riemann surface $\mathbb{C}$. Set $z = x + iy$. The induced metric $ds^2 = 4c^2 \sinh^2 x (dx^2 + dy^2)$ degenerates where $x = 0$. Take a $p \in \mathbb{C}$ which satisfies $\mathrm{Re}(p) = 0$. Define $\beta := \tanh^{-2} x$. Then

$$\lim_{\substack{z \to p \\ z \in W}} \beta ds^2 = 4c^2 (dx^2 + dy^2).$$

So all singularities are admissible and hence this is a CMC 1 face. Since the singular set is non-compact, this CMC 1 face is neither complete nor of finite type.
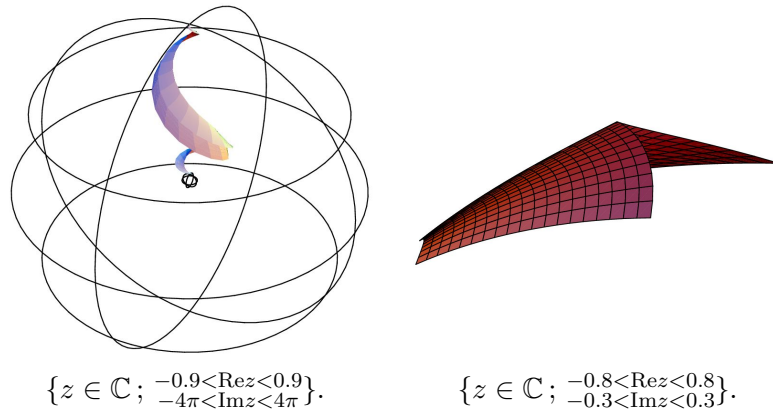

$$\{z \in \mathbb{C} \,; \, {}^{-0.9 < \text{Re}z < 0.9}_{-4\pi < \text{Im}z < 4\pi}\}. \qquad\qquad \{z \in \mathbb{C} \,; \, {}^{-0.8 < \text{Re}z < 0.8}_{-0.3 < \text{Im}z < 0.3}\}.$$

Fig. 4.   Pictures of Example 5.3, where $c = 1$.

Example 5.4   The CMC 1 face associated to the catenoid cousin in $\mathbb{H}^3$ is given by the Weierstrass data $(g, \omega) = (z^\mu, (1 - \mu^2)dz/4\mu z^{\mu+1})$ with $\mu \in \mathbb{R}^+ \setminus \{1\}$ on the Riemann surface $\mathbb{C} \setminus \{0\}$. The induced metric

$$ds^2 = \left(\frac{|z|^\mu - |z|^{-\mu}}{|z|} \cdot \frac{1 - \mu^2}{4\mu} \right)^2 dz d\bar{z}$$

degenerates where $|z| = 1$. Take a $p \in \mathbb{C}$ which satisfies $|p| = 1$. Define

$$\beta := \left(\frac{|z|^\mu + |z|^{-\mu}}{|z|^\mu - |z|^{-\mu}} \right)^2 .$$

Then

$$\lim_{\substack{z \to p \\ z \in W}} \beta ds^2 = 4 \left(\frac{1 - \mu^2}{4\mu} \right)^2 dz d\bar{z}.$$

So all singularities are admissible and hence this is a CMC 1 face. Moreover, it is easily seen that this CMC 1 face is complete and of finite type. Since the eigenvalues of the monodromy representation at each end are $-e^{\mu \pi i}$, $-e^{-\mu \pi i} \in \mathbb{S}^1$, each end of this CMC 1 face is an elliptic end. Since $\text{Ord}_0 Q = \text{Ord}_\infty Q = -2$, each end of this CMC 1 face is regular. Also, each end is of

this CMC 1 face is embedded, so this CMC 1 face satisfies equality in the inequality (4.5).
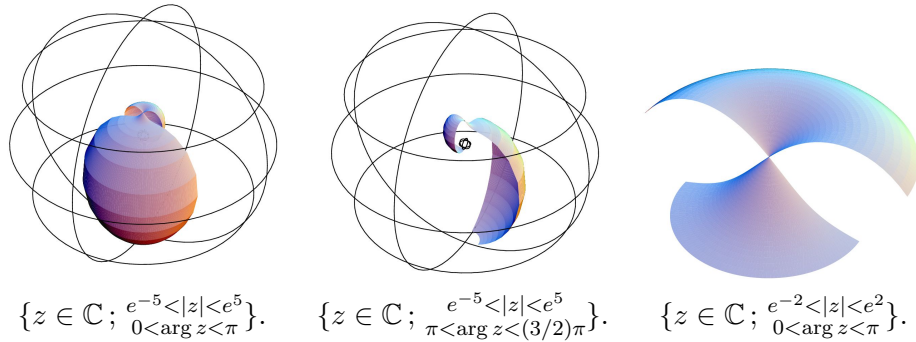

$$\left\{z \in \mathbb{C}\,;\, \begin{smallmatrix} e^{-5} < |z| < e^5 \\ 0 < \arg z < \pi \end{smallmatrix}\right\}. \qquad \left\{z \in \mathbb{C}\,;\, \begin{smallmatrix} e^{-5} < |z| < e^5 \\ \pi < \arg z < (3/2)\pi \end{smallmatrix}\right\}. \qquad \left\{z \in \mathbb{C}\,;\, \begin{smallmatrix} e^{-2} < |z| < e^2 \\ 0 < \arg z < \pi \end{smallmatrix}\right\}.$$

Fig. 5.   Pictures of Example 5.4, where $\mu = 0.8$.

$$\left\{z \in \mathbb{C}\,;\, \begin{smallmatrix} e^{-5} < |z| < e^5 \\ 0 < \arg z < \pi \end{smallmatrix}\right\}. \qquad \left\{z \in \mathbb{C}\,;\, \begin{smallmatrix} e^{-5} < |z| < e^5 \\ \pi < \arg z < (3/2)\pi \end{smallmatrix}\right\}. \qquad \left\{z \in \mathbb{C}\,;\, \begin{smallmatrix} e^{-2} < |z| < e^2 \\ 0 < \arg z < \pi \end{smallmatrix}\right\}.$$
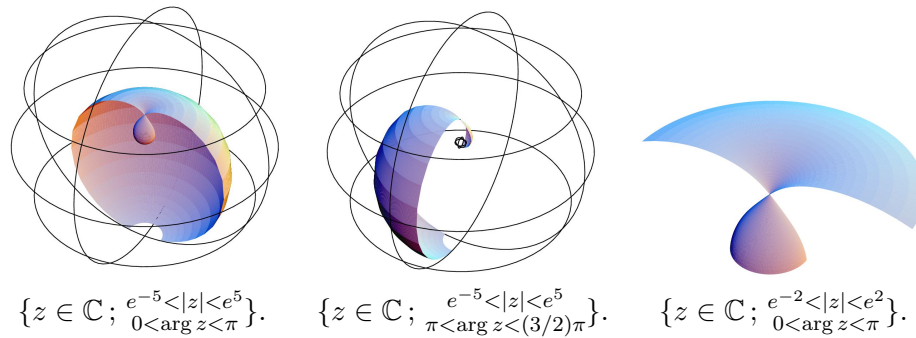
Fig. 6.   Pictures of Example 5.4, where $\mu = 1.2$.

To produce further examples, we consider a relationship between CMC 1 faces and CMC 1 immersions in the hyperbolic space $\mathbb{H}^3$, and shall give a method for transferring from CMC 1 immersions in $\mathbb{H}^3$ to CMC 1 faces in $\mathbb{S}_1^3$.

Let $\hat{f} \colon M = \overline{M} \setminus \{p_1, \ldots, p_n\} \to \mathbb{H}^3$ be a reducible CMC 1 immersion, whose first fundamental form $d\hat{s}^2$ has finite total curvature and is complete, where we define $\hat{f}$ to be *reducible* if there exists a holomorphic null lift F of $\hat{f}$ such that the image of the monodromy representation is in

$$U(1) = \left\{\left. \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right| \theta \in \mathbb{R} \right\}.$$

Let (g, ω) be the Weierstrass data associated to F, that is, (g, ω) satisfies

$$F^{-1}dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega.$$

Then $|g|$ and $|\omega|$ are single-valued on M, as seen in the proof of Proposition 2.6. Assume that the absolute value of the secondary Gauss map is not equal to 1 at all ends $p_1, \ldots, p_n$. Define $f := Fe_3F^*$. Then f is defined on M as well. Furthermore, we have the following proposition:

Proposition 5.5 *The CMC 1 face $f\colon M \to \mathbb{S}^3_1$ defined as above, using $\hat{f}$ and its lift F with monodromy in $U(1)$, is complete and of finite type with only elliptic ends. Moreover, an end of $\hat{f}$ is embedded if and only if the corresponding end of f is embedded.*

Proof. Fix an end p_j and assume $|g(p_j)| < 1$. Then we can take a neighborhood U_j such that $|g|^2 < 1-\varepsilon$ holds on U_j, where $\varepsilon \in (0, 1)$ is a constant. In this case,

$$ds^2 = (1 - |g|^2)^2\omega\bar{\omega} \geq \varepsilon^2\omega\bar{\omega} \geq \frac{\varepsilon^2}{4}(1 + |g|^2)^2\omega\bar{\omega} = \frac{\varepsilon^2}{4}d\hat{s}^2$$

holds on U_j. Since $d\hat{s}^2$ is complete at p_j, ds^2 is also complete. Moreover, the Gaussian curvatures K and $K_{d\hat{s}^2}$ satisfy

$$Kds^2 = \frac{4dgd\bar{g}}{(1 - |g|^2)^2}$$
$$\leq \left(\frac{2}{\varepsilon} - 1\right)^2 \frac{4dgd\bar{g}}{(1 + |g|^2)^2} = \left(\frac{2}{\varepsilon} - 1\right)^2 (-K_{d\hat{s}^2})d\hat{s}^2.$$

Hence ds^2 is of finite total curvature at the end p_j.

On the other hand, if $|g(p_j)| > 1$, we can choose the neighborhood U_j such that $|g|^{-2} < 1 - \varepsilon$ holds on U_j. Then

$$ds^2 = (1 - |g|^{-2})^2|g|^4\omega\bar{\omega} \geq \varepsilon^2|g|^4\omega\bar{\omega} \geq \frac{\varepsilon^2}{5}(1 + |g|^2)^2\omega\bar{\omega} = \frac{\varepsilon^2}{5}d\hat{s}^2.$$

Hence ds^2 is complete at p_j. Moreover, since

$$Kds^2 = \frac{4dgd\bar{g}}{(1 - |g|^2)^2}$$
$$\leq \left(\frac{2}{\varepsilon} - 1\right)^2 \frac{4dgd\bar{g}}{(1 + |g|^2)^2} = \left(\frac{2}{\varepsilon} - 1\right)^2 (-K_{d\hat{s}^2})d\hat{s}^2,$$

ds^2 is of finite total curvature. The proof of the final sentence of the proposition follows from the proof of Theorem 3.1, by showing $m_1 = 1$ for both f and $\hat{f}$. $\qquad\square$

Moreover, [UY1, Theorem 3.3] shows that for each $\lambda \in \mathbb{R} \setminus \{0\}$, $(\lambda g, \lambda^{-1}\omega)$ induces a CMC 1 immersion $\hat{f}_\lambda \colon M \to \mathbb{H}^3$, where (g, ω) is stated in Proposition 5.5. Thus we have the following theorem:

Theorem 5.6   *Let $\hat{f} \colon M \to \mathbb{H}^3$ be a reducible complete CMC 1 immersion of finite total curvature with n ends. Then there exists the holomorphic null lift F so that the image of the monodromy representation is in $U(1)$. Let (g, ω) be the Weierstrass data associated to F. Then there exist m $(0 \le m \le n)$ positive real numbers $\lambda_1, \ldots, \lambda_m \in \mathbb{R}^+$ such that $f_\lambda \colon M \to \mathbb{S}_1^3$, induced from the Weierstrass data $(\lambda g, \lambda^{-1}\omega)$, is a complete CMC 1 face of finite type with only elliptic ends for any $\lambda \in \mathbb{R} \setminus \{0, \pm\lambda_1, \ldots, \pm\lambda_m\}$.*

Proof.   Let $\hat{f}$, F and (g, ω) be as above and set $M = \overline{M} \setminus \{p_1, \ldots, p_n\}$. Then by [UY1, Theorem 3.3], there exists a 1-parameter family of reducible complete CMC 1 immersions $\hat{f}_\lambda \colon M \to \mathbb{H}^3$ of finite total curvature with n ends. Define $\lambda_j \in \mathbb{R}^+ \cup \{0, \infty\}$ $(j = 1, \ldots, m)$ as

$$\lambda_j = \begin{cases} 0 & \text{if } |g(p_j)| = \infty, \\ \infty & \text{if } |g(p_j)| = 0, \\ |g(p_j)|^{-1} & \text{otherwise.} \end{cases}$$

Then for any $\lambda \in (\mathbb{R} \cup \{\infty\}) \setminus \{0, \pm\lambda_1, \ldots, \pm\lambda_m\}$, $|\lambda g(p_j)| \ne 1$ for all p_j, and hence $f_\lambda \colon M \to \mathbb{S}_1^3$ induced from the Weierstrass data $(\lambda g, \lambda^{-1}\omega)$ is a complete CMC 1 face of finite type with only elliptic ends. $\qquad\square$

Complete CMC 1 immersions with low total curvature and low dual total curvature in $\mathbb{H}^3$ were classified in [RUY1, RUY2]. Applying Theorem 5.6 to the reducible examples in their classification, we have the following:

Corollary 5.7   *There exist the following twelve types of complete CMC 1 faces $f \colon M \to \mathbb{S}_1^3$ of finite type with elliptic ends:*

$$\begin{array}{llll} \mathbf{O}(0), & \mathbf{O}(-5), & \mathbf{O}(-2, -3), & \mathbf{O}(-1, -1, -2), \\ \mathbf{O}(-4), & \mathbf{O}(-6), & \mathbf{O}(-2, -4), & \mathbf{O}(-1, -2, -2), \\ \mathbf{O}(-2, -2), & \mathbf{O}(-1, -4), & \mathbf{O}(-3, -3), & \mathbf{O}(-2, -2, -2), \end{array}$$

where f is of type $\mathbf{O}(d_1, \ldots, d_n)$ when $M = (\mathbb{C} \cup \{\infty\}) \setminus \{p_1, \ldots, p_n\}$ and Q has order d_j at each end p_j.

Furthermore, reducible complete CMC 1 immersions of genus zero with an arbitrary number of regular ends and one irregular end and finite total curvature are constructed in [MU], using an analogue of the so-called UP-iteration. Applying Theorem 5.6 to their results, we have the following:

Corollary 5.8 *Set $M = \mathbb{C} \setminus \{p_1, \ldots, p_n\}$ for arbitrary $n \in \mathbb{N}$. Then there exist choices for $p_1, \ldots, p_n$ so that there exist complete CMC 1 faces $f\colon M \to \mathbb{S}_1^3$ of finite type with n regular elliptic ends and one irregular elliptic end.*

References

[AA]   Aiyama R. and Akutagawa K., *Kenmotsu-Bryant type representation formulas for constant mean curvature surfaces in $\mathbb{H}^3(-c^2)$ and $\mathbb{S}_1^3(c^2)$*. Ann. Global Anal. Geom. (1) **17** (1998), 49–75.

[Ak]   Akutagawa K., *On spacelike hypersurfaces with constant mean curvature in the de Sitter space*. Math. Z. **196** (1987), 13–19.

[B]   Bryant R., *Surfaces of Mean Curvature One in Hyperbolic Space*. Astérisque **154-155** (1987), 321–347.

[CHR]   Collin P., Hauswirth L. and Rosenberg H., *The geometry of finite topology Bryant surfaces*. Ann. of Math. (2) **153** (2001), 623–659.

[ER]   Estudillo F. J. M. and Romero A., *Generalized maximal surfaces in Lorentz-Minkowski space L^3*. Math. Proc. Cambridge Philos. Soc. (3) **111** (1992), 515–524.

[FRUYY] Fujimori S., Rossman W., Umehara M., Yamada K. and Yang S.-D., *Spacelike mean curvature one surfaces in de Sitter 3-space*. in preparation.

[FSUY]   Fujimori S., Saji K., Umehara M. and Yamada K., *Singularities of maximal surfaces*. `math. DG/0510366`, preprint.

[H]   Huber A., *On subharmonic functions and differential geometry in the large*. Comment. Math. Helv. **32** (1957), 13–72.

[JM]   Jorge L. and Meeks III W.H., *The topology of complete minimal surfaces of finite total Gaussian curvature*. Topology (2) **22** (1983), 203–221.

[KTUY]   Kokubu M., Takahashi M., Umeharara M. and Yamada K., *An analogue of minimal surface theory in* $\mathrm{SL}(n, \mathbf{C})/\mathrm{SU}(n)$. Trans. Amer. Math. Soc. **354** (2002), 1299–1325.

[KUY]   Kokubu M., Umehara M. and Yamada K., *Flat fronts in hyperbolic 3-space.* Pacific J. Math. **216** (2004), 149–175.

[KY]   Kim Y.-W. and Yang S.-D., *The goedesic reflection principle for spacelike constant mean curvature surfaces in de Sitter three-space.* preprint.

[L]   Lawson Jr H.B., Lectures on minimal submanifolds, Vol. I, Second edition, Mathematics Lecture Series, 9. Publish or Perish, Inc., (1980).

[LY]   Lee S. and Yang S.-D., *Spacelike constant mean curvature* 1 *trinoids in de Sitter three-space.* to appear in Osaka J. Math.

[MU]   McCune C. and Umehara M., *An analogue of the UP-iteration for constant mean curvature one surfaces in hyperbolic* 3-*space.* Differential Geom. Appl. **20** (2004), 197–207.

[O1]   Osserman R., *Global properties of minimal surfaces in* E^3 *and* E^n. Ann. of Math. (2) **80** (1964), 340–364.

[O2]   Osserman R., A survey of minimal surfaces, Dover Publications (1986).

[P]   Palmer B., *Spacelike constant mean curvature surfaces in pseudo-Riemannian space forms.* Ann. Global Anal. Geom. (3) **8** (1990), 217–226.

[R]   Ramanathan J., *Complete spacelike hypersurfaces of constant mean curvature in de Sitter space.* Indiana Univ. Math. J. (2) **36** (1987), 349–359.

[RUY1]   Rossman W., Umehara M. and Yamada K., *Mean curvature* 1 *surfaces in hyperbolic* 3-*space with low total curvature* I. Hiroshima Math. J. **34** (2004), 21–56.

[RUY2]   Rossman W., Umehara M. and Yamada K., *Mean curvature* 1 *surfaces in hyperbolic* 3-*space with low total curvature* II. Tohoku Math. J. (2) **55** (2003), 375–395.

[S]   Schoen R., *Uniqueness, symmetry, and embeddedness of minimal surfaces.* J. Differential Geom. **18** (1983), 791–809.

[UY1]   Umehara M. and Yamada K., *Complete surfaces of constant mean curvature* 1 *in the hyperbolic* 3-*space.* Ann. of Math. (2) **137** (1993), 611–638.

[UY2]   Umehara M. and Yamada K., *A duality on CMC-*1 *surfaces in hyperbolic space, and a hyperbolic analogue of the Osserman inequality.* Tsukuba J. Math. (1) **21** (1997), 229–237.

[UY3]   Umehara M. and Yamada K., *Surfaces of constant mean curvature c in* $H^3(-c^2)$ *with prescribed hyperbolic Gauss map.* Math. Ann. **304** (1996), 203–224.

[UY4]   Umehara M. and Yamada K., *Maximal surfaces with singularities in Minkowski space.* to appear in Hokkaido Math. J.

[Y]   Yang S.-D., *Björling formula for spacelike surfaces of constant mean curvature in the de Sitter space.* preprint.

[Yu]   Yu Z.-H., *The value distribution of the hyperbolic Gauss map.* Proc. Amer. Math. Soc. **125** (1997), 2997–3001.

Faculty of Mathematics
Kyushu University
Fukuoka 812-8581, Japan
E-mail: fujimori@math.kyushu-u.ac.jp