Hokkaido Mathematical Journal Vol. 35 (2006) p. 1–12

Numerical ranges of composition operators on l^2

B.S. Komal and Shallu Sharma

(Received September 1, 2003; Revised September 27, 2004)

Abstract. In this paper we obtain numerical ranges of composition operators on l^2.

Key words: numerical range, composition operator, orbit.

1. Preliminaries

Let $\mathbb{N}$ be the set of all positive integers and $l^2 = l^2(\mathbb{N})$ be the Hilbert space of all square summable sequences of complex numbers. Let T be a mapping of $\mathbb{N}$ into itself. Then we can define a composition transformation C_T from $l^2(\mathbb{N})$ into the space of all complex valued sequences by $C_T f = foT$ for every $f \in l^2(\mathbb{N})$. In the case C_T is bounded and the range of C_T is contained in $l^2(\mathbb{N})$, we call it a composition operator induced by T. It is shown in [9] that a composition trasformation $C_T\colon l^2(\mathbb{N}) \to l^2(\mathbb{N})$ is a bounded operator if and only if there exists a real number $M > 0$ such that $\#\,(T^{-1}(\{n\})) \leq M$ for every $n \in N$, where $\#(E)$ denotes the cardinality of the set E. Thus it is evident that a composition transformation C_T is unbounded if and only if the sequence $\{\#(T^{-1}(\{n\}))\}_{n=1}^{\infty}$ is an unbounded sequence. From Theorem 3.1 of Singh and Komal [9], it follows that C_T is surjective if and only if T is injective.

The numerical range of a bounded linear operator $A\colon H \to H$ from Hilbert space H into itself is defined as $W(A) = \{\langle Ax, x\rangle : \|x\| = 1\}$, where $\langle\, .\, ,\, .\,\rangle$ is the inner product of H. The numerical radius of A is defined as $w(A) = \sup\{|\langle Ax, x\rangle| : \|x\| = 1\}$. These concepts and their generalisations have been studied extensively because of their connections and applications to many different branches of Mathematics (e.g. see chapter I of Horn and Johnson [12]).

Let $T\colon N \to N$ be a mapping. Two positive integers m and n are said to be in the same orbit of T if there exit two positive integers r and s such that $T^r(m) = T^s(n)$. Here and else where, T^r denotes the composition of T with itself r times. If $n \in N$, then $O_T(n) = \{m \in N : T^r(m) = T^s(n)$ for

2000 Mathematics Subject Classification : Primary 47B99, Secondary 47B38.

some $r,\, s \in N\}$ is called the orbit of n with respect to T.

A mapping $T\colon N \to N$ is said to be antiperiodic at n, if $T^m(n) \neq n$ for every $m \in N$. If T is antiperiodic at every $n \in N$, then we say that T is purely antiperiodic. For an example, the mapping $T\colon N \to N$ defined by

$$T(n) = \begin{cases} n+2, & \text{if } n \text{ is even} \\ n, & \text{if } n \text{ is odd} \end{cases}$$

is antiperiodic at every even natural number but not antiperiodic at an odd natural number. If for a natural number n there exists $m \in N$ such that $T^m(n) = n$, then T is called periodic at n. If T is periodic at every $n \in N$, we say that T is purely periodic. The integer $m_n = \inf\{m : T^m(n) = n\}$ is called the period of T at n. The set $\{m_n : n \in N\}$ of periods of T is denoted by $P(T)$. For sake of convenience, we shall use $f_0(n)$ to denote the cardinality of the set $T^{-1}(\{n\})$. The smallest convex set containing the set $G \subset l^2(\mathbb{N})$ is called the convex hull of G and we shall denote it by $C_o(G)$. The spectrum of a bounded operator A on a Hilbert space H into itself is defined by $\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible}\}$. If $F \subset N$, then χ_F denotes the characteristics function of F. It is well known that $\{\chi_{\{n\}} : n \in N\}$ is an orthonormal basis for $l^2(\mathbb{N})$. The Banach algebra of all bounded linear operators from $l^2(\mathbb{N})$ into itself is denoted by $B(l^2(\mathbb{N}))$. For $E \subset N$, $l^2(E) = \{f \in l^2(\mathbb{N}) : f(m) = 0 \text{ for every } m \notin E\}$. The symbol $C_T|l^2(E)$ denotes the restriction of C_T to $l^2(E)$.

The composition operators have been the subject matter of a systematic study over the past three decades (e.g. see monographs Cowen and Maccluer [2], Shapiro [11] and Singh and Manhas [13]). For more information concerning composition operators on $l^2(\mathbb{N})$ and numerical ranges of operators, we refer to Carlson [1], Singh and Komal ([9] and [10]), Gustafson and Rao [3], Halmos [4], Matache [7] and Stout [14]. In this paper we compute the numerical ranges of composition operators on $l^2(\mathbb{N})$. It is shown that numerical range of an unbounded composition operator is the entire complex plane.

2. Numerical ranges of surjective composition operators

In this section we obtain the numerical ranges of surjective composition operators.

Lemma 2.1 *Let $T\colon N \to N$ be injective and antiperiodic at some $n \in N$ and let S be the restriction of T to $O_T(n)$. Then $W(C_S) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$, where C_S is a composition operator on $l^2(O_T(n))$ induced by S.*

Proof. If S is invertible, then by Theorem 2.3 of Singh and Komal [9], C_S is normal and therefore $\overline{W(C_S)} = C_o(\sigma(C_S))$ in view of theorm 1.4.5 of Gustafason and Rao [3]. Since $\sigma(C_S) = \{\lambda : |\lambda| = 1\}$, it follows that $\overline{W(C_S)} = \{\lambda : |\lambda| \leq 1\}$. If $|\lambda| = 1$ and $\lambda \in W(C_S)$, then $|\lambda| = |\langle C_s f, f\rangle| \leq \|C_S f\| \, \|f\| \leq 1$ for some unit vector $f \in l^2(O_T(n))$ implies that Schwartz inequality becomes an equality. Hence $C_S f$ and f are linearly dependent so that $C_S f = \mu f$ for some scalar μ. This contradicts the fact that C_S has no eigenvalue in the light of Lemma 2.1 of Komal and Pathania [6]. Thus $W(C_S) \subset \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. To prove the reverse inclusion, let $\lambda = re^{i\theta}$, where $0 < r < 1$, $0 \leq \theta \leq 2\pi$. For $O_T(n) = \{n_k\}_{k=1}^{\infty}$ where $S(n_k) = n_{k+1}$, define $f = \sum_{k=1}^{\infty} \sqrt{(1-r^2)}\, r^{k-1} e^{-i(k-1)\theta} \chi_{\{n_k\}}$. A simple computation reveals that $\|f\| = 1$ and $\langle C_S f, f\rangle = re^{i\theta}$. This proves that $\{\lambda : |\lambda| < 1\} \subset W(C_s)$. Hence $W(C_S) = \{\lambda : |\lambda| < 1\}$. Next, if S is not invertible in $O_T(n)$, then S is not surjective and hence $S^{-1}(\{n_1\}) = \phi$. Taking f as defined above, we see that $\langle C_S f, f\rangle = re^{i\theta}$, $0 < r < 1$, $0 \leq \theta \leq 2\pi$. This proves that $\{\lambda \in \mathbb{C} : |\lambda| < 1\} \subseteq W(C_S)$. Now S is injective. Therefore by corollary to Theorem 2.1.9 of Singh and Komal [9], $\|C_s\| = 1$. Hence $W(C_S) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. As proved earlier, if $|\lambda| = 1$, then $\lambda \notin W(C_S)$. Thus in this case also $W(C_S) = \{\lambda : |\lambda| < 1\}$. □

Lemma 2.2 *Let $T\colon N \to N$ be an injection and periodic at some $n \in N$. Then*

(i) $W(C_S) = \{1\}$, *if* $\#(O_T(n)) = 1$,
(ii) $W(C_S) = [-1, 1]$, *if* $\#(O_T(n)) = 2$,
(iii) $W(C_S)$ *is the closed polygonal region whose boundary is the regular polygon with* $\#(O_T(n))$ *sides inscribed in the unit circle having one of the vertex at* 1, *where S is the restriction of T to $O_T(n)$.*

Proof. If $\#(O_T(n)) = 1$, then S is the identity map and so trivially $W(C_S) = \{1\}$. Further if $\#(O_T(n)) \geq 2$, then S is periodic of period $k = \#(O_T(n))$. Clearly S is invertible. This implies that C_S is invertible and hence C_S is normal by Theorem 2.3 of Singh and Komal [9]. From corollary to Theorem 3.3 of singh and Komal [10], it follows that $\sigma(C_S) = \{\lambda \in \mathbb{C} : \lambda^k = 1\}$. By using normality of C_S, and the fact that range of an operator on a finite dimensional space is always closed, we have $W(C_S) = \overline{W(C_S)} = C_o(\sigma(C_S))$. We have thus proved that $W(C_S) = C_o\{\lambda \in \mathbb{C} : \lambda^k = 1\}$. In particular, for $k = 2$, $W(C_S) = [-1, 1]$. □

Theorem 2.3 *Let $T : N \to N$ be injective. Then*

(i) $W(C_T) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$, *if T is purely antiperiodic*

(ii) $W(C_T) = C_o\left(\bigcup_{n\in P(T)}\{\lambda \in \mathbb{C} : \lambda^n = 1\}\right)$, *if T is purely periodic.*

(iii) $W(C_T) = C_o\left(\{\lambda \in \mathbb{C} : |\lambda| < 1\} \cup \bigcup_{k\in P(T)}\{\lambda \in \mathbb{C} : \lambda^k = 1\}\right)$, *if T is periodic at some $n \in N$ and antiperiodic at some other point $m \in N$.*

Proof. The proof of (i) follows from Lemma 2.1. For the proof of (ii), write $l^2(\mathbb{N}) = \sum_{n\in P(T)} \oplus l^2(O_T(n))$ so that

$$\begin{aligned} W(C_T) &= C_o\Bigg(\bigcup_{n\in P(T)} W(C_T|l^2(O_T(n)))\Bigg) \\ &= C_o\Bigg(\bigcup_{n\in P(T)} \{\lambda \in \mathbb{C} : \lambda^n = 1\}\Bigg), \end{aligned}$$

where $C_T|l^2(O_T(n))$ is restriction of C_T to $l^2(O_T(n))$. For the proof of (iii), let $E = \{n \in N : T \text{ is periodic at } n\}$. Then $l^2(\mathbb{N}) = l^2(E) \oplus l^2(N - E)$. Hence that

$$\begin{aligned} W(C_T) &= C_o\big(W(C_T|l^2(E)) \cup W(C_T|l^2(N - E))\big) \\ &= C_o\Bigg(\bigcup_{k\in P(T)} \{\lambda \in \mathbb{C} : \lambda^k = 1\} \cup \{\lambda \in \mathbb{C} : |\lambda| < 1\}\Bigg) \end{aligned}$$

follows by using the above parts (i) and (ii). □

3. Numerical Ranges of Non-Surjective Composition Operators

In this section we obtain the numerical ranges of non-surjective composition operators.

Lemma 3.1 *Let $C_T \in B(l^2(\mathbb{N}))$. Then $W(C_T/l^2(E_n)) = \left\{\lambda \in \mathbb{C} : |\lambda| \leq \sqrt{f_0(n)}/2\right\}$, where n is such that $f_0(n) \geq 2$, $T(n) \neq n$; $T^2(n) \neq n$ and $E_n = T^{-1}(\{n\}) \cup \{n\}$.*

Proof. Suppose $f_0(n) = k \geq 2$ and $E_n = \{n, n_1 n_2 \dots n_k\}$. Clearly

$$f = f_n \chi_{\{n\}} + f_{n_1} \chi_{\{n_1\}} + \cdots + f_{n_k} \chi_{\{n_k\}} \in l^2(E_n),$$

and

$$\langle C_T f, f \rangle = \sum_{i=1}^{k} f_n \overline{f_{n_i}}.$$

Therefore $|\langle C_T f,\, f \rangle| \leq |f_n| \sum_{i=1}^{k} |f_{n_i}|$. In order to calculate the numerical radius of $C_T | l^2(E_n)$, we compute $\sup\big\{|f_n| \sum_{i=1}^{k} |f_{n_i}|\big\} \dots\dots (1)$ subject to the condition $|f_n|^2 + \sum_{i=1}^{k} |f_{n_i}|^2 = 1$. Let $r_i = |f_{n_i}|$ for $i = 1, 2, \dots, k$ and $r_n = |f_n|$. Consider the Lagranges function $F(r_1, r_2 \dots r_k, r_n, \lambda) = r_n \sum_{i=1}^{k} r_i - \lambda\big(\sum_{i=1}^{k} r_i^2 + r_n^2 - 1\big)$. Therefore, solving the equations $\partial F / \partial r_n = \sum_{i=1}^{k} r_i - 2\lambda r_n = 0$, $\partial F / \partial r_i = r_n - 2\lambda r_i = 0$, for $i = 1, 2, \dots, k$, the maximum expression (1) is $\sqrt{f_0(n)}/2$. By the definition the numerical radius of C_T restricted to $l^2(E_n)$ is the maximum expression (1). Therefore $w(C_T | l^2(E_n)) = \sqrt{f_0(n)}/2$. This proves that $W(C_T | l^2(E_n)) \subset \big\{\lambda : |\lambda| \leq \sqrt{f_0(n)}/2\big\}$. Taking $f = \cos\alpha\, \chi_{\{n\}} + (\sin\alpha/\sqrt{k})\, e^{-i\theta} \chi_{\{n_1, n_2 \dots n_k\}}$, we get $\|f\| = 1$ and

$$\langle C_T f, f \rangle = \frac{k \cos\alpha \sin\alpha\, e^{i\theta}}{\sqrt{k}} = \frac{\sqrt{k}}{2} \sin 2\alpha\, e^{i\theta} = r e^{i\theta},$$

where $r = (\sqrt{k}/2) \sin 2\alpha \leq \sqrt{k}/2$. This proves that $\big\{\lambda \in \mathbb{C} : |\lambda| \leq \sqrt{f_0(n)}/2\big\} \subset W(C_T)$. Thus $W(C_T | l^2(E_n)) = \big\{\lambda \in \mathbb{C} : |\lambda| \leq \sqrt{f_0(n)}/2\big\}$. $\square$

Lemma 3.2 *Let $C_T \in B(l^2(N))$. Then $W(C_T | l^2(E_n)) \subseteq \big\{\lambda \in \mathbb{C} : |\lambda| \leq \sqrt{f_0(n) + 3}/2\big\}$, where n is such that $f_0(n) \geq 2$, $T(n) \neq n$ but $T^2(n) = n$, and $E_n = T^{-1}(\{n\}) \cup \{n\}$.*

Proof. Suppose $f_0(n) = k \geq 2$ and $E_n = \{n, n_1 n_2 \dots n_k\}$. From the hypothesis, $T(n) = n_j$ for some j, $1 \leq j \leq k$. Clearly $f = f_n \chi_{\{n\}} + f_{n_1} \chi_{\{n_1\}} + \cdots + f_{n_k} \chi_{\{n_k\}} \in l^2(E_n)$, and $|\langle C_T f, f \rangle| \leq |f_n| \sum_{i=1}^{k} |f_{n_i}| + |f_n|\, |f n_j|$. We shall now compute $\sup\big\{|f_n| \sum_{i=1}^{k} |f_{n_i}| + |f_n|\, |f_{n_j}|\big\} \dots\dots (1)$ subject to the condition $\sum_{i=1}^{k} |f_{n_i}|^2 + |f_n|^2 = 1$. For this, consider the Lagrange's function $F(r_1, r_2 \dots, r_n, \lambda) = r_n \sum_{i=1}^{k} r_i + r_n r_j - \lambda\big(\sum_{i=1}^{k} r_i^2 + r_n^2 - 1\big)$, where $r_i = |f_{n_i}|$ and $r_n = |f_n|$. Solving the equations $\partial F / \partial r_n = \sum_{i=1}^{k} r_i + r_j - 2\lambda r_n = 0$, $\partial F / \partial r_j = 2 r_n - 2\lambda r_j = 0$, and $\partial F / \partial r_i = r_n - 2\lambda r_i = 0$, for $i = 1, 2, \dots, k$,

$i \neq j$, we find that the maximum expression (1) is equal to $\sqrt{f_0(n)+3}/2$. This gives $w(C_T|l^2(E_n)) = \sqrt{f_0(n)+3}/2$ which proves the lemma. □

Lemma 3.3 *Let $C_T \in B(l^2(\mathbb{N}))$. Then $W(C_T|l^2(E_n))$ is the closed elliptical disc with foci at 0 and 1 and major axis $= \sqrt{f_0(n)}$ and minor axis $= \sqrt{f_0(n)-1}$, where $f_0(n) \geq 2$ and $T(n) = n$, $E_n = T^{-1}\{n\}$.*

Proof. Let $f_0(n) = k \geq 2$ and $E_n = T^{-1}(\{n\}) = \{n, n_1, \ldots, n_{k-1}\}$. Then the range of $C_T|l^2(E_n)$ is the one dimensional subspace of $l^2(N)$ spanned by χ_{E_n}. For $h = \chi_{\{n,n_1,\ldots,n_{k-1}\}}/\sqrt{k}$, and $g = \sqrt{k}\,\chi_{\{n\}}$, $(C_T|l^2(E_n))(f) = \langle f, g\rangle h$ for every $f \in l^2(E_n)$.

Also $\|h\| = 1$. Hence by the two dimensional Case 2.4 and Proposition 2.5 of Bourdon and Shapiro [5], $W(C_T|l^2(E_n))$ is a closed elliptical disc with foci at 0 and 1, major axis $= \sqrt{f_0(n)}$ and minor axis $= \sqrt{f_0(n)-1}$. This proves the lemma. □

Theorem 3.4 *Let $C_T \in B(l^2(N))$ be such that T is purely antiperiodic but not injective. Then $W(C_T) \supseteq \{\lambda : |\lambda| \leq \|\sqrt{f_0}\|_\infty/2\}$, where $\|f_0\|_\infty = \sup\{f_0(n) : n \in N\}$.*

Proof. Let $n \in N$ be such that $f_0(n) = \sup\{f_0(m) : m \in N\} = \|f_0\|_\infty$. This is possible, since by Theorem 2.1 of Singh and Komal [9] range of f_0 is a finite set. By using Lemma 3.1 for any $m \in N$, we obtain $W(C_T|l^2(E_m)) = \{\lambda \in \mathbb{C} : |\lambda| \leq \sqrt{f_0(m)}/2\}$. In particular for $m = n$, W$(C_T|l^2(E_n)) = \{\lambda \in \mathbb{C} : |\lambda| \leq \|\sqrt{f_0}\|_\infty/2\}$. Hence $W(C_T) \supseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \|\sqrt{f_0}\|_\infty/2\}$. □

Theorem 3.5 *Let $T\colon N \to N$ be a mapping such that $C_T\colon D\ (\subseteq l^2(N)) \to l^2(N)$ is an unbounded operator, where D, the domain of C_T is a dense subspace of $l^2(\mathbb{N})$. Then $W(C_T) = \mathbb{C}$, the complex plane.*

Proof. Suppose C_T is an unbounded operator. Then by Theorem 2.1 of Singh and Komal [9], there exists an increasing sequence $\{n_k\}$ of positive integers such that $f_0(n_k) \to \infty$ as $k \to \infty$. Now two cases can arise:

Case 1: $T(n_k) \neq n_k$ for infinitely many values of k. In this case either $T^2(n_k) \neq n_k$ for infinitely many values of k or $T^2(n_k) = n_k$ for all but finitely many values of k. Now if $T^2(n_k) \neq n_k$ for infinitely many values of k, then by Lemma 3.1, $W(C_T|l^2(E_{n_k})) = \{\lambda \in \mathbb{C} : |\lambda| \leq \sqrt{f_0(n_k)}/2\}$ for infinitely many values of k. But $W(C_T|l^2(E_{n_k})) \subset W(C_T)$ for every k. Therefore $\mathbb{C} \subseteq W(C_T)$. Next suppose $T^2(n_k) = n_k$ for all but finitely many values of k. Let $T^{-1}\{n_k\} = \{p_{1k}, p_{2k} \ldots, p_{m_k k}\}$ and $T(n_k) = p_{jk}$ for some j, $1 \leq j \leq m_k$ as $T(n_k) \in T^{-1}(\{n_k\})$. Now $f_0(n_k) = m_k \to \infty$ as $k \to \infty$. Take

$$f = \frac{1}{\sqrt{2}}\,\chi_{\{n_k\}} + \frac{e^{i\theta}}{\sqrt{2(m_k+3)}}\,\chi_{\{p_{1k},\cdots p_{(j-1)k}, p_{(j+1)k}\cdots p_{m_k k}\}} + \sqrt{\frac{2}{(m_k+3)}}\,\chi_{\{p_{jk}\}},$$

for $0 \leq \theta \leq 2\pi$. A simple computation reveals that $\|f\| = 1$ and

$$\langle C_T f, f\rangle = \frac{m_k - 1}{\sqrt{2}} \frac{e^{i\theta}}{\sqrt{2(m_k+3)}} + \frac{2}{\sqrt{m_k+3}}.$$

Therefore, $2/\sqrt{m_k+3} + (m_k-1)/(2\sqrt{m_k+3})\,e^{i\theta} \in W(C_T)$ for $0 \leq \theta \leq 2\pi$ which implies that

$$\left\{z \in \mathbb{C} : \left|z - \frac{2}{\sqrt{m_k+3}}\right| \leq \frac{m_k-1}{2\sqrt{m_k+3}}\right\} \subset W(C_T) \ldots\ldots (*),$$

since numerical range of an operator is convex. We now show that $\mathbb{C} \subseteq W(C_T)$. Since $m_k \to \infty$, choose m_k so large that $\sqrt{m_k+3}/2 \geq |\lambda| + 4 \geq |\lambda| + 4/\sqrt{m_k+3}$. This implies that

$$\begin{aligned}\left|\lambda - \frac{2}{\sqrt{m_k+3}}\right| &\leq |\lambda| + \frac{2}{\sqrt{m_k+3}} \\ &\leq \frac{\sqrt{m_k+3}}{2} - \frac{4}{\sqrt{m_k+3}} + \frac{2}{\sqrt{m_k+3}} \\ &= \frac{m_k+1}{2\sqrt{m_k+3}}.\end{aligned}$$

Hence by (*) $\lambda \in W(C_T)$.

Case II: $T(n_k) = n_k$ for all but finitely many values of k. An application of Lemma 3.3 yields that the closed elliptical disc with foci at 0, 1, major axis $= \sqrt{f_0(n_k)}$ and minor axis $= \sqrt{f_0(n_k) - 1}$ is contained in $W(C_T)$. Since $f_0(n_k) \to \infty$, we conclude that $\mathbb{C} \subseteq W(C_T)$. This completes the proof of the theorem. □

We shall now give an example of an unbounded composition operator.

Example 3.6 Let $D = \{f \in l^2(\mathbb{N}) : f(n) = 0$ for all but finitely many values of $n\}$. Then D is a dense linear subspace of $l^2(\mathbb{N})$. Let $T\colon N \to N$

be defined as

$$T(p) = \begin{cases} 2^{2^m} + 1, & \text{if } p \in \left[2^{2^{m-1}} + 1, 2^{2^m}\right] \text{ for } m = 1,\, 2, \ldots \\ 3, & \text{if } p \in \{1, 2\} \end{cases}$$

where $[m, n]$ denotes the set of all positive integers p such that $m \leq p \leq n$. Then $f_0(p) = \#(T^{-1}(\{p\})) \to \infty$ as $p \to \infty$. Therefore $C_T \colon D \to l^2(N)$ is not bounded.

Theorem 3.7 *Let $C_T \in B(l^2(N))$ and T be such that $T^2(n) = T(n)$ for every $n \in N$. Then*
(i) $W(C_T) = \{1\}$, *if T is the identity map.*
(ii) $W(C_T) = C_0(\cup_{n \in T(N)} Dn)$, *if T is not the identity map, where Dn is the elliptical disc with foci at 0, 1, major axis $= \sqrt{f_0(n)}$ and minor axis $= \sqrt{f_0(n) - 1}$.*

Proof. The proof of part (i) is trivial. For the proof of part (ii) if $n \in T(N)$, write $E_n = T^{-1}(\{n\})$. Now $C_T(l^2(E_n)) \subset l^2(E_n)$ and $C_T(l^2(N - E_n)) \subseteq l^2(N - E_n)$. Therefore, $C_T = \sum_{n \in T(N)} \oplus (C_T | l^2(E_n))$ because $l^2(N) = \sum_{n \in T(N)} \oplus l^2(E_n)$. Hence $W(C_T) = C_0\big(\bigcup_{n \in T(N)}^{\infty} W(C_T | l^2(E_n))\big)$. But by Lemma 3.3, $W(C_T | l^2(E_n)) = D_n$. Therefore the proof of the theorem is complete. □

In [5] Bourdon and Shapiro considered the zero inclusion question i.e. for which T does W(C_T) contain the origin? This is proved in Theorem 3.1 of [5] that if T is not identity, $0 \in \overline{W(C_T)}$. We in the following theorem show that $W(C_T)$ always contains the origin when $T \neq I$.

Theorem 3.8 *Let $C_T \in B(l^2(N))$ and T be not the identity map. Then $0 \in W(C_T)$.*

Proof. If T is not surjective, then we can choose a positive integer $n_0 \notin T(N)$. Clearly $\|\chi_{\{n_0\}}\| = 1$ and $\langle C_T \chi_{\{n_0\}}, \chi_{\{n_0\}} \rangle = o$. Thus $0 \in W(C_T)$. Next, suppose T is surjective. Now if T is injective, then since $T \neq I$, so there exists $n \in N$ for which $T(n) \neq n$. Again $\langle C_T \chi_{\{n\}}, \chi_{\{n\}} \rangle = o$. Further, if T is not injective, then $T(n_1) = T(n_2) = n$ (say) for two distict positive integers n_1 and n_2. Atleast one of n_1 and n_2, say n_1 is distinct from n. By surjectivity of T, $T(n_0) = n_1$ for some $n_0 \in N$. Moreover, $n_1 \notin T^{-1}(\{n_1\})$; otherwise $T(n_1) = n_1$ and so $n_1 = n$ which contradicts the choice of n. Clearly $\langle C_T \chi_{\{n_1\}}, \chi_{\{n_1\}} \rangle = 0$. Thus $0 \in W(C_T)$. □

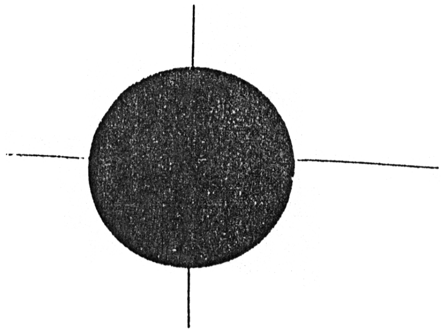

Fig. 3.1.

Example 3.9 Let $T\colon N \to N$ be defined by $T(n) = pn + q$, $p, q \in N$. Then $W(C_T) = \{\lambda : |\lambda| < 1\}$.

Example 3.10 Let $T\colon N \to N$ be defined by $T(n) = \sum_{j=0}^{k+1} 2^{2j}$ if $n \in \left[\sum_{j=0}^{k} 2^{2j}, \sum_{j=1}^{k+1} 2^{2j}\right]$ for $k = 0, 1, 2, 3, \ldots$. Then $C_T\colon D \to l^2(N)$ is an unbounded operator where D is given in Example 3.6. Then

$$r_k = \frac{\sqrt{f_0\left(\sum_{j=0}^{k+1} 2^{2j}\right)}}{2} = \frac{\sqrt{2^{2(k+1)}}}{2} = \frac{2^{(k+1)}}{2}.$$

Then in view of Lemma 3.1 the closed disc $\{\lambda \in \mathbb{C} : |\lambda| \le 2^k\} \subset W(C_T)$ for each $k = 0, 1, 2, \ldots$ It follows that $\mathbb{C} \subset W(C_T)$. Hence $W(C_T) = \mathbb{C}$.

Example 3.11 Let $T\colon N \to N$ be defined by

$$T(n) = \begin{cases} 1, & \text{if } n \in \{1,2\} \\ 1 + \sum\limits_{j=0}^{k} 2^{2^j}, & \text{if } n \in \left[1 + \sum\limits_{j=0}^{k} 2^{2^j}, \sum\limits_{j=0}^{k+1} 2^{2^j}\right] \\ & \text{for } k = 0, 1, 2, 3, 4, 5 \\ n, & \text{if } n > \sum\limits_{j=0}^{6} 2^{2^j} \end{cases}$$

Then $f_0(1) = 2$, $f_0\left(1 + \sum_{j=0}^{k} 2^{2^j}\right) = 2^{2^{k+1}}$ for $k = 0, 1, 2, \ldots, 5$. By Lemma 3.3, $W(C_T) = D_1 \cup \bigcup_{i=0}^{5} D_{m_k}$, where D_{m_k} is an elliptical disc with foci at 0, 1, major axis $= \sqrt{2}$ and minor axis $= 1$, D_{m_k} is an elliptical disc with foci at 0, 1, major axis $= 2^{2^k}$ and minor axis $= \sqrt{2^{2k+1} - 1}$, for

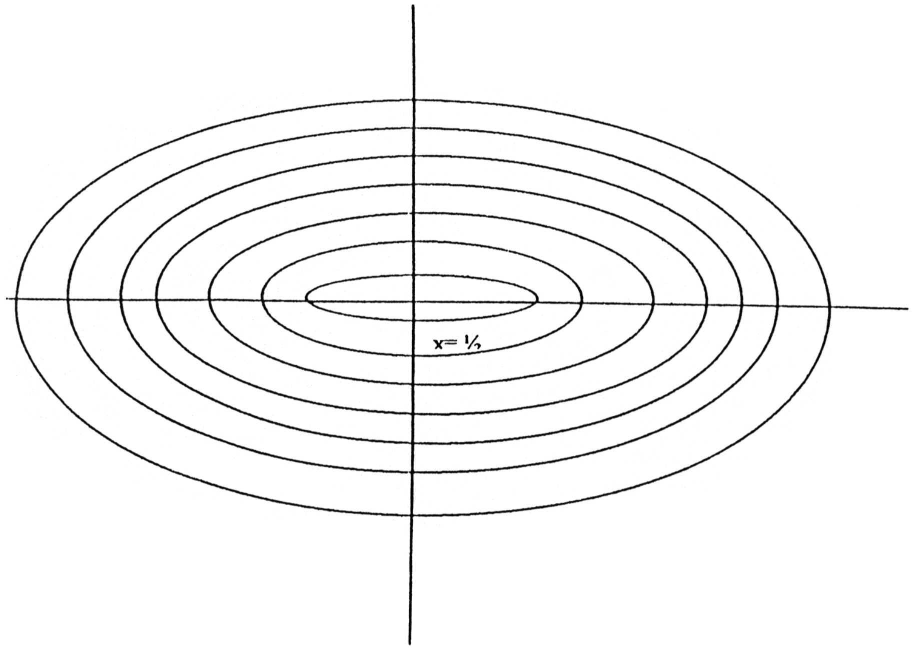

Fig. 3.2.

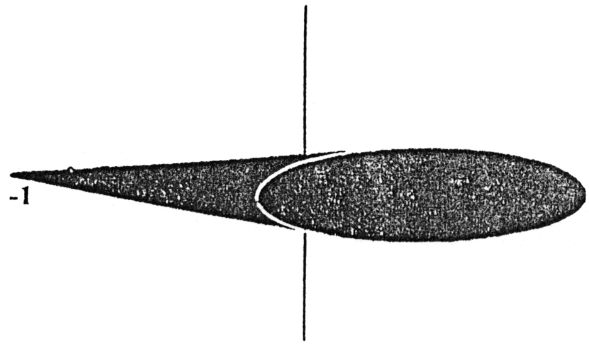

Fig. 3.3.

$k = 0, 1, 2, 3, 4, 5$ and $m_k = 1 + \sum_{j=0}^{k} 2^{2^j}$.

Example 3.12 (i)

$$T(n) = \begin{cases} n+1, & \text{if } n \text{ is odd} \\ n-1, & \text{if } n \text{ is even} \end{cases}$$

(ii)

$$T(n)=\begin{cases} n+1, & \text{if } n=1\\ n-1, & \text{if } n=2\\ \quad n\ \ , & \text{if } n \text{ is odd} \geq 3\\ n-1, & \text{if } n \text{ is even} \geq 3\end{cases}$$

$W(C_T) = C_0(\{-1\}\ U\ D_1)$, where D_1 is ellipse with foci at 0, 1 and major axis $= \sqrt{2}$ and minor axis $= 1$.

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B.S. Komal
Department of Mathematics,
University of Jammu,
Jammu- 180006. (INDIA)
E-mail: bskomal2@yahoo.co.in

S. Sharma
Department of Mathematics,
University of Jammu,
Jammu- 180006. (INDIA)
E-mail: shalluk3@yahoo.co.in