Hokkaido Mathematical Journal Vol. 34 (2005) p. 667–677

On a class of generalized difference sequence space defined by modulus function

Binod Chandra Tripathy, Sabita Mahanta and Mikail Et

(Received April 14, 2004)

Abstract. In this article we introduce the sequence space $m(f, \phi, \Delta^n, p, q)$, $1 \le p < \infty$, using modulus functions. We study its different properties like completeness, solidity etc. Also we obtain some inclusion results involving the space $m(f, \phi, \Delta^n, p, q)$.

Key words: completeness, modulus function, difference sequence space, seminorm, solid space, symmetric space.

1. Introduction

Throughout the article $w(X)$, $\ell_\infty(X)$, $\ell^p(X)$ denote the spaces of *all*, *bounded* and *p-absolutely summable* sequences respectively with elements in X, where (X, q) denote a seminormed space, seminormed by q. The zero sequence is denoted by $\bar{\theta} = (\theta, \theta, \theta, \ldots)$, where θ is the zero element of X.

The sequence space $m(\phi)$ was introduced by Sargent [11], who studied some of its properties and obtained its relationship with the space ℓ^p. Later on it was investigated from sequence space point of view by Çolak and Et [2], Et et al. [3], Rath and Tripathy [9], Tripathy [13], Tripathy and Sen [16] and others.

The notion of difference sequence space was introduced by Kızmaz [6] as follows:

$$X(\Delta) = \{x = (x_k) \in w\colon (\Delta x_k) \in X\},$$

for $X = \ell_\infty$, c and c_0, where $\Delta x_k = x_k - x_{k+1}$, for all $k \in \mathbb{N}$.

The notion of difference sequence spaces was further generalized by Et and Çolak [4] as follows:

$$X(\Delta^n) = \{x = (x_k) \in w\colon (\Delta^n x_k) \in X\},$$

for $X = \ell_\infty$, c and c_0, where $\Delta^n x_k = \Delta^{n-1} x_k - \Delta^{n-1} x_{k+1}$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$.

2000 Mathematics Subject Classification : 40A05, 40A25, 40F05.

The generalized difference has the following binomial representation:

$$\Delta^n x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+v}, \quad \text{for all } k \in \mathbb{N}. \tag{1}$$

Different types of difference sequence spaces have been studied by Et and Nuray [5], Tripathy ([14],[15]) and many others.

The notion of *modulus function* was introduced by Ruckle [10], defined as follows:

A real valued function $f\colon [0, \infty) \to [0, \infty)$ is called a *modulus* if

(i) $f(x) \geq 0$ for each x,

(ii) $f(x) = 0$ if and only if $x = 0$,

(iii) $f(x + y) \leq f(x) + f(y)$,

(iv) f is increasing and

(v) f is continuous from the right at 0.

It is immediate from (ii) and (iv) that f is continuous everywhere on $[0, \infty)$. Later on it was studied from sequence space point of view by Maddox [7], Nuray and Savaş [8], Bilgin [1], Savaş [12] and many others.

2. Definitions and Background

Let φ_s denotes the class of all subsets of $\mathbb{N}$, those do not contain more than s elements. Throughout $\{\phi_s\}$ represents a non-decreasing sequence of real numbers such that $s\phi_{s+1} \leq (s + 1)\phi_s$ for all $s \in \mathbb{N}$.

The sequence space $m(\phi)$ introduced by Sargent [11] is defined as follows:

$$m(\phi) = \left\{ (x_k) \in w\colon \|x_k\|_{m(\phi)} = \sup_{s\geq 1,\, \sigma \in \varphi_s} \frac{1}{\phi_s} \sum_{n \in \sigma} |x_n| < \infty \right\}.$$

In this article we introduced the following sequence space

$$m(f, \phi, \Delta^n, p, q) = \left\{ (x_k) \in w(X)\colon \sup_{s\geq 1,\, \sigma\in\varphi_s} \frac{1}{\phi_s} \left(\sum_{k\in\sigma} \big[f\big(q(\Delta^n x_k)\big)\big]^p \right)^{1/p} < \infty \right\}, \quad \text{for} \quad 1 \leq p < \infty.$$

We use the following existing sequence spaces in this article

$$\ell^p(f,\,\Delta^n,\,q)=\left\{(x_k)\in w(X)\colon\ \sum_{k=1}^{\infty}\big(\big[f\big(q(\Delta^n x)\big)\big]^p\big)^{1/p}<\infty\right\},$$

$$\ell_\infty(f,\,\Delta^n,\,q)=\Big\{(x_k)\in w(X)\colon\ \sup_{k\geq 1} f\big(q(\Delta^n x_k)\big)<\infty\Big\}.$$

A sequence space E is said to be *solid* (or *normal*) if $(\alpha_n x_n)\in E$, whenever $(x_n)\in E$ and for all scalars (α_n) with $|\alpha_n|\leq 1$ for all $n\in\mathbb{N}$.

A sequence space is said to be *monotone* if it contains the canonical preimages of all its step spaces.

A sequence space E is said to be *symmetric* if $(x_{\pi(n)})\in E$, whenever $(x_n)\in E$ where $\pi\,(n)$ is a permutation of $\mathbb{N}$.

A sequence space E is said to be *convergence free* if $(y_n)\in E$, whenever $(x_n)\in E$ and $y_n=0$ when $x_n=0$.

The following results will be used for establishing some results of this article.

Lemma 1 (Tripathy and Sen [16], Proposition 5) *$m(\phi,\,p)\subseteq m(\psi,\,p)$ if and only if $\sup_{s\geq 1}\phi_s/\psi_s<\infty$.*

Lemma 2 (Tripathy and Sen [16], Theorem 7) *$\ell^p\subseteq m(\phi,\,p)\subseteq \ell^\infty$ for all ϕ in Φ.*

Lemma 3 (Tripathy and Sen [16], Proposition 8) *$m(\phi,\,p)=\ell^p$ if and only if $\sup_{s\geq 1}\phi_s<\infty$ and $\sup_{s\geq 1}\phi_s^{-1}<\infty$.*

Lemma 4 (Tripathy and Sen [16], Proposition 9) *If $p<q$, then $m(\phi,\,p)\subset m(\phi,\,q)$.*

Lemma 5 (Tripathy and Sen [16], Proposition 10) *$m(\phi,\,p)\subseteq m(\psi,\,q)$. If $p<q$ and $\sup_{s\geq 1}\phi_s/\psi_s<\infty$.*

3. Main Results

In this article we prove some results involving the sequence space $m(f,\,\phi,\,\Delta^n,\,p,\,q)$.

Theorem 1 *The space $m(f,\,\phi,\,\Delta^n,\,p,\,q)$ is a linear space.*

Proof. Let (x_k), $(y_k)\in m(f,\,\phi,\,\Delta^n,\,p,\,q)$. Then we have

$$\sup_{s\geq 1,\,\sigma\in\varphi_s}\frac{1}{\phi_s}\left(\sum_{k\in\sigma}\big[f\big(q(\Delta^n x_k)\big)\big]^p\right)^{1/p}<\infty$$

and

$$\sup_{s\geq 1,\,\sigma\in\varphi_s}\frac{1}{\phi_s}\left(\sum_{k\in\sigma}\left[f\left(q(\Delta^n y_k)\right)\right]^p\right)^{1/p}<\infty.$$

Let $\alpha,\ \beta\in\mathbb{C}$. Now

$$\begin{aligned}
&\left\{\sum_{k\in\sigma}\left[f\Big(q\big(\Delta^n(\alpha x_k+\beta y_k)\big)\Big)\right]^p\right\}^{1/p}\\
&\leq\left\{\sum_{k\in\sigma}\left[f\left(|\alpha|q(\Delta^n x_k)+|\beta|q(\Delta^n y_k)\right)\right]^p\right\}^{1/p}\\
&\leq\left\{\sum_{k\in\sigma}\left[f\left(|\alpha|q(\Delta^n x_k)\right)\right]^p\right\}^{1/p}+\left\{\sum_{k\in\sigma}\left[f\left(|\beta|q(\Delta^n y_k)\right)\right]^p\right\}^{1/p}\\
&\leq(1+[\alpha])\left\{\sum_{k\in\sigma}\left[f\left(q(\Delta^n x_k)\right)\right]^p\right\}^{1/p}+(1+[\beta])\left\{\sum_{k\in\sigma}\left[f\left(q(\Delta^n y_k)\right)\right]^p\right\}^{1/p},
\end{aligned}$$

where $[\alpha]$ and $[\beta]$ denote the integer part of $|\alpha|$ and $|\beta|$.

$$\begin{aligned}
&\Rightarrow\sup_{s\geq 1,\,\sigma\in\varphi_s}\frac{1}{\phi_s}\left\{\sum_{k\in\sigma}\left[f\left(q\left(\Delta^n(\alpha x_k+\beta y_k)\right)\right)\right]^p\right\}^{1/p}<\infty\\
&\Rightarrow(\alpha x_k+\beta y_k)\in m(f,\,\phi,\,\Delta^n,\,p,\,q).
\end{aligned}$$

Thus $m(f,\,\phi,\,\Delta^n,\,p,\,q)$ is a linear space. □

Theorem 2 *The space $m(f,\,\phi,\,\Delta^n,\,p,\,q)$ is a paranormed space, paranormed by*

$$g_\Delta(x)=\sum_{k=1}^{n}q(x_k)+\sup_{s\geq 1,\,\sigma\in\varphi_s}\frac{1}{\phi_s}\left(\sum_{k\in\sigma}\left[f\left(q(\Delta^n x_k)\right)\right]^p\right)^{1/p}.$$

Proof. Clearly $g_\Delta(x)=g_\Delta(-x)$ for all $x\in m(f,\,\phi,\,\Delta^n,\,p,\,q)$ and $g_\Delta(\bar\theta)=0$, where $\bar\theta=(\theta,\,\theta,\,\theta,\ldots)$. Subadditivity of g_Δ follows from Theorem 1, Minkowski's inequality and the definition of f.

Next let λ be a non-zero scalar. The continuity of scalar multiplication follows from the equality.

$$g_\Delta(\lambda x)=\sum_{k=1}^{n}q(\lambda x_k)+\sup_{s\geq 1,\,\sigma\in\varphi_s}\frac{1}{\phi_s}\left(\sum_{k\in\sigma}\left[f\left(q(\Delta^n\lambda x_k)\right)\right]^p\right)^{1/p}$$

$$= |\lambda| \sum_{k=1}^{n} q(x_k) + \sup_{s\geq 1,\, \sigma\in\varphi_s} \frac{1}{\phi_s}\left(\sum_{k\in\sigma}\left[f\left(|\lambda| q(\Delta^n x_k)\right)\right]^p\right)^{1/p}.$$

This completes the proof of the theorem. □

Theorem 3 *Let $n \geq 1$. Then $m(f,\, \phi,\, \Delta^{n-1},\, p,\, q) \subset m(f,\, \phi,\, \Delta^n,\, p,\, q)$. In general, $m(f,\, \phi,\, \Delta^i,\, p,\, q) \subset m(f,\, \phi,\, \Delta^n,\, p,\, q)$ for $i = 0,\, 1,\, 2,\, \ldots,\, n-1$.*

Proof. Let $(x_k) \in m(f,\, \phi,\, \Delta^{n-1},\, p,\, q)$. Then

$$\sup_{s\geq 1,\, \sigma\in\varphi_s} \frac{1}{\phi_s}\left(\sum_{k\in\sigma}\left[f\left(q(\Delta^{n-1} x_k)\right)\right]^p\right)^{1/p} < \infty.$$

Since f is non-decreasing and satisfies triangular inequality,

$$\left(\sum_{k\in\sigma}\left[f\left(q(\Delta^{n} x_k)\right)\right]^p\right)^{1/p}$$

$$\leq \left(\sum_{k\in\sigma}\left[f\left(q(\Delta^{n-1} x_k)\right)\right]^p\right)^{1/p} + \left(\sum_{k\in\sigma}\left[f\left(q(\Delta^{n-1} x_{k+1})\right)\right]^p\right)^{1/p}$$

$$\Rightarrow \sup_{s\geq 1,\, \sigma\in\varphi_s} \frac{1}{\phi_s}\left(\sum_{k\in\sigma}\left[f\left(q(\Delta^{n} x_k)\right)\right]^p\right)^{1/p} < \infty$$

$$\Rightarrow (x_k) \in m(f,\, \phi,\, \Delta^n,\, p,\, q).$$

Hence $m(f,\, \phi,\, \Delta^{n-1},\, p,\, q) \subset m(f,\, \phi,\, \Delta^n,\, p,\, q)$.

Proceeding inductively we have

$$m(f,\, \phi,\, \Delta^i,\, p,\, q) \subset m(f,\, \phi,\, \Delta^n,\, p,\, q) \quad \text{for } i = 0,\, 1,\, 2,\, \ldots,\, n-1.$$

The above inclusion is strict. For that consider the following example. □

Example 1 Let $X = \ell_\infty$, $\phi_k = 1$ for all $k \in \mathbb{N}$. Let $n = 1$, $f(x) = x$ and $p = 1$. For $x_k = (x_k^i) \in \ell_\infty$, for all $k \in \mathbb{N}$, let $q(x_k) = \sup_{i\geq 2} |x_k^i|$. Define the sequence $(x_k^i)_{i=1}^\infty = (1)$, for all $k \in \mathbb{N}$. Then $(x_k) \in m(f,\, \phi,\, \Delta^n,\, p,\, q)$ but $(x_k) \notin m(f,\, \phi,\, \Delta^{n-1},\, p,\, q)$.

Theorem 4 *Let $(X,\, q)$ be complete, then $m(f,\, \phi,\, \Delta^n,\, p,\, q)$ is also complete.*

Proof. Let (x^i) be a Cauchy sequence in $m(f,\, \phi,\, \Delta^n,\, p,\, q)$, where $x^i = (x_k^i) = (x_1^i,\, x_2^i,\, x_3^i,\, \ldots) \in m(f,\, \phi,\, \Delta^n,\, p,\, q)$ for each $i \in \mathbb{N}$. Then for a given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $g_\Delta(x^i - x^j) < \varepsilon$, for all $i,\, j > n_0$.

$$\Rightarrow \sum_{k=1}^{n} q(x_k^i - x_k^j) + \sup_{s\geq 1,\,\sigma\in\varphi_s} \frac{1}{\phi_s}\left(\sum_{k\in\sigma}\big[f\big(q\big(\Delta^n(x_k^i - x_k^j)\big)\big)\big]^p\right)^{1/p} < \varepsilon,$$

$$\text{for all } i,\, j > n_0. \tag{2}$$

We have for all $i,\, j > n_0$, $\sum_{k=1}^{n} q(x_k^i - x_k^j) < \varepsilon$. Hence $(x_k^i)_{i=1}^{\infty}$ is a Cauchy sequence in $(X,\, q)$, for all $k = 1,\, 2,\, 3,\, \dots,\, n$. Thus $(x_k^i)_{i=1}^{\infty}$ is convergent for all $k = 1,\, 2,\, 3,\, \dots,\, n$. Let

$$\lim_{i\to\infty} x_k^i = x_k, \quad \text{for } k = 1,\, 2,\, 3,\, \dots,\, n. \tag{3}$$

Again from (2) we have

$$\sup_{s\geq 1,\,\sigma\in\varphi_s} \frac{1}{\phi_s}\left(\sum_{k\in\sigma}\big[f\big(q\big(\Delta^n(x_k^i - x_k^j)\big)\big)\big]^p\right)^{1/p} < \varepsilon,$$

$$\text{for all } i,\, j \geq n_0 \text{ and } k \in \mathbb{N}$$

$$\Rightarrow f\big(q(\Delta^n x_k^i - \Delta^n x_k^j)\big) < \varepsilon\phi_1 = \varepsilon_1, \quad \text{for all } i,\, j \geq n_0 \text{ and } k \in \mathbb{N}$$
$$\Rightarrow f\big(q(\Delta^n x_k^i - \Delta^n x_k^j)\big) < f(\varepsilon_2)$$
$$\Rightarrow q(\Delta^n x_k^i - \Delta^n x_k^j) < \varepsilon_2, \quad \text{by the continuity of } f.$$
$$\Rightarrow (\Delta^n x_k^i)_{i=1}^{\infty} \text{ is a Cauchy sequence in } (X,\, q), \text{ so it is convergent.}$$

Let

$$\lim_{i\to\infty} \Delta x_k^i = y_k \quad \text{for each } k \in \mathbb{N}. \tag{4}$$

Now from (1), (3) and (4) we have $\lim_{i\to\infty} x_{k+1}^i = x_{k+1}$ for $k \in \mathbb{N}$. Proceeding in this way we get $\lim_{i\to\infty} x_k^i = x_k$ in X. Taking limit as $j \to \infty$ in (2), we get

$$\sum_{k=1}^{n} q(x_k^i - x_k) + \sup_{s\geq 1,\,\sigma\in\varphi_s} \frac{1}{\phi_s}\left(\sum_{k\in\sigma}\big[f\big(q\big(\Delta^n(x_k^i - x_k)\big)\big)\big]^p\right)^{1/p} < \varepsilon$$
$$\Rightarrow (x_k^i - x_k) \in m(f,\, \phi,\, \Delta^n,\, p,\, q), \quad \text{for all } i > n_0.$$

Since $m(f,\, \phi,\, \Delta^n,\, p,\, q)$ is linear and (x_k^i) and $(x_k^i - x_k)$ are in $m(f,\, \phi,\, \Delta^n,\, p,\, q)$, so it follows that

$$(x_k) = (x_k^i) + (x_k^i - x_k) \in m(f,\, \phi,\, \Delta^n,\, p,\, q).$$

Hence $m(f,\,\phi,\,\Delta^n,\,p,\,q)$ is complete. This completes the proof of the theorem. $\square$

The following result is straightforward in view of the techniques applied for establishing the above result.

Proposition 5 *The space $m(f,\,\phi,\,\Delta^n,\,p,\,q)$ is a K-space.*

Theorem 6 *Let f, f_1 and f_2 be moduli. Then*
(i) $m(f_1,\,\phi,\,\Delta^n,\,p,\,q) \subseteq m(f\circ f_1,\,\phi,\,\Delta^n,\,p,\,q)$,
(ii) $m(f_1,\,\phi,\,\Delta^n,\,p,\,q) \cap m(f_2,\,\phi,\,\Delta^n,\,p,\,q) \subseteq m(f_1+f_2,\,\phi,\,\Delta^n,\,p,\,q)$.

Proof. (i) Let $(x_k) \in m(f_1,\,\phi,\,\Delta^n,\,p,\,q)$. Then

$$\sup_{s\geq 1,\,\sigma\in\varphi_s} \frac{1}{\phi_s}\sum_{k\in\sigma}\big[f_1\big(q(\Delta^n x_k)\big)\big]^p < \infty.$$

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $0 \leq t \leq \delta$. Write $t_k = f_1\big(q(\Delta^n x_k)\big)$ and for any $\sigma \in \varphi_s$ consider

$$\sum_{k\in\sigma}[f(t_k)]^p = \sum_1 [f(t_k)]^p + \sum_2 [f(t_k)]^p,$$

where the first summation is over $t_k \leq \delta$ and the second summation is over $t_k > \delta$. Since f is continuous, we have

$$\sum_1 [f(t_k)]^p < \varepsilon^p \phi_1 \tag{5}$$

and for $t_k > \delta$ we use the fact that

$$t_k < \frac{t_k}{\delta} < 1 + \Big[\frac{t_k}{\delta}\Big].$$

By the definition of f we have for $t_k > \delta$,

$$f(t_k) \leq f(1)\Big(1 + \Big[\frac{t_k}{\delta}\Big]\Big) < 2f(1)\frac{t_k}{\delta}.$$

Hence

$$\sum_2 [f(t_k)]^p < (2f(1)\delta^{-1})^p \sum [f(t_k)]^p. \tag{6}$$

By (5) and (6) we have $m(f_1,\,\phi,\,\Delta^n,\,p,\,q) \subset m(f\circ f_1,\,\phi,\,\Delta^n,\,p,\,q)$.

(ii) Let $(x_k) \in m(f_1, \phi, \Delta^n, p, q) \cap m(f_2, \phi, \Delta^n, p, q)$. Then

$$\sup_{s\geq 1,\, \sigma\in\varphi_s} \frac{1}{\phi_s}\left(\sum_{k\in\sigma}\big[f_1\big(q(\Delta^n x_k)\big)\big]^p\right)^{1/p} < \infty$$

and

$$\sup_{s\geq 1,\, \sigma\in\varphi_s} \frac{1}{\phi_s}\left(\sum_{k\in\sigma}\big[f_2\big(q(\Delta^n x_k)\big)\big]^p\right)^{1/p} < \infty.$$

The rest of the proof follows from the equality

$$\left(\sum_{k\in\sigma}\big[(f_1+f_2)\big(q(\Delta^n x_k)\big)\big]^p\right)^{1/p}$$
$$\leq \left(\sum_{k\in\sigma}\big[f_1\big(q(\Delta^n x_k)\big)\big]^p\right)^{1/p} + \left(\sum_{k\in\sigma}\big[f_2\big(q(\Delta^n y_k)\big)\big]^p\right)^{1/p}.$$

Using the same technique of Theorem 6 (i) it can be shown that $m(\phi, \Delta^n, p, q) \subseteq m(f, \phi, \Delta^n, p, q)$. □

Proposition 7 *The space $m(f, \phi, \Delta^n, p, q)$ is not monotone, for $n \geq 1$.*

Proof. This result follows from the following example. □

Example 2 Let $X = \ell_\infty$, $\phi_k = k$ for all $k \in \mathbb{N}$. Let $n = 1$, $f(x) = x$ and $p = 1$. For $x_k = (x_k^i) \in \ell_\infty$, for all $k \in \mathbb{N}$, let $q(x_k) = \sup_{i\geq 2} |x_k^i|$. Define the sequence $(x_k^i) = (k)$, for all $k \in \mathbb{N}$, $i \in \mathbb{N}$. Consider the step-space E of $m(f, \phi, \Delta^n, p, q)$, defined as:

Let $(x_k) \in m(f, \phi, \Delta^n, p, q)$, the $(y_k) \in E$ implies

$$y_k = \begin{cases} x_k, & \text{for } k \text{ even} \\ 0, & \text{otherwise.} \end{cases}$$

Then $(y_k) \notin m(f, \phi, \Delta^n, p, q)$. Hence $m(f, \phi, \Delta^n, p, q)$ is not monotone.

Following result follows from the above result.

Corollary 8 *The space $m(f, \phi, \Delta^n, p, q)$ is not solid, for $n \geq 1$.*

Proposition 9 *The space $m(f, \phi, \Delta^n, p, q)$ is not symmetric in general.*

Proof. The result follows from the following example. □

Example 3 Let $X = \mathbb{C}$, $\phi_k = k^{-1}$ for all $k \in \mathbb{N}$. Let $n = 1$, $q(x) = |x|$, $f(x) = x$ and $p = 1$. Let us consider the sequence (x_k) defined by $x_k = k$, for all $k \in \mathbb{N}$. Then $(x_k) \in m(f, \phi, \Delta^n, p, q)$. Consider the rearrangement of (x_k) defined as follows:

$$y_k = (x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10}, x_{64}, \ldots).$$

Then $(y_k) \notin m(f, \phi, \Delta^n, p, q)$. Hence $m(f, \phi, \Delta^n, p, q)$ is not symmetric.

Remark The space $m(f, \phi, p, q)$ is solid, monotone as well as symmetric.

Taking $y_k = f\big(q(\Delta^n x_k)\big)$ for all $k \in \mathbb{N}$, we have the following results those follows from the Lemmas listed in Section 2.

Proposition 10 *$m(f, \phi, \Delta^n, p, q) \subseteq m(f, \psi, \Delta^n, p, q)$ if and only if $\sup_{s \geq 1} \phi_s/\psi_s < \infty$.*

Corollary 11 *$m(f, \phi, \Delta^n, p, q) = m(f, \psi, \Delta^n, p, q)$ if and only if*

$$\sup_{s \geq 1} \frac{\phi_s}{\psi_s} < \infty \quad \textit{and} \quad \sup_{s \geq 1} \frac{\psi_s}{\phi_s} < \infty.$$

Proposition 12 *$\ell^p(f, \Delta^n, q) \subseteq m(f, \phi, \Delta^n, p, q) \subseteq \ell_\infty(f, \Delta^n, q)$.*

Proposition 13 *$m(f, \phi, \Delta^n, p, q) = \ell^p(f, \Delta^n, q)$ if and only if*

$$\sup_{s \geq 1} \phi_s < \infty \quad \textit{and} \quad \sup_{s \geq 1} \phi_s^{-1} < \infty.$$

Proposition 14 *If $p_1 < p_2$, then $m(f, \phi, \Delta^n, p_1, q) \subset m(f, \phi, \Delta^n, p_2, q)$.*

The following result follows from the above result.

Corollary 15 *$m(f, \phi, \Delta^n, q) \subset m(f, \phi, \Delta^n, p, q)$.*

Proposition 16 *$m(f, \phi, \Delta^n, p_1, q) \subset m(f, \psi, \Delta^n, p_2, q)$ if $p_1 < p_2$ and $\sup_{s \geq 1} \phi_s/\psi_s < \infty$.*

Corollary 17 *$m(f, \phi, \Delta^n, p, q) = \ell_\infty(f, \Delta^n, q)$ if $\sup_{s \geq 1} s/\psi_s < \infty$.*

Proof. $m(f, \phi, \Delta^n, p, q) = \ell_\infty(f, \Delta^n, q)$ if $p = 1$ and $\phi_k = k$, $(k = 1, 2, 3, \ldots)$. Hence from Proposition 14 it follows that $\ell_\infty(f, \Delta^n, q) \subseteq m(f, \phi, \Delta^n, p, q)$ if $\sup_{s \geq 1} s/\psi_s < \infty$. This completes the proof. ☐

The proof of the following result is straightforward.

Proposition 18 *Let f be a modulus function q_1 and q_2 be seminorms. Then*

(i) $m(f, \phi, \Delta^n, p, q_1) \cap m(f, \phi, \Delta^n, p, q_2) \subseteq m(f, \phi, \Delta^n, p, q_1 + q_2)$,

(ii) *If q_1 is stronger than q_2, then $m(f, \phi, \Delta^n, p, q_1) \subset m(f, \phi, \Delta^n, p, q_2)$,*

(iii) $\ell_\infty(f, \Delta^n, q_1) \cap \ell_\infty(f, \Delta^n, q_2) \subseteq \ell_\infty(f, \Delta^n, q_1 + q_2)$,

(iv) *If q_1 is stronger than q_2, then $\ell_\infty(f, \Delta^n, q_1) \subset \ell_\infty(f, \Delta^n, q_2)$,*

(v) $\ell^p(f, \Delta^n, q_1) \cap \ell^p(f, \Delta^n, q_2) \subseteq \ell^p(f, \Delta^n, q_1 + q_2)$

(vi) *If q_1 is stronger than q_2, then $\ell^p(f, \Delta^n, q_1) \subset \ell^p(f, \Delta^n, q_2)$.*

Proposition 19 *The space $m(f, \phi, \Delta^n, p, q)$ is not convergence free.*

Proof. The result follows from the following example. □

Example 4 Let $X = \ell_\infty$, $\phi_k = k^{-1}$, for all $k \in \mathbb{N}$. Let $n = 2$, $f(x) = x$ and $p = 2$. For $x_k = (x_k^i) \in \ell_\infty$, for all $k \in \mathbb{N}$, let $q(x_k) = \sup_{i\geq 2} |x_k^i|$. Define the sequence (x_k^i) as follows:

$$x_k = \begin{cases} k^{-1}, & \text{for } k \text{ even} \\ 0, & \text{for } k \text{ odd} \end{cases}$$

Then $(x_k) \in m(f, \phi, \Delta^n, p, q)$.

Let the sequence (y_k) be defined as

$$y_k = \begin{cases} k^2, & \text{for } k \text{ even} \\ 0, & \text{for } k \text{ odd} \end{cases}$$

Then $(y_k) \notin m(f, \phi, \Delta^n, p, q)$.

References

[1] Bilgin T., *The sequence space $\ell(p, f, q, s)$ on seminormed spaces.* Bull. Calcutta Math. Soc. **86** (1994), 295–304.

[2] Çolak R. and Et M., *On some difference sequence sets and their topological properties.* to appear in Bull. Malays. Math. Sci. Soc.

[3] Et M., Altin Y. and Altinok H., *The sequence space $m(f, \phi, q)$ on seminormed spaces.* Thai J. Math. (2) **1** (2003), 121–127.

[4] Et M. and Çolak R., *On some generalized difference sequence spaces.* Soochow J. Math. **21** (1995), 377–386.

[5] Et M. and Nuray F., *Δ^m-statistical convergence.* Indian J. Pure Appl. Math. (6) **32** (2001), 961–969.

[6] Kizmaz H., *On certain sequence spaces.* Canad. Math. Bull. **24** (1981), 164–176.

[7] Maddox I.J., *Sequence spaces defined by a modulus*. Math. Proc. Camb. Phil. Soc. **100** (1986), 161–166.

[8] Nuray F. and Savaş E., *Some new sequence spaces defined by a modulus function*. Indian J. Pure Appl. Math. (11) **24** (1993), 657–663.

[9] Rath D. and Tripathy B.C., *Characterization of certain matrix operators*. J. Orissa Math. Soc. **8** (1989), 121–134.

[10] Ruckle W.H., *FK spaces in which the sequence of coordinate vectors is bounded*. Canad. J. Math. **25** (1973), 973–978.

[11] Sargent W.L.C., *Some sequence spaces related to* ℓ^p *spaces*. J. London Math. Soc. **35** (1960), 161–171.

[12] Savaş E., *On some generalized sequence spaces defined by a modulus*. Indian J. Pure Appl. Math. (5) **30** (1999), 459–464.

[13] Tripathy B.C., *Matrix maps on the power series convergent on the unit Disc*. J. Analysis **6** (1998), 27–31.

[14] Tripathy B.C., *A class of difference sequences related to the p-normed space* ℓ^p. Demonstratio Math. (4) **36** (2003), 867–872.

[15] Tripathy B.C., *On some classes of difference paranormed sequences associated with multiplier sequences*. International J. Math. Sci. (1) **2** (2003), 159–166.

[16] Tripathy B.C. and Sen M., *On a new class of sequence related to the space* ℓ^p. Tamkang J. Math. (2) **33** (2002), 167–171.

B.C. Tripathy
Mathematical Sciences Division
Institute of Advanced Study in Science and Technology
Khanapara, Guwahati-781 022
India
E-mail: tripathybc@yahoo.com

S. Mahanta
Mathematical Sciences Division
Institute of Advanced Study in Science and Technology
Khanapara, Guwahati-781 022
India
E-mail: sabitamahanta@yahoo.co.in

M. Et
Department of Mathematics
Firat University
23119 Elazig
Turkey
E-mail: mikailet@yahoo.com