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Strongly almost $(V, \lambda)(\Delta^r)$-summable sequences defined by Orlicz functions

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Abstract. The purpose of this paper is to introduce the space of sequences that are strongly almost $(V, \lambda)(\Delta^r)$-summable with respect to an Orlicz function. We give some relations related to these sequence spaces. We also show that the space $[\hat{V}, \lambda, M](\Delta^r) \cap \ell_\infty(\Delta^r)$ may be represented as a $\hat{s}_\lambda(\Delta^r) \cap \ell_\infty(\Delta^r)$ space.

Key words: Almost (V, λ)-summability, Almost statistical convergence, Orlicz function.

1. Introduction

Let w be the set of all sequences of real or complex numbers and ℓ_∞, c and c_0 be respectively the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ with the usual norm $\|x\| = \sup |x_k|$, where $k \in \mathbb{N} = \{1, 2, \ldots\}$, the set of positive integers. The difference sequence spaces was introduced by Kızmaz [10] and the concept was generalized by Et and Çolak [4] as follows:

$$X(\Delta^r) = \{x \in w \colon \Delta^r x \in X\},$$

for $X = \ell_\infty$, c and c_0, where $r \in \mathbb{N}$, $\Delta^0 x = x$, $\Delta x = (x_k - x_{k+1})$, $\Delta^r x = (\Delta^{r-1} x_k - \Delta^{r-1} x_{k+1})$, and so $\Delta^r x_k = \sum_{v=0}^{r} (-1)^v \binom{r}{v} x_{k+v}$. These sequence spaces are BK-spaces with the norm $\|x\|_\Delta = \sum_{i=1}^{r} |x_i| + \|\Delta^r x\|_\infty$.

A sequence $x \in \ell_\infty$ is said to be almost convergent if all its Banach limits coincide and the set of all almost convergent sequences is denoted by $\hat{c}$. Lorentz [14] proved that $x \in \hat{c}$ if and only if $\lim_n (1/n) \sum_{k=1}^{n} x_{k+m}$ exists uniformly in m.

Several authors including Duran [2], King [9], Nanda [19], Et and Basarir [3], Malkowsky and Savas [17] and Altınok et al. [1] have studied almost convergent sequences. Maddox [15], [16] has defined x to be strongly almost convergent to a number ℓ if

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$$\lim_n \frac{1}{n}\sum_{k=1}^{n} |x_{k+m} - \ell| = 0, \quad \text{uniformly in } m.$$

By $[\hat{c}]$ we denote the space of all strongly almost convergent sequences. It is easy to see that $c \subset [\hat{c}] \subset \hat{c} \subset \ell_\infty$.

Orlicz [22] used the idea of Orlicz function to construct the space (L^M). Subsequently Lindenstrauss and Tzafriri [13] defined the sequence space ℓ_M as follows:

$$\ell_M = \left\{ x \in w \colon \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \quad \text{for some } \rho > 0 \right\}.$$

The space ℓ_M is a Banach space with the norm

$$\|x\| = \inf\left\{ \rho > 0 \colon \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

and this space is called an Orlicz sequence space. Lindenstrauss and Tzafriri proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p for some $p \geq 1$. For $M(t) = t^p$, $1 \leq p < \infty$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

An Orlicz function is a function $M\colon [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)$, $u \geq 0$ (for further details see Krasnoselskii and Rutitsky [11], Orlicz [21]).

It is well known that if M is a convex function and $M(0) = 0$, then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Definition 1 Any two Orlicz functions M_1 and M_2 are said to be equivalent if there are positive constants α and β, and x_0 such that $M_1(\alpha x) \leq M_2(x) \leq M_1(\beta x)$ for all x with $0 \leq x \leq x_0$ (see Kamthan and Gupta [8]).

Orlicz sequence spaces have been studied by Nung and Lee [20], Güngör and Et [7], Tripathy et al. [25] and many others.

Let $x \in w$ and $X, Y \subset w$. Then we shall write

$$E(X, Y) = \bigcap_{x \in X} x^{-1} * Y = \{a \in w \colon ax \in Y \text{ for all } x \in X\} \text{ [26]}.$$

The set $X^\alpha = E(X, \ell_1)$ is called Köthe-Toeplitz dual space or α-dual of X.

Let X be a sequence space. Then X is called

i) Solid (or normal), if $(\alpha_k x_k) \in X$ whenever $(x_k) \in X$ for all sequences (α_k) of scalar with $|\alpha_k| \leq 1$.

ii) Monotone provided X contains the canonical preimages of all its step-space.

iii) Perfect $X = X^{\alpha\alpha}$

iv) Symmetric if $(x_k) \in X$ implies $(x_{\pi(k)}) \in X$, where $\pi(k)$ is a permutation of $\mathbb{N}$.

v) A sequence algebra if (x_k), $(y_k) \in X$ implies $(x_k y_k) \in X$.

Remark It is well known that "X is perfect $\Longrightarrow$ X is normal $\Longrightarrow$ X is monotone".

The generalized de la Vallée-Pousin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $\lambda = (\lambda_n)$ is a non-decreasing sequence of positive numbers such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$, $\lambda_n \to \infty$ as $n \to \infty$ and $I_n = [n - \lambda_n + 1, n]$.

A sequence $x = (x_k)$ is said to be (V, λ)-summable to a number ℓ [12] if $t_n(x) \to \ell$ as $n \to \infty$. (V, λ)-summability reduces to $(C, 1)$ summability when $\lambda_n = n$ for all n.

The following inequality will be used throughout this paper. Let $p = (p_k)$ be a sequence of positive real numbers with $0 < p_k \leq \sup p_k = G$ and let $D = \max(1, 2^{G-1})$. For a_k, $b_k \in \mathbb{C}$, the set of complex numbers, we have

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\}. \tag{1}$$

2. Some new sequence spaces defined by an Orlicz function

In this section we introduce the concept of strongly almost $(V, \lambda)(\Delta^r)$-summable sequences with respect to an Orlicz function and examine some properties of the space of strongly almost $(V, \lambda)(\Delta^r)$-summable sequences with respect to an Orlicz function.

Definition 2 Let M be an Orlicz function and $p = (p_k)$ be any sequence of strictly positive real numbers. We define the following sets of sequences.

$$[\hat{V},\, \lambda,\, M,\, p](\Delta^r) = \left\{ x = (x_k)\colon \begin{array}{l} \lim_n \dfrac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\dfrac{|\Delta^r x_{k+m} - \ell|}{\rho} \right) \right]^{p_k} = 0 \\ \text{uniformly in } m, \text{ for some } \ell \text{ and } \rho > 0 \end{array} \right\},$$

$$[\hat{V},\, \lambda,\, M,\, p]_0(\Delta^r) = \left\{ x = (x_k)\colon \begin{array}{l} \lim_n \dfrac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\dfrac{|\Delta^r x_{k+m}|}{\rho} \right) \right]^{p_k} = 0 \\ \text{uniformly in } m, \text{ for some } \rho > 0 \end{array} \right\},$$

$$[\hat{V},\, \lambda,\, M,\, p]_\infty(\Delta^r) = \left\{ x = (x_k)\colon \begin{array}{l} \sup_{m,n} \dfrac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\dfrac{|\Delta^r x_{k+m}|}{\rho} \right) \right]^{p_k} < \infty \\ \text{for some } \rho > 0 \end{array} \right\}.$$

We denote $[\hat{V}, \lambda, M, p](\Delta^r)$, $[\hat{V}, \lambda, M, p]_0(\Delta^r)$ and $[\hat{V}, \lambda, M, p]_\infty(\Delta^r)$ by $[\hat{V}, \lambda, M](\Delta^r)$, $[\hat{V}, \lambda, M]_0(\Delta^r)$ and $[\hat{V}, \lambda, M]_\infty(\Delta^r)$, respectively, when $p_k = 1$ for all k. If $x \in [\hat{V}, \lambda, M](\Delta^r)$ then we say that x is strongly almost $(V, \lambda)(\Delta^r)$-summable with respect to the Orlicz function M.

Theorem 2.1 *Let M be an Orlicz function. Then $[\hat{V}, \lambda, M, p]_0(\Delta^r) \subset [\hat{V}, \lambda, M, p](\Delta^r) \subset [\hat{V}, \lambda, M, p]_\infty(\Delta^r)$ and the inclusions are strict.*

Proof. The inclusion $[\hat{V}, \lambda, M, p]_0(\Delta^r) \subset [\hat{V}, \lambda, M, p](\Delta^r)$ is obvious. Now let $x \in [\hat{V}, \lambda, M, p](\Delta^r)$. Then there exists some positive number ρ_1 such that

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|\Delta^r x_{k+m} - \ell|}{\rho_1} \right) \right]^{p_k} \to 0, \quad \text{uniformly in } m.$$

Define $\rho = 2\rho_1$. Since M is non decreasing and convex, we have

$$\begin{aligned} &\frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|\Delta^r x_{k+m}|}{\rho} \right) \right]^{p_k} \\ &\le \frac{1}{\lambda_n} \sum_{k \in I_n} \frac{1}{2^{p_k}} \left[M\left(\frac{|\Delta^r x_{k+m} - \ell|}{\rho_1} \right) + M\left(\frac{|\ell|}{\rho_1} \right) \right]^{p_k} \\ &\le \frac{D}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|\Delta^r x_{k+m} - \ell|}{\rho_1} \right) \right]^{p_k} + \frac{D}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|\ell|}{\rho_1} \right) \right]^{p_k} \end{aligned}$$

$$\leq \frac{D}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|\Delta^r x_{k+m}-\ell|}{\rho_1}\right)\right]^{p_k} + D\max\left\{1,\ \left[M\left(\frac{|\ell|}{\rho_1}\right)\right]^G\right\},$$

by (1). Thus $x \in [\hat{V},\lambda,M,p]_\infty(\Delta^r)$. To show the inclusions are strict consider the following example. □

Example 1 Let $M(x) = x$, $p_k = 1$ for all $k \in \mathbb{N}$ and $\lambda_n = n$ for all $n \in \mathbb{N}$. Then the sequence $x = (k^r)$ belongs to $[\hat{V},\lambda,M,p](\Delta^r)$ but does not belong to $[\hat{V},\lambda,M,p]_0(\Delta^r)$.

Theorem 2.2 *For any Orlicz function M and a bounded sequence $p = (p_k)$ of strictly positive real numbers, $[\hat{V},\lambda,M,p](\Delta^r)$, $[\hat{V},\lambda,M,p]_0(\Delta^r)$ and $[\hat{V},\lambda,M,p]_\infty(\Delta^r)$ are linear space over the field of complex numbers.*

Proof. We shall prove only for $[\hat{V},\lambda,M,p]_0(\Delta^r)$. The other cases can be proved similarly. Let $x, y \in [\hat{V},\lambda,M,p]_0(\Delta^r)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|\Delta^r x_{k+m}|}{\rho_1}\right)\right]^{p_k} \to 0$$

and

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|\Delta^r y_{k+m}|}{\rho_2}\right)\right]^{p_k} \to 0, \text{uniformly in } m.$$

Define $\rho_3 = \max(2|\alpha|\rho_1,\ 2|\beta|\rho_2)$. Since M is non-decreasing and convex and Δ^r linear

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|\Delta^r(\alpha x_{k+m}+\beta y_{k+m})|}{\rho_3}\right)\right]^{p_k}$$
$$\leq \frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|\alpha\Delta^r x_{k+m}|}{\rho_3}+\frac{|\beta\Delta^r y_{k+m}|}{\rho_3}\right)\right]^{p_k}$$
$$\leq \frac{1}{\lambda_n}\sum_{k\in I_n}\frac{1}{2^{p_k}}\left[M\left(\frac{|\Delta^r x_{k+m}|}{\rho_1}\right)+M\left(\frac{|\Delta^r y_{k+m}|}{\rho_2}\right)\right]^{p_k}$$
$$\leq \frac{D}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|\Delta^r x_{k+m}|}{\rho_1}\right)\right]^{p_k} + \frac{D}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|\Delta^r y_{k+m}|}{\rho_2}\right)\right]^{p_k} \to 0$$

as $n \to \infty$ uniformly in m. This proves that $[\hat{V},\lambda,M,p]_0(\Delta^r)$ is linear space. □

Theorem 2.3 *For any Orlicz function M and a bounded sequence $p = (p_k)$ of strictly positive real numbers, $[\hat{V},\ \lambda,\ M,\ p]_0(\Delta^r)$ is paranormed space (not necessarily total paranormed) with*

$$g(x) = \inf_{\substack{\rho>0\\ n\geq 1}} \left\{ \rho^{\frac{p_n}{H}} :\ \sup_k M\left(\frac{|\Delta^r x_{k+m}|}{\rho}\right) \leq 1, \textit{uniformly in } m \right\}$$

where $H = \max(1, \sup_k p_k)$.

Proof. Clearly $g(x) = g(-x)$. Since $M(0) = 0$, we get $\inf\{\rho^{p_n/H}\} = 0$ for $x = \theta$. Now let $x,\ y \in [\hat{V},\ \lambda,\ M,\ p]_0(\Delta^r)$ and let us choose $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\sup_k M\left(\frac{|\Delta^r x_{k+m}|}{\rho_1}\right) \leq 1, \quad \text{uniformly in } m$$

and

$$\sup_k M\left(\frac{|\Delta^r y_{k+m}|}{\rho_2}\right) \leq 1, \quad \text{uniformly in } m.$$

Let $\rho = \rho_1 + \rho_2$. Then we get

$$\begin{aligned}
&\sup_k M\left(\frac{|\Delta^r (x_{k+m} + y_{k+m})|}{\rho}\right) \\
&\leq \sup_k M\left(\frac{|\Delta^r x_{k+m}|}{\rho_1+\rho_2} + \frac{|\Delta^r y_{k+m}|}{\rho_1+\rho_2}\right) \\
&\leq \left(\frac{\rho_1}{\rho_1+\rho_2}\right) \sup_k M\left(\frac{|\Delta^r x_{k+m}|}{\rho_1+\rho_2}\right) \\
&\quad + \left(\frac{\rho_2}{\rho_1+\rho_2}\right) \sup_k M\left(\frac{|\Delta^r y_{k+m}|}{\rho_1+\rho_2}\right) \leq 1, \quad \text{uniformly in } m.
\end{aligned}$$

Therefore $g(x + y) \leq g(x) + g(y)$.

For the continuity of scalar multiplication let $l \neq 0$ be any complex number. Then by the definition we have

$$\begin{aligned}
g\,(lx) &= \inf\left\{ \rho^{p_n/H} :\ \sup_k M\left(\frac{|\Delta^r\,(l x_{k+m})|}{\rho}\right) \leq 1, \text{uniformly in } m \right\} \\
&= \inf\left\{ (|l|s)^{p_n/H} :\ \sup_k M\left(\frac{|\Delta^r x_{k+m}|}{s}\right) \leq 1, \text{uniformly in } m \right\}
\end{aligned}$$

where $s=\rho/|l|$. Since $|l|^{p_n}\le \max(1,\,|l|^H)$, we have

$$\begin{aligned} g(lx) &\le \max(1,\,|l|^H) \\ &\quad\times \inf\left\{ s^{p_n/H}\colon\ \sup_k M\left(\frac{|\Delta^r x_{k+m}|}{s}\right)\le 1,\ \text{uniformly in } m\right\} \\ &= \max(1,\,|l|^H)g(x) \end{aligned}$$

and therefore $g(rx)$ converges to zero when $g(x)$ converges to zero in $[\hat{V},\,\lambda,\,M,\,p]_0(\Delta^r)$.

Now let x be a fixed element in $[\hat{V},\,\lambda,\,M,\,p]_0(\Delta^r)$. Then there exists $\rho>0$ such that

$$g(x)=\inf\left\{ \rho^{p_n/H}\colon\ \sup_k M\left(\frac{|\Delta^r x_{k+m}|}{\rho}\right)\le 1,\quad \text{uniformly in } m\right\}.$$

Now

$$\begin{aligned} &g(lx) \\ &=\inf\left\{ \rho^{p_n/H}\colon\ \begin{matrix} \sup_k M\left(\frac{|l\Delta^r x_{k+m}|}{\rho}\right)\le 1,\ \rho>0 \\ \text{uniformly in } m \end{matrix}\right\}\to 0 \end{aligned}$$

as $l\to 0$. This completes the proof. □

Theorem 2.4 *Let M_1, M_2 be Orlicz functions Then we have*

i) $[\hat{V},\,\lambda,\,M_1,\,p]_0(\Delta^r)\cap[\hat{V},\,\lambda,\,M_2,\,p]_0(\Delta^r)\subset[\hat{V},\,\lambda,\,M_1+M_2,\,p]_0(\Delta^r)$,
ii) $[\hat{V},\,\lambda,\,M_1,\,p](\Delta^r)\cap[\hat{V},\,\lambda,\,M_2,\,p](\Delta^r)\subset[\hat{V},\,\lambda,\,M_1+M_2,\,p](\Delta^r)$,
iii) $[\hat{V},\,\lambda,\,M_1,\,p]_\infty(\Delta^r)\cap[\hat{V},\,\lambda,\,M_2,\,p]_\infty(\Delta^r)\subset[\hat{V},\,\lambda,\,M_1+M_2,\,p]_\infty(\Delta^r)$.

Proof. Omitted. □

The proof of the following result is a routine work.

Proposition 2.5 *Let M be an Orlicz function. Then we have*

$$[\hat{V},\,\lambda,\,M,\,p](\Delta^{r-1})\subset[\hat{V},\,\lambda,\,M,\,p]_0(\Delta^r).$$

Theorem 2.6 *Let M_1 and M_2 be two Orlicz functions. If M_1 and M_2 are equivalent then*

i) $[\hat{V},\,\lambda,\,M_1,\,p]_0(\Delta^r)=[\hat{V},\,\lambda,\,M_2,\,p]_0(\Delta^r)$,
ii) $[\hat{V},\,\lambda,\,M_1,\,p](\Delta^r)=[\hat{V},\,\lambda,\,M_2,\,p](\Delta^r)$,
iii) $[\hat{V},\,\lambda,\,M_1,\,p]_\infty(\Delta^r)=[\hat{V},\,\lambda,\,M_2,\,p]_\infty(\Delta^r)$.

Proof. Proof follows from Definition 1. □

Theorem 2.7 *Let $0 < p_k \leq t_k$ for each k and (t_k/p_k) be bounded, then*
i) $[\hat{V}, \lambda, M, t]_0(\Delta^r) \subset [\hat{V}, \lambda, M, p]_0(\Delta^r)$,
ii) $[\hat{V}, \lambda, M, t](\Delta^r) \subset [\hat{V}, \lambda, M, p](\Delta^r)$,
iii) $[\hat{V}, \lambda, M, t]_\infty(\Delta^r) \subset [\hat{V}, \lambda, M, p]_\infty(\Delta^r)$.

Proof. We prove it for (i) and the other cases will follow on applying similar technique. Let $x \in [\hat{V}, \lambda, M, t]_0(\Delta^r)$. Write $w_{k,m} = [M(|\Delta^r x_{k+m}|/\rho)]^{t_k}$ and $\mu_k = p_k/t_k$, so that $0 < \mu < \mu_k \leq 1$ for each k.

We define the sequences $(u_{k,m})$ and $(v_{k,m})$ as follows:

Let $u_{k,m} = w_{k,m}$ and $v_{k,m} = 0$ if $w_{k,m} \geq 1$, and let $u_{k,m} = 0$ and $v_{k,m} = w_{k,m}$ if $w_{k,m} < 1$. Then it is clear that for all $k \in \mathbb{N}$, we have $w_{k,m} = u_{k,m} + v_{k,m}$, $w_{k,m}^{\mu_k} = u_{k,m}^{\mu_k} + v_{k,m}^{\mu_k}$. Now it follows that $u_{k,m}^{\mu_k} \leq u_{k,m} \leq w_{k,m}$ and $v_{k,m}^{\mu_k} \leq v_{k,m}^{\mu}$. Therefore

$$\begin{aligned}\lambda_n^{-1} \sum_{k \in I_n} w_{k,m}^{\mu_k} &= \lambda_n^{-1} \sum_{k \in I_n} (u_{k,m}^{\mu_k} + v_{k,m}^{\mu_k}) \\ &\leq \lambda_n^{-1} \sum_{k \in I_n} w_{k,m} + \lambda_n^{-1} \sum_{k \in I_n} v_{k,m}^{\mu}.\end{aligned}$$

Since $\mu < 1$ so that $1/\mu > 1$, for each n

$$\begin{aligned}\lambda_n^{-1} \sum_{k \in I_n} v_{k,m}^{\mu} &= \sum_{k \in I_n} (\lambda_n^{-1} v_{k,m})^{\mu} (\lambda_n^{-1})^{1-\mu} \\ &\leq \left(\sum_{k \in I_n} [(\lambda_n^{-1} v_{k,m})^{\mu}]^{1/\mu} \right)^{\mu} \left(\sum_{k \in I_n} [(\lambda_n^{-1})^{1-\mu}]^{1/(1-\mu)} \right)^{1-\mu} \\ &= \left(\lambda_n^{-1} \sum_{k \in I_n} v_{k,m} \right)^{\mu}\end{aligned}$$

by Hölder's inequality, and thus

$$\lambda_n^{-1} \sum_{k \in I_n} w_{k,m}^{\mu_k} \leq \lambda_n^{-1} \sum_{k \in I_n} w_{k,m} + \left(\lambda_n^{-1} \sum_{k \in I_n} v_{k,m} \right)^{\mu}.$$

Hence $x \in [\hat{V}, \lambda, M, p]_0(\Delta^r)$. □

Theorem 2.8 *Let X stands for $[\hat{V}, \lambda, M, p]_0$ or $[\hat{V}, \lambda, M, p]$ or $[\hat{V}, \lambda, M, p]_\infty$. Then the inclusions $X(\Delta^{r-1}) \subset X(\Delta^r)$ are strict. In general $X(\Delta^i) \subset X(\Delta^r)$, for $i = 1, 2, \ldots, r-1$.*

Proof. We give the proof for $[\hat{V}, \lambda, M, p]_\infty$ only. The other cases can be proved in a similar way. Let $x \in [\hat{V}, \lambda, M, p]_\infty$, then we have

$$\sup_{m,n} \frac{1}{\lambda_n} \sum_{k \in I_n} \left[M \left(\frac{|\Delta^{r-1} x_{k+m}|}{\rho} \right) \right]^{p_k} < \infty$$

for some $\rho > 0$. Since M is non-decreasing and convex function we have

$$\begin{aligned}
&\frac{1}{\lambda_n} \sum_{k \in I_n} \left[M \left(\frac{|\Delta^r x_{k+m}|}{2\rho} \right) \right]^{p_k} \\
&= \frac{1}{\lambda_n} \sum_{k \in I_n} \left[M \left(\frac{|\Delta^{r-1} x_{k+m} - \Delta^{r-1} x_{k+m+1}|}{2\rho} \right) \right]^{p_k} \\
&\le \frac{D}{\lambda_n} \sum_{k \in I_n} \left[\frac{1}{2} M \left(\frac{|\Delta^{r-1} x_{k+m}|}{\rho} \right) \right]^{p_k} \\
&\qquad + \frac{D}{\lambda_n} \sum_{k \in I_n} \left[\frac{1}{2} M \left(\frac{|\Delta^{r-1} x_{k+m+1}|}{\rho} \right) \right]^{p_k} \\
&\le \frac{D}{\lambda_n} \sum_{k \in I_n} \left[M \left(\frac{|\Delta^{r-1} x_{k+m}|}{\rho} \right) \right]^{p_k} \\
&\qquad + \frac{D}{\lambda_n} \sum_{k \in I_n} \left[M \left(\frac{|\Delta^{r-1} x_{k+m+1}|}{\rho} \right) \right]^{p_k}.
\end{aligned}$$

Thus $[\hat{V}, \lambda, M, p]_\infty(\Delta^{r-1}) \subset [\hat{V}, \lambda, M, p]_\infty(\Delta^r)$.

The inclusion is strict. In fact the sequence $x = (k^{r-1})$, for example, belongs to $[\hat{V}, \lambda, M, p]_0(\Delta^r)$, but does not belong to $[\hat{V}, \lambda, M, p]_0(\Delta^{r-1})$ for $M(x) = x$, $\lambda_n = n$ for all $n \in \mathbb{N}$ and $p_k = 1$ for all $k \in \mathbb{N}$. (If $x = (k^{r-1})$, then $\Delta^r x_k = 0$ and $\Delta^{r-1} x_k = (-1)^{r-1}(r-1)!$ for all $k \in \mathbb{N}$). □

Theorem 2.9 (i) *The sequence spaces $[\hat{V}, \lambda, M, p]_0$ and $[\hat{V}, \lambda, M, p]_\infty$ are solid and hence are monotone.*

(ii) *The space $[\hat{V}, \lambda, M, p]$ is not monotone and as such is neither solid nor perfect.*

Proof. We give the proof for $[\hat{V}, \lambda, M, p]_0$. Let $x \in [\hat{V}, \lambda, M, p]_0$ and (α_k) be sequences of scalars such that $|\alpha_k| \le 1$ for all $k \in \mathbb{N}$. Then we have

$$\lambda_n^{-1} \sum_{k \in I_n} \left[M \left(\frac{|\alpha_{k+m} x_{k+m}|}{\rho} \right) \right]^{p_k} \le \lambda_n^{-1} \sum_{k \in I_n} \left[M \left(\frac{|x_{k+m}|}{\rho} \right) \right]^{p_k} \to 0, \quad (n \to \infty), \quad \text{uniformly in } m.$$

Hence $\alpha x \in [\hat{V},\, \lambda,\, M,\, p]_0$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$, whenever $x \in [\hat{V},\, \lambda,\, M,\, p]_0$. The spaces are monotone follows from the Remark. □

ii) The space $[\hat{V},\, \lambda,\, M,\, p]$ is not monotone follows from the following example.

Example 2 Let $p_k = 1$ and $\lambda_k = 1$, for all $k \in \mathbb{N}$ and $M(x) = x^p$, for some $p \geq 1$. Consider the sequence (x_k) defined as $x_k = 1$ for all $k \in \mathbb{N}$. Consider the J^{th} step space E_J for a sequence space E defined as, for $(x_k) \in E$, (y_k) is the J^{th} canonical preimage of (x_k) i.e. $(y_k) \in E_J$ implies $y_k = x_k$, if k is odd and $y_k = 0$, otherwise. Then $(y_k) \notin E$. Hence the space $[\hat{V},\, \lambda,\, M,\, p]$ is not monotone. The rest follows from the Remark.

Theorem 2.10 $[\hat{V},\, M,\, p]_\infty(\Delta^r) = \ell_\infty(M,\, p)(\Delta^r)$
where $\ell_\infty(M,\, p)(\Delta^r) = \{x\colon\ \sup_k [M(|\Delta^r x_k|/\rho)]^{p_k} < \infty\}$.

Proof. Write

$$t_{nm} = \frac{1}{n}\sum_{k=1}^{n}\left[M\left(\frac{|\Delta^r x_{k+m}|}{\rho}\right)\right]^{p_k} = \frac{1}{n}\sum_{k=m+1}^{m+n}\left[M\left(\frac{|\Delta^r x_k|}{\rho}\right)\right]^{p_k}.$$

We have,

$$\begin{aligned}\sup_{n,\,m} t_{nm} &= \sup_m \frac{\sup_k \left[M\left(\frac{|\Delta^r x_k|}{\rho}\right)\right]^{p_k}}{n}\sum_{k=m+1}^{m+n} 1 \\ &= \sup_k \left[M\left(\frac{|\Delta^r x_k|}{\rho}\right)\right]^{p_k} \qquad (4)\end{aligned}$$

and

$$\sup_{n,\,m} t_{nm} \geq \sup_m t_{1,\,m} = \sup_m \left[M\left(\frac{|\Delta^r x_{m+1}|}{\rho}\right)\right]^{p_{m+1}} \qquad (5)$$

The result follows from (4) and (5). □

In the following theorem, we consider the case when $\Delta^r x_k \to \ell$ implies $x_k \to \ell[\hat{V},\, \lambda,\, M,\, p](\Delta^r)$ and the uniqueness of a *strongly almost difference limit of* x with respect to an Orlicz function M. To prove the next theorem we need the following Lemma.

Lemma *Let $p_k > 0$ and $q_k > 0$. If $\liminf_k (p_k/q_k) > 0$ then $c_0(q)(\Delta^r) \subset c_0(p)(\Delta^r)$ (see Et and Basarir [2]).*

Theorem 2.11 *If $\liminf_k p_k > 0$, then $\Delta^r x_k \to \ell$ implies $x_k \to \ell[\hat{V},\,\lambda,\,M,\,p](\Delta^r)$ uniquely.*

Proof. Let $\liminf_k p_k = s > 0$ and $\Delta^r x_k \to \ell$. Then from above lemma follows that $x_k \to \ell[\hat{V},\,\lambda,\,M,\,p](\Delta^r)$.

Now we show that the limit is unique. Let $x_k \to \ell[\hat{V},\,\lambda,\,M,\,p](\Delta^r)$ and $x_k \to \ell_1[\hat{V},\,\lambda,\,M,\,p](\Delta^r)$. Then there exist ρ_1 and ρ_2 such that

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|\Delta^r x_{k+m}-\ell|}{\rho_1}\right)\right]^{p_k} \to 0$$

and

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|\Delta^r x_{k+m}-\ell_1|}{\rho_2}\right)\right]^{p_k} \to 0,$$

as $n \to \infty$. Let $\rho = \max 2(\rho_1,\,\rho_2)$. Then we have

$$\begin{aligned}&\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|\ell-\ell_1|}{\rho}\right)\right]^{p_k}\\ &\leq \frac{D}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|\Delta^r x_{k+m}-\ell|}{\rho_1}\right)\right]^{p_k} + \frac{D}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|\Delta^r x_{k+m}-\ell_1|}{\rho_2}\right)\right]^{p_k}\\ &\qquad \to 0, \text{ as } n\to\infty.\end{aligned}$$

Hence $|\ell - \ell_1| = 0$.
$\Rightarrow \ell = \ell_1$.
Thus the limit is unique. □

3. Δ^r_λ-Statistical convergence

The idea of statistical convergence was introduced by Fast [5] and studied by various authors ([6], [18], [23], [24]).

In this section we define almost Δ^r_λ-statistically convergent sequences and give some inclusion relations between Δ^r_λ-statistically convergent sequences and strongly almost $(V,\,\lambda)(\Delta^r)$-summable sequences with respect to an Orlicz function.

Definition 3 A sequence $x = (x_k)$ is said to be almost Δ^r_λ-statistically convergent to the number ℓ if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n \colon |\Delta^r x_{k+m} - \ell| \geq \varepsilon\}| = 0, \quad \text{uniformly in } m.$$

In this case we write $\hat{s}_\lambda(\Delta^r) - \lim x = \ell$ or $x_k \to \ell(\hat{s}_\lambda(\Delta^r))$.
In the special case $\lambda_n = n$, for all $n \in \mathbb{N}$ we shall write $\hat{s}(\Delta^r)$ instead of $\hat{s}_\lambda(\Delta^r)$.

Definition 4 A sequence $x = (x_k)$ is said to be strongly almost $\Delta^r_{\lambda p}$-convergent to the number ℓ if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |\Delta^r x_{k+m} - \ell|^p = 0, \quad \text{uniformly in } m.$$

In this case we write $[\hat{c}(\Delta^r_{\lambda p})] - \lim x = \ell$ or $x_k \to \ell[\hat{c}(\Delta^r_{\lambda p})]$ and

$$[\hat{c}(\Delta^r_{\lambda p})] = \left\{ x = (x_k) \colon \begin{array}{l} \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |\Delta^r x_{k+m} - \ell|^p = 0, \\ \hfill \text{uniformly in } m \end{array} \right\}.$$

In the case $p = 1$ we shall write $[\hat{c}(\Delta^r_\lambda)]$ and for the case $\lambda_n = n$ for all $n \in \mathbb{N}$ and $p = 1$ we shall write $[\hat{c}(\Delta^r)]$.

Theorem 3.1 *Let $\lambda = (\lambda_n)$ be the same as above, then*
i) $x_k \to \ell[\hat{c}(\Delta^r_{\lambda p})] \Rightarrow x_k \to \ell(\hat{s}_\lambda(\Delta^r))$,
ii) *If $x \in \ell_\infty(\Delta^r)$ and $x_k \to \ell(\hat{s}_\lambda(\Delta^r))$, then $x_k \to \ell[\hat{c}(\Delta^r_{\lambda p})]$,*
iii) $\hat{s}_\lambda(\Delta^r) \cap \ell_\infty(\Delta^r) = [\hat{c}(\Delta^r_{\lambda p})] \cap \ell_\infty(\Delta^r)$.

Proof. i) Let $\varepsilon > 0$ and $x_k \to \ell[\hat{c}(\Delta^r_{\lambda p})]$. Since

$$\begin{aligned} \sum_{k \in I_n} |\Delta^r x_{k+m} - \ell|^p &\geq \sum_{\substack{k \in I_n \\ |\Delta^r x_{k+m} - \ell| \geq \varepsilon}} |\Delta^r x_{k+m} - \ell|^p \\ &\geq \varepsilon^p |\{k \in I_n \colon |\Delta^r x_{k+m} - \ell| \geq \varepsilon\}|. \end{aligned}$$

Therefore $x_k \to \ell(\hat{s}_\lambda(\Delta^r))$.

ii) Suppose that $x_k \to \ell(\hat{s}_\lambda(\Delta^r))$ and $x \in \ell_\infty(\Delta^r)$, say that $|\Delta^r x_{k+m} - \ell| \leq K$. Let $\varepsilon > 0$ be given and N_ε such that

$$\lambda_n^{-1} \left| \left\{ k \in I_n \colon |\Delta^r x_{k+m} - \ell| \geq \left(\frac{\varepsilon}{2}\right)^{1/p} \right\} \right| \leq \frac{\varepsilon}{2K^p}$$

for all $n > N_\varepsilon$ and set $L_{nm} = \{k \in I_n \colon |\Delta^r x_{k+m} - \ell| \geq (\varepsilon/2)^{1/p}\}$.

Now for all $n > N_\varepsilon$ we have

$$\begin{aligned}
&\frac{1}{\lambda_n} \sum_{k \in I_n} |\Delta^r x_{k+m} - \ell|^p \\
&= \frac{1}{\lambda_n} \sum_{k \in L_{nm}} |\Delta^r x_{k+m} - \ell|^p + \frac{1}{\lambda_n} \sum_{k \notin L_{nm}} |\Delta^r x_{k+m} - \ell|^p \\
&\leq \frac{1}{\lambda_n} \left(\frac{\lambda_n \varepsilon}{2K^p} \right) K^p + \frac{1}{\lambda_n} \lambda_n \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence $x_k \to \ell[\hat{c}(\Delta^r_{\lambda p})]$.

iii) This immediately follows from (i) and (ii). □

It can be shown that $\hat{s}(\Delta^r) \subset \hat{s}_\lambda(\Delta^r)$ if and only if $\liminf_n \lambda_n/n > 0$ and $\hat{s}_\lambda(\Delta^r) \subset \hat{s}(\Delta^r)$ for all λ, since λ_n/n is bounded.

Theorem 3.2 *Let M be an Orlicz function. Then $[\hat{V}, \lambda, M, p](\Delta^r) \subset \hat{s}_\lambda(\Delta^r)$.*

Proof. Let $x \in [\hat{V}, \lambda, M, p](\Delta^r)$. Then there exists $\rho > 0$ such that

$$\frac{1}{\lambda_n} \sum_{k \in I_n} \left[M \left(\frac{|\Delta^r x_{k+m} - \ell|}{\rho} \right) \right]^{p_k} \to 0, \quad \text{as } n \to \infty.$$

Then given any $\varepsilon > 0$ we can write

$$\begin{aligned}
&\frac{1}{\lambda_n} \sum_{k \in I_n} \left[M \left(\frac{|\Delta^r x_{k+m} - \ell|}{\rho} \right) \right]^{p_k} \\
&= \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ |\Delta^r x_{k+m} - \ell| \geq \varepsilon}} \left[M \left(\frac{|\Delta^r x_{k+m} - \ell|}{\rho} \right) \right]^{p_k} \\
&\quad + \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ |\Delta^r x_{k+m} - \ell| < \varepsilon}} \left[M \left(\frac{|\Delta^r x_{k+m} - \ell|}{\rho} \right) \right]^{p_k} \\
&\geq \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ |\Delta^r x_{k+m} - \ell| \geq \varepsilon}} \left[M \left(\frac{|\Delta^r x_{k+m} - \ell|}{\rho} \right) \right]^{p_k} \\
&\geq \frac{1}{\lambda_n} \sum_{k \in I_n} [M(\varepsilon_1)]^{p_k}, \quad \text{where } \varepsilon_1 = \frac{\varepsilon}{\rho}
\end{aligned}$$

$$\geq \frac{1}{\lambda_n}\sum_{k\in I_n}\min\{[M(\varepsilon_1)]^{\inf p_k},\,[M(\varepsilon_1)]^G\}$$

$$\geq \frac{1}{\lambda_n}|\{k\in I_n\colon |\Delta^r x_{k+m}-\ell|\geq\varepsilon\}|\min\{[M(\varepsilon_1)]^{\inf p_k},\,[M(\varepsilon_1)]^G\}.$$

Hence $x\in\hat{s}_\lambda(\Delta^r)$. □

Theorem 3.3 *Let M be an Orlicz function. Then $[\hat{V},\,\lambda,\,M](\Delta^r)\cap\ell_\infty(\Delta^r) = \hat{s}_\lambda(\Delta^r)\cap\ell_\infty(\Delta^r)$.*

Proof. By Theorem 3.2, we need only show that $\hat{s}_\lambda(\Delta^r)\cap\ell_\infty(\Delta^r)\subset[\hat{V},\,\lambda,\,M](\Delta^r)\cap\ell_\infty(\Delta^r)$. Let $z_{k+m}=(\Delta^r x_{k+m}-\ell)\to 0(\hat{s}_\lambda)$. Since $x\in\ell_\infty(\Delta^r)$, there exists an integer $K>0$ such that $M(|z_{k+m}|/\rho)<K$. Then for a given $\varepsilon>0$ and for each n, we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n}M\left(\frac{|z_{k+m}|}{\rho}\right)$$

$$=\frac{1}{\lambda_n}\sum_{k\in I_n,\,|z_{k+m}|<\varepsilon}M\left(\frac{|z_{k+m}|}{\rho}\right)+\frac{1}{\lambda_n}\sum_{k\in I_n,\,|z_{k+m}|\geq\varepsilon}M\left(\frac{|z_{k+m}|}{\rho}\right)$$

$$\leq\lambda_n\frac{1}{\lambda_n}M\left(\frac{\varepsilon}{\rho}\right)+\frac{1}{\lambda_n}K|\{k\in I_n\colon |z_{k+m}|\geq\varepsilon\rho\}|.$$

Hence $x\in[\hat{V},\,\lambda,\,M](\Delta^r)\cap\ell_\infty(\Delta^r)$. □

Theorem 3.4 *The spaces $[\hat{V},\,\lambda,\,M,\,p]_0(\Delta^r)$, $[\hat{V},\,\lambda,\,M,\,p](\Delta^r)$, $[\hat{V},\,\lambda,\,M,\,p]_\infty(\Delta^r)$, $\hat{s}_\lambda(\Delta^r)$ and $\hat{s}_{0\lambda}(\Delta^r)$ are not solid for $r>0$.*

Proof. To show that the spaces are not solid in general, consider the following example. □

Example 3 Let $M(x)=x$, $\lambda_n=n$ for each $n\in\mathbb{N}$ and $p_k=1$ for all $k\in\mathbb{N}$. Then $x=(k^r)\in[\hat{V},\,\lambda,\,M,\,p](\Delta^r)$, $[\hat{V},\,\lambda,\,M,\,p]_\infty(\Delta^r)$ and $\hat{s}_\lambda(\Delta^r)$. Let $\alpha_k=(-1)^k$ for all $k\in\mathbb{N}$, then $\alpha x\notin[\hat{V},\,\lambda,\,M,\,p](\Delta^r)$, $[\hat{V},\,\lambda,\,M,\,p]_\infty(\Delta^r)$ and $\hat{s}_\lambda(\Delta^r)$. Hence $[\hat{V},\,\lambda,\,M,\,p](\Delta^r)$, $[\hat{V},\,\lambda,\,M,\,p]_\infty(\Delta^r)$ and $\hat{s}_\lambda(\Delta^r)$ are not solid for $r>0$. To show that $[\hat{V},\,\lambda,\,M,\,p]_0(\Delta^r)$ and $\hat{s}_{0\lambda}(\Delta^r)$ are not solid, consider the sequence $(x_k)=(k^{r-1})$ and $\alpha_k=(-1)^k$ for all $k\in\mathbb{N}$.

Theorem 3.5 *The spaces $[\hat{V},\,\lambda,\,M,\,p]_0(\Delta^r)$, $[\hat{V},\,\lambda,\,M,\,p](\Delta^r)$, $[\hat{V},\,\lambda,\,M,\,p]_\infty(\Delta^r)$, $\hat{s}_\lambda(\Delta^r)$ and $\hat{s}_{0\lambda}(\Delta^r)$ are not symmetric for $r>0$.*

Proof. To show that the spaces are not symmetric, consider the following example. □

Example 4 Let $M(x) = x$, $\lambda_n = n$ for each $n \in \mathbb{N}$ and $p_k = 1$ for all $k \in \mathbb{N}$. Then the sequence $x = (k^r) \in [\hat{V}, \lambda, M, p](\Delta^r)$, $[\hat{V}, \lambda, M, p]_\infty(\Delta^r)$ and $\hat{s}_\lambda(\Delta^r)$. Let

$$(y_k) = \{x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10}, \ldots\}.$$

Then $y \notin [\hat{V}, \lambda, M, p](\Delta^r)$, $[\hat{V}, \lambda, M, p]_\infty(\Delta^r)$ and $\hat{s}_\lambda(\Delta^r)$.

Now let us consider the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} 1, & \text{if } (2i-1)^2 \le k < (2i)^2, \quad i = 1, 2, \ldots \\ 4, & \text{otherwise.} \end{cases}$$

and let (y_k) be the same as above. Then $x \in \hat{s}_{0\lambda}(\Delta)$ but $y \notin \hat{s}_{0\lambda}(\Delta)$.

Theorem 3.6 *The sequence spaces $[\hat{V}, \lambda, M, p]_0(\Delta^r)$, $[\hat{V}, \lambda, M, p](\Delta^r)$, $[\hat{V}, \lambda, M, p]_\infty(\Delta^r)$, $\hat{s}_\lambda(\Delta^r)$ and $\hat{s}_{0\lambda}(\Delta^r)$ are not sequence algebra for $r > 0$.*

Proof. Under the restriction on M, λ and p as in the Example 4, consider $x = (k^{r-1})$ and $y = (k^{r-1})$, then $x, y \in [\hat{V}, \lambda, M, p]_0(\Delta^r)$, $[\hat{V}, \lambda, M, p](\Delta^r)$, $[\hat{V}, \lambda, M, p]_\infty(\Delta^r)$, $\hat{s}_\lambda(\Delta^r)$ and $\hat{s}_{0\lambda}(\Delta^r)$, but $x.y \notin [\hat{V}, \lambda, M, p]_0(\Delta^r)$, $[\hat{V}, \lambda, M, p](\Delta^r)$, $[\hat{V}, \lambda, M, p]_\infty(\Delta^r)$, $\hat{s}_\lambda(\Delta^r)$ and $\hat{s}_{0\lambda}(\Delta^r)$. □

From Theorem 3.5 and the Remark we may give the following corollary.

Corollary 3.7 *The sequence spaces $[\hat{V}, \lambda, M, p]_0(\Delta^r)$, $[\hat{V}, \lambda, M, p](\Delta^r)$, $[\hat{V}, \lambda, M, p]_\infty(\Delta^r)$, $\hat{s}_\lambda(\Delta^r)$ and $\hat{s}_{0\lambda}(\Delta^r)$ are not perfect for $r > 0$.*

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