Hokkaido Mathematical Journal Vol. 34 (2005) p. 689–717

Stability of discrete ground state

Tadahiro Miyao and Itaru Sasaki

(Received June 17, 2004; Revised October 29, 2004)

Abstract. We present new criteria for a self-adjoint operator to have a ground state. Although the proof which we give is easy, it has many applications. And in various models, it is easy to check the criteria. As an application, we consider models of "quantum particles" coupled to a massive Bose field and prove the existence of a ground state of them, where the particle Hamiltonian does not necessarily have compact resolvent.

Key words: ground state, discrete ground state, generalized spin-boson model, Fock space, Dereziński-Gérard model.

1. Introduction

Let T be a self-adjoint operator on a Hilbert space $\mathcal{H}$, and bounded from below. We say that T has a discrete ground state if the bottom of the spectrum of T is an isolated eigenvalue of T. In that case a non-zero vector in $\ker(T - E_0(T))$ is called a ground state of T. Let S be a symmetric operator on $\mathcal{H}$. Suppose that T has a discrete ground state and S is T-bounded. By the regular perturbation theory [8, XII], it is already known that $T + \lambda S$ has a discrete ground state for "sufficiently small" $\lambda \in \mathbb{R}$. Our aim is to present new criteria for $T + \lambda S$ to have a ground state.

In Section 2, we prove an existence theorem of a ground state which is useful to show the existence of a ground state of models of quantum particles coupled to a massive Bose field.

In Section 3, we consider the GSB model [2] with a self-interaction term of a Bose field, which we call the GSB + ϕ^2 model. We consider only the case where the Bose field is massive. The GSB model — an abstract system of quantum particles coupled to a Bose field — was proposed in [2]. In [2], A. Arai and M. Hirokawa proved the existence and uniqueness of the GSB model in the case where the particle Hamiltonian A has compact resolvent. Shortly after that, they proved the existence of a ground state of the GSB model in the case where A does not have necessarily compact resolvent [4, 3]. In this paper, using a theorem in Section 2, we prove the

---

2000 Mathematics Subject Classification : 81T10.

existence of a ground state of the $\mathrm{GSB} + \phi^2$ model in the case where A does not necessarily have compact resolvent.

In Section 4, we consider an extended version of the Nelson type model, which we call the Dereziński-Gérard model [5]. The Dereziński-Gérard model is introduced in [5], and J. Dereziński and C. Gérard prove existence of a ground state for their model under some conditions including that A has compact resolvent. In Section 4, we prove the existence of a ground state of the Dereziński-Gérard model in the case where A does not have compact resolvent. Our strategy to establish existence of a ground state is the same as in Section 3.

2. basic results

Let $\mathcal{H}$ be a separable complex Hilbert space. We denote by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ the scalar product on Hilbert space $\mathcal{H}$ and by $\| \cdot \|_{\mathcal{H}}$ the associated norm. Scalar product $\langle f, g \rangle_{\mathcal{H}}$ is linear in g and antilinear in f. We omit $\mathcal{H}$ in $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $\| \cdot \|_{\mathcal{H}}$, respectively if there is no danger of confusion. For a linear operator T in Hilbert space, we denote by $D(T)$ and $\sigma(T)$ the domain and the spectrum of T respectively. If T is self-adjoint and bounded from below, then we define

$$E_0(T) := \inf \sigma(T), \quad \Sigma(T) := \inf \sigma_{\mathrm{ess}}(T),$$

where $\sigma_{\mathrm{ess}}(T)$ is the essential spectrum of T. If T has no essential spectrum, then we set $\Sigma(T) = \infty$. For a self-adjoint operator T, we denote the form domain of T by $Q(T)$. In this paper, an eigenvector of a self-adjoint operator T with eigenvalue $E_0(T)$ is called a ground state of T (if it exists). We say that T has a ground state if $\dim \ker(T - E_0(T)) > 0$.

The basic results are as follows:

Theorem 2.1 *Let H be a self-adjoint operator on $\mathcal{H}$, and bounded from below. Suppose that there exists a self-adjoint operator W on $\mathcal{H}$ satisfying the following conditions* (i)-(iii):

(i) $D(H) \subset D(W)$.

(ii) *W is bounded from below, and* $\Sigma(W) > 0$.

(iii) $H - E_0(H) \geq W$ *on* $D(H)$.

Then H has purely discrete spectrum in the interval $[E_0(H), E_0(H)+\Sigma(W))$. In particular, H has a ground state.

Proof. For all $u_1, \ldots, u_{n-1} \in \mathcal{H}$, we have

$$\inf_{\substack{\Psi \in \mathrm{L.h.}[u_1, \ldots, u_{n-1}]^\perp \\ \|\Psi\|=1,\, u \in D(H)}} \langle \Psi, H\Psi \rangle - E_0(H) \geq \inf_{\substack{\Psi \in \mathrm{L.h.}[u_1, \ldots, u_{n-1}]^\perp \\ \|\Psi\|=1,\, u \in D(H)}} \langle \Psi, W\Psi \rangle,$$

where L.h.$[\cdots]$ denotes the linear hull of the vectors in $[\cdots]$. Since $D(H) \subset D(W)$, we have that

$$\inf_{\substack{\Psi \in \mathrm{L.h.}[u_1, \ldots, u_{n-1}]^\perp \\ \|\Psi\|=1,\, \Psi \in D(H)}} \langle \Psi, W\Psi \rangle \geq \inf_{\substack{\Psi \in \mathrm{L.h.}[u_1, \ldots, u_{n-1}]^\perp \\ \|\Psi\|=1,\, \Psi \in D(W)}} \langle \Psi, W\Psi \rangle.$$

Hence, for all $n \in \mathbb{N}$

$$\mu_n(H) - E_0(H) \geq \mu_n(W).$$

where

$$\mu_n(\sharp) := \sup_{u_1, \ldots, u_{n-1} \in \mathcal{H}} \inf_{\substack{\Psi \in \mathrm{L.h.}[u_1, \ldots, u_{n-1}]^\perp \\ \|\Psi\|=1,\, \Psi \in D(\sharp)}} \langle \Psi, \sharp\Psi \rangle, \quad (\sharp = H, W).$$

By the min-max principle ([8, Theorem XIII.1]), $\lim_{n\to\infty} \mu_n(H) = \Sigma(H)$ and $\lim_{n\to\infty} \mu_n(W) = \Sigma(W)$. Therefore we obtain

$$\Sigma(H) - E_0(H) \geq \Sigma(W) > 0.$$

This means that H has purely discrete spectrum in $[E_0(H), E_0(H) + \Sigma(W))$. □

Theorem 2.2 *Let H be a self-adjoint operator on $\mathcal{H}$, and bounded from below. Suppose that there exists a self-adjoint operator W on $\mathcal{H}$ satisfying the following conditions* (i)-(iii):

(i) $Q(H) \subset Q(W)$.

(ii) *W is bounded from below, and $\Sigma(W) > 0$.*

(iii) $H - E_0(H) \geq W$ *on* $Q(H)$.

Then H has purely discrete spectrum in the interval $[E_0(H), E_0(H) + \Sigma(W))$. In particular, H has a ground state.

Proof. Similar to the proof of Theorem 2.1. □

We apply Theorems 2.1 and 2.2 to a perturbation problem of a self-adjoint operator.

Theorem 2.3 *Let A be a self-adjoint operator on $\mathcal{H}$ with $E_0(A) = 0$, and let B be a symmetric operator on $D(A)$. Suppose that $A + B$ is self-adjoint on $D(A)$ and that there exist constants $a \in [0, 1)$ and $b \geq 0$ such that*

$$|\langle \psi, B\psi \rangle| \leq a\langle \psi, A\psi \rangle + b\|\psi\|^2, \quad \psi \in D(A).$$

Assume

$$\frac{b + E_0(A+B)}{1-a} < \Sigma(A). \tag{1}$$

Then $A + B$ has purely discrete spectrum in $[E_0(A+B), (1-a)\Sigma(A) - b)$. In particular, $A + B$ has a ground state.

Proof. By the assumption we have

$$A + B - E_0(A+B) \geq (1-a)A - b - E_0(A+B)$$

on $D(A)$, and $(1-a)\Sigma(A) - b - E_0(A+B) > 0$. Hence we can apply Theorem 2.1, to conclude that $A+B$ has purely discrete spectrum in $[E_0(A+B), (1-a)\Sigma(A) - b)$. In particular, $A + B$ has a ground state. □

Remark It is easily to see that $-b \leq E_0(A+B) \leq b$. Therefore condition (1) is satisfied if

$$\frac{2b}{1-a} < \Sigma(A).$$

Let $\mathcal{H}$, $\mathcal{K}$ be complex separable Hilbert spaces. Let A and B be self-adjoint operators on $\mathcal{H}$ and $\mathcal{K}$ respectively. Suppose that $E_0(A) = E_0(B) = 0$. We set

$$T_0 := A \otimes I + I \otimes B.$$

Let Z be a symmetric sesquilinear form on $Q(T_0)$, and assume that there exist constants $a_1 \in [0, 1)$, $a_2 \in [0, 1)$ and $b \geq 0$ such that, for all $\Psi \in Q(T_0)$

$$|Z(\Psi, \Psi)| \leq a_1\langle \Psi, A \otimes I\Psi \rangle_{\text{form}} + a_2\langle \Psi, I \otimes B\Psi \rangle_{\text{form}} + b\|\Psi\|^2,$$

where $\langle \Psi, A \otimes I\Psi \rangle_{\text{form}} = \|A^{1/2} \otimes I\Psi\|^2$. Therefore, by the KLMN theorem there exists a unique self-adjoint operator T on $\mathcal{H} \otimes \mathcal{K}$ such that $Q(T) = Q(T_0)$ and $T = T_0 + Z$ in the sense of sesquilinear form on $Q(T_0)$. We set

$$s := \min\{(1-a_1)\Sigma(A), (1-a_2)\Sigma(B)\}.$$

The following thorem is easy application of Theorem 2.2.

Theorem 2.4 *Assume*

$$s > b + E_0(T). \tag{2}$$

Then, T has purely discrete spectrum in the interval $[E_0(T), s - b)$. In particular, T has a ground state.

Proof. Similar to the proof of Theorem 2.3. □

Remark It is easy to see that $-b \leq E_0(T) \leq b$. Therefore the condition (2) is satisfied if

$$s > 2b.$$

Remark Theorem 2.4 is essentially same as [4, Theorem B.1]. But our proof is very simple.

3. Ground States of a General Class of Quantum Field Hamiltonians

We consider a model which is an abstract unification of some quantum field models of particles interacting with a Bose field. It is the GSB model [2] with a self-interaction term of the field.

Let $\mathcal{H}$ be a separable complex Hilbert space and $\mathcal{F}_{\mathrm{b}}$ be the Boson Fock space over $L^2(\mathbb{R}^d)$:

$$\mathcal{F}_{\mathrm{b}} := \bigoplus_{n=0}^{\infty} \left[\bigotimes_{s}^{n} L^2(\mathbb{R}^d) \right].$$

The Hilbert space of the quantum field model we consider is

$$\mathcal{F} := \mathcal{H} \otimes \mathcal{F}_{\mathrm{b}}.$$

Let $\omega \colon \mathbb{R}^d \to [0, \infty)$ be Borel measurable such that $0 < \omega(k) < \infty$ for almost every $k \in \mathbb{R}^d$ (a.e. k). We denote the multiplication operator by the function ω acting in $L^2(\mathbb{R}^d)$ by the same symbol ω. We set

$$H_{\mathrm{b}} := \mathrm{d}\Gamma_{\mathrm{b}}(\omega)$$

the second quantization of ω (e.g. [7, Section X.7]). We denote by $a(f)$, $f \in L^2(\mathbb{R}^d)$, the smeared annihilation operators on $\mathcal{F}_{\mathrm{b}}$. It is a densely defined closed linear operator on $\mathcal{F}_{\mathrm{b}}(\mathbb{R}^d)$ (e.g. [7, Section X.7]). The adjoint $a(f)^*$, called the creation operator, and the annihilation operator $a(g)$, $g \in L^2(\mathbb{R}^d)$

obey the canonical commutation relations

$$[a(f),\, a(g)^*] = \langle f,\, g\rangle, \quad [a(f),\, a(g)] = 0, \quad [a(f)^*,\, a(g)^*] = 0$$

for all $f,\, g \in L^2(\mathbb{R}^d)$ on the dense subspace

$$\mathcal{F}_0 := \{\psi = (\psi^{(n)})_{n=0}^{\infty} \in \mathcal{F}_{\mathrm{b}} | \text{there exists a number } n_0 \text{ such that } \psi^{(n)} = 0 \text{ for all } n \geq n_0\},$$

where $[X,\, Y] = XY - YX$. The symmetric operator

$$\phi(f) := \frac{1}{\sqrt{2}}[a(f)^* + a(f)],$$

called the Segal field operator, is essentially self-adjoint on $\mathcal{F}_0$ (e.g. [7, Section X.7]). We denote its closure by the same symbol. Let A be a positive self-adjoint operator on $\mathcal{H}$ with $E_0(A) = 0$. Then, the unperturbed Hamiltonian of the model is defined by

$$H_0 := A \otimes I + I \otimes H_{\mathrm{b}}$$

with domain $D(H_0) = D(A \otimes I) \cap D(I \otimes H_{\mathrm{b}})$. For $g_j,\, f_j \in L^2(\mathbb{R}^d)$ $j = 1,\, \ldots,\, J$, and $B_j (j = 1,\, \ldots,\, J)$ a symmetric operator on $\mathcal{H}$, we define a symmetric operator

$$H_1 := \sum_{j=1}^{J} B_j \otimes \phi(g_j),$$

$$H_2 := \sum_{j=1}^{J} I \otimes \phi(f_j)^2.$$

The Hamiltonian of the model we consider is of the form

$$H(\lambda,\, \mu) := H_0 + \lambda H_1 + \mu H_2,$$

where $\lambda \in \mathbb{R}$ and $\mu \geq 0$ are coupling parameters.

For $H(\lambda,\, \mu)$ to be self-adjoint, we shall need the following conditions [H.1]-[H.3]:

[H.1] $g_j \in D(\omega^{-1/2})$, $f_j \in D(\omega^{1/2}) \cap D(\omega^{-1/2})$, $j = 1,\, \ldots,\, J$.

[H.2] $D(A^{1/2}) \subset \cap_{j=1}^{J} D(B_j)$ and there exist constants $a_j \geq 0$, $b_j \geq 0$, $j = 1,\, \ldots,\, J$, such that,

$$\|B_j u\| \leq a_j \|A^{1/2} u\| + b_j \|u\|, \quad u \in D(A^{1/2}).$$

[H.3] $|\lambda|\sum_{j=1}^{J} a_j\|g_j/\sqrt{\omega}\| < 1$.

Proposition 3.1 *Assume* [H.1], [H.2] *and* [H.3]*. Then, $H(\lambda,\,\mu)$ is self-adjoint with $D(H(\lambda,\,\mu)) = D(H_0) \subset D(H_1) \cap D(H_2)$ and bounded from below. Moreover, $H(\lambda,\,\mu)$ is essentially self-adjoint on every core for H_0.*

Remark This proposition has no restriction of the coupling parameter $\mu \geq 0$.

To perform a finite volume approximation, we need an additional condition:

[H.4] The function $\omega(k)$ $(k \in \mathbb{R}^d)$ is continuous with

$$\lim_{|k|\to\infty} \omega(k) = \infty,$$

and there exist constants $\gamma > 0$, $C > 0$ such that

$$|\omega(k) - \omega(k')| \leq C|k - k'|^{\gamma}[1 + \omega(k) + \omega(k')], \quad k,\, k' \in \mathbb{R}^d.$$

Let

$$m := \inf_{k\in\mathbb{R}^d} \omega(k). \tag{3}$$

If A has compact resolvent, we can prove the extension of the previous theorem [2, Theorem 1.2].

Theorem 3.2 *Consider the case $m > 0$. Suppose that A has entire purely discrete spectrum. Assume Hypotheses* [H.1]-[H.4]*. Then, $H(\lambda,\,\mu)$ has purely discrete spectrum in the interval $[E_0(H(\lambda,\,\mu)),\, E_0(H(\lambda,\,\mu)) + m)$. In particular, $H(\lambda,\,\mu)$ has a ground state.*

Remark This theorem has no restriction of the coupling parameter $\mu \geq 0$.

Remark In the case $m > 0$, the condition [H.1] equivalent to the following:

$$g_j \in L^2(\mathbb{R}^d), \quad f_j \in D(\sqrt{\omega}), \quad j = 1,\, \ldots,\, J.$$

For a vector $v = (v_1,\, \ldots,\, v_J) \in \mathbb{R}^J$ and $h = (h_1,\, \ldots,\, h_J) \in \oplus_{j=1}^{J} L^2(\mathbb{R}^d)$, we define

$$M_v(h) = \sum_{j=1}^{J} v_j\|h_j\|.$$

We set

$$g = (g_1, \ldots, g_J) \in \bigoplus_{j=1}^{J} L^2(\mathbb{R}^d), \quad f = (f_1, \ldots, f_J) \in \bigoplus_{j=1}^{J} L^2(\mathbb{R}^d),$$

and

$$a = (a_1, \ldots, a_J), \quad b = (b_1, \ldots, b_J).$$

For θ, ϵ, ϵ', we introduce the following constants:

$$\begin{aligned} C_{\theta,\,\epsilon} &:= \theta M_a(g/\sqrt{\omega}) + \epsilon M_a(g), \\ D_{\theta,\,\epsilon'} &:= M_a(g/\sqrt{\omega})/2\theta + \epsilon' M_b(g/\sqrt{\omega}), \\ E_{\epsilon,\,\epsilon'} &:= M_a(g)/8\epsilon + M_b(g/\sqrt{\omega})/2\epsilon' + M_b(g)/\sqrt{2}. \end{aligned}$$

Let the condition [H.3] be satisfied. Then, we define

$$I_{\lambda,g} := \begin{cases} \left(\dfrac{|\lambda| M_a(g\sqrt{\omega})}{2}, \dfrac{1}{|\lambda| M_a(g/\sqrt{\omega})} \right), & |\lambda| M_a(g/\sqrt{\omega}) \neq 0 \\ [0, \infty], & |\lambda| M_a(g/\sqrt{\omega}) = 0 \end{cases}$$

It is easy to see that $[1/2, 1] \subset I_{\lambda,\,g}$. Therefore, for all $\theta \in I_{\lambda,\,g}$,

$$\begin{aligned} & 1 - \theta |\lambda| M_a(g/\sqrt{\omega}) > 0, \\ & 1 - \frac{|\lambda| M_a(g/\sqrt{\omega})}{2\theta} > 0. \end{aligned}$$

We define for $\theta \in I_{\lambda,\,g}$,

$$\mathsf{S}_\theta := \{(\epsilon, \epsilon') \mid \epsilon, \epsilon' > 0, |\lambda| C_{\theta,\,\epsilon} < 1, |\lambda| D_{\theta,\,\epsilon'} < 1\}.$$

Next we set

$$\tau_{\theta,\,\epsilon,\,\epsilon'} := (1 - |\lambda| C_{\theta,\,\epsilon}) \Sigma(A) - |\lambda| E_{\epsilon,\,\epsilon'},$$

and

$$\mathsf{T} := \big\{(\theta, \epsilon, \epsilon') \in \mathbb{R}^3 \mid \theta \in I_{\lambda,\,g}, (\epsilon, \epsilon') \in \mathsf{S}_\theta, \tau_{\theta,\,\epsilon,\,\epsilon'} > E_0(H(\lambda, \mu))\big\}.$$

Theorem 3.3 *Consider the case $m > 0$. Suppose that $\sigma_{\mathrm{ess}}(A) \neq \emptyset$. Assume Hypothesis* [H.1]-[H.4]*, and $\mathsf{T} \neq \emptyset$. Then, $H(\lambda, \mu)$ has purely discrete spectrum in the interval*

$$\big[E_0(H(\lambda, \mu)), \min\{m + E_0(H(\lambda, \mu)), \sup_{(\theta,\,\epsilon,\,\epsilon') \in \mathsf{T}} \tau_{\theta,\,\epsilon,\,\epsilon'}\}\big). \tag{4}$$

In particular, $H(\lambda, \mu)$ has a ground state.

Remark $\mathsf{T} \neq \emptyset$ is necessary condition for A to have a discrete ground state. Conversely, if A has a discrete ground state, then $\mathsf{T} \neq \emptyset$ holds for sufficiently small λ, μ. Therefore the condition $\mathsf{T} \neq \emptyset$ is a restriction for the coupling constants λ, μ.

3.1. Proof of Proposition 3.1

In what follows, we write simply

$$H := H(\lambda, \mu).$$

For $\mathcal{D}$ a dense subspace of $L^2(\mathbb{R}^d)$, we define

$$\mathcal{F}_{\text{fin}}(\mathcal{D}) := \text{L.h.}[\{\Omega, a(h_1)^* \cdots a(h_n)^*\Omega \,|\, n \in \mathbb{N}, h_j \in \mathcal{D}, j = 1, \ldots, n\}],$$

where $\Omega := (1, 0, 0, \ldots)$ is the Fock vacuum in $\mathcal{F}_{\text{b}}$. We introduce a dense subspace in $\mathcal{F}$

$$\mathcal{D}_\omega := D(A) \hat{\otimes} \mathcal{F}_{\text{fin}}(D(\omega)),$$

where $\hat{\otimes}$ denotes algebraic tensor product. The subspace $\mathcal{D}_\omega$ is a core for H_0.

Let

$$H_{\text{GSB}} := H_0 + \lambda H_1$$

be a GSB Hamiltonian. The Hamiltonian H and H_{GSB} has the following relation:

Proposition 3.4 *Let $D(A) \subset D(B_j)$, $j = 1, \ldots, J$ and $f_j \in D(\omega^{1/2})$. Assume that H_{GSB} is bounded from below. Then, for all $\Psi \in D_\omega$,*

$$\|(H_{\text{GSB}} - E_0)\Psi\|^2 + \|\mu H_2 \Psi\|^2 \leq \|(H - E_0)\Psi\|^2 + D\|\Psi\|^2, \tag{5}$$

where $D = \mu \sum_{j=1}^J \|\omega^{1/2} f_j\|^2$ and

$$E_0 := \inf_{\substack{\Psi \in D(H_{\text{GSB}}) \\ \|\Psi\| = 1}} \langle \Psi, H_{\text{GSB}} \Psi \rangle.$$

Proof. It is enough to show (5) in the case $\lambda = \mu = 1$. First we consider the case where $f_j \in D(\omega)$. Inequality (5) is equivalent to

$$-2\,\text{Re}\big\langle (H_{\text{GSB}} - E_0)\Psi, H_2\Psi \big\rangle \leq D\|\Psi\|^2. \tag{6}$$

By $H_{\rm GSB} - E_0 \geq 0$, we have

$$\begin{aligned}
&\big\langle (H_{\rm GSB} - E_0)\Psi,\, I \otimes \phi(f_j)^2\Psi \big\rangle \\
&\qquad\qquad = \big\langle [I \otimes \phi(f_j),\, (H_{\rm GSB} - E_0)]\Psi,\, I \otimes \phi(f_j)\Psi \big\rangle \\
&\qquad\qquad\quad + \big\langle (H_{\rm GSB} - E_0) I \otimes \phi(f_j)\Psi,\, I \otimes \phi(f_j)\Psi \big\rangle \\
&\qquad\qquad \geq \big\langle [I \otimes \phi(f_j),\, H_{\rm GSB} - E_0]\Psi,\, I \otimes \phi(f_j)\Psi \big\rangle.
\end{aligned}$$

Therefore we have

$$2\,{\rm Re}\big\langle (H_{\rm GSB} - E_0)\Psi,\, \phi(f_j)^2\Psi \big\rangle \geq -\|\sqrt{\omega} f_j\|^2 \|\Psi\|^2.$$

This means inequality (6). Next, we set $f_j \in D(\sqrt{\omega})$. Then, there exists a sequence $\{f_{jn}\}_{n=0}^{\infty} \subset D(\omega)$ such that $f_{jn} \to f_j$, $\omega^{1/2} f_{jn} \to \omega^{1/2} f_j$ $(n \to \infty)$. By limiting argument, (6) holds with $f_j \in D(\omega^{1/2})$. □

Lemma 3.5 *Suppose that $H_{\rm GSB}$ is self-adjoint with $D(H_{\rm GSB}) = D(H_0)$, essentially self-adjoint on $\mathcal{D}_\omega$, and bounded from below. Let $f_j \in D(\omega^{1/2}) \cap D(\omega^{-1/2})$. Then H is self-adjoint with $D(H) = D(H_0)$ and essentially self-adjoint on any core for $H_{\rm GSB}$ with*

$$\|(H_{\rm GSB} - E_0)\Psi\|^2 + \|\mu H_2 \Psi\|^2 \leq \|(H - E_0)\Psi\|^2 + D\|\Psi\|^2, \qquad \Psi \in D(H_0).$$

Proof. It is well known that $D(H_{\rm b}) \subset D(\phi(f_j)^2)$, and $\phi(f_j)^2$ is $H_{\rm b}$-bounded (e.g. [1, Lemma 13-16]). Namely, there exist constants $\eta \geq 0$, $\theta \geq 0$ such that

$$\left\| \sum_{j=1}^{J} \phi(f_j)^2 \psi \right\| \leq \eta \|H_{\rm b}\psi\| + \theta\|\psi\|, \quad \psi \in D(H_{\rm b}). \tag{7}$$

Since $H_{\rm GSB}$ is self-adjoint on $D(H_0)$, by the closed graph theorem, we have

$$\|H_0 \Psi\| \leq \lambda \|H_{\rm GSB}\Psi\| + \nu\|\Psi\|, \quad \Psi \in D(H_0), \tag{8}$$

where λ and ν are non-negative constant independent of Ψ. Hence

$$\|H_2\Psi\| \leq \eta\lambda \|H_{\rm GSB}\Psi\| + (\eta\nu + \theta)\|\Psi\|, \quad \Psi \in D(H_0).$$

We fix a positive number μ_0 such that $\mu_0 < 1/(\mu\lambda)$. Then, by the Kato-Rellich theorem, $H(\lambda, \mu_0)$ is self-adjoint on $D(H_{\rm GSB})$, bounded from below and essentially self-adjoint on any core for $H_{\rm GSB}$. For a constant a $(0 < a < 1)$, we set $\mu_n := (1+a)^n \mu_0$. Since $H_{\rm GSB}$ is self-adjoint on $D(H_0)$, for each $j = 1, \ldots, J$ we have $D(A) \subset D(B)$. Thus by Proposition 3.4, for all $\Psi \in \mathcal{D}_\omega$

$$\|(H_{\mathrm{GSB}} - E_0)\Psi\|^2 + \|\mu_n H_2 \Psi\|^2 \leq \|(H(\lambda, \mu_n) - E_0)\Psi\|^2 + D\|\Psi\|^2.$$

If $H(\lambda, \mu_n)$ is self-adjoint on $D(H_{\mathrm{GSB}})$, bounded from below and essentially self-adjoint on any core for H_{GSB}, then $H(\lambda, \mu_{n+1})$ has the same property. On the other hand, we have $\mu_n \to \infty$ $(n \to \infty)$. Hence we conclude that H is self-adjoint with $D(H) = D(H_{\mathrm{GSB}})$, bounded from below and essentially self-adjoint on any core for H_{GSB}. □

Now, we assume conditions [H.1], [H.2] and [H.3].

Then H_{GSB} is self-adjoint on $D(H_0)$, bounded from below and essentially self-adjoint on any core for H_0 (see [2]). Hence, the assumptions of Lemma 3.5 hold. Thus Proposition 3.1 follows. □

3.2. Proofs of Theorems 3.2 and 3.3

Throughout this subsection, we assume Hypotheses [H.1]-[H.4] and $m > 0$.

For a parameter $V > 0$, we define the set of lattice points by

$$\begin{aligned}\Gamma_V &:= \frac{2\pi\mathbb{Z}^d}{V} \\ &:= \left\{ k = (k_1, \ldots, k_d) \,\middle|\, k_j = \frac{2\pi n_j}{V},\ n_j \in \mathbb{Z},\ j = 1, \ldots, d \right\}\end{aligned}$$

and we denote by $l^2(\Gamma_V)$ the set of l^2 sequences over Γ_V. For each $k \in \Gamma_V$ we introduce

$$C(k, V) := \left[k_1 - \frac{\pi}{V}, k_1 + \frac{\pi}{V}\right) \times \cdots \times \left[k_d - \frac{\pi}{V}, k_d + \frac{\pi}{V}\right) \subset \mathbb{R}^d,$$

the cube centered about k. By the map

$$U : l^2(\Gamma_V) \ni \{h_l\}_{l \in \Gamma_V} \mapsto (V/2\pi)^{d/2} \sum_{l \in \Gamma_V} h_l \chi_{l, V}(\,\cdot\,) \in L^2(\mathbb{R}^d),$$

we identify $l^2(\Gamma_V)$ with a subspace in $L^2(\mathbb{R}^d)$, where $\chi_{l, V}(\,\cdot\,)$ is the characteristic function of the cube $C(l, V) \subset \mathbb{R}^d$. It is easy to see that $l^2(\Gamma_V)$ is a closed subspace of $L^2(\mathbb{R}^d)$. Let

$$\mathcal{F}_{\mathrm{b, V}} := \mathcal{F}_{\mathrm{b}}(l^2(\Gamma_V)) = \bigoplus_{n=0}^{\infty} \left[\bigotimes_s^n l^2(\Gamma_V) \right],$$

the boson Fock space over $l^2(\Gamma_V)$. We can identify $\mathcal{F}_{\mathrm{b,\,V}}$ the closed subspace of $\mathcal{F}_{\mathrm{b}}$ by the operator $\Gamma(U) := \oplus_{n=0}^{\infty} \otimes^n U$, where we define $\otimes^0 U = 0$. For each $k \in \mathbb{R}^d$, there exists a unique point $k_V \in \Gamma_V$ such that $k \in C(k_V, V)$. Let

$$\omega_V(k) := \omega(k_V), \quad k \in \mathbb{R}^d$$

be a lattice approximate function of $\omega(k)$ and let

$$H_{\mathrm{b,\,V}} := \mathrm{d}\Gamma(\omega_V)$$

be the second quantization of ω_V. We define a constant

$$C_V := Cd^{\gamma}\Big(\frac{\pi}{V}\Big)\Big(\frac{1}{2m} + 1\Big),$$

where C and γ were defined in [H.4]. In what follows we assume that

$$C_V < 1.$$

This is satisfied for all sufficiently large V.

Lemma 3.6 ([2, Lemma 3.1]) *We have*

$$D(H_{\mathrm{b,\,V}}) = D(H_{\mathrm{b}}),$$

and

$$\|(H_{\mathrm{b}} - H_{\mathrm{b,\,V}})\Psi\| = \frac{2C_V}{1 - C_V}\|H_{\mathrm{b}}\Psi\|, \quad \Psi \in D(H_{\mathrm{b}}).$$

First we consider the case where g_j's and f_j's are continuous, and finally, by limiting argument, we treat a general case. For a constant $K > 0$, we define $g_{j,\,K}$, $f_{j,\,K}$, and $g_{j,\,K,V}$, $f_{j,\,K,V}$ as follows:

$$\begin{aligned}
g_{j,\,K}(k) &:= \chi_K(k_1)\cdots\chi_K(k_d)g_j(k),\\
g_{j,\,K,V}(k) &:= \sum_{\substack{\ell\in\Gamma_V,\,|\ell_i|<K\\ i=1,\,\dots,\,d}} g_j(\ell)\chi_{\ell,\,V}(k),\\
f_{j,\,K}(k) &:= \chi_K(k_1)\cdots\chi_K(k_d)f_j(k),\\
f_{j,\,K,V}(k) &:= \sum_{\substack{\ell\in\Gamma_V,\,|\ell_i|<K\\ i=1,\,\dots,\,d}} f_j(\ell)\chi_{\ell,\,V}(k),
\end{aligned}$$

where χ_K denotes the characteristic function of $[-K,\,K]$.

Lemma 3.7 *For all $j = 1, \ldots, J$,*

$$\lim_{V\to\infty} \|g_{j,K,V} - g_{j,K}\| = 0, \quad \lim_{V\to\infty} \|g_{j,K,V}/\sqrt{\omega_V} - g_{j,K}/\sqrt{\omega}\| = 0,$$
$$\lim_{K\to\infty} \|g_{j,K} - g_j\| = 0, \quad \lim_{K\to\infty} \|g_{j,K}/\sqrt{\omega} - g_j/\sqrt{\omega}\| = 0,$$
$$\lim_{V\to\infty} \|f_{j,K,V} - f_{j,K}\| = 0, \quad \lim_{V\to\infty} \|f_{j,K,V}/\sqrt{\omega_V} - f_{j,K}/\sqrt{\omega}\| = 0,$$
$$\lim_{K\to\infty} \|f_{j,K} - f_j\| = 0, \quad \lim_{K\to\infty} \|f_{j,K}/\sqrt{\omega} - f_j/\sqrt{\omega}\| = 0,$$
$$\lim_{K\to\infty} \|\sqrt{\omega} f_{j,K} - \sqrt{\omega} f_j\| = 0, \quad \lim_{V\to\infty} \|\sqrt{\omega_V} f_{j,K,V} - \sqrt{\omega} f_{j,K}\| = 0.$$

Proof. Similar to the proof of [2, Lemma 3.10]. □

We introduce a new operator:

$$H_{0,V} := A \otimes I + I \otimes H_{\mathrm{b,V}},$$
$$H_{1,K} := \sum_{j=1}^{J} B_j \otimes \phi(g_{j,K}),$$
$$H_{1,K,V} := \sum_{j=1}^{J} B_j \otimes \phi(g_{j,K,V}),$$
$$H_{2,K} := \sum_{j=1}^{J} I \otimes \phi(f_{j,K})^2,$$
$$H_{2,K,V} := \sum_{j=1}^{J} I \otimes \phi(f_{j,K,V})^2,$$

and define

$$H_K := H_0 + \lambda H_{1,K} + \mu H_{2,K},$$
$$H_{K,V} := H_{0,V} + \lambda H_{1,K,V} + \mu H_{2,K,V}.$$

Lemma 3.8 (i) *H_K is self-adjoint with $D(H_K) = D(H_0) \subset D(H_{1,K}) \cap D(H_{2,K})$, bounded from below, and essentially self-adjoint on any core for H_0.*
(ii) *For all large V, $H_{K,V}$ is self-adjoint with $D(H_{K,V}) = D(H_0) \subset D(H_{1,K,V}) \cap D(H_{2,K,V})$, bounded from below, and essentially self-adjoint on any core for $H_{0,V}$.*

Proof. Similar to the proof of Proposition 3.1. □

Lemma 3.9 *For all $z \in \mathbb{C}\backslash\mathbb{R}$, and $K > 0$,*

$$\lim_{K\to\infty} \|(H_K - z)^{-1} - (H - z)^{-1}\| = 0,$$
$$\lim_{V\to\infty} \|(H_{K,V} - z)^{-1} - (H_K - z)^{-1}\| = 0.$$

Proof. Similar to the proof of [2, Lemma 3.5]. □

The following fact is well known:

Lemma 3.10 *The operator $H_{\mathrm{b,V}}$ is reduced by $\mathcal{F}_{\mathrm{b,V}}$ and $H_{\mathrm{b,V}}\lceil\mathcal{F}_{\mathrm{b,V}}$ equal to the second quantization of $\omega_V\lceil l^2(\Gamma_V)$ on $\mathcal{F}_{\mathrm{b,V}}$.*

Lemma 3.11 *$H_{K,V}$ is reduced by $\mathcal{F}_V$.*

Proof. Similar to the proof of [2, Lemma 3.7]. □

Lemma 3.12 *We have*

$$H_{K,V}\lceil\mathcal{F}_V^{\perp} \geq E_0(H_{K,V}) + m.$$

Proof. Similar to the proof of [2, Lemma 3.10]. □

Lemma 3.13 *Let T_n and T be a self-adjoint operators on a separable Hilbert space and bounded from below. Suppose that $T_n \to T$ in norm resolvent sense as $n \to \infty$ and T_n has purely discrete spectrum in the interval $[E_0(T_n), E_0(T_n) + c_n)$ with some constant c_n. If $c := \limsup_{n\to\infty} c_n > 0$, then T has purely discrete spectrum in $[E_0(T), E_0(T) + c)$.*

Proof. There exists a sequence $\{c_{n_j}\}_{j=1}^{\infty} \subset \{c_n\}_{n=1}^{\infty}$ so that $c_{n_j} \to c(j \to \infty)$. So, for all $\epsilon > 0$ and for sufficiently large j, the spectrum of T_{n_j} in $[E_0(T_{n_j}), E_0(T_{n_j}) + c - \epsilon)$ is discrete. Therefore, applying [2, Lemma 3.12], we find that the spectrum of T in $[E_0(T), E_0(T) + c - \epsilon)$ is discrete. Since $\epsilon > 0$ is arbitrary, we get the conclusion. □

Now, if A has compact resolvent, by a method similar to the proof of [2, Theorem 1.2], we can prove Theorem 3.2. Therefore, we only prove Theorem 3.3.

The following inequality is known [2, (2.12)]:

$$|\langle\Psi, H_1\Psi\rangle| \leq C_{\theta,\epsilon}\langle\Psi, A\otimes I\Psi\rangle + D_{\theta,\epsilon}\langle\Psi, I\otimes H_{\mathrm{b}}\Psi\rangle + E_{\epsilon,\epsilon'}\|\Psi\|^2,$$

where $\Psi \in D(H_0)$ is arbitrary. Thus we have,

$$H \geq (1 - |\lambda|C_{\theta,\epsilon})A\otimes I + (1 - |\lambda|D_{\theta,\epsilon'})I\otimes H_{\mathrm{b}} + \mu H_2 - |\lambda|E_{\epsilon,\epsilon'}.$$

Let $I_{\lambda,g}(K)$, $C_{\theta,\epsilon}(K)$, $D_{\theta,\epsilon}(K)$ and $E_{\epsilon,\epsilon'}(K)$ are $I_{\lambda,g}$, $C_{\theta,\epsilon}$, $D_{\theta,\epsilon}$, $E_{\epsilon,\epsilon'}$ with g_j, f_j replaced by $g_{j,K}$, $f_{j,K}$ respectively, and let $I_{\lambda,g}(K,V)$, $C_{\theta,\epsilon}(K,V)$, $D_{\theta,\epsilon}(K,V)$ and $E_{\epsilon,\epsilon'}(K,V)$ are $I_{\lambda,g}$, $C_{\theta,\epsilon}$, $D_{\theta,\epsilon}$, $E_{\epsilon,\epsilon'}$ with g_j, f_j and ω replaced by $g_{j,K,V}$, $f_{j,K,V}$ and ω_V respectively. Then we have

Lemma 3.14 *The following operator inequalities hold*:

$$\begin{aligned} H_K &\geq (1-|\lambda|C_{\theta,\epsilon}(K))A\otimes I + (1-|\lambda|D_{\theta,\epsilon'}(K))I\otimes H_{\mathrm{b}} \\ &\quad + \mu H_{2,K} - |\lambda|E_{\epsilon,\epsilon'}(K) \quad \textit{on} \quad D(H_0), \\ H_{K,V} &\geq (1-|\lambda|C_{\theta,\epsilon}(K,V))A\otimes I + (1-|\lambda|D_{\theta,\epsilon'}(K,K))I\otimes H_{\mathrm{b,V}} \\ &\quad + \mu H_{2,K,V} - |\lambda|E_{\epsilon,\epsilon'}(K,V) \quad \textit{on} \quad D(H_0). \end{aligned}$$

Proof. Similar to the calculation of [2, (2.12)]. □

By Lemma 3.7, we have

$$\lim_{V\to\infty} C_{\theta,\epsilon}(K,V) = C_{\theta,\epsilon}(K), \qquad \lim_{K\to\infty} C_{\theta,\epsilon}(K) = C_{\theta,\epsilon}, \tag{9}$$

$$\lim_{V\to\infty} D_{\theta,\epsilon'}(K,V) = D_{\theta,\epsilon'}(K), \quad \lim_{K\to\infty} D_{\theta,\epsilon'}(K) = D_{\theta,\epsilon'}, \tag{10}$$

$$\lim_{V\to\infty} E_{\epsilon,\epsilon'}(K,V) = E_{\epsilon,\epsilon'}(K), \quad \lim_{K\to\infty} E_{\epsilon,\epsilon'}(K) = E_{\epsilon,\epsilon'}. \tag{11}$$

Let $(\theta, \epsilon, \epsilon') \in \mathsf{T}$, namely

$$\tau_{\theta,\epsilon,\epsilon'} = (1-|\lambda|C_{\theta,\epsilon})\Sigma(A) - |\lambda|E_{\epsilon,\epsilon'} > E_0(H).$$

Formulas (9)–(11) and Lemma 3.9 imply that for all large V there exists a constant $K_0 > 0$ such that for all $K > K_0$,

$$(1-|\lambda|C_{\theta,\epsilon}(K,V))\Sigma(A) - |\lambda|E_{\epsilon,\epsilon'}(K,V) > E_0(H_{K,V}), \tag{12}$$

$$|\lambda|C_{\theta,\epsilon}(K,V) < 1, \quad |\lambda|D_{\theta,\epsilon'}(K,V) < 1. \tag{13}$$

By Lemma 3.11, $H_{K,V}$ is reduced by $\mathcal{F}_V$. Therefore, $H_{K,V}$ satisfies the following inequality:

$$\begin{aligned} H_{K,V}\lceil\mathcal{F}_V \geq\ & (1-|\lambda|C_{\theta,\epsilon}(K,V))A\otimes I\lceil\mathcal{F}_V \\ & + (1-|\lambda|D_{\theta,\epsilon'}(K,V))I\otimes H_{\mathrm{b,V}}\lceil\mathcal{F}_V \\ & - |\lambda|E_{\epsilon,\epsilon'}(K,V). \end{aligned} \tag{14}$$

Since $H_{\mathrm{b,V}}\lceil\mathcal{F}_{\mathrm{b,V}}$ has compact resolvent, the bottom of essential spectrum of the right hand side of (14) is equal to

$$(1-|\lambda|C_{\theta,\epsilon}(K,V))\Sigma(A) - |\lambda|E_{\epsilon,\epsilon'}(K,V).$$

By Lemma 3.12, we have $E_0(H_{K,V}\lceil\mathcal{F}_V) = E_0(H_{K,V})$. Thus, applying Theorem 2.1 with $H_{K,V}\lceil\mathcal{F}_V$, we have that $H_{K,V}\lceil\mathcal{F}_V$ has purely discrete spectrum in $[E_0(H_{K,V}), (1-|\lambda|C_{\theta,\epsilon}(K, V))\Sigma_A - E_{\epsilon,\epsilon'}(K, V))$. Since this fact and Lemma 3.12, $H_{K,V}$ has purely discrete spectrum in

$$\big[E_0(H_{K,V}),\\ \min\{E_0(H_{K,V}) + m, (1-|\lambda|C_{\theta,\epsilon}(K, V))\Sigma_A - E_{\epsilon,\epsilon'}(K, V)\}\big).$$

By Lemma 3.9 and Lemma 3.13, we have that for all sufficiently large $K > 0$, H_K has purely discrete spectrum in $[E_0(H_K), \min\{E_0(H_K) + m, (1 - |\lambda|C_{\theta,\epsilon}(K))\Sigma(A) - |\lambda|E_{\epsilon,\epsilon'}(K)\})$. Similarly, H has purely discrete spectrum in $[E_0(H(\lambda, \mu)), \min\{m + E_0(H(\lambda, \mu)), \tau_{\theta,\epsilon,\epsilon'}\})$. Since $(\theta, \epsilon, \epsilon') \in \mathsf{T}$ is arbitrary, H has purely discrete spectrum in (4). Finally, we have to consider the case where g_j's and f_j's are not necessarily continuous. But, that argument were already discussed in [4]. So we skip that argument. □

4. Ground State of the Dereziński-Gérard Model

We consider a model discussed by J. Dereziński and C. Gérard [5]. We take the Hilbert space of the particle system is taken to be

$$\mathcal{H} = L^2(\mathbb{R}^N).$$

The Hilbert space for the Dereziński-Gérard (DG) model is given by

$$\mathcal{F} := \mathcal{H} \otimes \mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^d)).$$

We identify $\mathcal{F}$ as

$$\bigoplus_{n=0}^{\infty}\Big[\mathcal{H} \otimes \bigotimes_s^n L^2(\mathbb{R}^d)\Big].$$

Hence, if we denote that $\Psi \in (\Psi^{(n)})_{n=0}^{\infty} \in \mathcal{F}$, each $\Psi^{(n)}$ belongs to $\mathcal{H} \otimes [\otimes_s^n L^2(\mathbb{R}^d)]$. We denote by $\mathsf{B}(\mathcal{K}, \mathcal{J})$ the set of bounded linear operators from $\mathcal{K}$ to $\mathcal{J}$. For $v \in \mathsf{B}(\mathcal{H}, \mathcal{H} \otimes L^2(\mathbb{R}^d))$, we define an operator $\widetilde{a}^*(v)$ by

$$(\widetilde{a}^*(v)\Psi)^{(0)} := 0,$$
$$(\widetilde{a}^*(v)\Psi)^{(n)} := \sqrt{n}(I_{\mathcal{H}} \otimes S_n)(v \otimes I_{\otimes_s^{n-1}L^2(\mathbb{R}^d)})\Psi^{(n-1)}, \quad (n \geq 1),$$
$$\Psi \in D(\widetilde{a}^*(v)) := \left\{\Psi = (\Psi^{(n)})_{n=0}^{\infty} \in \mathcal{F} \;\middle|\; \sum_{n=0}^{\infty} \|(\widetilde{a}^*(v)\Psi)^{(n)}\|^2 < \infty\right\}.$$

We set

$$\mathcal{D}_0 := \{\Psi = (\Psi^{(n)})_{n=0}^{\infty} \in \mathcal{F} \mid \text{there exists a constant } n_0 \in \mathbb{N}, \\ \text{such that, for all } n \geq n_0,\ \Psi^{(n)} = 0\}.$$

Throughout this section, we write simply $I_n := I_{\otimes_s^n L^2(\mathbb{R}^d)}$. It is easy to see that:

Proposition 4.1 *$\widetilde{a}^*(v)$ is a closed linear operator and $\mathcal{D}_0$ is a core for $\widetilde{a}^*(v)$.*

So we set

$$\widetilde{a}(v) := (\widetilde{a}^*(v))^*$$

the adjoint operator of $\widetilde{a}^*(v)$.

Proposition 4.2 *The operator $\widetilde{a}(v)$ has the following properties*:

$$D(\widetilde{a}(v)) = \left\{\Psi = (\Psi^{(n)})_{n=0}^{\infty} \,\middle|\, \sum_{n=0}^{\infty}(n+1)\|(I_{\mathcal{H}} \otimes S_n)(v^* \otimes I_n)\Psi^{(n+1)}\|^2 < \infty\right\} \quad (15)$$

$$(\widetilde{a}(v)\Psi)^{(n)} = \sqrt{n+1}I_{\mathcal{H}} \otimes S_n(v^* \otimes I_n)\Psi^{(n+1)},\ \Psi \in D(\widetilde{a}(v)), \quad (16)$$

and $\mathcal{D}_0$ is a core for $\widetilde{a}(v)$.

Proof. For $\Phi \in \mathcal{F}$, $\Psi \in D(\widetilde{a}^*(v))$,

$$\begin{aligned}\langle \Phi, \widetilde{a}^*(v)\Psi\rangle &= \sum_{n=1}^{\infty}\langle \Phi^{(n)}, \sqrt{n}(I_{\mathcal{H}} \otimes S_n)(v \otimes I_{n-1})\Psi^{(n-1)}\rangle \\ &= \sum_{n=0}^{\infty}\sqrt{n+1}\langle v^* \otimes I_n\Phi^{(n+1)}, \Psi^{(n)}\rangle \\ &= \sum_{n=0}^{\infty}\left\langle \sqrt{n+1}(I_{\mathcal{H}} \otimes S_n)(v^* \otimes I_n)\Phi^{(n+1)}, \Psi^{(n)}\right\rangle.\end{aligned}$$

This implies (15) and (16). It is easy to prove that $\mathcal{D}_0$ is a core for $\widetilde{a}(v)$. $\square$

An analogue of the Segal field operator is defined by

$$\widetilde{\phi}(v) := \frac{1}{\sqrt{2}}(\widetilde{a}(v) + \widetilde{a}^*(v)).$$

Let A be a non-negative self-adjoint operator on $\mathcal{H}$ with $E_0(A) = 0$. Then the Hamiltonian of the DG model is defined by

$$H_{\mathrm{DG}} := A \otimes I + I \otimes H_{\mathrm{b}} + \widetilde{\phi}(v).$$

We call it the *Dereziński-Gérard Hamiltonian.* Here H_{b} is the second quantization of ω introduce in Section 3. Let

$$H_0 := A \otimes I + I \otimes H_{\mathrm{b}}.$$

Throughout this section we assume the following conditions:

[DG.1] There is a Borel measurable function $v(x, k) \in \mathbb{C}$, $(x \in \mathbb{R}^N$, $k \in \mathbb{R}^d)$, such that

$$(vf)(x, k) = v(x, k)f(x), \quad f \in L^2(\mathbb{R}^d).$$

We need also the following assumption:

[DG.2]

$$\operatorname*{ess.sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} \left| \frac{v(x, k)}{\sqrt{\omega(k)}} \right|^2 \mathrm{d}k < \infty.$$

Proposition 4.3 *Assume* [DG.1] *and* [DG.2]*. Then H_{DG} is self-adjoint with $D(H_{\mathrm{DG}}) = D(H_0)$, and essentially self-adjoint on any core for H_0.*

For a finite volume approximation, we introduce the following hypotheses:

[DG.3] There exists a nonnegative function $\widetilde{v} \in L^2(\mathbb{R}^d)$ and function $\widetilde{o}\colon \mathbb{R} \to \mathbb{R}$, such that

$$\operatorname*{ess.sup}_{x \in \mathbb{R}^n} |v(x, k) - v(x, \ell)| \le \widetilde{v}(k)\widetilde{o}(|k - \ell|), \quad \text{a.e. } k,\ \ell \in \mathbb{R}^d$$
$$\lim_{t \downarrow 0} \widetilde{o}(t) = 0.$$

[DG.4]

$$\operatorname*{ess.sup}_{x \in \mathbb{R}^n} \int_{([-K, K]^d)^{\mathrm{c}}} |v(x, k)|^2 \mathrm{d}k = \mathsf{o}(K^0).$$

where

$$([-K,\,K]^d)^{\mathrm{c}} := \mathbb{R}^d \setminus (I \times \cdots \times I), \quad I := [-K,\,K]$$

and, $\mathsf{o}(t^0)$ satisfies $\lim_{t\to 0} \mathsf{o}(t^0) = 0$.

Let m be defined by (3). Let

$$D := \frac{1}{2} \inf_{0<\epsilon'<\|v\|/\|v/\sqrt{\omega}\|^2} \Big(\epsilon' + \frac{1}{\epsilon'}\Big). \tag{17}$$

Here, $v/\sqrt{\omega}$ is a multiplication operator by the function $v(x,\,k)/\sqrt{\omega(k)}$ from $L^2(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N) \otimes L^2(\mathbb{R}^d)$. In the case $m > 0$, we can establish the existence of a ground state of H_{DG}:

Theorem 4.4 *Let $m > 0$. Suppose that* [DG.1]-[DG.4] *and* [H.4] *hold, and suppose*

$$\Sigma(A) - \|v\|D - E_0(H_{\mathrm{DG}}) > 0.$$

Then, H_{DG} has purely discrete spectrum in

$$[E_0(H_{\mathrm{DG}}),\, \min\{E_0(H_{\mathrm{DG}}) + m,\, \Sigma(A) - \|v\|D\}).$$

In particular H_{DG} has a ground state.

Remark In the case where A has compact resolvent, this theorem has been proved in [5]. A new aspect here is in that A does not necessarily have compact resolvent. Also our method is different from that in [5].

4.1. Proof of Proposition 4.3

Lemma 4.5 *Let $M(x) = (\int_{\mathbb{R}^d} |v(x,k)|^2 \mathrm{d}k)^{1/2}$, $x \in \mathbb{R}^N$ and $M\colon L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ be a multiplication operator by the function $M(x)$. Then*

$$\|vf\|^2 = \|Mf\|^2, \quad f \in L^2(\mathbb{R}^N).$$

In particular, $\|v\| = \|M\| = (\operatorname{ess.sup}_{x\in\mathbb{R}^N} \int_{\mathbb{R}^d} |v(x,\,k)|^2 \mathrm{d}k)^{1/2}$ hold.

Proof. By the Fubini's theorem, we have

$$\begin{aligned}\|vf\|^2 &= \int_{\mathbb{R}^d} \mathrm{d}k \int_{\mathbb{R}^N} \mathrm{d}x |v(x,\,k)|^2 |f(x)|^2 \\ &= \int_{\mathbb{R}^N} \Big(|f(x)|^2 \int_{\mathbb{R}^d} |v(x,\,k)|^2 \mathrm{d}k\Big)\, \mathrm{d}x.\end{aligned}$$

This means the result. □

The adjoint v^* has the following form:

Lemma 4.6 *For all $g \in \mathcal{H} \otimes L^2(\mathbb{R}^d)$,*

$$(v^* g)(x) = \int_{\mathbb{R}^d} v(x, k)^* g(x, k) \mathrm{d}k, \quad \text{a.e. } x \in \mathbb{R}^d. \tag{18}$$

Proof. For all $f \in \mathcal{H}$, we have

$$\begin{aligned}\langle g, vf \rangle &= \int \mathrm{d}x \int \mathrm{d}k g(x, k)^* v(x, k) f(x) \\ &= \int \mathrm{d}x \left(\int g(x, k)^* v(x, k) \mathrm{d}k \right) f(x).\end{aligned}$$

Since f is arbitrary, this proves (18). ☐

Lemma 4.7 $\widetilde{a}(v)$ *is*

$$D(\widetilde{a}(v)) = \left\{ \Psi \in \mathcal{F} \,\middle|\, \sum_{n=0}^{\infty} (n+1) \int_{\mathbb{R}^{N+dn}} \mathrm{d}x \mathrm{d}k_1 \cdots \mathrm{d}k_n \right. \\ \left. \left| \int_{\mathbb{R}^d} \mathrm{d}k v(k, x)^* \Psi^{(n+1)}(x, k, k_1, \ldots, k_n) \right|^2 < \infty \right\}$$

$$(\widetilde{a}(v)\Psi)^{(n)}(x, k_1, \ldots, k_n) \\ = \sqrt{n+1} \int_{\mathbb{R}^d} v(x, k)^* \Psi^{(n+1)}(x, k, k_1, \ldots, k_n), \\ \text{a.e.} \quad (\Psi \in D(\widetilde{a}(v)))$$

Proof. Using Lemma 4.6, we have

$$(v^* \otimes I_n) \Psi^{(n+1)}(x, k_1, \ldots, k_n) \\ = \int_{\mathbb{R}^d} v^*(x, k) \Psi^{(n+1)}(x, k, k_1, \ldots, k_n) \mathrm{d}k. \tag{19}$$

This is invariant for all permutations of $k_1, \ldots, k_n$. Therefore, using Proposition 4.2, we get

$$(\widetilde{a}(v)\Psi)^{(n)}(x, k_1, \ldots, k_n) \\ = \sqrt{n+1} \int_{\mathbb{R}^d} v(x, k)^* \Psi^{(n+1)}(x, k, k_1, \ldots, k_n) \mathrm{d}k.$$

☐

Lemma 4.8 *Suppose that* [DG.1] *and* [DG.2] *hold. Then,* $D(\widetilde{a}(v)) \supset D(I \otimes H_{\mathrm{b}}^{1/2})$ *and*

$$\|\widetilde{a}(v)\Phi\| \le \|v/\sqrt{\omega}\| \|I \otimes H_{\mathrm{b}}^{1/2}\Phi\|, \quad \Phi \in D(I \otimes H_{\mathrm{b}}^{1/2}).$$

Proof. By (19), we have for all $\Phi \in D(\widetilde{a}(v))$

$$\|(\widetilde{a}(v)\Phi)^{(n)}\|^2 = (n+1)\int_{\mathbb{R}^{dn+N}} \mathrm{d}x\mathrm{d}k_1 \cdots \mathrm{d}k_n \left| \int_{\mathbb{R}^d} \sqrt{\omega(k)} \times \frac{1}{\sqrt{\omega(k)}} v(x, k)^* \Phi^{(n+1)}(x, k, k_1, \ldots, k_n)\mathrm{d}k \right|^2.$$

Using the Schwarz inequality, one has

$$\left| \int_{\mathbb{R}^d} \sqrt{\omega(k)} \frac{1}{\sqrt{\omega(k)}} v(x, k)^* \Phi^{(n+1)}(x, k, k_1, \ldots, k_n)\mathrm{d}k \right|^2$$
$$\le \int_{\mathbb{R}^d} \left| \frac{v(x, k)^*}{\sqrt{\omega(k)}} \right|^2 \mathrm{d}k \cdot \int_{\mathbb{R}^d} \omega(k) |\Phi^{(n+1)}(x, k, k_1, \ldots, k_n)|^2 \mathrm{d}k.$$

Hence, for every $\Phi \in \mathcal{D}_0 \cap D(I \otimes H_{\mathrm{b}}^{1/2})$, we have

$$\begin{aligned}
&\|(\widetilde{a}(v)\Phi)^{(n)}\|^2 \\
&\le \left(\operatorname*{ess.sup}_x \int_{\mathbb{R}^d} \left| \frac{v(x, k)^*}{\sqrt{\omega(k)}} \right|^2 \mathrm{d}k \right) (n+1) \times \\
&\quad \int_{\mathbb{R}^{dn+N}} \mathrm{d}x\mathrm{d}k_1 \cdots \mathrm{d}k_n \mathrm{d}k\omega(k) |\Phi^{(n+1)}(x, k, k_1, \ldots, k_n)|^2 \\
&= \left(\operatorname*{ess.sup}_x \int_{\mathbb{R}^d} \left| \frac{v(x, k)^*}{\sqrt{\omega(k)}} \right|^2 \mathrm{d}k \right) \times \\
&\quad \int_{\mathbb{R}^{dn+N}} \mathrm{d}x\mathrm{d}k_1 \cdots \mathrm{d}k_{n+1} \sum_{j=1}^{n+1} \omega(k_j) |\Phi^{(n+1)}(x, k_1, \ldots, k_{n+1})|^2 \\
&= \left\| \frac{v}{\sqrt{\omega}} \right\| \|(I \otimes H_{\mathrm{b}}^{1/2}\Phi)^{(n+1)}\|^2.
\end{aligned}$$

Therefore

$$\|\widetilde{a}(v)\Phi\| \le \left\| \frac{v}{\sqrt{\omega}} \right\| \|(I \otimes H_{\mathrm{b}}^{1/2}\Phi)\|^2.$$

Since, $\mathcal{D}_0 \cap D(I \otimes H_{\mathrm{b}}^{1/2})$ is a core for $I \otimes H_{\mathrm{b}}^{1/2}$, one can extend this inequality to all $\Phi \in D(I \otimes H_{\mathrm{b}}^{1/2})$, and $D(I \otimes H_{\mathrm{b}}^{1/2}) \subset D(\widetilde{a}(v))$ holds. $\square$

Lemma 4.9 *On $\mathcal{D}_0$, $\widetilde{a}(v)$ and $\widetilde{a}^*(v)$ satisfy the following commutation relation*:

$$[\widetilde{a}(v),\, \widetilde{a}(v)^*] = \int_{\mathbb{R}^d} |v(\,\cdot\,,\, k)|^2 \mathrm{d}k.$$

where the right hand side is a multiplication operator by the function: $x \mapsto \int_{\mathbb{R}^d} |v(x,\, k)|^2 \mathrm{d}k$.

Proof. Let $\Phi \in \mathcal{D}_0$. By the definition of $\widetilde{a}^*(v)$, and using Proposition 4.2, we get

$$\begin{aligned}
([\widetilde{a}^*(v),\, \widetilde{a}(v)]\Phi)^{(n)} &= (\widetilde{a}(v)\widetilde{a}(v)^*\Phi)^{(n)} - (\widetilde{a}(v)^*\widetilde{a}(v)\Phi)^{(n)} \\
&= \sqrt{n+1} I_{\mathcal{H}} \otimes S_n(v^* \otimes I_n)(\widetilde{a}(v)^*\Phi)^{(n+1)} \\
&\quad - \sqrt{n}(I \otimes S_n)(v \otimes I_{n-1})(\widetilde{a}(v)\Phi)^{(n-1)}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&([\widetilde{a}^*(v),\, \widetilde{a}(v)]\Phi)^{(n)}(x,\, k_1,\, \dots,\, k_n) \\
&= (n+1)\int_{\mathbb{R}^d} v(x,\, k)^*(I \otimes S_{n+1}(v \otimes I_{n-1})\Phi^{(n)})(x,\, k,\, k_1,\, \dots,\, k_n)\mathrm{d}k \\
&\quad - n\frac{1}{n}\sum_{j=1}^n v(x,\, k_j)(v^* \otimes I_{n-1}\Phi^{(n)})(x,\, k_1,\, \dots,\, \widehat{k_j},\, \dots,\, k_n) \\
&= \int_{\mathbb{R}^d} \mathrm{d}k\, v(x,\, k)^*\big(v(x,\, k)\Phi^{(n)}(x,\, k_1,\, \dots,\, k_n) \\
&\quad + \sum_{j=1}^n v(x,\, k_j)\Phi^{(n)}(x,\, k,\, k_1,\, \dots,\, \widehat{k_j},\, \dots,\, k_n)\big) \\
&\quad - \sum_{j=1}^n v(x,\, k_j)\int_{\mathbb{R}^d} \mathrm{d}k v(x,\, k)^*\Phi^{(n)}(x,\, k,\, k_1,\, \dots,\, \widehat{k_j},\, \dots,\, k_n) \\
&= \left(\int_{\mathbb{R}^d} |v(x,k)|^2\right)\Phi(x,\, k_1,\, \dots,\, k_n).
\end{aligned}$$

Here '$\widehat{\ }$' indicates the omission of the object wearing the hat. $\square$

Lemma 4.10 *Assume,* [DG.1] *and* [DG.2]. *Then* $D(I \otimes H_{\mathrm{b}}^{1/2}) \subset D(\widetilde{a}^*(v))$ *and for all* $\Phi \in D(I \otimes H_{\mathrm{b}}^{1/2})$,

$$\|\widetilde{a}^*(v)\Phi\|^2 \leq \|v/\sqrt{\omega}\|^2 \|I \otimes H_{\mathrm{b}}^{1/2}\Phi\|^2 + \|v\|^2\|\Phi\|^2. \tag{20}$$

Proof. For all $\Phi \in \mathcal{D}_0 \cap D(I \otimes H_{\mathrm{b}}^{1/2})$, we have

$$\begin{aligned}\|\widetilde{a}^*(v)\Phi\|^2 &= \langle \Phi, \widetilde{a}(v)\widetilde{a}^*(v)\Phi\rangle \\ &= \langle \Phi, \widetilde{a}^*(v)\widetilde{a}(v)\Phi\rangle + \left\langle \left(\int_{\mathbb{R}^d} |v(\,\cdot\,, k)|^2 \right)\Phi, \Phi \right\rangle \\ &\leq \|\widetilde{a}(v)\Phi\|^2 + \|v\|^2\|\Phi\|^2.\end{aligned}$$

Thus we can apply Lemma 4.8 to obtain the result. □

Now we can prove Proposition 4.3:

Proof of Proposition 4.3. By Lemma 4.8 and 4.10, the operator $\widetilde{\phi}(v)$ is $I \otimes H_{\mathrm{b}}^{1/2}$-bounded. Hence $\widetilde{\phi}(v)$ is infinitesimally small with respect to $I \otimes H_{\mathrm{b}}$. Namely, for all $\epsilon > 0$, there exists a constant $c_\epsilon > 0$, such that,

$$\|\widetilde{\phi}(v)\Phi\| \leq \epsilon \|I \otimes H_{\mathrm{b}}\Phi\| + c_\epsilon\|\Phi\|, \quad \Phi \in D(I \otimes H_{\mathrm{b}}).$$

Since $A \geq 0$, we have

$$\|\widetilde{\phi}(v)\Phi\| \leq \epsilon \|H_0\Phi\| + c\|\Phi\|, \quad \Phi \in D(H_0).$$

Thus we can apply the Kato-Rellich theorem to obtain the conclusion of Proposition 4.3. □

4.2. Proof of Theorem 4.4

In this subsection we suppose that the assumption of Theorem 4.4 holds. Let $\mathcal{F}_{\mathrm{b,V}}$, ω_V, $H_{\mathrm{b,V}}$, $H_{0,V}$, $\mathcal{F}_V$, Γ_V, $\chi_{\ell,V}(k)$ be an object already defined in Section 3, respectively. Suppose that χ_K is a characteristic function of $[-K, K]$.

For a parameter $K > 0$, we define $v_K \in \mathsf{B}(\mathcal{H}, \mathcal{H} \otimes L^2(\mathbb{R}^d))$ by

$$(v_K f)(x, k) := \chi_{[-K,K]}(k)v(x, k)f(x).$$

and $v_{K,V} \in \mathsf{B}(\mathcal{H}, \mathcal{H} \otimes L^2(\mathbb{R}^d))$ by

$$(v_{K,V} f)(x, k) := \sum_{\substack{\ell \in \Gamma_V,\, |\ell_i| < K \\ i=1,\ldots,d}} \chi_{\ell,V}(k)v(x, \ell)f(x).$$

Lemma 4.11 *The following hold*:

$$\|v_K - v_{K,V}\| \to 0 \ (V \to \infty), \ \|v_K - v\| \to 0 \ (K \to \infty). \tag{21}$$

$$\left\| \frac{v_K}{\sqrt{\omega}} - \frac{v_{K,V}}{\sqrt{\omega_V}} \right\| \to 0 \ (V \to \infty), \ \left\| \frac{v}{\sqrt{\omega}} - \frac{v_K}{\sqrt{\omega}} \right\| \to 0 \ (K \to \infty). \tag{22}$$

Proof. By [DG.3] and [DG.4], we have

$$\begin{aligned}
\|v_K - v_{K,V}\|^2 &= \operatorname*{ess.sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} \left| \chi_K(k) v(x, k) - \sum_{\substack{\ell \in \Gamma_V \\ |\ell_i| < K}} v(x, \ell) \chi_{\ell, V}(k) \right|^2 \mathrm{d}k \\
&= \operatorname*{ess.sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} \sum_{\substack{\ell \in \Gamma_V \\ |\ell_i| < K}} \chi_{\ell, V}(k) |v(x, k) - v(x, \ell)|^2 \mathrm{d}k \\
&\le \operatorname*{ess.sup}_{x \in \mathbb{R}^N} \int_{\mathbb{R}^d} \sum_{\substack{\ell \in \Gamma_V \\ |\ell_i| < K}} \chi_{\ell, V}(k) |\widetilde{v}(k)|^2 \widetilde{o}(|k - \ell|)^2 \mathrm{d}k \\
&\le \int_{\mathbb{R}^d} \sum_{\substack{\ell \in \Gamma_V \\ |\ell_i| < K}} \chi_{\ell, V}(k) |\widetilde{v}(k)|^2 \widetilde{o}(|k - \ell|)^2 \mathrm{d}k.
\end{aligned}$$

It follows from the property of $\widetilde{o}$ that for every $\epsilon > 0$, there exists a constant $V_0 > 0$ such that, for all $V > V_0$,

$$\chi_{\ell, V}(k) \widetilde{o}(|k - \ell|)^2 \le \epsilon \chi_{\ell, V}(k).$$

Therefore,

$$\|v_K - v_{K,V}\|^2 \le \epsilon \int_{\mathbb{R}^d} \sum_{\substack{\ell \in \Gamma_V \\ |\ell_i| < K}} \chi_{\ell, V}(k) |\widetilde{v}(k)|^2 \mathrm{d}k = \epsilon \|\widetilde{v}\|^2_{L^2(\mathbb{R}^d)}.$$

Hence the first one of (21) holds. The second one is a direct result of condition [DG.4]:

$$\begin{aligned}
\|v_K - v\|^2 &= \operatorname*{ess.sup}_{x} \int_{\mathbb{R}^d} |\chi_K(k) - 1|^2 |v(x, k)|^2 \mathrm{d}k \\
&= \operatorname*{ess.sup}_{x} \int_{([-K, K]^d)^c} |v(x, k)|^2 \mathrm{d}k = \mathsf{o}(K^0) \to 0 \ (K \to \infty).
\end{aligned}$$

Using [H.4], one can easily check (22) □

We introduce two operators:

$$H_{\mathrm{DG}}(K) := A \otimes I + I \otimes H_{\mathrm{b}} + \widetilde{\phi}(v_K),$$
$$H_{\mathrm{DG}}(K,\, V) := A \otimes I + I \otimes H_{\mathrm{b,\,V}} + \widetilde{\phi}(v_{K,\,V}).$$

Lemma 4.12 (i) *$H_{\mathrm{DG}}(K)$ is self-adjoint with $D(H_{\mathrm{DG}}(K)) = D(H_0)$, bounded from below, and essentially self-adjoint on any core for H_0.*
(ii) *For sufficiently large $V > 0$, $H_{\mathrm{DG}}(K,\, V)$ is self-adjoint with domain $D(H_{\mathrm{DG}}(K,\, V)) = D(H_0)$, bounded from below, and essentially self-adjoint on any core for H_0.*

Proof. Similar to the proof of Proposition 4.3. □

Lemma 4.13 *For all $z \in \mathbb{C}\backslash\mathbb{R}$,*

$$\lim_{V\to\infty} \|(H_{\mathrm{DG}}(K,\, V) - z)^{-1} - (H_{\mathrm{DG}}(K) - z)^{-1}\| = 0,$$
$$\lim_{K\to\infty} \|(H_{\mathrm{DG}}(K) - z)^{-1} - (H_{\mathrm{DG}} - z)^{-1}\| = 0.$$

Proof. Similar to the proof of [2, Lemma 3.5]. □

Lemma 4.14 *The operator $H_{\mathrm{DG}}(K,\, V)$ is reduced by $\mathcal{F}_V$.*

Proof. We identify $v(x,\, \ell)$ with multiplication operator by $v(\,\cdot\,,\, \ell)$. By abuse of symbols, we denote $\chi_{\ell,\,V}(\,\cdot\,)$ by $\chi_{\ell,\,V}(k)$. Then

$$\begin{aligned}(\widetilde{a}^*(v(x,\, \ell)\chi_{\ell,\,V}(k))\Phi)^{(n)} &= \sqrt{n}(I \otimes S_n)(v(x,\, \ell)\chi_{\ell,\,V}(k) \otimes I)\Phi^{(n-1)} \\ &= \sqrt{n}v(x,\, \ell)S_n(\chi_{\ell,\,V} \otimes \Phi^{(n-1)}) \\ &= \chi(x,\, \ell)\sqrt{n}S_n(\chi_{\ell,\,V} \otimes \Phi^{(n-1)}).\end{aligned}$$

Hence, we have

$$\widetilde{a}^*(v(x,\, \ell)\chi_{\ell,\,V}(k))\Phi = v(x,\, \ell) \otimes a^*(\chi_{\ell,\,V})\Phi.$$

Therefore, we get

$$\widetilde{a}^*(v_{K,\,V}) = \sum_{\substack{\ell\in\Gamma_V \\ |\ell_i|<K}} v(\,\cdot\,,\, \ell) \otimes a^*(\chi_{\ell,\,V}). \tag{23}$$

Hence, its adjoint is

$$\widetilde{a}(v_{K,\,V}) = \sum_{\substack{\ell\in\Gamma_V \\ |\ell_i|<K}} v(\,\cdot\,,\, \ell)^* \otimes a(\chi_{\ell,\,V}). \tag{24}$$

This means that the operator $H_{\mathrm{DG}}(K, V)$ is a special case of the GSB Hamiltonian (see [2]). Hence, by [2, Lemma 3.7], $H_{\mathrm{DG}}(K, V)$ is reduced by $\mathcal{F}_V$. $\square$

Lemma 4.15 $H_{\mathrm{DG}}(K, V)\lceil \mathcal{F}_V^{\perp} \geq E_0(H_{\mathrm{DG}}(K, V)) + m$

Proof. Similar to the proof of [2, Lemma 3.10]. $\square$

Lemma 4.16 *For all $\Phi \in D(I \otimes H_{\mathrm{b}}^{1/2})$, and for all $\epsilon' > 0$,*

$$|\langle \Phi, \widetilde{\phi}(v)\Phi\rangle| \leq \frac{\epsilon'}{\|v\|}\left\|\frac{v}{\sqrt{\omega}}\right\|^2 \|I \otimes H_{\mathrm{b}}^{1/2}\|^2 + \frac{\|v\|}{2}\left(\epsilon' + \frac{1}{\epsilon'}\right)\|\Phi\|^2.$$

Proof. For all $\Phi \in D(I \otimes H_{\mathrm{b}}^{1/2})$, $\epsilon' > 0$,

$$\begin{aligned}
&|\langle \Phi, \widetilde{\phi}(v)\Phi\rangle| \\
&\leq \frac{1}{\sqrt{2}}\left(\epsilon\|\widetilde{a}(v)\Phi\|^2 + \frac{1}{4\epsilon}\|\Phi\|^2 + \epsilon\|\widetilde{a}^*(v)\Phi\|^2 + \frac{1}{4\epsilon}\|\Phi\|^2\right) \\
&\leq \frac{1}{\sqrt{2}}\left(2\epsilon\left\|\frac{v}{\sqrt{\omega}}\right\|^2 \|I \otimes H_{\mathrm{b}}^{1/2}\Phi\|^2 + \epsilon\|v\|^2\|\Phi\|^2 + \frac{1}{2\epsilon}\|\Phi\|^2\right) \\
&= \sqrt{2}\epsilon\left\|\frac{v}{\sqrt{\omega}}\right\|^2 \|I \otimes H_{\mathrm{b}}^{1/2}\Phi\|^2 + \frac{\|v\|}{2}\left(\sqrt{2}\epsilon\|v\| + \frac{1}{\sqrt{2}\epsilon\|v\|}\right)\|\Phi\|^2,
\end{aligned}$$

where we have used Lemma 4.8 and 4.10. Let $\sqrt{2}\epsilon\|v\| =: \epsilon'$. Then, for all $\epsilon' > 0$, we have

$$|\langle \Phi, \widetilde{\phi}(v)\Phi\rangle| \leq \frac{\epsilon'}{\|v\|}\left\|\frac{v}{\sqrt{\omega}}\right\|^2 \|I \otimes H_{\mathrm{b}}^{1/2}\Phi\|^2 + \frac{\|v\|}{2}\left(\epsilon' + \frac{1}{\epsilon'}\right)\|\Phi\|^2.$$

$\square$

Proof of Theorem 4.4. From (23) and (24), $H_{\mathrm{DG}}(K, V)$ is equal to the special case of the GSB model. Therefore, $H_{\mathrm{DG}}(K, V)\lceil \mathcal{F}_V$ has the same form with $H_{\mathrm{DG}}(K, V)$. Using Lemma 4.16 we have on $D(H_0) \cap \mathcal{F}_V$

$$\begin{aligned}
&H_{\mathrm{DG}}(K, V) \\
&= A \otimes I + I \otimes H_{\mathrm{b,V}} + \widetilde{\phi}(v_{K,V}) \\
&\geq A \otimes I + I \otimes H_{\mathrm{b,V}} \\
&\qquad - \frac{\epsilon'}{\|v_{K,V}\|}\left\|\frac{v_{K,V}}{\sqrt{\omega_V}}\right\|^2 I \otimes H_{\mathrm{b,V}} - \frac{\|v_{K,V}\|}{2}\left(\epsilon' + \frac{1}{\epsilon'}\right)
\end{aligned}$$

$$= A \otimes I + \left(1 - \frac{\epsilon'}{\|v_{K,V}\|}\left\|\frac{v_{K,V}}{\sqrt{\omega_V}}\right\|^2\right) I \otimes H_{\mathrm{b,V}}$$

$$- \frac{\|v_{K,V}\|}{2}\left(\epsilon' + \frac{1}{\epsilon'}\right), \tag{25}$$

where $\epsilon' > 0$ is an arbitrary constant. By Lemma 3.10, $H_{\mathrm{b,V}} \lceil \mathcal{F}_{\mathrm{b,V}}$ has compact resolvent. Thus, for $\epsilon' > 0$ satisfying

$$1 - \frac{\epsilon'}{\|v_{K,V}\|}\left\|\frac{v_{K,V}}{\sqrt{\omega_V}}\right\|^2 > 0, \tag{26}$$

the bottom of the essential spectrum of (25) is equal to

$$\Sigma(A) - \frac{\|v_{k,V}\|}{2}\left(\epsilon' + \frac{1}{\epsilon'}\right).$$

Let, D_K and $D_{K,V}$ be D with v replaced by v_K, $v_{K,V}$, respectively. It is easy to see that

$$\lim_{K\to\infty} D_K = D, \quad \lim_{V\to\infty} D_{K,V} = D_K.$$

By Lemma 4.13, one has

$$\lim_{K\to\infty} E_0(H_{\mathrm{DG}}(K)) = E_0(H_{\mathrm{DG}}),$$

$$\lim_{V\to\infty} E_0(H_{\mathrm{DG}}(K, V)) = E_0(\mathrm{DG}(K)).$$

From the assumption of Theorem 4.4, for all $K > 0$, there exists a constant V_0 such that for $V > V_0$,

$$\Sigma(A) - \frac{\|v_{K,V}\|}{2} D_{K,V} - E_0(H_{\mathrm{DG}}(K, V)) > 0.$$

By the definition of $D_{K,V}$, for all $K > 0$ and $V > V_0$, and for all ϵ' which satisfies (26), we have

$$\Sigma(A) - \frac{\|v_{K,V}\|}{2}\left(\epsilon' + \frac{1}{\epsilon'}\right) > E_0(H_{\mathrm{DG}}(K, V)).$$

Therefore, by Theorem 2.1, we have that $H_{\mathrm{DG}}(K, V)\lceil \mathcal{F}_V$ has purely discrete spectrum in

$$[E_0(H_{\mathrm{DG}}(K, V)), \Sigma(A) - \|v_{K,V}\| D_{K,V}).$$

This fact and Lemma 4.15 mean that $H_{\mathrm{DG}}(K, V)$ has purely discrete spectrum in

$$\big[E_0(H_{\mathrm{DG}}(K, V)),\\ \min\{E_0(H_{\mathrm{DG}}(K, V)) + m, \Sigma(A) - \|v_{K,V}\| D_{K,V}\}\big).$$

Finally, we use Lemma 3.13 and Lemma 4.13, to conclude that H_{DG} has purely discrete spectrum in the interval

$$\big[E_0(H_{\mathrm{DG}}), \min\{E_0(H_{\mathrm{DG}}) + m, \Sigma(A) - \|v\| D\}\big)$$

□

Acknowledgments We would like to thank Professor A. Arai of Hokkaido University for proposing a problem, discussions and helpful comments.

References

[1] Arai A., *Fock Spaces and Quantum Fields.* (in Japanese), Nippon-Hyouronsha, Tokyo, 2000.

[2] Arai A. and Hirokawa M., *On the existence and uniqueness of ground states of a generalized spin-boson model.* J. Funct. Anal. **151** (1997), 455–503.

[3] Arai A. and Hirokawa M., *Ground states of a general class of quantum field Hamiltonians.* Rev. Math. Phys. **8** (2000), 1085–1135.

[4] Arai A. and Kawano H., *Enhanced binding in a general class of quantum field models.* Rev. Math. Phys. **4** (2003), 387–423.

[5] Dereziński J. and Gérard C., *Asymptotic completeness in quantum field theory. Massive Pauli-Fierz Hamiltonians.* Rev. Math. Phys. **11** (1999), 383–450.

[6] Reed M. and Simon B., *Methods of Modern Mathematical Physics Vol.* I. Academic Press, New York, 1972.

[7] Reed M. and Simon B., *Methods of Modern Mathematical Physics Vol.* II. Academic Press, New York, 1975.

[8] Reed M. and Simon B., *Methods of Modern Mathematical Physics Vol.* IV. Academic Press, New York, 1978.

[9] Simon B. and Hoegh-Krohn R., *Hypercontractive semigroups and two dimensional self coupled Bose fields.* J. Funct. Anal. **9** (1972), 121–180.

T. Miyao
Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan
E-mail: s993165@math.sci.hokudai.ac.jp

I. Sasaki
Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan
E-mail: i-sasaki@math.sci.hokudai.ac.jp