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Trace formulae of p-hyponormal operators II

(Dedicated to Professors Atsushi Inoue and Takahiko Nakazi on their sixtieth birthdays)

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Abstract. Let T be a semi-hyponormal operator on a complex separable Hilbert space. In this paper, we give Helton-Howe type trace formulae of smooth functions associated with the polar decomposition $T = U|T|$ and improve Theorem 10 of [5] by this result.

Key words: Hilbert space, operator, trace, principal function.

1. Introduction

Let $\mathcal{H}$ be a complex separable Hilbert space and $\{e_j\}$ be an orthonormal basis of $\mathcal{H}$. For a bounded positive semi-definite operator A, $\mathrm{Tr}(A)$ denotes its trace, that is, $\mathrm{Tr}(A) = \sum_j (Ae_j, e_j)$. Let $\mathcal{C}_1$ denote the trace class. For operators A, B, their commutator is denoted by $[A, B]$ $(= AB - BA)$. Let $T = X + iY$ be the Cartesian decomposition of a hyponormal operator T on $\mathcal{H}$ with trace class self-commutator $[T^*, T]$ $(= 2i[X, Y])$. For a polynomial $p(x, y) = \sum a_{ij}x^i y^j$ of two variables, an operator $p(X, Y)$ is defined by $p(X, Y) = \sum a_{ij}X^i Y^j$. For polynomials $p(x, y)$, $q(x, y)$, it is well-known that Helton-Howe type trace formula holds: There exists a summable function g on $\mathbf{R}^2$ such that, for polynomials $p(x, y)$, $q(x, y)$,

$$\mathrm{Tr}\big([p(X, Y), q(X, Y)]\big) = \frac{1}{2\pi i} \iint J(p, q)(x, y) g(x, y)\, dx\, dy,$$

where $J(p, q)(x, y)$ be the Jacobian of p, q. The function g is called the principal function of T. This function g gives much information about the structure of T (see, for example, [2], [9], [10], [11], [12]). In particular, g satisfies that $g(x, y) = 0$ for $x + iy \notin \sigma(T)$. Let $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$ and $T = U|T|$ be the polar decomposition of T. An operator $T \in B(\mathcal{H})$ is called p-hyponormal if $(T^*T)^p \geq (TT^*)^p$ ([1], [3]). If $p = 1$ and $1/2$, then T is called hyponormal and semi-hyponomal, respec-

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tively. By Löwner-Heinz inequality, if $0 < q \leq p \leq 1$ and T is p-hyponormal, then T is q-hyponormal. Under the assumption $|T| - U|T|U^* \in \mathcal{C}_1$, in [2], [5] and [12] Helton-Howe type trace formulae associated with the polar decomposition were studied for operators $\phi(|T|, U)$ and $\psi(|T|, U)$. In [5], for polynomials $p(\,\cdot\,,\,\cdot\,)$ and $q(\,\cdot\,,\,\cdot\,)$ we gave Helton-Howe type trace formula of $p(|T|, U)$ and $q(|T|, U)$ under the assumption that U is unitary. In this paper, using elementary polynomial approximation, we extend this result to Helton-Howe type trace formulae for smooth functions and generalize Theorem 10 of [5] by it.

For $T \in \mathcal{C}_1$, $\|T\|_1$ denotes the trace norm of T. Throughout this paper let $T = U|T| \in B(\mathcal{H})$ be the polar decomposition of T. Hence it satisfies $\ker(U) = \ker(T)$.

2. Trace formula for smooth functions

We denote by $\mathcal{A}$ the linear space of all Laurent polynomials $\mathcal{P}(r, z)$ with polynomial coefficients such that $\mathcal{P}(r, z) = \sum_{k=-N}^{N} p_k(r) z^k$, where N is a non-negative integer and every $p_k(r)$ is a polynomial of one variable. For $T = U|T|$ with unitary U, put $\mathcal{P}(|T|, U) = \sum_{k=-N}^{N} p_k(|T|)U^k$.

We denote by $J(\phi, \psi)$ the Jacobian of functions $\phi(r, z)$ and $\psi(r, z)$ defined on $\mathbf{R} \times \mathbf{C}$, that is,

$$J(\phi, \psi)(r, e^{i\theta}) = \frac{\partial \phi}{\partial r}(r, e^{i\theta}) \cdot \frac{\partial \psi}{\partial z}(r, e^{i\theta}) - \frac{\partial \phi}{\partial z}(r, e^{i\theta}) \cdot \frac{\partial \psi}{\partial r}(r, e^{i\theta}).$$

Let $\mathbf{T} = \{e^{i\theta} | 0 \leq \theta < 2\pi\}$, Σ be the set of all Borel sets in $\mathbf{T}$ and m be a measure on the measure space $(\mathbf{T}, \Sigma)$ such that $dm(\theta) = (1/2\pi)d\theta$. Then we have

Theorem A ([12, Chapter 7, Theorem 3.3],[5, Theorem 9]) *Let $T \in B(\mathcal{H})$ be semi-hyponormal and $T = U|T|$ be the polar decomposition of T. Assume that U is unitary and $[U, |T|] \in \mathcal{C}_1$. Then there exists a summable function g_T such that, for $\mathcal{P}(r, z)$, $\mathcal{Q}(r, z) \in \mathcal{A}$, it holds*

$$\begin{aligned}&\mathrm{Tr}\big([\mathcal{P}(|T|, U),\ \mathcal{Q}(|T|, U)]\big)\\ &\qquad = \iint J(\mathcal{P}, \mathcal{Q})(r, e^{i\theta}) e^{i\theta} g_T(e^{i\theta}, r)\, dr\, dm(\theta).\end{aligned}$$

The function g_T in Theorem A is called the principal function assoicated with the polar decomosition $T = U|T|$.

We denote by $C^\infty(\mathbf{R})$ the set of all smooth functions on $\mathbf{R}$ and by $C_c^\infty(\mathbf{R})$ the set of all functions in $C^\infty(\mathbf{R})$ with compact support. We denote by $\mathcal{B}$ the linear space of all Laurent polynomials $\phi(r, z)$ such that $\phi(r, z) = \sum_{k=-N}^{N} f_k(r)z^k$, where every $f_k \in C^\infty(\mathbf{R})$. For $T = U|T|$ with unitary U, put $\phi(|T|, U) = \sum_{k=-N}^{N} f_k(|T|)U^k$ for $\phi \in \mathcal{B}$.

In [2], Carey and Pincus proved a more general version of Theorem A. It requires complicate calculations. Using polynomial approximation, we improve Theorem A in the following form.

Theorem 1 *Let $T \in B(\mathcal{H})$ be semi-hyponormal and $T = U|T|$ be the polar decomposition of T. Assume that U is unitary and $[U, |T|] \in \mathcal{C}_1$. Then, for $\phi, \psi \in \mathcal{B}$, it holds*

$$\begin{aligned}\operatorname{Tr}\big([\phi(|T|, U), \psi(|T|, U)]\big)& \\ &= \iint J(\phi, \psi)(r, e^{i\theta})e^{i\theta} g_T(e^{i\theta}, r)\,dr\,dm(\theta).\end{aligned}$$

In order to prove Theorem 1, we need some results. Let $\mathcal{S}$ denote the Schwarz space of rapidly decreasing functions on $\mathbf{R}$ at infinity. For $f \in \mathcal{S}$, put

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist} f(s)ds.$$

The function $\hat{f}$ is called the Fourier transform of f. It is known that $\hat{f} \in \mathcal{S}$ and $f(x) = (1/\sqrt{2\pi}) \int e^{ixt}\hat{f}(t)dt$. Let H be a self-adjoint operator. Let $\{E_x\}$ be the spectral resolution of H. Then

$$\begin{aligned} f(H) &= \int f(x)dE_x = \int \left(\frac{1}{\sqrt{2\pi}} \int e^{ixt}\hat{f}(t)dt\right) dE_x \\ &= \frac{1}{\sqrt{2\pi}} \int \left(\int e^{ixt}dE_x\right) \hat{f}(t)dt = \frac{1}{\sqrt{2\pi}} \int e^{itH}\hat{f}(t)dt. \end{aligned}$$

First, we prepare the following proposition (see [9, p. 158 (3.3)]).

Proposition 2 *Let A, $\{B_j\}_{j=1,\dots,n}$ be operators such that $[A, B_j] \in \mathcal{C}_1$ and $\|B_j\| \le r$ for all j ($j = 1, 2, \dots, n$). Then*

$$\big\|[A, B_1B_2\cdots B_n]\big\|_1 \le nr^{n-1} \max_j \big\|[A, B_j]\big\|_1.$$

Let $T = U|T| \in B(\mathcal{H})$ be the polar decomposition of T. Assume that

U is unitary and $[U,\, |T|] \in \mathcal{C}_1$. Then we have

$$[U,\, e^{it|T|}] = \sum_{n=1}^{\infty} \frac{(it)^n}{n!}[U,\, |T|^n].$$

By Proposition 2,

$$\big\|[U,\, e^{it|T|}]\big\|_1 \leq \sum_{n=1}^{\infty} \frac{|t|^n n \|T\|^{n-1}}{n!}\big\|[U,\, |T|]\big\|_1 \leq |t| \cdot \big\|[U,\, |T|]\big\|_1 e^{|t|\cdot\|T\|}.$$

Definition 1 Under the assumption above, we define a constant c_T of an operator $T = U|T|$ satisfying $[U,\, |T|] \in \mathcal{C}_1$ by

$$c_T = \max_{|t|\leq 1}\big\|[U,\, e^{it|T|}]\big\|_1.$$

Proof of the next proposition is based on an idea of the proof of [9, Lemma 3.2].

Proposition 3 *Let $T = U|T| \in B(\mathcal{H})$ be the polar decomposition of T. Assume that U is unitary and $[U,\, |T|] \in \mathcal{C}_1$. Then, for $f \in \mathcal{S}$ and an integer n, it holds*

$$\big\|[U^n,\, f(|T|)]\big\|_1 \leq |n| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c_T(|t|+1)|\hat{f}(t)|dt,$$

where c_T is the constant of Definition 1.

Proof. The proposition is clear for $n = 0$. Let $n > 0$ and $t \in \mathbf{R}$. Then by Proposition 2 we have

$$\big\|[U^n,\, e^{it|T|}]\big\|_1 \leq n \cdot \big\|[U,\, e^{it|T|}]\big\|_1.$$

Let m be a positive integer such that $m-1 < |t| \leq m$ and put $V = e^{i(t/m)|T|}$. Then we have

$$\big\|[U,\, e^{it|T|}]\big\|_1 = \big\|[U,\, V^m]\big\|_1 \leq m \cdot \|[U,\, V]\|_1 \leq m \cdot c_T \leq c_T(|t|+1).$$

Let $n < 0$. Then

$$\begin{aligned}\big\|[U^n,\, e^{it|T|}]\big\|_1 &= \big\|[e^{-it|T|},\, U^{*n}]\big\|_1 \\ &= \big\|[U^{-n},\, e^{i(-t)|T|}]\big\|_1 \leq (-n)\cdot c_T(|t|+1).\end{aligned}$$

Therefore, we have

$$\big\|[U^n,\, e^{it|T|}]\big\|_1 \leq |n| c_T(|t|+1).$$

Then

$$\begin{aligned}\big\|[U^n,\, f(|T|)]\big\|_1 &= \left\|[U^n,\, \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{it|T|}\hat{f}(t)dt]\right\|_1 \\ &= \left\|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[U^n,\, e^{it|T|}]\hat{f}(t)dt\right\|_1 \\ &\le \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\big\|[U^n,\, e^{it|T|}]\big\|_1 \big|\hat{f}(t)\big|dt \\ &\le |n|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} c_T\big(|t|+1\big)\big|\hat{f}(t)\big|dt.\end{aligned}$$

□

Proposition 4 *Let F be a compact set of $\mathbf{R}$ and $f \in C^\infty(\mathbf{R})$. Then there exist a function $f_1 \in C_c^\infty(\mathbf{R})$, a sequence $\{p_n\}$ of polynomials and a sequence $\{\gamma_n\}$ in $C_c^\infty(\mathbf{R})$ such that*

$$\begin{aligned}&f(x) = f_1(x), \quad p_n(x) = \gamma_n(x) \quad \textit{for} \quad x \in F, \\ &\sup_{y\in F}|f_1(y) - \gamma_n(y)| \to 0 \quad (n \to \infty), \\ &\sup_{y\in F}|f_1^{(1)}(y) - \gamma_n^{(1)}(y)| \to 0 \quad (n \to \infty), \\ &\sup_{t\in\mathbf{R}}|\hat{f}_1(t) - \hat{\gamma}_n(t)| \to 0 \quad (n \to \infty),\end{aligned}$$

and

$$\sup_{t\in\mathbf{R}}|t|^3 \cdot |\hat{f}_1(t) - \hat{\gamma}_n(t)| \to 0 \quad (n \to \infty).$$

Proof. Let $F \subset [a,\, b]$ and we choose $\phi \in C_c^\infty(\mathbf{R})$ such that $\phi(x) = 1$ for $x \in F$ and real numbers a_1 and b_1 such that $[a_1,\, b_1]$ contains the support of ϕ. Put $c = a_1 - 1$, $d = b_1 + 1$, $f_0 = f\phi$ and $f_1 = f_0\phi$. Then we have $f_0(c) = f_0^{(1)}(c) = f_0^{(2)}(c) = f_0^{(3)}(c) = 0$. For a continuous function h on $[c,\, d]$, we define $|||h|||$ by

$$|||h||| = \sup_{c\le x\le d}|h(x)|.$$

There exists a sequence $\{p_{n3}\}_{n=1}^\infty$ of polynomials such that

$$\lim_{n\to\infty}|||p_{n3} - f_0^{(3)}||| = 0.$$

We define polynomials p_{n2}, p_{n1} and p_n ($n = 1, 2, \ldots$) as follows:

$$p_{n2}(x) = \int_c^x p_{n3}(t)dt, \quad p_{n1}(x) = \int_c^x p_{n2}(t)dt \quad \text{and} \quad p_n(x) = \int_c^x p_{n1}(t)dt.$$

Then it is easy to check

$$\lim_{n\to\infty} |||p_n - f_0||| = 0 \quad \text{and} \quad \lim_{n\to\infty} |||p_{nj} - f_0^{(j)}||| = 0 \quad (j = 1, 2, 3).$$

Therefore, we have

$$\lim_{n\to\infty} |||p_n^{(j)} - f_0^{(j)}||| = 0 \quad (j = 0, 1, 2, 3).$$

Put $\gamma_n = p_n\phi$ ($n = 1, 2, \ldots$) and it is easy to see that

$$f(x) = f_0(x) = f_1(x) \quad \text{and} \quad p_n(x) = \gamma_n(x) \quad (x \in F).$$

Hence it is sufficient to prove two equalities for Fourier transforms. For $t \in \mathbf{R}$, it holds

$$\begin{aligned} |\hat{f}_1(t) - \hat{\gamma}_n(t)| &= \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itx}(f_0(x) - p_n(x))\phi(x)dx \right| \\ &= \left| \frac{1}{\sqrt{2\pi}} \int_c^d e^{-itx}(f_0(x) - p_n(x))\phi(x)dx \right| \\ &\le \frac{1}{\sqrt{2\pi}} |||f_0 - p_n||| \cdot |||\phi||| \end{aligned}$$

and

$$\begin{aligned} &(it)^3(\hat{f}_1(t) - \hat{\gamma}_n(t)) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itx}\big((f_0(x) - p_n(x))\phi(x)\big)^{(3)}dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itx}\Big(\big(f_0(x) - p_n(x)\big)\phi^{(3)}(x) \\ &\qquad + 3\big(f_0^{(1)}(x) - p_{n1}(x)\big)\phi^{(2)}(x) \\ &\qquad + 3\big(f_0^{(2)}(x) - p_{n2}(x)\big)\phi^{(1)}(x) + \big(f_0^{(3)}(x) - p_{n3}(x)\big)\phi(x)\Big)dx \\ &= \frac{1}{\sqrt{2\pi}} \int_c^d e^{-itx}\Big(\big(f_0(x) - p_n(x)\big)\phi^{(3)}(x) \end{aligned}$$

$$
\begin{aligned}
&+3\big(f_0^{(1)}(x)-p_n^{(1)}(x)\big)\phi^{(2)}(x)\\
&+3\big(f_0^{(2)}(x)-p_n^{(2)}(x)\big)\phi^{(1)}(x)+\big(f_0^{(3)}(x)-p_n^{(3)}(x)\big)\phi(x)\Big)dx.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&\big|t^3\big(\hat{f}_1(t)-\hat{\gamma}_n(t)\big)\big|\\
&\le \frac{d-c}{\sqrt{2\pi}}\big(|||f_0-p_n|||\cdot|||\phi^{(3)}|||+3|||f_0^{(1)}-p_{n1}|||\cdot|||\phi^{(2)}|||\\
&\qquad\qquad +3|||f_0^{(2)}-p_{n2}|||\cdot|||\phi^{(1)}|||+|||f_0^{(3)}-p_{n3}|||\cdot|||\phi|||\big).
\end{aligned}
$$

□

Proposition 5 *Let $T=U|T|$ be the polar decomposition. Assume that U is unitary and $[U,\,|T|]\in\mathcal{C}_1$. Then, for $f,\,g\in\mathcal{S}$ and integers $m,\,n$, it holds*

$$
\begin{aligned}
&\|[f(|T|)U^n,\,g(|T|)U^m]\|_1\\
&\le |n|\cdot|||f|||\frac{1}{2\pi}\int_{-\infty}^{\infty}c_T(|t|+1)|\hat{g}(t)|dt\\
&\quad+|m|\cdot|||g|||\frac{1}{2\pi}\int_{-\infty}^{\infty}c_T(|t|+1)|\hat{f}(t)|dt
\end{aligned}
$$

where c_T is the constant of Definition 1 *and $|||h|||=\sup_{x\in\sigma(|T|)}|h(x)|$.*

Proof. By Proposition 3, we have

$$
\begin{aligned}
&\big\|f(|T|)U^ng(|T|)U^m-f(|T|)g(|T|)U^{m+n}\big\|_1\\
&=\big\|f(|T|)[U^n,\,g(|T|)]U^m\big\|_1\\
&\le\big\|f(|T|)\big\|\cdot\big\|[U^n,\,g(|T|)]\big\|_1\|U^m\|\\
&\le |n|\cdot|||f|||\frac{1}{2\pi}\int_{-\infty}^{\infty}c_T(|t|+1)|\hat{g}(t)|dt.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\big\|g(|T|)U^mf(|T|)U^n-g(|T|)f(|T|)U^{m+n}\big\|_1\\
&=\big\|g(|T|)[U^m,\,f(|T|)]U^n\big\|_1\\
&\le\big\|g(|T|)\big\|\cdot\big\|[U^m,\,f(|T|)]\big\|_1\|U^n\|\\
&\le |m|\cdot|||g|||\frac{1}{2\pi}\int_{-\infty}^{\infty}c_T(|t|+1)|\hat{f}(t)|dt.
\end{aligned}
$$

Therefore, we have

$$\begin{aligned}&\big\|[U^n f(|T|),\, U^m g(|T|)]\big\|_1\\ &\le |n|\cdot |||f|||\frac{1}{2\pi}\int_{-\infty}^{\infty} c_T(|t|+1)|\hat{g}(t)|dt\\ &\quad + |m|\cdot |||g|||\frac{1}{2\pi}\int_{-\infty}^{\infty} c_T(|t|+1)|\hat{f}(t)|dt.\end{aligned}$$

□

Proof of Theorem 1. It is sufficient to prove the theorem for functions $\phi(r,\, z) = f(r)z^k$ and $\psi(r,\, z) = g(r)z^m$ of $f,\, g \in C^\infty(\mathbf{R})$ and integers $k,\, m$. Let a compact set F of Proposition 4 be $\sigma(|T|)$. For f (resp. g) and $\sigma(|T|)$, we choose $f_1 \in C_c^\infty(\mathbf{R})$ (resp. $g_1 \in C_c^\infty(\mathbf{R})$), a sequence $\{p_n\}$ (resp. $\{q_n\}$) of polynomials and a sequence $\{\gamma_n\} \subset C_c^\infty(\mathbf{R})$ (resp. $\{\eta_n\} \subset C_c^\infty(\mathbf{R})$) just as Proposition 4. We denote $\sup_{x\in\sigma(|T|)} |h(x)|$ by $|||h|||$. Let c_T be the constant of Definition 1. Then we have

$$\begin{aligned}&\big\|[f_1(|T|)U^k,\, g_1(|T|)U^m] - [\gamma_n(|T|)U^k,\, \eta_n(|T|)U^m]\big\|_1\\ &= \big\|[(f_1(|T|) - \gamma_n(|T|))U^k,\, g_1(|T|)U^m]\\ &\qquad + [\gamma_n(|T|)U^k,\, (g_1(|T|) - \eta_n(|T|))U^m]\big\|_1\\ &\le |k|\cdot |||f_1 - \gamma_n|||\frac{1}{2\pi}\int_{-\infty}^{\infty} c_T(|t|+1)|\hat{g}_1(t)|dt\\ &\quad + |m|\cdot |||g_1|||\frac{1}{2\pi}\int_{-\infty}^{\infty} c_T(|t|+1)|\hat{f}_1(t) - \hat{\gamma}_n(t)|dt\\ &\quad + |k|\cdot |||\gamma_n|||\frac{1}{2\pi}\int_{-\infty}^{\infty} c_T(|t|+1)|\hat{g}_1(t) - \hat{\eta}_n(t)|dt\\ &\quad + |m|\cdot |||g_1 - \eta_n|||\frac{1}{2\pi}\int_{-\infty}^{\infty} c_T(|t|+1)|\hat{\gamma}_n(t)|dt.\end{aligned}$$

Put

$$\begin{aligned}A_n &= |||f_1 - \gamma_n|||\frac{1}{2\pi}\int_{-\infty}^{\infty} c_T(|t|+1)|\hat{g}_1(t)|dt,\\ B_n &= |||g_1|||\frac{1}{2\pi}\int_{-\infty}^{\infty} c_T(|t|+1)|\hat{f}_1(t) - \hat{\gamma}_n(t)|dt,\\ C_n &= |||\gamma_n|||\frac{1}{2\pi}\int_{-\infty}^{\infty} c_T(|t|+1)|\hat{g}_1(t) - \hat{\eta}_n(t)|dt\end{aligned}$$

and

$$D_n = |||g_1 - \eta_n||| \frac{1}{2\pi} \int_{-\infty}^{\infty} c_T(|t|+1)|\hat{\gamma}_n(t)|dt.$$

By Proposition 4, we have $\lim_{n\to\infty} A_n = 0$, $\lim_{n\to\infty} B_n = 0$, $\lim_{n\to\infty} C_n = 0$ and $\lim_{n\to\infty} D_n = 0$. In fact, it is easy to see that

$$\lim_{n\to\infty} A_n = 0.$$

For any $R > 0$, it holds

$$\begin{aligned} B_n \leq |||g_1||| \Big(&\frac{1}{2\pi} \int_{|t|\leq R} c_T(|t|+1)|\hat{f}_1(t) - \hat{\gamma}_n(t)|dt \\ &+ \frac{1}{2\pi} \int_{|t|>R} c_T \frac{|t|+1}{|t|^3} \cdot |t|^3 |\hat{f}_1(t) - \hat{\gamma}_n(t)|dt \Big). \end{aligned}$$

Hence we have

$$\lim_{n\to\infty} B_n = 0.$$

Similarly, we have

$$\lim_{n\to\infty} C_n = 0.$$

Finally, it holds

$$\begin{aligned} D_n &\leq |||g_1 - \eta_n||| \frac{1}{2\pi} \int_{-\infty}^{\infty} c_T(|t|+1)(|\hat{\gamma}_n(t) - f_1(t)| + |f_1(t)|)dt \\ &\leq |||g_1 - \eta_n||| \Big(\frac{1}{2\pi} \int_{|t|\leq R} c_T(|t|+1)|\hat{\gamma}_n(t) - f_1(t)|dt \\ &\qquad + \frac{1}{2\pi} \int_{|t|>R} c_T \frac{|t|+1}{|t|^3} \cdot |t|^3 |\hat{\gamma}_n(t) - f_1(t)|dt \\ &\qquad + \frac{1}{2\pi} \int_{-\infty}^{\infty} c_T(|t|+1)|f_1(t)|dt \Big), \end{aligned}$$

so that

$$\lim_{n\to\infty} D_n = 0.$$

Therefore, it holds

$$\|[f_1(|T|)U^k,\, g_1(|T|)U^m] - [\gamma_n(|T|)U^k,\, \eta_n(|T|)U^m]\|_1 \longrightarrow 0$$

$$(n \to \infty).$$

Since $|\operatorname{Tr}(A)| \le \|A\|_1$ for $A \in \mathcal{C}_1$ ([8, Theorem III.8.5]) and $[f_1(|T|)U^k,\, g_1(|T|)U^m] \in \mathcal{C}_1$ by Proposition 5, we have

$$\operatorname{Tr}\big([f_1(|T|)U^k,\, g_1(|T|)U^m]\big) = \lim_{n\to\infty} \operatorname{Tr}\big([\gamma_n(|T|)U^k,\, \eta_n(|T|)U^m]\big).$$

Then for every n, it holds $p_n(x) = \gamma_n(x)$ and $q_n(x) = \eta_n(x)$ for $x \in \sigma(|T|)$. Let $\mathcal{P}_n(x,\, y) = p_n(x)y^k$ and $\mathcal{Q}_n(x,\, y) = q_n(x)y^m$. Then it holds

$$\operatorname{Tr}\big([\gamma_n(|T|)U^k,\, \eta_n(|T|)U^m]\big) = \operatorname{Tr}\big([\mathcal{P}_n(|T|,\, U),\, \mathcal{Q}_n(|T|,\, U)]\big).$$

Since p_n and q_n are polynomials, by Theorem A we have

$$\begin{aligned}&\operatorname{Tr}\big([\mathcal{P}_n(|T|,\, U),\, \mathcal{Q}_n(|T|,\, U)]\big)\\ &\qquad = \iint J(\mathcal{P}_n,\, \mathcal{Q}_n)(r,\, e^{i\theta})e^{i\theta}g_T(e^{i\theta},\, r)\,dr\,dm(\theta).\end{aligned}$$

Since $f(x) = f_1(x)$ and $g(x) = g_1(x)$ for $x \in \sigma(|T|)$, $f(|T|) = f_1(|T|)$ and $g(|T|) = g_1(|T|)$. By Proposition 4, we have unifromly $\lim_{n\to\infty} p_n(x) = f(x)$, $\lim_{n\to\infty} q_n(x) = g(x)$, $\lim_{n\to\infty} p_n'(x) = f'(x)$ and $\lim_{n\to\infty} q_n'(x) = g'(x)$ for $x \in \sigma(|T|)$. Then we have $\lim_{n\to\infty} \mathcal{P}_n(x,\, y) = \phi(x,\, y)$ and $\lim_{n\to\infty} \mathcal{Q}_n(x,\, y) = \psi(x,\, y)$ for $x \in \sigma(|T|)$. Therefore, we have

$$\begin{aligned}&\operatorname{Tr}\big([f(|T|)U^k,\, g(|T|)U^m]\big)\\ &= \operatorname{Tr}\big([f_1(|T|)U^k,\, g_1(|T|)U^m]\big)\\ &= \lim_{n\to\infty} \operatorname{Tr}\big([\mathcal{P}_n(|T|,\, U),\, \mathcal{Q}_n(|T|,\, U)]\big)\\ &= \lim_{n\to\infty} \iint J(\mathcal{P}_n,\, \mathcal{Q}_n)(r,\, e^{i\theta})e^{i\theta}g_T(e^{i\theta},\, r)\,dr\,dm(\theta)\\ &= \lim_{n\to\infty} \int_{\mathbf{T}} \left(\int_{\sigma(|T|)} J(\mathcal{P}_n,\, \mathcal{Q}_n)(r,\, e^{i\theta})e^{i\theta}g_T(e^{i\theta},\, r)\,dr\right) dm(\theta)\\ &= \int_{\mathbf{T}} \left(\int_{\sigma(|T|)} J(\phi,\, \psi)(r,\, e^{i\theta})e^{i\theta}g_T(e^{i\theta},\, r)\,dr\right) dm(\theta)\\ &= \iint J(\phi,\, \psi)(r,\, e^{i\theta})e^{i\theta}g_T(e^{i\theta},\, r)\,dr\,dm(\theta).\end{aligned}$$

□

Next we apply this result to p-hyponormal operators $(0 < p < 1/2)$.

Definition 2 Let $T = U|T|$ be p-hyponormal with unitary U. Put $S = U|T|^{2p}$. Then S is semi-hyponormal with unitary U. Hence there exists the Pincus principal function g_S of S and we define the principal function g_T of T by

$$g_T(e^{i\theta}, r) = g_S(e^{i\theta}, r^{1/(2p)})$$

(see [5, Definition 3]).

We finally give a generalization of Theorem 10 of [5].

Theorem 6 *Let $T = U|T|$ be an invertible p-hyponormal operator. If $|T|^{2p} - U|T|^{2p}U^* \in \mathcal{C}_1$, then for $\mathcal{P}(r, z)$, $\mathcal{Q}(r, z) \in \mathcal{A}$ it holds*

$$\begin{aligned}\mathrm{Tr}\big([\mathcal{P}(|T|, U), \mathcal{Q}(|T|, U)]\big) \\ = \iint J(\mathcal{P}, \mathcal{Q})(r, e^{i\theta})e^{i\theta}g_T(e^{i\theta}, r)drdm(\theta).\end{aligned}$$

Proof. Since T is invertible, U is unitary. Define $S = U|T|^{2p}$. Then S is semi-hyponormal with unitary U and it holds $\sigma(S) = \{r^{2p}e^{i\theta} : re^{i\theta} \in \sigma(T)\}$ ([6, Theorem 3]). Since $0 \notin \sigma(|T|)$, we have $0 \notin \sigma(|S|)$ and hence $f(r) = r^{1/(2p)}$ is smooth on $\sigma(|S|)$. Then we choose a function $\eta \in C_c^\infty(\mathbf{R})$ such that $\eta(x) = f(x)$ for $x \in \sigma(|S|)$. Put $\widetilde{\mathcal{P}}(r, z) = \mathcal{P}(\eta(r), z)$ and $\widetilde{\mathcal{Q}}(r, z) = \mathcal{Q}(\eta(r), z)$. Hence we have $\widetilde{\mathcal{P}}, \widetilde{\mathcal{Q}} \in \mathcal{B}$. By Theorem 1, it holds

$$\begin{aligned}\mathrm{Tr}\big([\widetilde{\mathcal{P}}(|S|, U), \widetilde{\mathcal{Q}}(|S|, U)]\big) \\ = \iint J(\widetilde{\mathcal{P}}, \widetilde{\mathcal{Q}})(s, e^{i\theta})e^{i\theta}g_S(e^{i\theta}, s)dsdm(\theta).\end{aligned}$$

Since $g_S(e^{i\theta}, s) = 0$ for $se^{i\theta} \notin \sigma(S)$ (see, for example, [5, Theroem 1, Definition 3]),

$$\begin{aligned}\iint J(\widetilde{\mathcal{P}}, \widetilde{\mathcal{Q}})(s, e^{i\theta})e^{i\theta}g_S(e^{i\theta}, s)dsdm(\theta) \\ = \iint_{\sigma(S)} J(\widetilde{\mathcal{P}}, \widetilde{\mathcal{Q}})(s, e^{i\theta})e^{i\theta}g_S(e^{i\theta}, s)dsdm(\theta).\end{aligned}$$

Since $|S| = |T|^{2p}$, we have

$$\mathrm{Tr}\big([\widetilde{\mathcal{P}}(|S|, U), \widetilde{\mathcal{Q}}(|S|, U)]\big) = \mathrm{Tr}\big([\mathcal{P}(|T|, U), \mathcal{Q}(|T|, U)]\big).$$

Hence

$$\operatorname{Tr}\big([\mathcal{P}(|T|, U), \mathcal{Q}(|T|, U)]\big) = \iint_{\sigma(S)} J(\widetilde{\mathcal{P}}, \widetilde{\mathcal{Q}})(s, e^{i\theta})e^{i\theta}g_S(e^{i\theta}, s)dsdm(\theta).$$

For $se^{i\theta} \in \sigma(S)$, it holds

$$J(\widetilde{\mathcal{P}}, \widetilde{\mathcal{Q}})(s, e^{i\theta}) = J(\mathcal{P}, \mathcal{Q})(s^{1/(2p)}, e^{i\theta})\frac{1}{2p}s^{1/(2p)-1} \quad \text{and}$$
$$g_T(e^{i\theta}, r) = g_S(e^{i\theta}, r^{1/(2p)}).$$

From the translation $s = r^{2p}$, we have

$$\begin{aligned}
&\operatorname{Tr}\big([\mathcal{P}(|T|, U), \mathcal{Q}(|T|, U)]\big)\\
&= \iint_{\sigma(S)} J(\widetilde{\mathcal{P}}, \widetilde{\mathcal{Q}})(s, e^{i\theta})e^{i\theta}g_S(e^{i\theta}, s)dsdm(\theta)\\
&= \iint_{\sigma(S)} J(\mathcal{P}, \mathcal{Q})(s^{1/(2p)}, e^{i\theta})\frac{1}{2p}s^{1/(2p)-1}e^{i\theta}g_T(e^{i\theta}, s^{1/(2p)})dsdm(\theta)\\
&= \iint_{\sigma(T)} J(\mathcal{P}, \mathcal{Q})(r, e^{i\theta})e^{i\theta}g_T(e^{i\theta}, r)drdm(\theta)\\
&= \iint J(\mathcal{P}, \mathcal{Q})(r, e^{i\theta})e^{i\theta}g_T(e^{i\theta}, r)drdm(\theta).
\end{aligned}$$

□

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