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Long range scattering for the Wave-Schrödinger system with large wave data and small Schrödinger data

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Abstract. We study the theory of scattering for the Wave-Schrödinger system with Yukawa type coupling in space dimension 3. We prove in particular the existence of modified wave operators for that system with no size restriction on the wave data in the framework of a direct method which requires smallness of the Schrödinger data, and we determine the asymptotic behaviour in time of solutions in the range of the wave operators.

Key words: long range scattering, modified wave operators, Wave-Schrödinger system.

1. Introduction

This paper is devoted to the theory of scattering and more precisely to the construction of modified wave operators for the Wave-Schrödinger system $(\mathrm{WS})_3$ in space dimension 3, namely

$$\begin{cases} i\partial_t u = -\dfrac{1}{2}\Delta u + Au \\ \square A = -|u|^2 \end{cases} \tag{1.1}$$

where u and A are respectively a complex valued and a real valued function defined in space time $\mathbb{R}^{3+1}$, Δ is the Laplacian in $\mathbb{R}^3$ and $\square = \partial_t^2 - \Delta$ is the d'Alembertian in $\mathbb{R}^{3+1}$. That system is Lagrangian and admits a number of formally conserved quantities, among which the L^2 norm of u and the energy

$$E(u,\, A) = \int dx \left\{ \frac{1}{2}\left(|\nabla u|^2 + (\partial_t A)^2 + |\nabla A|^2\right) + A|u|^2 \right\}. \tag{1.2}$$

The Cauchy problem for the $(\mathrm{WS})_3$ system is known to be globally well posed in the energy space $X_e = H^1 \oplus \dot{H}^1 \oplus L^2$ for $(u,\, A,\, \partial_t A)$ [1].

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A large amount of work has been devoted to the theory of scattering for nonlinear equations and systems centering on the Schrödinger equation, in particular for nonlinear Schrödinger (NLS) equations, Hartree equations, Klein-Gordon-Schrödinger (KGS), Wave-Schrödinger (WS) and Maxwell-Schrödinger (MS) systems. As in the case of the linear Schrödinger equation, one must distinguish the short range case from the long range case. In the former case, ordinary wave operators are expected and in a number of cases proved to exist, describing solutions where the Schrödinger function behaves asymptotically like a solution of the free Schrödinger equation. In the latter case, ordinary wave operators do not exist and have to be replaced by modified wave operators including a suitable phase in their definition. In that respect, the $(\mathrm{WS})_3$ system (1.1) belongs to the borderline (Coulomb) long range case, because of the t^{-1} decay in L^∞ norm of solutions of the wave equation. Such is the case also for the Hartree equation with $|x|^{-1}$ potential. Both are simplified models for the more complicated Maxwell-Schrödinger system $(\mathrm{MS})_3$ in $\mathbb{R}^{3+1}$, which belongs to the same case, as well as the Klein-Gordon-Schrödinger system $(\mathrm{KGS})_2$ in $\mathbb{R}^{2+1}$.

The construction of modified wave operators for the previous long range equations and systems has been tackled by two methods. The first one was initiated in [10] on the example of the NLS equation in $\mathbb{R}^{1+1}$ and subsequently applied to the NLS equation in $\mathbb{R}^{2+1}$ and $\mathbb{R}^{3+1}$ and to the Hartree equation [2], to the $(\mathrm{KGS})_2$ system [11] [12] [13] [14], to the $(\mathrm{WS})_3$ system [15] and to the $(\mathrm{MS})_3$ system [16] [18]. That method is rather direct, starting from the original equation or system. It will be sketched below. It is restricted to the (Coulomb) limiting long range case, and requires a smallness condition on the asymptotic state of the Schrödinger function. Early applications of the method required in addition a support condition on the Fourier transform of the Schrödinger asymptotic state and a smallness condition of the Klein-Gordon or Maxwell field in the case of the $(\mathrm{KGS})_2$ or $(\mathrm{MS})_3$ system respectively [11] [18]. The support condition was subsequently removed for the $(\mathrm{KGS})_2$ and $(\mathrm{MS})_3$ system and the method was applied to the $(\mathrm{WS})_3$ system without a support condition, at the expense of adding a correction term to the Schrödinger asymptotic function [12] [15] [16]. Finally the smallness condition of the KG field was removed for the $(\mathrm{KGS})_2$ system, first with and then without a support condition [13] [14]. All the previous papers on $(\mathrm{KGS})_2$, $(\mathrm{WS})_3$ and $(\mathrm{MS})_3$ use spaces of fairly regular solutions, with at least H^2 regularity for the Schrödinger function.

In the present paper, we reconsider the same problem for the $(WS)_3$ system in the framework of the previous method. Our purpose is twofold. First we show that no smallness condition is required on the wave field. Second, we treat the problem in function spaces that are as large as possible, namely with regularity as low as possible, and with convergence in time as slow as possible. In particular we treat the problem with regularity of the Schrödinger function at the level of L^2 only. Furthermore, only a weak convergence in time of the solutions to their asymptotic form is needed, namely $t^{-\lambda}$ with $\lambda > 3/8$. Under such a weak condition, neither a support condition nor a correction term for the asymptotic Schrödinger function is needed, as long as $\lambda \leq 1/2$. We also treat the same problem at the level of H^1 and H^2 for the Schrödinger function. This does not require any reinforcement of the smallness condition of the Schrödinger asymptotic state or of the time decay.

In a subsequent paper, we shall treat the same problem for the $(MS)_3$ system in the Coulomb gauge in the framework of the present method. Again no smallness condition will be required for the Maxwell field, and a weak time decay $t^{-\lambda}$ with $\lambda > 3/8$ will be sufficient, so that no support condition or correction term will be needed. (On that problem see [16] [18]).

For completeness and although we shall not make use of that fact in the present paper, we mention that the same problem for the Hartree equation and for the $(WS)_3$ and $(MS)_3$ system can also be treated by a more complex method where one first applies a phase-amplitude separation to the Schrödinger function. The main interest of that method is to remove the smallness condition on the Schrödinger function, and to go beyond the Coulomb limiting case for the Hartree equation. That method has been applied in particular to the $(WS)_3$ system and to the $(MS)_3$ system in a special case [5] [6] [7].

We now sketch briefly the method of construction of the modified wave operators initiated in [10]. That construction basically consists in solving the Cauchy problem for the system (1.1) with infinite initial time, namely in constructing solutions (u, A) with prescribed asymptotic behaviour at infinity in time. We restrict our attention to time going to $+\infty$. That asymptotic behaviour is imposed in the form of suitable approximate solutions (u_a, A_a) of the system (1.1). The approximate solutions are parametrized by data $(u_+, A_+, \dot{A}_+)$ which play the role of (actually would be in simpler e.g. short range cases) initial data at time zero for a simpler evolution. One then looks for exact solutions (u, A) of the system (1.1), the difference of which with the given asymptotic ones tends to zero at infinity in time in a suitable sense, more precisely, in suitable norms. The wave operator is then defined traditionally as the map $\Omega_+ \colon (u_+, A_+, \dot{A}_+) \to (u, A, \partial_t A)(0)$. However what really matters is the solution (u, A) in the neighborhood of infinity in time, namely in some interval $[T, \infty)$, and we shall restrict our attention to the construction of such solutions. Continuing such solutions down to $t = 0$ is a somewhat different question, connected with the global Cauchy problem at finite times, which we shall not touch here. That problem is well controlled for the $(\mathrm{WS})_3$ system, but not for instance for the $(\mathrm{MS})_3$ system.

The construction of solutions (u, A) with prescribed asymptotic behaviour (u_a, A_a) is performed in two steps.

Step 1. One looks for (u, A) in the form $(u, A) = (u_a + v, A_a + B)$. The system satisfied by (v, B) is

$$\begin{cases} i\partial_t v = -\dfrac{1}{2}\Delta v + Av + Bu_a - R_1 \\ \square B = -(|v|^2 + 2\,\mathrm{Re}\,\overline{u}_a v) - R_2 \end{cases} \tag{1.3}$$

where the remainders R_1, R_2 are defined by

$$\begin{cases} R_1 = i\partial_t u_a + \dfrac{1}{2}\Delta u_a - A_a u_a \\ R_2 = \square A_a + |u_a|^2. \end{cases} \tag{1.4}$$

It is technically useful to consider also the partly linearized system for functions (v', B')

$$\begin{cases} i\partial_t v' = -\dfrac{1}{2}\Delta v' + Av' + Bu_a - R_1 \\ \square B' = -(|v|^2 + 2\,\mathrm{Re}\,\overline{u}_a v) - R_2. \end{cases} \tag{1.5}$$

The first step of the method consists in solving the system (1.3) for (v, B), with (v, B) tending to zero at infinity in time in suitable norms, under assumptions on (u_a, A_a) of a general nature, the most important of which being decay assumptions on the remainders R_1 and R_2. That can be done as follows. One first solves the linearized system (1.5) for (v', B') with given (v, B) and initial data $(v', B')(t_0) = 0$ for some large finite t_0. One then takes the limit $t_0 \to \infty$ of that solution, thereby obtaining a solution (v', B') of (1.5) which tends to zero at infinity in time. That construction defines a map ϕ: $(v, B) \to (v', B')$. One then shows by a contraction method that the map ϕ has a fixed point. That first step will be performed in Section 2.

Step 2. The second step of the method consists in constructing approximate asymptotic solutions (u_a, A_a) satisfying the general estimates needed to perform Step 1. With the weak time decay allowed by our treatment of Step 1, one can take the simplest version of the asymptotic form used in previous works [5] [6] [15]. Thus we choose

$$u_a = MD \exp(-i\varphi)w_+ \tag{1.6}$$

where

$$M \equiv M(t) = \exp\left(\frac{ix^2}{2t}\right), \tag{1.7}$$

$$D(t) = (it)^{-3/2} D_0(t), \quad (D_0(t)f)(x) = f\left(\frac{x}{t}\right), \tag{1.8}$$

φ is a real phase to be chosen below and $w_+ = Fu_+$. We furthermore choose A_a so that $R_2 = 0$, namely $A_a = A_0 + A_1$ where A_0 is the solution of the free wave equation $\square A_0 = 0$ given by

$$A_0 = \cos \omega t A_+ + \omega^{-1} \sin \omega t \dot{A}_+ \tag{1.9}$$

where $\omega = (-\Delta)^{1/2}$, and where

$$A_1(t) = \int_t^\infty dt' \omega^{-1} \sin(\omega(t - t'))|u_a(t')|^2. \tag{1.10}$$

Substituting (1.6) into (1.10) yields

$$A_1(t) = t^{-1} D_0(t) \widetilde{A}_1 \tag{1.11}$$

where

$$\widetilde{A}_1 = -\int_1^\infty d\nu \nu^{-3} \omega^{-1} \sin(\omega(\nu - 1)) D_0(\nu)|w_+|^2. \tag{1.12}$$

In particular $\widetilde{A}_1$ is constant in time. We finally choose φ by imposing $\partial_t \varphi = t^{-1} \widetilde{A}_1$, $\varphi(1) = 0$, namely

$$\varphi = (\ell n t) \widetilde{A}_1. \tag{1.13}$$

We shall show in Section 3 that the previous choice fulfills the conditions needed for Step 1, under suitable assumptions on the asymptotic state

$(u_+, A_+, \dot{A}_+)$.

In order to state our results we introduce some notation. We denote by F the Fourier transform and by $\| \cdot \|_r$ the norm in $L^r \equiv L^r(\mathbb{R}^3)$, $1 \le r \le \infty$. For any nonnegative integer k and for $1 \le r \le \infty$, we denote by W_r^k the Sobolev spaces

$$W_r^k = \left\{ u\colon \|u; W_r^k\| = \sum_{\alpha\colon 0 \le |\alpha| \le k} \|\partial_x^\alpha u\|_r < \infty \right\}$$

where α is a multiindex, so that $H^k = W_2^k$. We shall need the weighted Sobolev spaces $H^{k,s}$ defined for k, $s \in \mathbb{R}$ by

$$H^{k,s} = \left\{ u\colon \|u; H^{k,s}\| = \|(1+x^2)^{s/2}(1-\Delta)^{k/2} u\|_2 < \infty \right\}$$

so that $H^k = H^{k,0}$. For any interval I, for any Banach space X and for any q, $1 \le q \le \infty$, we denote by $L^q(I, X)$ (resp. $L^q_{\text{loc}}(I, X)$) the space of L^q integrable (resp. locally L^q integrable) functions from I to X if $q < \infty$ and the space of measurable essentially bounded (resp. locally essentially bounded) functions from I to X if $q = \infty$. For any $h \in \mathcal{C}([1, \infty), \mathbb{R}^+)$, non increasing and tending to zero at infinity and for any interval $I \subset [1, \infty)$, we define the spaces

$$\begin{aligned}
X(I) = \big\{ (v, B)\colon\ & v \in \mathcal{C}(I, L^2),\ \|(v, B); X(I)\| \equiv \operatorname*{Sup}_{t \in I} h(t)^{-1} \\
& \big(\|v(t)\|_2 + \|v; L^{8/3}(J, L^4)\| \\
& + \|B; L^4(J, L^4)\| \big) < \infty \big\}, \qquad (1.14)\\
X_1(I) = \big\{ (v, B)\colon\ & v \in \mathcal{C}(I, H^1),\ \nabla B, \partial_t B \in \mathcal{C}(I, L^2), \\
& \|(v, B); X_1(I)\| \equiv \operatorname*{Sup}_{t \in I} h(t)^{-1} \big(\|v(t); H^1\| \\
& + \|v; L^{8/3}(J, W_4^1)\| + \|B; L^4(J, L^4)\| \\
& + \|\nabla B(t)\|_2 + \|\partial_t B(t)\|_2 \big) < \infty \big\}, \qquad (1.15)\\
X_2(I) = \big\{ (v, B)\colon\ & v \in \mathcal{C}(I, H^2) \cap \mathcal{C}^1(I, L^2),\ \nabla B,\ \partial_t B \in \mathcal{C}(I, L^2), \\
& \|(v, B); X_2(I)\| \equiv \operatorname*{Sup}_{t \in I} h(t)^{-1} \big(\|v(t); H^2\| \\
& + \|\partial_t v(t)\|_2 + \|v; L^{8/3}(J, L^4)\| \\
& + \|\partial_t v; L^{8/3}(J, L^4)\| + \|B; L^4(J, L^4)\| \\
& + \|\nabla B(t)\|_2 + \|\partial_t B(t)\|_2 \big) < \infty \big\} \qquad (1.16)
\end{aligned}$$

where $J = [t, \infty) \cap I$.

We can now state our results.

Proposition 1.1 *Let* $h(t) = t^{-1/2}$. *Let* u_a *be defined by* (1.6) *with* $w_+ = Fu_+ \in L^4$ *and* $c_4 = \|w_+\|_4$ *sufficently small and with* φ *defined by* (1.12) (1.13). *Let* A_a *be defined by* $A_a = A_0 + A_1$ *with* A_0 *and* A_1 *defined by* (1.9) *and* (1.10)-(1.12).

(1) *Let* $u_+ \in H^{0,2}$, *let* $A_+, \omega^{-1}\dot{A}_+ \in L^2$ *and* $\nabla^2 A_+, \nabla\dot{A}_+ \in L^1$. *Then there exists* T, $1 \le T < \infty$ *and there exists a unique solution* (u, A) *of the system* (1.1) *such that* $(u - u_a, A - A_a) \in X([T, \infty))$.

(2) *Let* $u_+ \in H^{0,3} \cap H^{1,2}$, *let* $A_+, \omega^{-1}\dot{A}_+ \in H^1$ *and* $\nabla^2 A_+, \nabla\dot{A}_+ \in W_1^1$. *Then there exists* T, $1 \le T < \infty$, *and there exists a unique solution* (u, A) *of the system* (1.1) *such that* $(u - u_a, A - A_a) \in X_1([T, \infty))$. *Furthermore* A *satisfies the estimate*

$$\|\nabla(A - A_a)(t)\|_2 \vee \|\partial_t(A - A_a)(t)\|_2 \le Ct^{-3/4} \tag{1.17}$$

for some constant C *and for all* $t \ge T$.

(3) *Let* $u_+ \in H^{1,3} \cap H^{2,2}$, *let* $A_+, \omega^{-1}\dot{A}_+ \in H^1$ *and* $\nabla^2 A_+, \nabla\dot{A}_+ \in W_1^1$. *Then there exists* T, $1 \le T < \infty$ *and there exists a unique solution* (u, A) *of the system* (1.1) *such that* $(u - u_a, A - A_a) \in X_2([T, \infty))$. *Furthermore* $u - u_a \in L^{8/3}([T, \infty), W_4^2)$ *and* (u, A) *satisfies the estimates* (1.17) *and*

$$\|\Delta(u - u_a); L^{8/3}([t, \infty), L^4)\| \le Ct^{-1/2} \tag{1.18}$$

for some constant C *and for all* $t \ge T$.

Remark 1.1 The only smallness condition bears on c_4 and appears at the level of the L^2 theory in Part (1) of Proposition 1.1. In particular there is no smallness condition on A_a. Furthermore no additional smallness condition is required for the theories at the level of H^1 and H^2.

2. The Cauchy problem at infinite initial time

In this section we perform the first step of the construction of solutions of the system (1.1) as described in the introduction, namely we construct solutions (v, B) of the system (1.3) defined in a neighborhood of infinity in time and tending to zero at infinity under suitable regularity and decay assumptions on the asymptotic functions (u_a, A_a) and on the remainders R_i.

As mentioned in the introduction, we offer three theories with u (or v) at the level of regularity of L^2, H^1 and H^2 respectively. As a preliminary to that study, we need to solve the Cauchy problem with finite initial time for the linearized system (1.5). That system consists of two independent equations. The second one is simply a wave equation with an inhomogeneous term and the Cauchy problem with finite or infinite initial time for it is readily solved under suitable assumptions on the inhomogeneous term, which will be fulfilled in the applications. The first one is a Schrödinger equation with time dependent real potential and time dependent inhomogeneity which we rewrite in a more concise form and with slightly different notation as

$$i\partial_t v = -\frac{1}{2}\Delta v + Vv + f. \tag{2.1}$$

We first give some preliminary results on the Cauchy problem with finite initial time for that equation at the level of regularity of L^2, H^1 and H^2. Those results rely in an essential way on the well known Strichartz inequalities for the Schrödinger equation [3] [9] [19] which we recall for completeness. We define

$$U(t) = \exp\left(i\left(\frac{t}{2}\right)\Delta\right). \tag{2.2}$$

A pair of Hölder exponents (q, r) will be called admissible if $0 \le 2/q = 3/2 - 3/r \le 1$. For any r, $1 \le r \le \infty$, we define $\bar{r}$ by $1/r + 1/\bar{r} = 1$.

Lemma 2.1 *The following inequalities hold.*

(1) *For any admissible pair (q, r) and for any $u \in L^2$*

$$\|U(t)u; L^q(\mathbb{R}, L^r)\| \le C\|u\|_2. \tag{2.3}$$

(2) *Let I be an interval and let $t_0 \in I$. Then for any admissible pairs (q_i, r_i), $i = 1, 2$,*

$$\left\| \int_{t_0}^{t} dt' U(\,\cdot\, - t') f(t'); L^{q_1}(I, L^{r_1}) \right\| \le C\|f; L^{\bar{q}_2}(I, L^{\bar{r}_2})\|. \tag{2.4}$$

The basic result on the Cauchy problem for (2.1) can be stated as follows.

Proposition 2.1 *Let I be an interval and let $t_0 \in I$.*

(1) *Let $V \in L^1_{\mathrm{loc}}(I, L^\infty) + L^p_{\mathrm{loc}}(I, L^s) + L^\infty_{\mathrm{loc}}(I, L^{3/2+\varepsilon})$ for some p, $1 \le p < \infty$ with $2/p = 2 - 3/s$ and for some $\varepsilon > 0$, and let $f \in L^1(I, L^2) +$*

$L^2(I, L^{6/5})$. Let $v_0 \in L^2$. Then there exists a unique solution v of (2.1) with $v(t_0) = v_0$ such that $v \in L^q_{\rm loc}(I, L^r)$ for all admissible pairs (q, r). Furthermore $v \in \mathcal{C}(I, L^2)$ and for all $t \in I$, v satisfies the equality

$$\|v(t)\|_2^2 - \|v_0\|_2^2 = \int_{t_0}^t dt' 2\,{\rm Im}\langle v(t'),\, f(t')\rangle. \tag{2.5}$$

(2) *Let V and f satisfy the assumptions of Part (1) and in addition $\nabla V \in L^1_{\rm loc}(I, L^3 + L^\infty) + L^4_{\rm loc}(I, L^{6/5})$ and $\nabla f \in L^1_{\rm loc}(I, L^2) + L^2_{\rm loc}(I, L^{6/5})$. Let $v_0 \in H^1$. Then the solution v of (2.1) obtained in Part (1) satisfies in addition $\nabla v \in L^q_{\rm loc}(I, L^r)$ for all admissible pairs (q, r). Furthermore $v \in \mathcal{C}(I, H^1)$ and for all $t \in I$, v satisfies the equality*

$$\begin{aligned}\|\nabla v(t)\|_2^2 &- \|\nabla v_0\|_2^2 \\ &= \int_{t_0}^t dt' 2\,{\rm Im}\langle \nabla v(t'),\, (\nabla V)(t')v(t') + \nabla f(t')\rangle.\end{aligned} \tag{2.6}$$

(3) *Let V satisfy $V \in \mathcal{C}(I, L^2 + L^\infty)$, $\partial_t V \in L^1_{\rm loc}(I, L^2 + L^\infty) + L^2_{\rm loc}(I, L^{6/5})$, and let f satisfy $f \in \mathcal{C}(I, L^2)$, $\partial_t f \in L^1_{\rm loc}(I, L^2) + L^2_{\rm loc}(I, L^{6/5})$. Let $v_0 \in H^2$. Then the solution v of (2.1) obtained in Part (1) satisfies in addition $\partial_t v \in L^q_{\rm loc}(I, L^r)$ for all admissible pairs (q, r). Furthermore $v \in \mathcal{C}(I, H^2) \cap \mathcal{C}^1(I, L^2)$ and for all $t \in I$, v satisfies the equality*

$$\begin{aligned}&\|\partial_t v(t)\|_2^2 - \left\| -\frac{1}{2}\Delta v_0 + V(t_0)v_0 + f(t_0)\right\|_2^2 \\ &= \int_{t_0}^t dt' 2\,{\rm Im}\langle \partial_t v(t'),\, (\partial_t V)(t')v(t') + \partial_t f(t')\rangle.\end{aligned} \tag{2.7}$$

If in addition $V \in L^{q_0}_{\rm loc}(I, L^{r_0} + L^\infty)$ and $f \in L^{q_0}_{\rm loc}(I, L^{r_0})$ for some admissible pair (q_0, r_0), then $\Delta v \in L^q_{\rm loc}(I, L^r)$ for all admissible pairs (q, r) with $2 \le r \le r_0$.

The proof is a variation of that given in [8] [19], using extensively Lemma 2.1.

For any interval J, let

$$Z(J) = \mathcal{C}(J, L^2) \cap L^2(I, L^6). \tag{2.8}$$

The local Cauchy problem for (2.1) is treated by a contraction method applied to the integral equation associated with (2.1). The relevant spaces for the contraction have $v \in Z(J)$ for Part (1), v, $\nabla v \in Z(J)$ for Part (2) and v, $\partial_t v \in Z(J)$, $v \in \mathcal{C}(J, H^2)$ for Part (3), for suitable small J. The extension of local solutions to global ones is easy because the problem is linear.

We now begin the construction of solutions of the system (1.3). For any T, t_0 with $1 \le T < t_0 \le \infty$, we denote by I the interval $I = [T, t_0]$ and for any $t \in I$, we denote by J the interval $J = [t, t_0]$. In all this section, we denote by h a function in $\mathcal{C}([1, \infty), \mathbb{R}^+)$ such that for some $\lambda > 0$, the function $\overline{h}(t) \equiv t^\lambda h(t)$ is non increasing and tends to zero as $t \to \infty$.

We shall make repeated use of the following lemma.

Lemma 2.2 *Let $1 \le q$, $q_k \le \infty$ $(1 \le k \le n)$ be such that*

$$\mu \equiv \frac{1}{q} - \sum_k \frac{1}{q_k} \ge 0.$$

Let $f_k \in L^{q_k}(I)$ satisfy

$$\| f_k; L^{q_k}(J) \| \le N_k h(t) \tag{2.9}$$

for $1 \le k \le n$, for some constants N_k and for all $t \in I$.

Let $\rho \ge 0$ be such that $n\lambda + \rho > \mu$. Then the following inequality holds for all $t \in I$

$$\left\| \left(\prod_k f_k \right) t^{-\rho}; L^q(J) \right\| \le C \left(\prod_k N_k \right) h(t)^n t^{\mu-\rho} \tag{2.10}$$

where

$$C = \left(1 - 2^{-q(n\lambda+\rho-\mu)}\right)^{-1/q}. \tag{2.11}$$

Proof. For $t \in I$, we define $I_j = [t2^j, t2^{j+1}] \cap I$ so that $J = \bigcup_{j \ge 0} I_j$. We then rewrite $L^q(J) = \ell^q_j(L^q(I_j))$. We estimate

$$\begin{aligned}
\left\| \left(\prod_k f_k \right) t^{-\rho}; L^q(J) \right\| &\le \left\| \left(\prod_k \| f_k; L^{q_k}(I_j) \| \right) \| t^{-\rho}; L^{1/\mu}(I_j) \|; \ell^q_j \right\| \\
&\le \left(\prod_k N_k \right) \| h(t2^j)^n (t2^j)^{-\rho+\mu}; \ell^q_j \| \\
&\le \left(\prod_k N_k \right) \overline{h}(t)^n t^{-n\lambda-\rho+\mu} \| 2^{j(-n\lambda-\rho+\mu)}; \ell^q_j \|
\end{aligned}$$

from which (2.10) follows. $\square$

Remark 2.1 In some special cases, the dyadic decomposition is not needed for the proof of Lemma 2.2. For instance if all the q_k are infinite, one can estimate

$$\begin{aligned} \|h(t)^n t^{-\rho}\|_q &\leq \overline{h}(t)^n \|t^{-\rho-n\lambda}\|_q \\ &\leq C\overline{h}(t)^n t^{-\rho-n\lambda+1/q} = Ch(t)^n t^{-\rho+\mu} \end{aligned} \tag{2.12}$$

by a direct application of Hölder's inequality in J. The same situation occurs if $\rho > \mu$.

In addition to the Strichartz inequalities for the Schrödinger equation (Lemma 2.1), we shall need special cases of the Strichartz inequalities for the wave equation [4] [9]. Let I be an interval, let $t_0 \in I$ and let $B(t_0) = \partial_t B(t_0) = 0$. Then

$$\|B; L^4(I,\, L^4)\| \leq C\|\Box B; L^{4/3}(I,\, L^{4/3})\|, \tag{2.13}$$

$$\mathop{\mathrm{Sup}}_{t\in I} (\|\nabla B(t)\|_2 \vee \|\partial_t B(t)\|_2) \leq \|\Box B; L^1(I,\, L^2)\|. \tag{2.14}$$

We now construct solutions of the system (1.3) at the level of regularity of L^2 for v. The result can be stated as follows.

Proposition 2.2 *Let h be defined as above with $\lambda = 3/8$ and let $X(\,\cdot\,)$ be defined by* (1.14)*. Let $u_a \in L^\infty([1,\, \infty),\, L^4)$, $A_a \in L^\infty([1,\, \infty),\, L^\infty)$, $R_1 \in L^\infty([1,\, \infty),\, L^2)$ and $R_2 \in L^{4/3}([1,\, \infty),\, L^{4/3})$ satisfy the estimates*

$$\|u_a(t)\|_4 \leq c_4 t^{-3/4}, \tag{2.15}$$

$$\|A_a(t)\|_\infty \leq a t^{-1}, \tag{2.16}$$

$$\|R_1; L^1([t,\, \infty),\, L^2)\| \leq r_1 h(t), \tag{2.17}$$

$$\|R_2; L^{4/3}([t,\, \infty),\, L^{4/3})\| \leq r_2 h(t), \tag{2.18}$$

for some constants c_4, a, r_1, r_2 with c_4 sufficiently small and for all $t \geq 1$. Then there exists T, $1 \leq T < \infty$ and there exists a unique solution $(v,\, B)$ of the system (1.3) *in the space $X([T,\, \infty))$.*

Proof. We follow the sketch given in the introduction. Let $1 \leq T < \infty$ and let $(v,\, B) \in X([T,\, \infty))$. In particular $(v,\, B)$ satisfies

$$\|v(t)\|_2 \leq N_0 h(t) \tag{2.19}$$

$$\|v; L^{8/3}([t,\, \infty),\, L^4)\| \leq N_1 h(t) \tag{2.20}$$

$$\|B; L^4([t,\, \infty),\, L^4)\| \leq N_2 h(t) \tag{2.21}$$

for some constants N_i and for all $t \geq T$. We first construct a solution (v', B') of the system (1.5) in $X([T, \infty))$. For that purpose, we take t_0, $T < t_0 < \infty$ and we solve the system (1.5) in $X(I)$ where $I = [T, t_0]$ with initial condition $(v', B')(t_0) = 0$. Let (v'_{t_0}, B'_{t_0}) be the solution thereby obtained. The existence of v'_{t_0} follows from Proposition 2.1, part (1) with $V = A$ and $f = Bu_a - R_1$. We want to take the limit of (v'_{t_0}, B'_{t_0}) as $t_0 \to \infty$ and for that purpose we need estimates of (v'_{t_0}, B'_{t_0}) in $X(I)$ that are uniform in t_0. Omitting the subscript t_0 for brevity we define

$$N'_0 = \operatorname*{Sup}_{t \in I} h(t)^{-1} \|v'(t)\|_2 \tag{2.22}$$

$$N'_1 = \operatorname*{Sup}_{t \in I} h(t)^{-1} \|v'; L^{8/3}(J, L^4)\| \tag{2.23}$$

$$N'_2 = \operatorname*{Sup}_{t \in I} h(t)^{-1} \|B'; L^4(J, L^4)\| \tag{2.24}$$

where $J = [t, \infty) \cap I$. We first estimate N'_0. From (2.5) we obtain

$$\begin{aligned} \|v'(t)\|_2 &\leq \|Bu_a - R_1; L^1(J, L^2)\| \\ &\leq \big\| \|B\|_4 \|u_a\|_4 + \|R_1\|_2; L^1(J) \big\| \\ &\leq C_0(c_4 N_2 + r_1) h(t) \end{aligned} \tag{2.25}$$

by Lemma 2.2, so that

$$N'_0 \leq C_0(c_4 N_2 + r_1). \tag{2.26}$$

We next estimate N'_1. By Lemma 2.1

$$\begin{aligned} \|v'; L^{8/3}(J, L^4)\| \leq C\big(&\|A_a v'; L^1(J, L^2)\| + \|Bv'; L^{8/5}(J, L^{4/3})\| \\ &+ \|Bu_a - R_1; L^1(J, L^2)\|\big). \end{aligned} \tag{2.27}$$

The last norm has already been estimated by (2.25) while

$$\begin{aligned} \|A_a v'; L^1(J, L^2)\| &\leq \big\| \|A_a\|_\infty \|v'\|_2; L^1(J) \big\| \\ &\leq C a N'_0 h(t), \\ \|Bv'; L^{8/5}(J, L^{4/3})\| &\leq \big\| \|B\|_4 \|v'\|_2; L^{8/5}(J) \big\| \\ &\leq C N_2 N'_0 \overline{h}(t) h(t) \end{aligned} \tag{2.28}$$

by Lemma 2.2. Substituting (2.28) into (2.27) and using (2.26), we obtain

$$N'_1 \leq C_1 \left(c_4 N_2 + r_1\right) \left(1 + a + N_2 \overline{h}(T)\right). \tag{2.29}$$

We finally estimate N'_2. From (2.13), we obtain

$$\begin{aligned}\|B'; L^4(J,\, L^4)\| &\leq C\| |v|^2 + 2\,\mathrm{Re}\,\overline{u}_a v + R_2; L^{4/3}(J,\, L^{4/3})\| \\ &\leq C\big(\big\| \|v\|_2\,(\|v\|_4 + \|u_a\|_4)\,; L^{4/3}(J)\big\| + r_2 h(t)\big) \\ &\leq C_2\left(c_4 N_0 + r_2 + N_0 N_1 \overline{h}(t)\right) h(t) \qquad (2.30)\end{aligned}$$

by Lemma 2.2, so that

$$N'_2 \leq C_2\left(c_4 N_0 + r_2 + N_0 N_1 \overline{h}(T)\right). \qquad (2.31)$$

It follows from (2.26) (2.29) and (2.31) that $(v',\, B')$ is bounded in $X(I)$ uniformly in t_0.

We now take the limit $t_0 \to \infty$ of $(v'_{t_0},\, B'_{t_0})$, restoring the subscript t_0 for that part of the argument. Let $T < t_0 < t_1 < \infty$ and let $(v'_{t_0},\, B'_{t_0})$ and $(v'_{t_1},\, B'_{t_1})$ be the corresponding solutions of (1.5). From the L^2 norm conservation of the difference $v'_{t_0} - v'_{t_1}$ and from (2.25), it follows that for all $t \in [T,\, t_0]$

$$\|v'_{t_0}(t) - v'_{t_1}(t)\|_2 = \|v'_{t_1}(t_0)\|_2 \leq C_0(c_4 N_2 + r_1) h(t_0) \qquad (2.32)$$

while from (2.13) (2.30) and the initial conditions

$$\begin{aligned}&\|B'_{t_0} - B'_{t_1}; L^4([T,\, t_0],\, L^4)\| \\ &\leq C\| |v|^2 + 2\,\mathrm{Re}\,\overline{u}_a v + R_2; L^{4/3}([t_0,\, t_1],\, L^{4/3})\| \\ &\leq C_2\left(c_4 N_0 + r_2 + N_0 N_1 \overline{h}(T)\right) h(t_0). \qquad (2.33)\end{aligned}$$

It follows from (2.32) (2.33) that there exists $(v',\, B') \in L^\infty_{\mathrm{loc}}([T,\, \infty),\, L^2) \oplus L^4_{\mathrm{loc}}([T,\, \infty),\, L^4)$ such that $(v'_{t_0},\, B'_{t_0})$ converges to $(v',\, B')$ in that space when $t_0 \to \infty$. From the uniformity in t_0 of the estimates (2.25) (2.30), it follows that $(v',\, B')$ satisfies the same estimates in $[T,\, \infty)$ namely that $(v',\, B')$ satisfies (2.26) (2.31) with N'_i defined by (2.22)-(2.24) with $I = [T,\, \infty)$. Furthermore it follows from (2.29) by a standard compactness argument that $(v',\, B') \in X([T,\, \infty))$ and that v' also satisfies (2.29). Clearly $(v',\, B')$ satisfies the system (1.5).

From now on, I denotes the interval $[T,\, \infty)$. The previous construction defines a map ϕ: $(v,\, B) \to (v',\, B')$ from $X(I)$ to itself. The next step consists in proving that the map ϕ is a contraction on a suitable closed bounded set $\mathcal{R}$ of $X(I)$. We define $\mathcal{R}$ by the conditions (2.19)-(2.21) for some constants N_i and for all $t \in I$. We first show that for a suitable choice of N_i and for sufficiently large T, the map ϕ maps $\mathcal{R}$ into $\mathcal{R}$. By (2.26)

(2.29) (2.31) it suffices for that purpose that

$$\begin{cases} (N_0' \le) C_0 \left(c_4 N_2 + r_1 \right) \le N_0 \\ (N_1' \le) C_1 \left(c_4 N_2 + r_1 \right) \left(1 + a + N_2 \overline{h}(T) \right) \le N_1 \\ (N_2' \le) C_2 \left(c_4 N_0 + r_2 + N_0 N_1 \overline{h}(T) \right) \le N_2 . \end{cases} \tag{2.34}$$

We fulfill those conditions by choosing the N_i according to

$$\begin{cases} N_0 = C_0 \left(c_4 N_2 + r_1 \right) \\ N_1 = C_1 \left(c_4 N_2 + r_1 \right) (2 + a) \\ N_2 = C_2 \left(c_4 N_0 + r_2 + 1 \right) \end{cases} \tag{2.35}$$

which is possible under the smallness condition $C_0 C_2 c_4^2 < 1$, and by taking T sufficiently large so that

$$N_2 \overline{h}(T) \le 1, \quad N_0 N_1 \overline{h}(T) \le 1. \tag{2.36}$$

We next show that the map ϕ is a contraction on $\mathcal{R}$. Let $(v_i, B_i) \in \mathcal{R}$, $i = 1, 2$, and let $(v_i', B_i') = \phi\big((v_i, B_i)\big)$. For any pair of functions (f_1, f_2) we define $f_\pm = (1/2)(f_1 \pm f_2)$ so that $(fg)_\pm = f_+ g_\pm + f_- g_\mp$. In particular $u_+ = u_a + v_+$, $u_- = v_-$, $A_+ = A_a + B_+$ and $A_- = B_-$. Corresponding to (1.5), (v_-', B_-') satisfies the system

$$\begin{cases} i\partial_t v_-' = -\dfrac{1}{2}\Delta v_-' + A_+ v_-' + B_- u_a + B_- v_+' \\ \square B_-' = -2\,\mathrm{Re}\left(\overline{u}_a + \overline{v}_+ \right) v_- . \end{cases} \tag{2.37}$$

Since $\mathcal{R}$ is convex and stable under ϕ, (v_+, B_+) and (v_+', B_+') belong to $\mathcal{R}$, namely satisfy (2.19)-(2.21). Let N_{i-} and N_{i-}' be the seminorms of (v_-, B_-) and (v_-', B_-') corresponding to (2.22)-(2.24), namely the constants obtained by replacing (v', B', N_i') by (v_-, B_-, N_{i-}) and (v_-', B_-', N_{i-}') in (2.22)-(2.24). We have to estimate the N_{i-}' in terms of the N_{i-}. The estimates are essentially the same as those of N_i' in terms of N_i with minor differences: the contribution of the remainders disappear, the linear terms are the same, and the quadratic terms are in general obtained by polarization. The only exceptions to that rule are the $B_- v_+'$ term in the estimate of N_{0-}' and the $\overline{v}_+ v_-$ term in the estimate of N_{2-}'. Those terms are estimated as follows

$$\begin{aligned} \| B_- v_+' ; L^1(J, L^2) \| &\le \big\| \| B_- \|_4 \| v_+' \|_4 ; L^1(J) \big\| \\ &\le C N_{2-} N_1 \overline{h}(t) h(t), \end{aligned} \tag{2.38}$$

$$\begin{aligned}\|\overline{v}_+ v_-; L^{4/3}(J,\, L^{4/3})\| &\le \big\| \|v_+\|_4 \|v_-\|_2; L^{4/3}(J) \big\| \\ &\le C N_1 N_{0-} \overline{h}(t) h(t).\end{aligned} \tag{2.39}$$

We finally obtain

$$\begin{cases} N'_{0-} \le C_0 \left(c_4 + N_1 \overline{h}(T)\right) N_{2-} \\ N'_{1-} \le C \left((c_4 + N_1 \overline{h}(T)) N_{2-} + (a + N_2 \overline{h}(T)) N'_{0-}\right) \\ \qquad \le C_1 \left(c_4 + N_1 \overline{h}(T)\right) \left(1 + a + N_2 \overline{h}(T)\right) N_{2-} \\ N'_2 \le C_2 \left(c_4 + N_1 \overline{h}(T)\right) N_{0-}. \end{cases} \tag{2.40}$$

It follows from (2.40) that the map ϕ is a contraction for the pair of semi norms N_{0-}, N_{2-} under the condition $C_0 C_2 (c_4 + N_1 \overline{h}(T))^2 < 1$ which is the combination of a smallness condition for c_4 together with a condition that T be sufficiently large. The semi norm N_{1-} does not take part in the contraction, but is controlled separately by the previous ones. The constants C_0 and C_2 appearing in (2.40) can be taken to be the same as in (2.34) because they are determined by the linear terms, which are the same in both cases. There might occur additional constants coming from the nonlinear terms. They have been omitted. This completes the proof of the existence part of the Proposition. Uniqueness follows from (2.40) with $N'_{i-} = N_{i-}$. □

We now turn to the construction of solutions of the system (1.3) at the level of regularity of H^1 for v. The result can be stated as follows.

Proposition 2.3 *Let h be defined as previously with $\lambda = 3/8$ and let $X_1(\,\cdot\,)$ be defined by* (1.15). *Let u_a, A_a, R_1, R_2 satisfy the conditions* (2.15)-(2.18) *and in addition*

$$\|u_a(t)\|_\infty \le c t^{-3/2}, \quad \|\nabla u_a(t)\|_4 \le c t^{-3/4}, \tag{2.41}$$

$$\|\nabla A_a\|_\infty \le a t^{-1}, \tag{2.42}$$

$$\|\nabla R_1; L^1([t,\,\infty),\, L^2)\| \le r_1 h(t), \tag{2.43}$$

$$\|R_2; L^1([t,\,\infty),\, L^2)\| \le r_2 t^{-1/2} h(t) \tag{2.44}$$

for some constants c_4, c, a, r_1, r_2 with c_4 sufficiently small and for all $t \ge 1$. Then there exists T, $1 \le T < \infty$ and there exists a unique solution $(v,\, B)$ of the system (1.3) *in the space $X_1([T,\,\infty))$. Furthermore B satisfies the estimate*

$$\|\nabla B(t)\|_2 \vee \|\partial_t B(t)\|_2 \le C\big(t^{-1/2} + t^{1/4} h(t)\big) h(t) \tag{2.44e}$$

for some constant C and for all $t \ge T$.

Proof. The proof follows closely that of Proposition 2.2. Let $1 \le T < \infty$ and let $(v,\, B) \in X_1([T,\, \infty))$. In particular $(v,\, B)$ satisfies (2.19)-(2.21) and in addition

$$\|\nabla v(t)\|_2 \le N_3 h(t) \tag{2.45}$$

$$\|\nabla v; L^{8/3}([t,\, \infty),\, L^4)\| \le N_4 h(t) \tag{2.46}$$

$$\|\nabla B(t)\|_2 \vee \|\partial_t B(t)\|_2 \le N_5 h(t) \tag{2.47}$$

for some constants N_i and for all $t \ge T$. We first construct a solution $(v',\, B')$ of the system (1.5) in $X_1([T,\, \infty))$. For that purpose, we take t_0, $T < t_0 < \infty$ and we solve the system (1.5) in $X_1(I)$ where $I = [T,\, t_0]$ with initial condition $(v',\, B')(t_0) = 0$. Let $(v'_{t_0},\, B'_{t_0})$ be the solution thereby obtained. The existence of v'_{t_0} follows from Proposition 2.1, part (2) with $V = A$, and $f = Bu_a - R_1$. We want to take the limit of $(v'_{t_0},\, B'_{t_0})$ as $t_0 \to \infty$ and for that purpose we need estimates of $(v'_{t_0},\, B'_{t_0})$ in $X_1(I)$ that are uniform in t_0. Omitting the subscript t_0 for brevity we define N'_i, $0 \le i \le 5$, by (2.22)-(2.24) and by

$$N'_3 = \operatorname*{Sup}_{t\in I} h(t)^{-1} \|\nabla v'(t)\|_2 \tag{2.48}$$

$$N'_4 = \operatorname*{Sup}_{t\in I} h(t)^{-1} \|\nabla v'; L^{8/3}(J,\, L^4)\| \tag{2.49}$$

$$N'_5 = \operatorname*{Sup}_{t\in I} h(t)^{-1} \big(\|\nabla B'(t)\|_2 \vee \|\partial_t B'(t)\|_2\big) \tag{2.50}$$

where $J = [t,\, \infty) \cap I$. We have already estimated N'_i, $0 \le i \le 2$, in the proof of Proposition 2.2. We next estimate $\nabla v'$, starting from the equation

$$\begin{aligned} i\partial_t \nabla v' = -\frac{1}{2}\Delta \nabla v' + A\nabla v' + (\nabla A) v' & \\ + B\nabla u_a + (\nabla B) u_a - \nabla R_1 . & \end{aligned} \tag{2.51}$$

We first estimate N'_3. From (2.6) we obtain

$$\begin{aligned} \|\nabla v'(t)\|_2^2 \le \Big\| \|\nabla v'\|_2 (\|\nabla A_a\|_\infty \|v'\|_2 + \|B\|_4 \|\nabla u_a\|_4 & \\ + \|\nabla B\|_2 \|u_a\|_\infty + \|\nabla R_1\|_2) & \\ + \|\nabla v'\|_4 \|\nabla B\|_2 \|v'\|_4 ; L^1(J) \Big\| & \end{aligned}$$

$$\leq C\big(N_3'\big(aN_0' + cN_2 + cN_5 t^{-1/2} + r_1\big) \\ + N_4' N_5 N_1' t^{-1/8}\overline{h}(t)\big)h(t)^2 \qquad (2.52)$$

by Lemma 2.2, so that

$$N_3' \leq C_3\big(aN_0' + cN_2 + cN_5 T^{-1/2} + r_1 \\ + \big(N_1' N_4' N_5 T^{-1/8}\overline{h}(T)\big)^{1/2}\big). \qquad (2.53)$$

We next estimate N_4'. From Lemma 2.1, we obtain

$$\begin{aligned}
&\|\nabla v'; L^{8/3}(J, L^4)\| \\
&\leq C\big(\|A_a \nabla v' + (\nabla A_a)v' + B\nabla u_a + (\nabla B)u_a - \nabla R_1; L^1(J, L^2)\| \\
&\qquad + \|B\nabla v' + (\nabla B)v'; L^{8/5}(J, L^{4/3})\|\big) \\
&\leq C\big(\big\|\|A_a\|_\infty \|\nabla v'\|_2 + \|\nabla A_a\|_\infty \|v'\|_2 + \|B\|_4 \|\nabla u_a\|_4 \\
&\qquad + \|\nabla B\|_2 \|u_a\|_\infty + \|\nabla R_1\|_2; L^1(J)\big\| \\
&\qquad + \big\|\|B\|_4 \|\nabla v'\|_2 + \|\nabla B\|_2 \|v'\|_4; L^{8/5}(J)\big\|\big)
\end{aligned} \qquad (2.54)$$

so that by Lemma 2.2

$$N_4' \leq C_4\big(a(N_3' + N_0') + c(N_2 + N_5 T^{-1/2}) \\ + r_1 + N_2 N_3' \overline{h}(T) + N_5 N_1' T^{-1/8}\overline{h}(T)\big). \qquad (2.55)$$

We finally estimate N_5'. From (2.14) we obtain

$$\begin{aligned}
&\|\nabla B(t')\|_2 \vee \|\partial_t B(t')\|_2 \\
&\leq \||v|^2 + 2\,\mathrm{Re}\,\overline{u}_a v + R_2; L^1(J, L^2)\| \\
&\leq \big\|\|v\|_4^2 + 2\|u_a\|_\infty \|v\|_2 + \|R_2\|_2; L^1(J)\big\| \\
&\leq C_5\big((cN_0 + r_2)t^{-1/2} + N_1^2 t^{1/4} h(t)\big)h(t)
\end{aligned} \qquad (2.56)$$

by Lemma 2.2 so that

$$N_5' \leq C_5\big((cN_0 + r_2)T^{-1/2} + N_1^2 T^{-1/8}\overline{h}(T)\big). \qquad (2.57)$$

We next take the limit $t_0 \to \infty$ in the same way as in Proposition 2.2. From now on we take $I = [T, \infty)$.

From the previous estimates it follows that the map ϕ defined in Proposition 2.2, when restricted to $X_1(I)$ and more precisely to the subset $\mathcal{R}_1$ of $X_1(I)$ defined by (2.19)-(2.21) and (2.45)-(2.47) satisfies the estimates (2.53)

(2.55) (2.57) in addition to the previous estimates (2.26) (2.29) (2.31) for $(v', B') = \phi((v, B))$ with $(v, B) \in \mathcal{R}_1$. We next show that for a suitable choice of the N_i, $0 \le i \le 5$, and for T sufficiently large, ϕ maps $\mathcal{R}_1$ into itself. We have already chosen N_0, N_1, N_2. We now have to choose N_3, N_4, N_5 so as to ensure that the RHS of (2.53) (2.55) (2.57) do not exceed N_3, N_4 and N_5 respectively. We choose

$$\begin{cases} N_3 = C_3 \left(aN_0 + cN_2 + r_1 + 1 \right) \\ N_4 = C_4 \left(a \left(N_3 + N_0 \right) + cN_2 + r_1 + 1 \right) \\ N_5 = C_5 \end{cases} \tag{2.58}$$

and we take T sufficiently large so that the terms not considered in the RHS of (2.53) (2.55) (2.57) do not exceed 1. Since $\mathcal{R}_1$ is closed in the topology of $X(I)$, ϕ has a fixed point in $\mathcal{R}_1$ by the contraction argument of Proposition 2.2. Finally the estimate (2.44e) follows from (2.56). $\square$

We finally turn to the construction of solutions of the system (1.3) at the level of regularity of H^2 for v. The result can be stated as follows.

Proposition 2.4 *Let h be defined as previously with $\lambda = 3/8$ and let $X_2(\cdot)$ be defined by* (1.16). *Let u_a, A_a, R_1, R_2 satisfy the conditions* (2.15)-(2.18) *and in addition*

$$\|u_a(t)\|_\infty \le ct^{-3/2}, \quad \|\partial_t u_a(t)\|_4 \le ct^{-3/4}, \tag{2.59}$$

$$\|\partial_t A_a\|_\infty \le at^{-1}, \tag{2.60}$$

$$\|\partial_t R_1; L^1([t, \infty), L^2)\| \le r_1 h(t), \tag{2.61}$$

$$\|R_2; L^1([t, \infty), L^2)\| \le r_2 t^{-1/2} h(t) \qquad (2.44) \equiv (2.62)$$

for some constants c_4, c, a, r_1, r_2 with c_4 sufficiently small and for all $t \ge 1$. Then there exists T, $1 \le T < \infty$ and there exists a unique solution (v, B) of the system (1.3) *in the space $X_2([T, \infty))$. Furthermore B satisfies the estimate* (2.44e). *If in addition R_1 satisfies the estimate*

$$\|R_1; L^{8/3}([t, \infty), L^4)\| \le r_1 h(t) \tag{2.63}$$

for all $t \ge 1$, then $\Delta v \in L^{8/3}([T, \infty), L^4)$ and v satisfies the estimate

$$\|\Delta v; L^{8/3}([t, \infty), L^4)\| \le Ch(t) \tag{2.64}$$

for some constant C and for all $t \ge T$.

Proof. The proof is essentially the same as that of Proposition 2.3 with ∇v and $\nabla v'$ replaced everywhere by $\partial_t v$ and $\partial_t v'$ and with additional estimates of $\Delta v'$. The existence of v'_{t_0} with the required properties follows from Proposition 2.1, part (3) and the subset $\mathcal{R}_2$ of $X_2(I)$ invariant under ϕ is now defined by the conditions (2.19)-(2.21) and in addition

$$\|\partial_t v(t)\|_2 \le N_3 h(t) \tag{2.65}$$

$$\|\partial_t v; L^{8/3}([t,\infty), L^4)\| \le N_4 h(t) \tag{2.66}$$

$$\|\nabla B(t)\|_2 \vee \|\partial_t B(t)\|_2 \le N_5 h(t) \tag*{(2.47) ≡ (2.67)}$$

$$\|\Delta v(t)\|_2 \le N_6 h(t) \tag{2.68}$$

for all $t \in I$. The estimates associated with (2.65) (2.66) (2.67) are again (2.53) (2.55) (2.57) except for the fact that we have to estimate in addition

$$\|\partial_t v'(t_0)\|_2 = \|(Bu_a - R_1)(t_0)\|_2 . \tag{2.69}$$

For that purpose, we need pointwise estimates in time of R_1 and B. From (2.61) it follows that $R_1 \in \mathcal{C}([1,\infty), L^2)$ and that

$$\|R_1(t)\|_2 \le \|\partial_t R_1; L^1([t,\infty), L^2)\| \le r_1\ h(t) \tag{2.70}$$

for all $t \ge 1$, while by the definition of $\mathcal{R}_2$ and by Lemma 2.2

$$\|B(t)\|_3^3 \le 3\Big\| \|B\|_4^2 \|\partial_t B\|_2; L^1(J)\Big\| \le C N_2^2 N_5 t^{1/2} h(t)^3 \tag{2.71}$$

since $\|B(t)\|_3 \to 0$ as $t \to \infty$, which can be proved by using a finite time version of (2.71) together with the fact that $\|B(t)\|_6 \to 0$ as $t \to \infty$ by (2.47). Therefore

$$\|B(t)\|_3 \le \widetilde{N}_2 t^{1/6} h(t) \tag{2.72}$$

for all $t \in I$, with $\widetilde{N}_2 = C N_2^{2/3} N_5^{1/3}$.

We then estimate

$$\|Bu_a\|_2 \le \|B\|_3 \|u_a\|_6 \le c\widetilde{N}_2 t^{-5/6} h(t). \tag{2.73}$$

From (2.69) (2.70) (2.73) and the preceding remarks, it follows that N'_3 now defined by

$$N'_3 = \mathop{\mathrm{Sup}}_{t\in I} h(t)^{-1} \|\partial_t v'(t)\|_2$$

satisfies an estimate obtained from (2.53) by adding an extra term $c\widetilde{N}_2 T^{-5/6}$.

We have in addition to estimate $\Delta v'$. From (1.5) we obtain

$$\|\Delta v'\|_r \le 2\big(\|\partial_t v'\|_r + \|A_a\|_\infty \|v'\|_r + \|Bv'\|_r + \|Bu_a\|_r + \|R_1\|_r\big). \quad (2.74)$$

For $r = 2$, we estimate

$$\begin{aligned}\|Bv'\|_2 &\le C\|B\|_3 \|v'\|_2^{1/2} \|\Delta v'\|_2^{1/2}\\ &\le \frac{1}{4}\|\Delta v'\|_2 + C\widetilde{N}_2^2 N'_0 t^{1/3} h(t)^3\end{aligned}$$

so that from (2.74) with $r = 2$ and from (2.70) (2.73)

$$\|\Delta v'\|_2 \le 4\big(N'_3 + at^{-1}N'_0 + r_1 + c\widetilde{N}_2 t^{-5/6} + C\widetilde{N}_2^2 N'_0 t^{1/3} h(t)^2\big) h(t) \quad (2.75)$$

which suffices for the needs of the proof.

If R_1 satisfies (2.63), we can in addition derive (2.64) for v'. We estimate

$$\begin{aligned}\|Bu_a; L^{8/3}(J,\, L^4)\| &\le c\|B; L^4(J,\, L^4)\| \|t^{-3/2}\|_8\\ &\le cN_2 t^{-11/8} h(t)\end{aligned} \quad (2.76)$$

$$\begin{aligned}\|Bv'; L^{8/3}(J,\, L^4)\| &\le \|B; L^4(J,\, L^4)\| \|v'; L^8(J,\, L^\infty)\|\\ &\le C\|B; L^4(J,\, L^4)\| \|v'; L^8(J,\, L^2)\|^{1/24}\\ &\quad \times \|v'; L^{8/3}(J,\, L^4)\|^{1/3} \|\Delta v'; L^\infty(J,\, L^2)\|^{5/8}.\end{aligned} \quad (2.77)$$

From (2.74) with $r = 4$ and from (2.76) (2.77), we obtain

$$\|\Delta v'; L^{8/3}(J, L^4)\| \le 2\big(N'_4 + at^{-1}N'_1 + r_1 + cN_2 t^{-11/8} + CN_2 N_0'^{1/24} N_1'^{1/3} N'_6 h(t)^2\big) h(t) \quad (2.78)$$

where in the same way as before

$$N'_6 = \operatorname*{Sup}_{t\in I} h(t)^{-1} \|\Delta v'(t)\|_2.$$

and N'_6 is estimated by (2.75). □

Remark 2.2 There is some flexibility in the choice of the function spaces used here. For instance we have included the Strichartz norms $L^q(L^r)$ in the restricted range $2 \le r \le 4$. One could equally well use the full range $2 \le r \le 6$. Conversely one could omit the L^2 norm of $\partial_t B$ in the H^1

theory of Proposition 2.3 and/or the L^2 norm of ∇B in the H^2 theory of Proposition 2.4.

3. Remainder estimates and completion of the proof

In this section we first prove that the choice of the asymptotic functions (u_a, A_a) made in the introduction satisfies the assumptions of Propositions 2.2-2.4 under suitable assumptions on the asymptotic state $(u_+, A_+, \dot{A}_+)$. We then combine those results with those of Section 2 to complete the proof of Proposition 1.1.

We first supplement the definition of (u_a, A_a) with some additional properties of a general character. In addition to the representation (1.11) (1.12) for A_1, we need a representation of $\partial_t A_1$. From (1.10) it follows that

$$\partial_t A_1(t) = \int_t^\infty dt' \cos\left(\omega(t-t')\right) |u_a(t')|^2 \tag{3.1}$$

so that upon substitution of (1.6) we obtain

$$\partial_t A_1(t) = t^{-2} D_0(t) \widetilde{\widetilde{A}}_1 \tag{3.2}$$

where

$$\widetilde{\widetilde{A}}_1 = \int_1^\infty d\nu \nu^{-3} \cos(\omega(\nu-1)) D_0(\nu) |w_+|^2. \tag{3.3}$$

On the other hand from (1.11)

$$\nabla A_1(t) = t^{-2} D_0(t) \nabla \widetilde{A}_1. \tag{3.4}$$

We shall also need the commutation relations

$$\nabla MD = MD\left(ix + t^{-1}\nabla\right) \tag{3.5}$$

$$i\partial_t MD = MD\left(\frac{1}{2}x^2 + i\partial_t - it^{-1}\left(x\cdot\nabla + \frac{3}{2}\right)\right) \tag{3.6}$$

$$\left(i\partial_t + \frac{1}{2}\Delta\right) MD = MD\left(i\partial_t + (2t^2)^{-1}\Delta\right). \tag{3.7}$$

From (3.5) (3.6) and (1.13), it follows that

$$\nabla u_a = MD\exp(-i\varphi)\left(ixw_+ + t^{-1}\nabla w_+ - it^{-1}\ell nt(\nabla\widetilde{A}_1)w_+\right) \tag{3.8}$$

$$\partial_t u_a = MD\exp(-i\varphi)\left(\frac{1}{2}x^2 w_+ - it^{-1}\left(x\cdot\nabla + \frac{3}{2}\right)w_+\right.$$

$$+\, t^{-1}\widetilde{A}_1 w_+ - t^{-1}\ell nt(x\cdot\nabla\widetilde{A}_1)w_+\Big). \qquad (3.9)$$

We now consider the remainder R_1 defined by (1.4). (We recall that the choice (1.9) (1.10) yields $R_2 = 0$). From (3.7) (1.13) we obtain

$$R_1 = MD\exp(-i\varphi)\widetilde{R}_1 - A_0 u_a \qquad (3.10)$$

where

$$\widetilde{R}_1 = (2t^2)^{-1}\big(\Delta w_+ - i\ell nt(2(\nabla\widetilde{A}_1)\cdot\nabla w_+ + (\Delta\widetilde{A}_1)w_+) - (\ell nt)^2|\nabla\widetilde{A}_1|^2 w_+\big). \qquad (3.11)$$

From (3.5) (3.6) (3.10) and (1.13) we obtain

$$\nabla R_1 = MD\exp(-i\varphi)(ix + t^{-1}\nabla - it^{-1}\ell nt\nabla\widetilde{A}_1)\widetilde{R}_1 - \nabla(A_0 u_a) \qquad (3.12)$$

$$i\partial_t R_1 = MD\exp(-i\varphi)\Big(\frac{1}{2}x^2 + i\partial_t - it^{-1}\Big(x\cdot\nabla + \frac{3}{2}\Big) + t^{-1}\widetilde{A}_1 - t^{-1}\ell nt(x\cdot\nabla\widetilde{A}_1)\Big)\widetilde{R}_1 - i\partial_t(A_0 u_a). \qquad (3.13)$$

Finally, since $\widetilde{A}_1$ is independent of t, $\partial_t\widetilde{R}_1$ takes the explicit form

$$\partial_t\widetilde{R}_1 = t^{-3}\Big(-\Delta w_+ + i\Big(\ell nt - \frac{1}{2}\Big)\big(2(\nabla\widetilde{A}_1)\cdot\nabla w_+ + (\Delta\widetilde{A}_1)w_+\big) + \ell nt(\ell nt - 1)|\nabla\widetilde{A}_1|^2 w_+\Big). \qquad (3.14)$$

In order to ensure the assumptions of Propositions 2.2-2.4 on u_a, A_a, R_1, we shall use a number of general norm estimates. We first consider u_a. From (1.6) (3.8) (3.9) we obtain

$$\|u_a\|_r \le t^{-\delta(r)}\|w_+\|_r, \qquad (3.15)$$

$$\|\nabla u_a\|_r \le t^{-\delta(r)}\big(\|xw_+\|_r + t^{-1}\|\nabla w_+\|_r + t^{-1}\ell nt\|\nabla\widetilde{A}_1\|_\infty\|w_+\|_r\big), \qquad (3.16)$$

$$\|\partial_t u_a\|_r \le t^{-\delta(r)}\Big(\frac{1}{2}\|x^2 w_+\|_r + t^{-1}\|x\cdot\nabla w_+\|_r + t^{-1}\Big(\|\widetilde{A}_1\|_\infty + \frac{3}{2}\Big)\|w_+\|_r + t^{-1}\ell nt\|\nabla\widetilde{A}_1\|_\infty\|xw_+\|_r\Big) \qquad (3.17)$$

where $\delta(r) = 3/2 - 3/r$.

We next turn to A_1. From (1.11) (3.4) (3.2) we obtain

$$\|A_1\|_\infty = t^{-1}\|\widetilde{A}_1\|_\infty, \tag{3.18}$$

$$\|\nabla A_1\|_\infty = t^{-2}\|\nabla\widetilde{A}_1\|_\infty, \tag{3.19}$$

$$\|\partial_t A_1\|_\infty = t^{-2}\|\widetilde{\widetilde{A}}_1\|_\infty. \tag{3.20}$$

The L^∞ estimates of $\widetilde{A}_1$ and $\widetilde{\widetilde{A}}_1$ will be obtained through Sobolev inequalities from the L^2 estimates

$$\begin{aligned}\|\omega^{m+1}\widetilde{A}_1\|_2 \vee \|\omega^m\widetilde{\widetilde{A}}_1\|_2 &\le \int_1^\infty d\nu\nu^{-3/2-m}\|\omega^m|w_+|^2\|_2\\ &= \left(m+\frac{1}{2}\right)^{-1}\|\omega^m|w_+|^2\|_2\end{aligned} \tag{3.21}$$

which follow readily from (1.12) (3.3). Finally we shall estimate R_1 and its derivatives as follows:

$$\|R_1\|_2 \le \|\widetilde{R}_1\|_2 + t^{-3/2}\|A_0\|_2\|w_+\|_\infty \tag{3.22}$$

where we have used (3.15) and where

$$\begin{aligned}\|\widetilde{R}_1\|_2 \le (2t^2)^{-1}\big(&\|\Delta w_+\|_2 + \ell nt(2\|\nabla\widetilde{A}_1\|_6\|\nabla w_+\|_3\\ &+ \|\Delta\widetilde{A}_1\|_2\|w_+\|_\infty) + (\ell nt)^2\|\nabla\widetilde{A}_1\|_6^2\|\nabla w_+\|_6\big),\end{aligned} \tag{3.23}$$

$$\begin{aligned}\|\nabla R_1\|_2 \le\ & \|x\widetilde{R}_1\|_2 + t^{-1}\|\nabla\widetilde{R}_1\|_2 + t^{-1}\ell nt\|\nabla\widetilde{A}_1\|_\infty\|\widetilde{R}_1\|_2\\ &+t^{-3/2}\|\nabla A_0\|_2\|w_+\|_\infty + \|A_0\|_2\|\nabla u_a\|_\infty,\end{aligned} \tag{3.24}$$

$$\begin{aligned}\|\partial_t R_1\|_2 \le\ & \frac{1}{2}\|x^2\widetilde{R}_1\|_2 + \|\partial_t\widetilde{R}_1\|_2 + t^{-1}\|x\cdot\nabla\widetilde{R}_1\|_2\\ &+ t^{-1}\left(\|\widetilde{A}_1\|_\infty + \frac{3}{2}\right)\|\widetilde{R}_1\|_2 + t^{-1}\ell nt\|\nabla\widetilde{A}_1\|_\infty\|x\widetilde{R}_1\|_2\\ &+ t^{-3/2}\|\partial_t A_0\|_2\|w_+\|_\infty + \|A_0\|_2\|\partial_t u_a\|_\infty,\end{aligned} \tag{3.25}$$

where ∇u_a and $\partial_t u_a$ are estimated in L^∞ by (3.16) (3.17) with $r=\infty$.

In order to estimate A_0, we need some general estimates of solutions of the free wave equation.

Lemma 3.1 *Let A_0 be defined by* (1.9). *Let $k \ge 0$ be an integer. Let A_+ and $\dot{A}_+$ satisfy the conditions*

$$A_+,\, \omega^{-1}\dot{A}_+ \in H^k, \quad \nabla^2 A_+,\, \nabla\dot{A}_+ \in W_1^k. \tag{3.26}$$

Then A_0 satisfies estimates

$$\begin{cases} \|A_0(t); W_r^k\| \le a_0 t^{-1+2/r}, \\ \|\partial_t A_0(t); W_r^{k-1}\| \le a_0 t^{-1+2/r} \quad \textit{for } k \ge 1. \end{cases} \tag{3.27}$$

for $2 \le r \le \infty$ and for all $t \in \mathbb{R}$, where a_0 depends on A_+, $\dot{A}_+$ through the norms associated with (3.26).

A proof can be found in [17].

We are now in a position to derive the estimates required in Propositions 2.2-2.4. We recall that $w_+ = Fu_+$ and that $\delta(r) = 3/2 - 3/r$.

Proposition 3.1

(1) *Let $u_+ \in H^{0,2}$ and let $(A_+, \dot{A}_+)$ satisfy* (3.26) *with $k = 0$. Then the following estimates hold*

$$\|u_a\|_r \le t^{-\delta(r)} \|Fu_+\|_r \quad \textit{for } 2 \le r \le \infty, \tag{3.28}$$

$$\|A_a(t)\|_\infty \le a t^{-1}, \tag{3.29}$$

$$\|R_1(t)\|_2 \le r_1 t^{-3/2}. \tag{3.30}$$

(2) *Let $u_+ \in H^{0,3} \cap H^{1,2}$ and let $(A_+, \dot{A}_+)$ satisfy* (3.26) *with $k = 1$. Then the estimates* (3.28)-(3.30) *hold and in addition*

$$\|\nabla u_a\|_r \le c t^{-\delta(r)} \quad \textit{for } 2 \le r \le \infty, \tag{3.31}$$

$$\|\nabla A_a(t)\|_\infty \le a t^{-1}, \tag{3.32}$$

$$\|\nabla R_1(t)\|_2 \le r_1 t^{-3/2}. \tag{3.33}$$

(3) *Let $u_+ \in H^{1,3} \cap H^{2,2}$ and let $(A_+, \dot{A}_+)$ satisfy* (3.26) *with $k = 1$. Then the estimates* (3.28)-(3.33) *hold and in addition*

$$\|\partial_t u_a\|_r \le c t^{-\delta(r)} \quad \textit{for } 2 \le r \le \infty, \tag{3.34}$$

$$\|\partial_t A_a(t)\|_\infty \le a t^{-1}, \tag{3.35}$$

$$\|\partial_t R_1(t)\|_2 \le r_1 t^{-3/2}. \tag{3.36}$$

Proof. Part (1). The assumption on u_+ is equivalent to $w_+ \in H^2$, which by (3.21) implies that $\nabla \widetilde{A}_1 \in H^2$. The estimate (3.28) is a rewriting of (3.15) and is ensured by the fact that $H^2 \subset L^\infty$. The estimate (3.29) follows from (3.27) as regards the A_0 part and from (3.18) and the previous remarks as regards the A_1 part. Finally (3.30) follows from (3.22) (3.23) (3.27) and Sobolev inequalities.

Part (2). The assumption on u_+ is equivalent to $w_+ \in H^3 \cap H^{2,1}$ which by (3.21) implies that $\nabla \widetilde{A}_1 \in H^3$. Then (3.31) follows from (3.16) while (3.32) follows from (3.27) as regards the A_0 part and from (3.19) and the previous remarks as regards the A_1 part. Finally (3.33) follows from (3.24). In fact, in addition to terms previously estimated, we have to estimate $\|x\widetilde{R}_1\|_2$ and $\|\nabla \widetilde{R}_1\|_2$. The estimate of $x\widetilde{R}_1$ is obtained from (3.23) by replacing Δw_+, ∇w_+ and w_+ by $x\Delta w_+$, $x\nabla w_+$ and xw_+ respectively. The estimate of $\nabla \widetilde{R}_1$ is obtained from (3.23) by distributing ∇ among $\widetilde{A}_1$ and w_+, thereby generating norms which are controlled by the assumption $w_+ \in H^3$.

Part (3). The assumption on u_+ is equivalent to $w_+ \in H^{3,1} \cap H^{2,2}$ which by (3.21) implies that $\nabla \widetilde{A}_1 \in H^3$ and $\widetilde{\widetilde{A}}_1 \in H^2$. Then (3.34) follows from (3.17) while (3.35) follows from (3.27) as regards the A_0 part and from (3.20) and the previous remarks as regards the A_1 part. Finally (3.36) follows from (3.25). In fact, in addition to terms previously estimated, we have to estimate $\|x^2\widetilde{R}_1\|_2$, $\|x \cdot \nabla \widetilde{R}_1\|_2$ and $\|\partial_t \widetilde{R}_1\|_2$. In the same way as in the proof of Part (2), the first two estimates are obtained from that of $\|\widetilde{R}_1\|_2$ by absorbing x^2 or x by w and distributing the gradient among w_+ and $\widetilde{A}_1$, while $\|\partial_t \widetilde{R}_1\|_2$ is estimated in the same way as $\|\widetilde{R}_1\|_2$ from the explicit expression (3.14). □

We can now complete the proof of Proposition 1.1.

Proof of Proposition 1.1. From Parts (1), (2) and (3) of Proposition 3.1, together with the fact that $R_2 = 0$, it follows that the assumptions of Propositions 2.2, 2.3 and 2.4 respectively are satisfied with $h(t) = t^{-1/2}$ and $c_4 = \|w_+\|_4$. In particular (3.30) (3.33) imply (2.63) since

$$\|R_1(t)\|_4 \le C\|R_1(t); H^1\| \le Ct^{-3/2} \tag{3.37}$$

so that

$$\|R_1; L^{8/3}([t, \infty), L^4\| \le Ct^{-9/8}. \tag{3.38}$$

The estimate (1.17) follows from (2.44e) with $h(t) = t^{-1/2}$. □

Remark 3.1 The $t^{-3/2}$ decay of R_1 comes from the free wave term $A_0 u_a$. That term could be partly cancelled by the correcting term used in [15], thereby producing a $t^{-2}(\ell nt)^2$ decay of R_1 allowing for $h(t) = t^{-1}(\ell nt)^2$.

Remark 3.2 The regularity assumptions on u_+ or w_+ are dictated by the term Δw_+ in $\widetilde{R}_1$. They could be somewhat weakened by eliminating that term through the choice

$$w(t) = U\left(\frac{1}{t}\right)^* w_+$$

but that choice would either generate a more complicated and less explicit φ or produce a non vanishing R_2.

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