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Nonlinear nonlocal Ott-Sudan-Ostrovskiy type equations on a segment

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Abstract. We study the global existence and large time asymptotic behavior of solutions to the initial-boundary value problems for the nolinear nonlocal equation on a segment $(0, a)$

$$\begin{cases} u_t + uu_x + \dfrac{C_1}{\pi}\partial_x \displaystyle\int_0^x \frac{u_s(s, t)}{\sqrt{x-s}}ds = 0, \quad t > 0, \\ u(x, 0) = u_0(x), \\ u(a, t) = 0, \quad t > 0 \end{cases} \tag{0.1}$$

and

$$\begin{cases} u_t + uu_x + \dfrac{C_1}{\pi} \displaystyle\int_0^x \frac{u_{ss}(s, t)}{\sqrt{x-s}}ds = 0, \quad t > 0, \\ u(x, 0) = u_0(x), \\ u(a, t) = u_x(0, t) = 0, \quad t > 0 \end{cases} \tag{0.2}$$

where the constant C_1 is choosen by a dissipative condition, such that $\operatorname{Re} C_1 p^{3/2} > 0$ for $\operatorname{Re} p = 0$. We prove that if the initial data $u_0 \in \mathbf{L}^\infty(0, a)$ is small enough, then there exists a unique solution of problems (0.1) and (0.2) $u \in \mathbf{C}\ [0, +\infty); \mathbf{L}^2(0, a)\ \cap\ \mathbf{C}\ \mathbf{R}^+; \mathbf{H}^1(0, a)$. Moreover there exists a constant A such that the solution has the following large time asymptotics uniformly with respect to $x \in (0, a)$

$$u(x, t) = At^{-2/3}\Lambda(xt^{-2/3}) + O(t^{-(2(1+\delta))/3}),$$

where $\delta \in (0, 2/3)$ and

$$\Lambda(s) = \frac{e^{-i\pi/4}\sqrt{2}}{2\pi i} \int_0^{+i\infty} \exp(sz - C_1 z^{3/2})dz, \ \ s > 0.$$

Key words: dissipative nonlinear evolution equation, large time asymptotics, Ott-Sudan-Ostrovskiy equation.

1. Introduction

We study the initial-boundary value problem for the nonlinear nonlocal Ott-Sudan-Ostrovskiy type equations

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$$\begin{cases} u_t + uu_x + \dfrac{C_1}{\pi}\partial_x \displaystyle\int_0^x \frac{u_s(s,\,t)}{\sqrt{x-s}}ds = 0, \quad t > 0,\ x \in (0,\,a), \\ u(x,\,0) = u_0(x), \quad x \in (0,\,a), \\ u(a,\,t) = 0, \quad t > 0 \end{cases} \tag{1.1}$$

and

$$\begin{cases} u_t + uu_x + \dfrac{C_1}{\pi} \displaystyle\int_0^x \frac{u_{ss}(s,\,t)}{\sqrt{x-s}}ds = 0, \quad t > 0,\ x \in (0,\,a), \\ u(x,\,0) = u_0(x), \quad x \in (0,\,a), \\ u(a,\,t) = 0, \quad u_x(0,\,t) = 0, \quad t > 0, \end{cases} \tag{1.2}$$

where the constant C_1 is chosen by dissipative condition, such that $\operatorname{Re} C_1 p^{3/2} > 0$ for $\operatorname{Re} p = 0$.

There are many physical problems, which are described in unified way by nonlinear nonlocal equation

$$u_t + uu_x + \int_0^x q(x-s)\partial_s^n u(s,\,t)ds = 0. \tag{1.3}$$

It represents, as particular cases, many equations that are of great physical interest, for example, the Korteveg-de Vries equation

$$u_t + uu_x + \alpha u_{xxx} = 0$$

and Korteveg-de Vries-Burgers equation

$$u_t + uu_x + u_{xx} + \alpha u_{xxx} = 0.$$

Both are well-known in the theory of surface waves in water and in nonlinear acoustics for fluids with gas bubbles. Ott, Sudan and Ostrovskiy proposed the following generalization of Korteveg-de Vries equation (see [6])

$$u_t + uu_x + \alpha u_{xxx} + \frac{1}{\pi}\int_0^x \frac{u_s(x,\,t)}{\sqrt{x-s}}ds = 0.$$

For the general theory of nonlinear pseudodifferential equations on a half-line we refer to the book [4]. (For the case of the Cauchy problem we refer to [3], [6].)

Up to now the theory of nonlinear nonlocal initial-boundary value problems on a segment is not developed well due to it's difficulty. There are many open natural questions which we need to study. First of them is how many boundary data should be posed in the initial-boundary value problems for it's correct solvability. There are some results in the case of nonlinear differential equations [1], [2]. However, as far as we know there are few results in the case of nonlinear pseudodifferential equations. In paper [7] we considered the case of pseudodifferential operator $\mathbb{K}$ of order $\alpha \in (0, 1)$ taking as an example the nonlocal Schrödinger equation on a segment

$$\begin{cases} u_t + i\,|u|^2\,u + \mathbb{K}u = 0, & t > 0,\ x \in (0, a), \\ u(x, 0) = u_0(x), & x \in (0, a). \end{cases} \tag{1.4}$$

We have introduced the pseudodifferential operator on a segment with symbol $K(p) = Cp^\alpha$ as follows

$$\mathbb{K}u = \theta_a(x) \times \mathcal{L}^{-1}\left\{K(p)\left(\mathcal{L}\{u\}(p, t) - \sum_{j=1}^{[\alpha]} \frac{\partial_x^{j-1}u(0, t) - e^{-pa}\partial_x^{j-1}u(a, t)}{p^j}\right)\right\},$$

where by $[\alpha]$ we denote the integer part of the real number α and we define by $\theta_a(x)$ the step function

$$\theta_a(x) = \begin{cases} 1, & x \in (0, a) \\ 0, & x \notin (0, a) \end{cases}.$$

Here the Laplace transform and the inverse Laplace transform we denote by symbols $\mathcal{L}$ and $\mathcal{L}^{-1}$ respectively. We proved in [7] that if the initial data $u_0 \in \mathbf{L}^\infty(0, a)$ and $\|u_0\|_{\mathbf{L}^\infty} < \varepsilon$, then there exists a unique solution $u \in \mathbf{C}\big([0, \infty); \mathbf{L}^\infty(0, a)\big)$ of the initial-boundary value problem (1.4). Moreover there exists a constant A such that the solution has the following large time asymptotics

$$u(x, t) = At^{-1/\alpha}\Lambda\left(\frac{x}{t^{1/\alpha}}\right) + O\left(t^{-(1+\delta)/\alpha}\right),$$

where $\Lambda(x) = 1/(2\pi i)\int_{-i\infty}^{i\infty} e^{-z^\alpha + zx}dz$.

In paper [5] we studied the Whitham equation on a segment in the case of $\alpha \in (3/2, 2)$

$$\begin{cases} u_t + uu_x + \mathbb{K}u = 0, & t > 0,\ x \in (0, a), \\ u(x, 0) = u_0(x), & x \in (0, a). \end{cases} \tag{1.5}$$

We proved that there exists a unique solution $u \in \mathbf{C}\big([0, \infty); \mathbf{L}^2(0,a)\big) \cap \mathbf{C}\big([0, \infty); \mathbf{H}^1(0, a)\big)$ of problem (1.5) for small initial data $u_0 \in \mathbf{L}^\infty(0, a)$.

Here we denote by

$$\mathbf{H}^1(0,\,a) = \{f,\,\partial_x f \in \mathbf{L}^2(0,\,a); \|f\|_{\mathbf{H}^1} = \|f\|_{\mathbf{L}^2} + \|\partial_x f\|_{\mathbf{L}^2} < +\infty\}$$

the Sobolev space. Note, that in both cases $\alpha \in (0,\,1)$ and $\alpha \in (3/2,\,2)$ we did not put any boundary data in the problems (1.5) and (1.4).

In the present paper we consider cases of nonlocal operators of type

$$L_1 u = \partial_x \int_0^x \frac{u_s(s,\,t)}{\sqrt{x-s}} ds$$

and

$$L_2 u = \int_0^x \frac{u_{ss}(s,\,t)}{\sqrt{x-s}} ds,$$

which have the same order $\alpha = 3/2$, so on the face of it should have the same number of the boundary data. However this is not the case. Indeed, we will prove that we have to put in the problem (1.1) for it's correct solvability one boundary value $u(a,\,t)$, whereas in the problem (1.2) - two boundary data $u(a,\,t)$ and $u_x(0,\,t)$. For simplicity we take homogeneous boundary data however we believe that the methods developed in this paper also work for non homogeneous boundary data.

Since the nonlinearity uu_x represents the so-called derivative loss, we have to use the smoothing properties of the strongly dissipative linear operator. Therefore the methods of papers [5] and [7] do not work directly. We adopt here approach based on the estimates of the Green function. Another difficulty of nonlocal equation on a segment is that the symbol $K(p)$ is not analytic in the left-half complex plane. Therefore we can not apply the Laplace theory directly, we use a methods of paper [5] to construct the Green function.

To state the results of the present paper precisely we give some notations. We introduce the following function space

$$\mathbf{Z}_{T,\,\gamma} = \big\{\phi(x,\,t) \in \mathbf{C}\big([0,\,T]; \mathbf{L}^2(0,\,a)\big) \cap \mathbf{C}\big((0,\,T]; \mathbf{H}^1(0,\,a)\big); \|\phi\|_{\mathbf{Z}_{T,\,\gamma}} < +\infty\big\}$$

with the norm

$$\|\phi\|_{\mathbf{Z}_{T,\,\gamma}} = \sup_{t\in[0,\,T]} \big(\|\phi(t)\|_{\mathbf{L}^2} + t^{1/3+\gamma}\|\phi_x(t)\|_{\mathbf{L}^2}\big).$$

Now we state the results. First of all we formulate the local existence of the solutions of the initial-boundary value problems (1.1), (1.2).

Theorem 1 *Suppose that the initial data* $u_0(x) \in \mathbf{L}^\infty(0, a)$*. Then for some* $T > 0$ *there exist unique solutions* $u(x, t) \in \mathbf{Z}_{T,\gamma}$ *to the initial-boundary value problems* (1.1), (1.2)*, where* $\gamma > 0$ *is small enough.*

In the next theorem we give some sufficient conditions for global existence of solutions.

Theorem 2 *Suppose that the initial data* $u_0(x) \in \mathbf{L}^\infty(0, a)$ *is small enough. Then there exist unique solutions* $u \in \mathbf{C}\big([0, +\infty); \mathbf{L}^2(0, a)\big) \cap \mathbf{C}(\mathbf{R}^+; \mathbf{H}^1(0, a))$ *to the initial-boundary value problems* (1.1), (1.2)*. Furthermore these solutions* u *have the following asymptotics for large time*

$$u(x, t) = At^{-2/3}\Lambda\big(xt^{-2/3}\big) + O\big(t^{-2(1+\delta)/3}\big),$$

where $\delta \in (0, 2/3)$ *and*

$$\Lambda(s) = \frac{e^{-i\pi/4}\sqrt{2}}{2\pi i}\int_0^{+i\infty} e^{sz - C_1 z^{3/2}} dz, \quad s > 0$$

and

$$A = \int_0^a u_0(x)dx + \int_0^{+\infty} d\tau \int_0^a u u_x dx < \infty.$$

Remark 1 Note that due to the condition $u_x(0, t) = 0$ the solutions for the initial-boundary value problems (1.1), (1.2) have the same large time asymptotics. However we expect that solutions of (1.2) and (1.1) could have different large time asymptotic behavior in the case non homogeneous boundary data $u_x(0, t) = h(t)$.

Remark 2 Note that the problem (1.1) with the boundary data at the point $x = 0$ instead of $x = a$ is not correctly posed. The problem (1.2) with the boundary data $u_x(a, t) = u_x(0, t) = 0$ or $u(a, t) = u(0, t) = 0$ also is not correctly posed.

2. Preliminaries

We denote operator

$$\mathbb{P}[g(p, t)] = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{(q-p)a} - 1}{q - p} g(q, t)dq.$$

In the next lemmas we give the proofs of the main properties of the operator $\mathbb{P}$ (see also paper [5]). Recall that by $\mathcal{L}^{-1}$ we denote the inverse Laplace transformation.

Lemma 1 *Let the function $g(p)$ be analytic for all complex p, except $p \in \Gamma$ and satisfy the following estimate*

$$|g(p)| < \frac{1 + |e^{-pa}|}{(1 + |p|)^{\delta}},$$

where $\delta > 0$. Then

$$\mathbb{P}[g(p)] = g(p) + \int_{\Gamma} \frac{e^{(q-p)a}}{q-p} g(q,\, t)dq \tag{2.1}$$

and

$$\mathcal{L}^{-1}\{\mathbb{P}[g(p,\, t)]\} = \theta_a(x)\mathcal{L}^{-1}\{g(p,\, t)\}. \tag{2.2}$$

Proof. Let us consider the case $\operatorname{Re} p > 0$. Using the Cauchy theorem we obtain

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{(q-p)a}}{q-p} g(q,\, t)dq = \frac{1}{2\pi i} \int_{\Gamma} \frac{e^{(q-p)a}}{q-p} g(q,\, t)dq$$

and

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-p} g(q,\, t)dq = -g(p,\, t).$$

In the same way we have for $\operatorname{Re} p \leq 0$

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{(q-p)a}}{q-p} g(q,\, t)dq = g(p,\, t) + \frac{1}{2\pi i} \int_{\Gamma} \frac{e^{(q-p)a}}{q-p} g(q,\, t)dq$$

and

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{1}{q-p} g(q,\, t)dq = 0.$$

Therefore statement (2.1) is proved. By a direct calculation using the Cauchy theorem we have for $x > a$

$$\int_{-i\infty}^{i\infty} e^{px} \int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} \frac{e^{(q-p)a} - 1}{q-p} g(q,\, t)dqdp = 0.$$

Also for $x \leq a$

$$\int_{-i\infty}^{i\infty} dp e^{p(x-a)} \int_{\Gamma} \frac{e^{qa}}{q-p} g(q,\, t) dq = 0.$$

Therefore using representation (2.1) we obtain formula (2.2). Lemma 1 is proved. □

Denote by $K(p) = C_1 p^{3/2}$. We make a cut along the contour Γ

$$\Gamma = \left\{ z \in \mathbb{C},\, \arg z \in (-2\pi + \beta,\, \beta],\, \beta \in \left(\frac{\pi}{2},\, \frac{3\pi}{2} \right) \right\}. \tag{2.3}$$

Lemma 2 *Let the function $g(p)$ be analytic for all complex p, except $p \in \Gamma$. Suppose that function $\phi(p)$ has the following asymptotics for $|p| > 1$*

$$g(p) = \frac{a - e^{-pa} b}{p} + O\left(\frac{1 + |e^{-pa}|}{|p|^{2+\gamma}} \right). \tag{2.4}$$

Then the function $g_1(p) = \mathbb{P}[g(p)]$ has the same main term of the asymptotics as the function $\phi(p)$, i.e.

$$g_1(p) = \mathbb{P}[g(p)] = \frac{a - e^{-pa} b}{p} + O\left(\frac{1 + |e^{-pa}|}{|p|^{2}} \right). \tag{2.5}$$

Moreover

$$\mathbb{P}\left[K(p) \frac{1}{2\pi i} \int_{\Gamma} \frac{e^{(q-p)a}}{q-p} g(q) dq \right] = 0. \tag{2.6}$$

Proof. We have

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{e^{(q-p)a}}{q-p} \frac{e^{-qa} b}{q} dq = -\frac{e^{-pa}}{p} b.$$

Therefore using the asymptotic formula for the function $\phi(p)$ and applying the Cauchy theorem we obtain

$$\begin{aligned} &\frac{1}{2\pi i} \int_{\Gamma} \frac{e^{(q-p)a}}{q-p} g(q) dq \\ &= \frac{1}{2\pi i} \int_{\Gamma} \frac{e^{(q-p)a}}{q-p} \left(g(q) + \frac{e^{-qa} b}{q} \right) dq + \frac{e^{-pa}}{p} b \\ &= \frac{1}{2\pi i} \frac{e^{-pa}}{p} \int_{\Gamma} e^{qa} \left(g(q) + \frac{e^{-qa} b}{q} \right) dq + \frac{e^{-pa}}{p} b \end{aligned} \tag{2.7}$$

$$- \frac{1}{2\pi i}\frac{e^{-pa}}{p}\int_\Gamma \frac{e^{qa}}{q-p}q\left(g(q)+\frac{e^{-qa}b}{q}\right)dq.$$

Using

$$\begin{aligned}&\lim_{x\to a-0}\mathcal{L}^{-1}\{g(p)\}\\ &=\lim_{x\to a-0}\int_{-i\infty}^{i\infty}e^{px}\left(\frac{a-e^{-pa}b}{p}+O\left(\frac{1+|e^{-pa}|}{|p|^{1+\gamma}}\right)\right)dp\\ &=-b\end{aligned}$$

via formula (2.4) we have

$$\begin{aligned}&\lim_{x\to a-0}\mathcal{L}^{-1}\{\mathbb{P}g\} \qquad (2.8)\\ &=\lim_{x\to a-0}\frac{1}{2\pi i}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon}e^{px}\frac{1}{2\pi i}\int\frac{e^{(q-p)a}-1}{q-p}\left(g(q,\,t)+\frac{e^{-qa}}{q}b(t)\right)dq\\ &=\lim_{x\to a-0}\frac{1}{2\pi i}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon}e^{p(x-a)}\frac{1}{2\pi i}\int_\Gamma\frac{e^{qa}}{q-p}\left(g(q,\,t)+\frac{e^{-qa}}{q}b(t)\right)dq\\ &\quad+\lim_{x\to a-0}\frac{1}{2\pi i}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon}e^{px}\left(g(p,\,t)+\frac{e^{-pa}}{p}b(t)\right)dp\\ &=\frac{1}{2\pi i}\int_\Gamma e^{qa}\left(g(q,\,t)+\frac{e^{-qa}}{q}b(t)\right)dq=-b.\end{aligned}$$

Putting (2.8) into (2.7) we obtain

$$\begin{aligned}&\frac{1}{2\pi i}\int_\Gamma\frac{e^{(q-p)a}}{q-p}g(q,\,t)dq \qquad (2.9)\\ &=-\frac{1}{2\pi i}\frac{e^{-pa}}{p}\int_\Gamma\frac{e^{qa}}{q-p}q\left(g(q,\,t)+\frac{e^{-qa}b(t)}{q}\right)dq.\end{aligned}$$

Since for $q\in\Gamma$

$$g(q)+\frac{e^{-qa}b}{q}=O\left(\frac{|e^{-qa}|}{|q|^{2+\gamma}}\right)$$

we can prove that

$$\frac{e^{-pa}}{p}\int_\Gamma\frac{e^{qa}}{q-p}q\left(g(q)+\frac{e^{-qa}b}{q}\right)dq=O\left(\frac{|e^{-pa}|}{|p|^2}\right),$$

and as a consequence of (2.9)

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{e^{(q-p)a}}{q-p}g(q,\,t)dq = O\left(\frac{|e^{-pa}|}{|p|^2}\right). \tag{2.10}$$

Therefore via the representation (2.1) we have

$$\begin{aligned}\phi_1(p) &= g(p) + \frac{1}{2\pi i}\int_{\Gamma}\frac{e^{(q-p)a}}{q-p}g(q)dq \\ &= \frac{a - e^{-pa}b}{p} + O\left(\frac{1+|e^{-pa}|}{|p|^2}\right).\end{aligned}$$

The first statement of Lemma is proved. Via estimate (2.10) we have

$$\left|K(p)\frac{1}{2\pi i}\int_{\Gamma}\frac{e^{(q-p)a}}{q-p}g(q)dq\right| < \frac{1+|e^{-pa}|}{(1+|p|)^{\delta}}.$$

Therefore from (2.1) and formulas (2.9), (2.10) we see that

$$\begin{aligned}&\mathcal{L}^{-1}\mathbb{P}\left[K(p)\frac{1}{2\pi i}\int_{\Gamma}\frac{e^{(q-p)a}}{q-p}g(q)dq\right] \\ &= \theta_a(x)\mathcal{L}^{-1}\left\{\frac{e^{-pa}}{p}K(p)\frac{1}{2\pi i}\int_{\Gamma}\frac{e^{qa}}{q-p}q\left(g(q)+\frac{e^{-qa}}{q}b\right)dq\right\} \\ &= \theta_a(x)\frac{1}{2\pi i}\int_{\Gamma}\frac{e^{qa}}{q-p}q\left(g(q)+\frac{e^{-qa}}{q}b\right)dq\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}\frac{e^{p(x-a)}K(p)}{p(p-q)}dp \\ &= 0.\end{aligned}$$

As a consequence we get (2.6). Lemma 2 is proved. □

Remark 3 If the function $g(p)$ is analytic for all $p \in \mathbb{C}$, then

$$\mathbb{P}[g(p)] = g(p).$$

We defined the pseudodifferential operator $\mathbb{K}u$ by

$$\mathbb{K}u = \theta_a(x)\mathcal{L}^{-1}\left\{K(p)\left(\widehat{u}(p,\,t)) - \frac{u(0,\,t) - e^{-pa}u(a,\,t)}{p}\right)\right\}. \tag{2.11}$$

Lemma 3 *We have for all* $x \in (0,\,a)$

$$\int_0^x \frac{u_{ss}(s,\,t)}{\sqrt{x-s}}ds = \mathbb{K}u - \frac{u_x(0,\,t)}{\sqrt{x}}$$

and

$$\partial_x \int_0^x \frac{u_s(s,\, t)}{\sqrt{x-s}} ds = \mathbb{K}u.$$

Proof. We have by direct calculations

$$\partial_x \int_0^x \frac{u_s(x,\, t)}{\sqrt{x-s}} ds = \frac{u_x(0,\, t)}{\sqrt{x}} + \int_0^x \frac{u_{ss}}{\sqrt{x-s}} ds.$$

Taking the Laplace transformation we get

$$\mathcal{L}\left\{\int_0^x \frac{u_{ss}}{\sqrt{x-s}} ds\right\}$$

$$= \mathbb{P}\left\{\frac{p^2}{\sqrt{p}}\left(\widehat{u}(p,\, t)) - \sum_{j=1}^{2} \frac{\partial_x^{j-1} u(0,\, t) - e^{-pa} \partial_x^{j-1} u(a,\, t)}{p}\right)\right\}.$$

Taking the inverse Laplace transformation and using the result of Lemma 1 we have

$$\int_0^x \frac{u_{ss}}{\sqrt{x-s}} ds = \theta_a(x) \mathcal{L}^{-1}\left\{K(p)\left(\widehat{u}(p,\, t) - \frac{u(0,\, t) - e^{-pa} u(a,\, t)}{p}\right)\right\}$$

$$-\theta_a(x) \mathcal{L}^{-1}\left\{\frac{\partial_x u(0,\, t) - e^{-pa} \partial_x u(a,\, t)}{\sqrt{p}}\right\}.$$

Since

$$\theta_a(x) \mathcal{L}^{-1}\left\{\frac{\partial_x u(0,\, t) - e^{-pa} \partial_x u(a,\, t)}{\sqrt{p}}\right\}$$

$$= \theta_a(x) \frac{\partial_x u(0,\, t)}{\sqrt{x}} - \frac{1}{2\pi i} \theta_a(x) \int_{-i\infty}^{i\infty} e^{p(x-a)} \frac{1}{\sqrt{p}} dp$$

$$= \theta_a(x) \frac{\partial_x u(0,\, t)}{\sqrt{x}}$$

we easily get the statement of the lemma. □

Lemma 4 *Let $K(p) = C_1 p^{3/2}$. Then there exists only one inverse function $\phi(\xi) = K^{-1}(-\xi)$, such that $\phi(\xi)$ is analytic for $\operatorname{Re} \xi \geq 0$ and*

$$\operatorname{Re} \phi(\xi) > 0.$$

Proof. The function $\xi = -C_1 p^{3/2}$ defined in $\arg p \in [-2\pi + \beta, \beta)$ has different inverse functions

$$\left|\frac{\xi}{C_1}\right|^{2/3} \exp \frac{2}{3} i \left(\pi + 2\pi l + \arg \xi - \arg C_1\right)$$

in the domain $\operatorname{Re} \xi \geq 0$, where l is an integer, such that

$$-2\pi + \beta \leq \frac{2}{3}(\pi + 2\pi l + \arg \xi - \arg C_1) < \beta \tag{2.12}$$

for all $-\pi/2 \leq \arg \xi \leq \pi/2$. From (2.12) we get the following estimate

$$-2 + \phi \leq l \leq -\frac{1}{2} + \phi$$

for $\phi = -\arg \xi/(2\pi) + \arg C_1/(2\pi) + 3\beta/(4\pi)$. The value $\arg C_1$ is defined by the dissipation condition $\operatorname{Re} K(p) > 0$ for $\operatorname{Re} p = 0$. This implies

$$\frac{3}{8} \leq \frac{\arg C_1}{2\pi} \leq \frac{5}{8}. \tag{2.13}$$

Then for $\psi = \arg C_1/(2\pi) - 5/8$ we get

$$-\frac{1}{4} \leq \psi \leq 0.$$

Thus the integers l must satisfy the inequalities

$$-\alpha + \phi - \frac{1}{2} \leq l \leq -\frac{1}{2} + \phi$$

and

$$-\frac{\arg \xi}{2\pi} - \frac{11}{8} + \psi + \frac{3\beta}{4\pi} \leq l \leq \frac{1}{8} - \frac{\arg \xi}{2\pi} + \psi + \frac{3\beta}{4\pi}$$

for all $\arg \xi/(2\pi) \in [-1/4, 1/4]$. Therefore

$$-\frac{9}{8} + \frac{3\beta}{4\pi} \leq l \leq -\frac{3}{8} + \frac{3\beta}{4\pi}.$$

Since $\beta \in (\pi/2, 3\pi/2)$ we have

$$\frac{3}{8} < \frac{3\beta}{4\pi} < \frac{9}{8}.$$

Hence there exists one inverse function $\phi(\xi) = K^{-1}(-\xi)$, which is analytic in $\operatorname{Re} \xi \geq 0$. Lemma 4 is proved. □

3. Linear problem

We consider the following linear initial-boundary value problem

$$\begin{cases} u_t + \dfrac{C_1}{\pi}\partial_x \displaystyle\int_0^x \frac{u_s(s,\,t)}{\sqrt{x-s}}ds = f(x,\,t), \quad t>0,\ x\in(0,\,a), \\ u(x,\,0) = u_0(x), \quad x\in(0,\,a), \\ u(a,\,t) = 0, \quad t>0. \end{cases} \tag{3.1}$$

To derive an integral representation for solutions of the problem (3.1) we suppose that there exists a solution $u(x,\,t)$ of problem (3.1), which is prolonged by zero outside of the interval $(0,\,a)$, that is

$$u(x,\,t) = 0 \quad \text{for all } x \notin [0,\,a].$$

Via Lemma 3 we have for the Laplace transforms

$$\mathcal{L}\left\{\frac{1}{\pi}\partial_x \int_0^x \frac{u_s(s,\,t)}{\sqrt{x-s}}ds\right\}$$
$$= \mathbb{P}\left[K(p)\left(\widehat{u}(p,\,t) - \frac{u(0,\,t)-e^{-pa}u(a,\,t)}{p}\right)\right]$$

and

$$\widehat{u}(p,\,t) = \mathbb{P}\big[\widehat{u}(p,\,t)\big], \quad \widehat{f}(p,\,t) = \mathbb{P}\big[\widehat{f}(p,\,t)\big].$$

Applying the Laplace transformation with respect to x to problem (3.1) we obtain

$$\begin{cases} \mathbb{P}\left[\widehat{u}_t + K(p)\left(\widehat{u}(p,\,t) - \dfrac{u(0,\,t)-e^{-qa}u(a,\,t)}{q}\right) - \widehat{f}(p,\,t)\right] = 0, \\ \qquad\qquad\qquad\qquad t>0,\ x\in(0,\,a), \\ \widehat{u}(p,\,0) = u_0(p), \\ u(a,\,t) = 0. \end{cases} \tag{3.2}$$

We look for the solution of (3.2) in the form

$$\widehat{u}(p,\,t) = \mathbb{P}[u_1(p,\,t)]. \tag{3.3}$$

The substitution of the representation (3.3) into (3.2) yields

$$\begin{cases} \mathbb{P}\left[u_{1t} + K(p)\left(u_1(p,\,t) - \dfrac{u(0,\,t) - e^{-pa}u(a,\,t)}{p}\right) - \widehat{f}(p,\,t)\right] \\ +\mathbb{P}\left[\dfrac{1}{2\pi i}\displaystyle\int_\Gamma \dfrac{e^{(q-p)a}}{q-p}(u_{1t}(q,\,t) + K(p)u_1(q,\,t))dq\right] = 0,\ t > 0, \\ \widehat{u}_1(p,\,0) = u_0(p) \\ u(a,\,t) = 0, \quad t > 0. \end{cases} \tag{3.4}$$

Now we prove that under some conditions (see (3.9) below) we can define the function $u_1(p,\,t)$ as the solution to the following problem

$$\begin{cases} \widehat{u}_{1t} + K(p)\left(u_1(p,\,t) - \dfrac{u(0,\,t) - e^{-pa}u(a,\,t)}{p}\right) = \widehat{f}(p,\,t) \\ u_1(p,\,0) = \widehat{u}_0(p), \\ u(a,\,t) = 0. \end{cases} \tag{3.5}$$

Indeed, integrating equation (3.5) with respect to time, we write $u_1(p,\,t)$ as

$$\widehat{u}_1(p,\,t) = e^{-K(p)t}\widehat{u}_0(p) + \int_0^t e^{-K(p)(t-\tau)} f_1(p,\,\tau)d\tau, \tag{3.6}$$

where

$$f_1(p,\,\tau) = \widehat{f}(p,\,t) - K(p)\left(u_1(p,\,t) - \frac{u(0,\,t) - e^{-pa}u(a,\,t)}{p}\right).$$

In order to get the integral formula for solutions of (3.4), we need to know the boundary values $u(0,\,t)$ and $u(a,\,t)$. We will find them using the following growth condition

$$|\hat{u}_1(p,\,t)| \le M(1 + |p|)^{-\delta}(1 + |e^{-pa}|) \quad \text{for all } |p| \ge 1, \tag{3.7}$$

with some $M,\,\delta > 0$, which guarantee us that $u_1(p,\,t)$ has the following asymptotic behavior for $|p| > 1$

$$\widehat{u}_1(p,\,t) = \frac{u(0,\,t) - e^{-pa}u(a,\,t)}{p} + t^{-1+\gamma}O\left(\frac{1 + e^{-pa}}{p(K(p))^{1-\gamma}}\right) \tag{3.8}$$

and as consequence of Lemma 2

$$\mathbb{P}\left[\frac{1}{2\pi i}\int_\Gamma \frac{e^{(q-p)a}}{q-p}(u_{1t}(q,\,t) + K(p)u_1(q,\,t))dq\right] = 0.$$

Also under condition (3.7) inverse Laplace transform $u(x,\,t)$ vanish for all $x < 0$ and $x > a$ (see Lemma 1). It is easy to prove that condition (3.7) is fulfilled in domains $\operatorname{Re} K(p) > 0$. In domains, where $\operatorname{Re} K(p) < 0$, we rewrite formula (3.6) as

$$\widehat{u}(p, t) = e^{-K(p)t}\left(\widehat{u}_0(p) + \int_0^{+\infty} e^{K(p)\tau} f_1(p, \tau) d\tau\right) - \int_t^{+\infty} e^{-K(p)(t-\tau)} f_1(p, \tau) d\tau.$$

Clearly the last integral

$$\int_t^{+\infty} e^{-K(p)(t-\tau)} f_1(p, \tau) d\tau$$

satisfies condition (3.7) for all $|p| \geq 1$, such that $\operatorname{Re} K(p) < 0$. However the first summand with exponentially growing factor $e^{-K(p)t}$ does not satisfy condition (3.7), therefore we have to put the following necessary and sufficient condition

$$\hat{u}_0(p) + \int_0^{+\infty} e^{K(p)\tau} f_1(p, \tau) d\tau = 0 \tag{3.9}$$

for all $|p| > 1$ in the domains, where $\operatorname{Re} K(p) < 0$.

We use the equation (3.9) to find the boundary values $u(0, t)$ and $u(a, t)$ involved in formula (3.6).

Via Lemma 4 taking the root $\phi(\xi)$ of equation $K(p) = -\xi$ we transform the half-complex plane $\operatorname{Re} \xi > 0$ to domain, where $\operatorname{Re} K(p) < 0$. We have

$$\phi(\xi) = C_1^{-1} |\xi|^{2/3} e^{i(-2\pi/3 + \arg \xi/\alpha)}. \tag{3.10}$$

Note that in right-half complex plane $\operatorname{Re} \xi > 0$

$$\operatorname{Re} \phi > 0.$$

The condition (3.9) can be written as equation

$$\widehat{u}_0(\phi) + \widehat{\widehat{f}}(\phi, \xi) - \xi \frac{\widehat{u}(0, \xi) - e^{-\phi_l a} \widehat{u}(a, \xi)}{\phi_l} = 0 \tag{3.11}$$

for $\operatorname{Re} \xi > 0$, where functions $\widehat{u}(0, \xi)$ and $\widehat{u}(a, \xi)$ are the Laplace transforms of the boundary data $u(0, t)$ and $u(a, t)$ with respect to time, and

$$\widehat{u}_0(\phi) = \int_0^a e^{-\phi y} u_0(y) dy, \quad \widehat{\widehat{f}}(\phi, \xi) = \int_0^{+\infty} \int_0^a e^{-(\phi y + \xi t)} f(y, t) dy dt.$$

From (3.11) we obtain

$$\widehat{u}(0,\,\xi)=\frac{\phi\Big(\widehat{u}_0(\phi)+\widehat{\widehat{f}}(\phi,\,\xi)\Big)}{\xi} \tag{3.12}$$

and the Laplace transforms $\widehat{u}(0,\,\xi)$ satisfy the growth condition

$$|\hat{u}(\,\cdot\,,\,\xi)|\le M(1+|\xi|)^{\beta}\quad\text{for all } |\xi|\ge 1 \tag{3.13}$$

with some $M,\ \beta>0$, which is sufficient for the existence of the inverse Laplace transform $u(\,\cdot\,,\,t)$.

Taking the inverse Laplace transform of (3.12) we obtain

$$u(0,\,t)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}e^{\xi t}\frac{\phi\Big(\widehat{u}_0(\phi)+\widehat{\widehat{f}}(\phi,\,\xi)\Big)}{\xi}d\xi. \tag{3.14}$$

Thus in the supposition that there exists a solution of the problem (3.1) we get the integral representation (5.2) for this solution

$$\begin{aligned}u(x,\,t)&=\theta_a(x)\mathcal{L}^{-1}\{u_1\} \\ &=\theta_a(x)\frac{1}{2\pi i}\left(\int_{-i\infty}^{i\infty}dpe^{px}e^{-K(p)t}\widehat{u}_0(p)\right.\\ &\quad\left.+\int_{-i\infty}^{i\infty}dpe^{px}\int_0^t e^{-K(p)(t-\tau)}\left(f(p,\,\tau)+\frac{K(p)}{p}u(0,\,\tau)\right)d\tau\right),\end{aligned} \tag{3.15}$$

where the functions $u(0,\,\tau)$ were defined by formula (3.14).

Now we prove that the function $u(x,\,t)$ given by formula (3.15) is a solution to problem (3.1). Taking the Laplace transformation of (3.15) and using the asymptotic representation (4.4) of $u_1(p,\,t)$ we get

$$\widehat{u}(p,\,t)=\mathbb{P}\left[u_1(p,\,t)\right], \tag{3.16}$$

where the function $u_1(p,\,t)$ is defined by (3.6), i.e.

$$\widehat{u}_1(p,\,t)=e^{-K(p)t}\widehat{u}_0(p)+\int_0^t e^{-K(p)(t-\tau)}f_1(p,\,\tau)d\tau.$$

By virtue of formula (4.4) and Lemma 2 the Laplace transform $\widehat{u}(p,\,t)$ has the following asymptotic representation for $|p|>1$

$$\widehat{u}(p,\,t)=\frac{u(0,\,t)-e^{-pa}u(a,\,t)}{p}+t^{-1+\gamma}O\left(\frac{1+|e^{-pa}|}{p^2}\right).$$

Therefore substituting (3.16) into the definition of the pseudodifferential operator $\mathbb{K}u$ (see formula (2.11)) and using

$$\mathbb{P}\left[K(p)\int_{\Gamma}\frac{e^{(q-p)a}}{q-p}u_1(q,\,t)dq\right]=0$$

we obtain

$$\begin{aligned}\mathbb{K}u &= \mathcal{L}^{-1}\mathbb{P}\left[K(p)\left(\mathbb{P}[u_1(p,\,t)]-\frac{u(0,\,t)-e^{-pa}u(a,\,t)}{p}\right)\right]\\ &= \mathcal{L}^{-1}\mathbb{P}\left[K(p)\left(u_1-\frac{u(0,\,t)-e^{-pa}u(a,\,t)}{p}\right)\right]\\ &\quad+\mathbb{P}\left[K(p)\int_{\Gamma}\frac{e^{(q-p)a}}{q-p}u_1(q,\,t)dq\right]\\ &= \mathcal{L}^{-1}\mathbb{P}\left[K(p)\left(u_1-\frac{u(0,\,t)-e^{-pa}u(a,\,t)}{p}\right)\right].\end{aligned}$$

Since $\mathcal{L}^{-1}\mathbb{P} = \theta_a(x)\mathcal{L}^{-1}$ and

$$u_{1t}+K(p)u_1=f(p,\,t)+\frac{K(p)}{p}\left(\frac{u(0,\,t)-e^{-pa}u(a,\,t)}{p}\right)$$

we obtain

$$\begin{aligned}\mathbb{K}u &= \theta_a(x)\mathcal{L}^{-1}\left\{K(p)\left(u_1-\frac{u(0,\,t)-e^{-pa}u(a,\,t)}{p}\right)\right\}\\ &= \theta_a(x)\mathcal{L}^{-1}\left\{-u_{1t}+\widehat{f}(p,\,t)\right\}=-u_t(x,\,t)+f(x,\,t).\end{aligned}$$

So the function $u(x,\,t)$ given by (3.15) satisfies equation $u_t(x,\,t)+\mathbb{K}u = f(x,\,t)$. Also clearly that the initial condition of the problem (3.1) is fulfilled

$$u(x,\,0)=\theta_a(x)\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dpe^{px}\widehat{u}_0(p)=u_0(x).$$

Thus there exists a solution to the problem (3.1), which is given by formula (3.15). The uniqueness follows from the fact that all solutions have representation (3.15).

Using representations (3.14) we have (for simplicity we put $f(x,\,t)=0$)

$$I_1=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}dpe^{px}\int_0^t e^{-K(p)(t-\tau)}K(p)\frac{u(0,\,\tau)}{p}d\tau$$

$$= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dpe^{px}\frac{K(p)}{p}e^{-K(p)t}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\phi\widehat{u}_0(\phi)}{\xi}d\xi$$
$$\times\int_0^t d\tau e^{(K(p)+\xi)\tau}.$$

Integrating with respect to τ, substituting the Laplace transform $\widehat{u}_0(\phi)$ and using

$$\int_{-i\infty}^{i\infty}\frac{\phi e^{-\phi(\xi)y}}{\xi}\frac{1}{K(p)+\xi}d\xi = 0$$

we obtain

$$I = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dpe^{px}\frac{K(p)}{p}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\xi t}\frac{\phi}{\xi}\frac{1}{K(p)+\xi}d\xi$$
$$= -\frac{1}{4\pi^2}\int_0^a dyu_0(y)\int_{-i\infty}^{i\infty} d\xi e^{\xi t}\frac{\phi e^{-\phi(\xi)y}}{\xi}\int_{-i\infty}^{i\infty} dpe^{px}\frac{K(p)}{p(K(p)+\xi)}d\xi.$$

Therefore we have the following integral representation for solutions $u(x, t)$ of problem (3.1)

$$u(x, t) = \int_0^a u_0(y)\mathcal{G}(x, y, t)dy \tag{3.17}$$
$$+\int_0^t d\tau\int_0^a f(y, \tau)\mathcal{G}(x, y, t-\tau)d\tau,$$

where the Green function $G(x, y, t)$ is defined by

$$\mathcal{G}(x, y, t) = \theta_a(x)\frac{1}{2\pi i}\left(\int_{-i\infty}^{i\infty} e^{-K(p)t+p(x-y)}\right.$$
$$\left. +\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\xi e^{\xi t}\frac{\phi e^{-\phi(\xi)y}}{\xi}\int_{-i\infty}^{i\infty} dpe^{px}\frac{K(p)}{p(K(p)+\xi)}\right)$$

Thus we have proved the following result.

Theorem 3 *Let the initial data* $u_0 \in L^1(0, a)$ *and a source* $f(x, t) \in \mathbf{L}^1_{\text{loc}}(0, \infty; \mathbf{L}^1(0, a))$. *Then there exists a unique solution* $u(x, t)$ *of the initial-boundary value problem* (3.1)*, which has representation* (3.17)*.*

Remark 4 By virtue of Lemma 3 we have

$$\partial_x\int_0^x\frac{u_s(s, t)}{\sqrt{x-s}}ds = \int_0^x\frac{u_{ss}(s, t)}{\sqrt{x-s}}ds$$

if $u_x(0,\,t) = 0$. Therefore Theorem 3 is true for the following initial-boundary value problem

$$\begin{cases} u_t + \dfrac{C_1}{\pi}\displaystyle\int_0^x \frac{u_{ss}(s,\,t)}{\sqrt{x-s}}ds = f, \quad t>0,\; x\in(0,\,a),\\ u(x,\,0) = u_0(x), \quad x\in(0,\,a),\\ u(a,\,t) = u_x(0,\,t) = 0, \quad t>0. \end{cases}$$

4. Asymptotics of the Green function

Consider the following function for $x \in (0,\,a)$, $y \in (0,\,a)$, $t \geq 0$

$$\mathcal{G}(x,\,y,\,t) = G(x,\,t) + F(x,\,y,\,t), \tag{4.1}$$

where

$$G(x,\,t) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} e^{px-K(p)t}dp \tag{4.2}$$

and

$$F(x,\,y,\,t) = -\frac{1}{4\pi^2}\int_{-i\infty}^{+i\infty} dpe^{px}p^{-1}K(p)\int_{-i\infty}^{+i\infty} e^{\xi t-\phi(\xi)y} \times\xi^{-1}\phi(\xi)(K(p)+\xi)^{-1}d\xi. \tag{4.3}$$

Here the function $\phi(\xi)$ is defined by

$$\phi(\xi) = K^{-1}(\xi), \quad \operatorname{Re}\xi > 0.$$

Note that $\operatorname{Re}\phi(\xi) > 0$ for all $\operatorname{Re}\xi > 0$.

Lemma 5 *The following asymptotics for $t \to \infty$ uniformly with respect to $x,\,y \in (0,\,a)$*

$$\mathcal{G}(x,\,y,\,t) = t^{-2/3}\Lambda\big(xt^{-2/3}\big) + O\big(t^{-2(1+\delta)/3}\big) \tag{4.4}$$

is true, where $\delta \in (0,\,1]$ and

$$\Lambda(s) = \frac{e^{-i\pi/4}\sqrt{2}}{2\pi i}\int_0^{+i\infty} e^{sz-K(z)}dz.$$

Moreover there exists a constant $C > 0$ such that for $\mu \in [0,\,2/3)$ and $\gamma > 0$

$$\sup_{t>0,\,y\in[0,\,a]} y^{\mu(3/2)}t^{1/3+\gamma-\mu}(1+t)^{-2\gamma} \times\big(\|\mathcal{G}(\,\cdot\,,\,y,\,t)\|_{\mathbf{L}^2} + t^{2/3}\|\mathcal{G}_x(\,\cdot\,,\,y,\,t)\|_{\mathbf{L}^2}\big) \leq C. \tag{4.5}$$

Proof. First we prove formula (4.4). We write the representation for function $G(x,\, t)$ (see (4.2)) as follows

$$G(x-y,\, t) = G(x,\, t) + (G(x-y,\, t) - G(x,\, t)). \tag{4.6}$$

Making a change of the variables $K(p)t = K(z)$ we easily find that

$$G(x,\, t) = t^{-2/3}\Lambda(xt^{-2/3}). \tag{4.7}$$

Using the estimate $|e^{-py} - 1| \le C|py|^{\delta}$ for $y > 0$ and $p \in (-i\infty,\, i\infty)$ and making the same change of the variables we get for $\delta \in (0,\, 1)$, $y \in (0,\, a)$

$$\begin{aligned}|G(x-y,\, t) - G(x,\, t)| &\le \left| \int_{-i\infty}^{+i\infty} e^{-K(p)t+px}(1-e^{-py})dp \right| \\ &\le Ct^{-(1+\delta)/\alpha} \int_{-i\infty}^{+i\infty} e^{\operatorname{Re} K(z)}|z|^{\delta}|dz| \\ &\le Ct^{-2(1+\delta)/3}.\end{aligned} \tag{4.8}$$

Therefore from (4.6)–(4.8) we obtain asymptotics for large time $t \to \infty$

$$G(x,\, y,\, t) = t^{-2/3}\Lambda(xt^{-2/3}) + O(t^{-2(1+\delta)/3}). \tag{4.9}$$

We write the representation of the function $F(x,\, y,\, t)$ (see (4.3))

$$F(x,\, y,\, t) = F(x,\, 0,\, t) + (F(x,\, y,\, t) - F(x,\, 0,\, t)). \tag{4.10}$$

Making a change of the variables $\xi t = q$ and $K(p)t = K(z)$ we get

$$F(x,\, 0,\, t) = \int_{-i\infty}^{i\infty} d\xi e^{\xi t}\frac{\phi}{\xi}\int_{-i\infty}^{i\infty} dpe^{px}\frac{K(p)}{p(K(p)+\xi)} = t^{-1/\alpha}\Lambda_2\Big(\frac{x}{t^{1/\alpha}}\Big),$$

where

$$\Lambda_2(s) = \int_{-i\infty}^{i\infty} e^{q}\phi'(q)dq \int_{-i\infty}^{i\infty} dze^{zs}\frac{K(z)}{z(K(z)+q)}. \tag{4.11}$$

Since $\operatorname{Re} K(z) > 0$ for $\operatorname{Re} z < 0$ via the Cauchy theorem we have for $s > 0$

$$\int_{-i\infty}^{i\infty} dze^{zs}\frac{K(z)}{z(K(z)+q)} = \int_{\Gamma} dze^{zs}\frac{K(z)}{z(K(z)+q)}, \tag{4.12}$$

where the contour Γ was defined by formula (2.3).

Also using $1/K'(\phi) = -\phi'$ (by definition $K(\phi) = -q$) we obtain

$$\int_{-i\infty}^{i\infty} e^q \phi(q) dq \int_\Gamma dz \frac{1}{z(K(z)+q)} = -2\pi i \int_{-i\infty}^{i\infty} dq e^q \frac{1}{K'(\phi)}$$
$$= 2\pi i \int_{-i\infty}^{i\infty} dq e^q \phi'(q)$$

Therefore we obtain the following estimate of the function $\Lambda_2(s)$

$$\begin{aligned}\Lambda_2(s) &= \int_{-i\infty}^{i\infty} e^q \phi'(q) dq \int_\Gamma dp e^{zs} \frac{K(z)}{z(K(z)+q)} \qquad (4.13)\\ &= 2\pi i \int_{-i\infty}^{i\infty} e^q \phi'(q) dq - \int_{-i\infty}^{i\infty} e^q \phi(q) dq \int_\Gamma dp e^{zs} \frac{1}{z(K(z)+q)}\\ &= 2\pi i \int_{-i\infty}^{i\infty} e^q \phi'(q) dq\\ &\quad - \int_{-i\infty}^{i\infty} e^q \phi(q) dq \int_\Gamma dz (e^{zs}-1) \frac{1}{z(K(z)+q)}\\ &\quad - \int_{-i\infty}^{i\infty} e^q \phi(q) dq \int_\Gamma dz \frac{1}{z(K(z)+q)} = O(t^{-\delta/\alpha}).\end{aligned}$$

We have

$$\frac{z^{-1}K(z)}{K(z)+q} = \frac{1}{z+1} + \frac{-q}{(z+1)(K(z)+q)} + \frac{z^{-1}K(z)}{(z+1)(K(z)+q)}$$

and $\int_{-i\infty}^{i\infty} e^{zx}(z+1)^{-1} dz = 2\pi i e^{-x}$. Then changing the variables $K(p)t = K(z)$ and $\xi t = q$, using $\phi(q) = rq^{2/3}$, where r is some complex constant we get

$$\begin{aligned}F(x,y,t) = Ct^{-1/\alpha}\Bigg(&2\pi i e^{-\tilde{x}} \int_{-i\infty}^{+i\infty} e^{q - rq^{1/\alpha}\tilde{y}} q^{1/\alpha - 1} dq\\ &+ \int_{-i\infty}^{+i\infty} dz e^{z\tilde{x}} (z+1)^{-1} \int_{-i\infty}^{+i\infty} e^{q - rq^{1/\alpha}\tilde{y}} q^{1/\alpha} (K(z)+q)^{-1} dq \qquad (4.14)\\ &+ \int_{-i\infty}^{+i\infty} dz e^{z\tilde{x}} z^{\alpha-1} (z+1)^{-1} \int_{-i\infty}^{+i\infty} e^{q - rq^{1/\alpha}\tilde{y}} q^{1/\alpha - 1} (K(z)+q)^{-1} dq\Bigg),\end{aligned}$$

where $\tilde{x} = xt^{-1/\alpha}$, $\tilde{y} = yt^{-1/\alpha}$. Differentiating the representation (4.14) of the function $F(x,\,y,\,t)$ with respect to y we get

$$F_y(x,y,t) = -Ct^{-2/\alpha}\left(2\pi i e^{-\tilde{x}}\int_{-i\infty}^{+i\infty} e^{q-rq^{1/\alpha}\tilde{y}}q^{2/\alpha-1}dq \right.$$
$$+\int_{-i\infty}^{+i\infty} e^{z\tilde{x}}(z+1)^{-1}dz\int_{-i\infty}^{+i\infty} e^{q-rq^{1/\alpha}\tilde{y}}q^{2/\alpha}(C_\alpha z^\alpha+q)^{-1}dq$$
$$\left.+\int_{-i\infty}^{+i\infty} e^{z\tilde{x}}z^{\alpha-1}(z+1)^{-1}dz\int_{-i\infty}^{+i\infty} e^{q-q^{1/\alpha}\tilde{y}}q^{2/\alpha-1}(C_\alpha z^\alpha+q)^{-1}dq\right).$$

It is easy to see that we can change the contour of integration into

$$\mathcal{C}_1 = \left\{z = \rho e^{\pm i\beta_1},\ \rho \geq 0,\ \beta_1 = \frac{\pi}{2} + \epsilon_1\right\} \tag{4.15}$$

and

$$\mathcal{C}_2 = \left\{q = \rho e^{\pm i\beta_2},\ \rho \geq 0,\ \beta_2 = \frac{\pi}{2} + \epsilon_2\right\}, \tag{4.16}$$

where ϵ_1 and ϵ_2 are fixed small positive constants. Then since $\operatorname{Re} q$, $\operatorname{Re} z < 0$ and $\operatorname{Re} rq^{2/3}\tilde{y} > 0$ for all $z \in \mathcal{C}_1$, $q \in \mathcal{C}_2$ we have

$$e^{\operatorname{Re} z\tilde{x}} \leq C|z|^{-\mu_2}\tilde{x}^{-\mu_2} \tag{4.17}$$
$$e^{\operatorname{Re} q} \leq C|q|^{-\gamma} \tag{4.18}$$

and

$$e^{-\operatorname{Re} rq^{2/3}\tilde{y}} \leq C|q|^{-\mu_1}\tilde{y}^{-\mu_1(3/2)}, \tag{4.19}$$

where $\mu_1,\,\mu_2,\,\gamma \geq 0$. Also it is easy to see that for all $z \in \mathcal{C}_1$, $q \in \mathcal{C}_2$ and $\nu \in [0,\,1]$

$$|K(z)+q|^{-1} \leq C|z|^{-\nu(3/2)}|q|^{\nu-1}. \tag{4.20}$$

Using the inequalities (4.17)–(4.20) with $\mu_1,\,\mu_2 = 0$, $1 - 1/\alpha < \nu < 1/\alpha$ and $0 < \gamma < 2/\alpha + \nu - 1$ for $|q| \leq 1$ and $\gamma > 2/\alpha + \nu$ for $|q| > 1$ we get

$$\|F_y(\,\cdot\,,\,\cdot\,,t)\|_{\mathbf{L}^\infty} \leq Ct^{-2/\alpha}\left(\int_{\mathcal{C}_2}|q|^{2/\alpha-1-\gamma}|dq| \right.$$
$$+\int_{-i\infty}^{+i\infty}|z|^{-\alpha\nu}|z+1|^{-1}|dz|\int_{\mathcal{C}_2}|q|^{2/\alpha-1+\nu-\gamma}|dq|$$
$$\left.+\int_{-i\infty}^{+i\infty}|z|^{\alpha-1-\alpha\nu}|z+1|^{-1}|dz|\int_{\mathcal{C}_2}|q|^{2/\alpha-2+\nu-\gamma}|dq|\right)$$

$$\leq Ct^{-2/\alpha}. \tag{4.21}$$

Also for $\mu_1 \in [0, 2/3)$ and $\mu_2 \in [0, 2/3)$, choosing $\nu \in [0, 1]$ such that $1/3 < \mu_2 + \nu < 2/3$ and $\gamma > 0$ such that $\mu_1 + \gamma < 2/3$ for $|q| \leq 1$ and $\mu_1 + \gamma > 2/3 + \nu + 1$ for $|q| > 1$ we have

$$\begin{aligned} |F(x,y,t)| &\leq Ct^{-2/3+\mu_1+\mu_2} y^{-\mu_1(3/2)} x^{-\mu_2(3/2)} \\ &\times \Bigg(\int_{\mathcal{C}_2} |q|^{2/3-1-\mu_1-\gamma} |dq| \\ &+ \int_{\mathcal{C}_1} |dz||z+1|^{-1}|z|^{-\nu(3/2)-\mu_2(3/2)} \int_{\mathcal{C}_2} |q|^{2/3+\nu-1-\mu_1-\gamma}|dq| \\ &+ \int_{\mathcal{C}_1} |dz||z|^{\alpha-1-\nu(3/2)-\mu_2(3/2)}|z+1|^{-1} \int_{\mathcal{C}_2} |q|^{2/3-2+\nu-\mu_1-\gamma}|dq| \Bigg) \\ &\leq Ct^{-2/3+\mu_1+\mu_2} y^{-\mu_1(3/2)} x^{-\mu_2(3/2)}. \end{aligned} \tag{4.22}$$

Therefore using (4.22) with $\mu_1 = 0$, $\mu_2 = 0$

$$\|F(\,\cdot\,,\,\cdot\,,\,t)\|_{\mathbf{L}^\infty} \leq Ct^{-2/3}. \tag{4.23}$$

From (4.21) and (4.23) we get for $\delta \in [0,\,1]$, $y \in [0,\,a]$.

$$\begin{aligned} &|F(x,\,y,\,t) - F(x,\,0,\,t)| \\ &\leq C\|F_y(\,\cdot\,,\,\cdot\,,\,t)\|^{\delta}_{\mathbf{L}^\infty} \|F(\,\cdot\,,\,\cdot\,,\,t)\|^{1-\delta}_{\mathbf{L}^\infty} y^{\delta} \leq Ct^{-2(1+\delta)/3}. \end{aligned} \tag{4.24}$$

By virtue of (4.10), (4.11), (4.13) and (4.24) we estimate $F(x,\,y,\,t)$ as

$$F(x,\,y,\,t) = O(t^{-2(1+\delta)/3}). \tag{4.25}$$

By formulas (4.9) and (4.25) we find

$$\mathcal{G}(x,\,y,\,t) = t^{-2/3}\Lambda(xt^{-2/3}) + O(t^{-2(1+\delta)/3}),$$

where

$$\Lambda(s) = \frac{e^{-i(\pi/4)}\sqrt{2}}{2\pi i} \int_0^{+i\infty} e^{sz-K(z)}dz.$$

Now we prove second part of Lemma 5. Differentiating (4.14) with respect to x and using (4.17)–(4.20) we get &

$$\begin{aligned} &|F_x(x,\,y,\,t)| \\ &\leq Ct^{-4/3+\mu_1+\mu_2} y^{-\mu_1\alpha} x^{-\mu_2\alpha} \Bigg(\int_{\mathcal{C}_2} |q|^{1/\alpha-1-\mu_1-\gamma}|dq| \end{aligned}$$

$$+ \int_{\mathcal{C}_1} |dz||z+1|^{-1}|z|^{1-\nu(3/2)-\mu_2(3/2)} \int_{\mathcal{C}_2} |q|^{3/2+\nu-1-\mu_1-\gamma}|dq|$$

$$+ \int_{\mathcal{C}_1} |dz||z|^{3/2-\nu(3/2)-\mu_2(3/2)}|z+1|^{-1} \int_{\mathcal{C}_2} |q|^{2/3-2+\nu-\mu_1-\gamma}|dq|\Bigg).$$

For any $\mu_1 \in [0,\, 2/3)$ and $\mu_2 \in (0,\, 2/3)$ we can choose $\nu \in (1/3,\, 1)$ and $\gamma > 0$ in such a way that the inequalities $1 < \mu_2 + \nu < 5/3$ and $\mu_1 + \gamma < 2/3$ in the domain $|q| \le 1$ or $\mu_1 + \gamma > 2/3 + \nu + 1$ in the domains $|q| > 1$ are valid. Then we obtain

$$|F_x(x,\, y,\, t)| \le Ct^{-4/3+\mu_1+\mu_2} y^{-\mu_1(3/2)} x^{-\mu_2(3/2)}. \tag{4.26}$$

Choosing in formulas (4.22) and (4.26) $\mu_2\alpha = 1/2 - \gamma$ we obtain for $\mu \in [0,\, 2/3)$ and $\gamma > 0$

$$\sup_{t\geqslant 0,\, y\in[0,\, a]} y^{\mu(3/2)} t^{1/3+\gamma-\mu}(1+t)^{-2\gamma}$$
$$\times \left(\|F(\,\cdot\,,\, y,\, t)\|_{\mathbf{L}^2} + t^{2/3}\|F_x(\,\cdot\,,\, y,\, t)\|_{\mathbf{L}^2}\right) \le C.$$

The Laplace transform of the function $G(x,\, t)$ is equal to $\widehat{G}(p,\, t) = e^{p^\alpha t}$. So making a change of variable $K(z) = K(p)t$ we get

$$\|G(t)\|_{\mathbf{L}^2} = \|\widehat{G}(t)\|_{\mathbf{L}^2(\operatorname{Re} p=0)} \le Ct^{-1/(2\alpha)} \left(\int_{-i\infty}^{+i\infty} e^{\operatorname{Re} K(z)}|dz|\right)^{1/2}$$
$$\le Ct^{-1/(2\alpha)}.$$

We have

$$G_x(x,\, t) = \int_{-i\infty}^{i\infty} e^{px-K(p)t} p dp.$$

Then changing the contour of integration into $\mathcal{C}_1$ such that $\operatorname{Re} K(z) > 0$ and $\operatorname{Re} z < 0$ and so for $\mu,\, \mu_1 \ge 0$ we have

$$e^{\operatorname{Re}(-K(z)+zxt^{-2/3})} \le C|z|^{-\mu-\mu_1(3/2)} x^{-\mu} t^{2\mu/3}.$$

Therefore we obtain

$$|G_x(x,\, t)| \le t^{2(-2+\mu)/3} x^{-\mu} \int_{\mathcal{C}_1} |z|^{1-\mu-\mu_1\alpha}|dz| \le Ct^{(-2+\mu)/\alpha} x^{-\mu},$$

where $\mu \in [0,\, 2)$ and we choose μ_1 such that $\mu + \mu_1\alpha < 2$ for $|z| \le 1$ and

$\mu + \mu_1\alpha > 2$ for $|z| \geq 1$. Choosing $\mu = 1/2 - \alpha\gamma$ we obtain

$$\sup_{t\geq 0} t^{-1+\gamma}(1+t)^{-2\gamma}\|G_x(t)\|_{\mathbf{L}^2} \leq C$$

Lemma 5 is proved. □

5. Local existence

In this section we prove Theorem 1 by the contraction mapping principle. Let u be a solution of the following linear problem

$$\begin{cases} u_t + \mathbb{N}(w) + \dfrac{C_1}{\pi}\partial_x \displaystyle\int_0^x \frac{u_{ss}(s,\, t)}{\sqrt{x-s}}ds = 0, \quad t > 0,\ x > 0, \\ u(x,\, 0) = u_0(x), \quad x > 0, \end{cases} \tag{5.1}$$

where $\mathbb{N}(w) = iww_x$ is well defined since $w \in \mathbf{Z}_{T,\,\gamma,\,\rho}$, where

$$\mathbf{Z}_{T,\,\gamma,\,\rho} = \{w \in \mathbf{Z}_{T,\,\gamma};\, \|w\|_{\mathbf{Z}_{T,\,\gamma}} \leq \rho\}.$$

Note that initial value problem (5.1) defines a mapping $\mathbb{M}$ by $u = \mathbb{M}(w)$ and we will show that $\mathbb{M}$ is the contraction mapping from $\mathbf{Z}_{T,\,\gamma,\,\rho}$ into itself for a sufficiently small $T > 0$.

From Section 3 we get

$$\begin{aligned} u(x,\, t) = &\int_0^a u_0(y)\mathcal{G}(x,\, y,\, t)dy \\ &+ \int_0^t d\tau \int_0^a f(y,\, \tau)\mathcal{G}(x,\, y,\, t-\tau)d\tau, \end{aligned} \tag{5.2}$$

where the function $\mathcal{G}(x,\, y,\, t)$ is defined by (4.1).

Let us prove the following estimate

$$\|u\|_{\mathbf{Z}_{T,\,\gamma}} = \sup_{t\in[0,\,T]} (t^{1/3+\gamma}\|u_x(t)\|_{\mathbf{L}^2} + \|u(t)\|_{\mathbf{L}^2}) \leq C\lambda, \tag{5.3}$$

where

$$\lambda = \|u_0\|_{\mathbf{L}^\infty} + T^\mu\|w\|^2_{\mathbf{Z}_{T,\,\gamma}},$$

$0 < \mu < 1/3$ and $\gamma > 0$.

By (5.2) we get

$$\|u(t)\|_{\mathbf{L}^2} \leq C\int_0^a |u_0(y)|\|\mathcal{G}(\,\cdot\,,\, y,\, t)\|_{\mathbf{L}^2}dy \tag{5.4}$$

$$+ C \int_0^t d\tau \int_0^a |\mathbb{N}(w)(y, \tau)| \|\mathcal{G}(\cdot, y, t-\tau)\|_{\mathbf{L}^2} dy.$$

From Lemma 5 for $\mu \in [0, 2/3)$ and $t \leq 1$ we have

$$\|\mathcal{G}(\cdot, y, t)\|_{\mathbf{L}^2} \leq C t^{-1/3+\mu-\gamma} y^{-\mu(3/2)}. \tag{5.5}$$

Therefore using (5.5) with $\mu(3/2) = 1/2 - \gamma(3/2)$, where $0 < \gamma < 1/3$ we obtain

$$\int_0^a |u_0(y)| \|\mathcal{G}(\cdot, y, t)\|_{\mathbf{L}^2} dy \leq C \int_0^a |u_0(y)| y^{-1/2+\gamma\alpha} dy$$
$$\leq C \|u_0\|_{\mathbf{L}^\infty} \int_0^a y^{-1/2+\gamma\alpha} dy \leq C \|u_0\|_{L^\infty}. \tag{5.6}$$

Since $w \in \mathbf{Z}_{T,\gamma}$ we have for $t \leq 1$

$$\|\mathbb{N}(w)(t)\|_{\mathbf{L}^1} \leq \int_0^a |w(x, t) w_x(x, t)| dx \tag{5.7}$$
$$\leq C \|w(t)\|_{\mathbf{L}^2} \|w_x(t)\|_{\mathbf{L}^2} \leq C \|w\|^2_{\mathbf{Z}_{T,\gamma}} t^{-1/3-\gamma}$$

Therefore using (5.5) with $\mu = 0$, we obtain

$$\int_0^t d\tau \int_0^a |\mathbb{N}(w)(y, \tau)| \|\mathcal{G}(\cdot, y, t-\tau)\|_{\mathbf{L}^2} dy \tag{5.8}$$
$$\leq C \int_0^t ((t-\tau)^{-\gamma-1/3} \|\mathbb{N}(w)(\tau)\|_{\mathbf{L}^1}) d\tau$$
$$\leq C \|w\|^2_{\mathbf{Z}_{T,\gamma}} \int_0^t \tau^{-1/3-\gamma} (t-\tau)^{-1/3-\gamma} d\tau \leq C \|w\|^2_{\mathbf{Z}_{T,\gamma}} T^{\mu},$$

where $0 < \mu_1 < 1/3$. Substituting (5.6) and (5.8) into (5.4) we obtain

$$\|u(t)\|_{\mathbf{L}^2} \leq C(\|u_0\|_X + \|w\|^2_{\mathbf{Z}_{T,\gamma}} T^{\mu_1}) \leq C\lambda, \tag{5.9}$$

where $0 < \mu_1 < 1/3$. Differentiating (5.4) with respect to x we have

$$\|u_x\|_{\mathbf{L}^2} \leq C \int_0^a |u_0(y)| \|\mathcal{G}_x(\cdot, y, t)\|_{\mathbf{L}^2} dy$$
$$+ C \int_0^t d\tau \int_0^a |\mathbb{N}(w)(y, \tau)| \|\mathcal{G}_x(\cdot, y, t-\tau)\|_{\mathbf{L}^2} dy. \tag{5.10}$$

Via Lemma 5 we have

$$\|\mathcal{G}_x(\cdot, y, t)\|_{\mathbf{L}^2} \leq C t^{-1+\mu_1} y^{-\mu_1(3/2)} \tag{5.11}$$

and choosing $\mu_1(3/2) = 1 - \gamma(3/2)$ we get

$$\sup_{t\in[0,T]} t^{1/3+\gamma} \int_0^a |u_0(y)| \|\mathcal{G}_x(\,\cdot\,,\, y,\, t)\|_{\mathbf{L}^2} dy \tag{5.12}$$
$$\leq C \int_0^a |u_0(y)| y^{-1+\gamma\alpha} dy \leq C\lambda.$$

Using (5.11) with $\mu_1(3/2) = 1/2 - \gamma(3/2)$ we obtain

$$\sup_{t\in[0,T]} t^{1/3+\gamma} \int_0^t d\tau \int_0^a |\mathbb{N}(w)(y,\,\tau)| \|\mathcal{G}_x(\,\cdot\,,\, y,\, t-\tau)\|_{\mathbf{L}^2} dy \tag{5.13}$$
$$\leq C \sup_{t\in[0,T]} t^{1/3+\gamma} \int_0^t d\tau (t-\tau)^{-2/3-\gamma} \int_0^a |\mathbb{N}(w)| y^{-1/2+\gamma\alpha} dy$$
$$\leq C \sup_{t\in[0,T]} t^{1/3+\gamma} \|w\|^2_{\mathbf{Z}_{T,\,\gamma}} \int_0^t d\tau (t-\tau)^{-2/3-\gamma} \tau^{-1/3-\gamma}$$
$$\leq C \|w\|^2_{\mathbf{Z}_{T,\,\gamma}} T^{\mu} \leq C\lambda.$$

From (5.12) and (5.13) we have

$$\sup_{t\in[0,T]} t^{1/3+\gamma} \|u_x(t)\|_{\mathbf{L}^2} \leq C\lambda. \tag{5.14}$$

Therefore we get (5.3), namely,

$$\|u\|_{\mathbf{Z}_{T,\,\gamma}} = \sup_{t\in[0,T]} t^{1/3+\gamma} \|u_x(t)\|_{\mathbf{L}^2} + \|u(t)\|_{\mathbf{L}^2}$$
$$\leq C(\|u_0\|_{\mathbf{L}^\infty} + T^{\mu} \|w\|^2_{\mathbf{Z}_{T,\,\gamma}}). \tag{5.15}$$

We choose $T^{\mu} \leq 1/(2C\rho)$ and $\|u_0\|_{\mathbf{L}^\infty} \leq \rho/(2C)$. Then we have $\|u\|_{\mathbf{Z}_{T,\,\gamma}} \leq \rho$. Thus the mapping $\mathbb{M}$ transforms the closed ball in $\mathbf{Z}_{T,\,\gamma}$ with a center at the origin and a radius ρ into itself. Analogously we can prove the estimate $\|u - \widetilde{u}\|_{\mathbf{Z}_{T,\,\gamma}} < \|w - \tilde{w}\|_{\mathbf{Z}_{T,\,\gamma}}$ for small T. Therefore the mapping $\mathbb{M}$ is a contraction mapping in $\mathbf{Z}_{T,\,\gamma}$ and there exists a unique solution $u(x,\,t)$ of the initial-value problem (5.1). Theorem 1 is proved.

Remark 5 By virtue of estimates (5.15) we see that if the norm of the initial data $\|u_0\|_{\mathbf{L}^\infty} < \epsilon$, then there exists a time $T > 1$ such that the solution is also sufficiently small

$$\sup_{t\in[0,T]} (\|u(t)\|_{\mathbf{L}^2} + t^{1/3+\gamma} \|u_x(t)\|_{\mathbf{L}^2}) < C\epsilon.$$

Remark 6 By virtue of Remark 4 Theorem 1 is true for the following initial-boundary value problem (1.2).

6. Proof of Theorem 2

Consider the initial-boundary value problem (5.1) with small initial data

$$\|u_0\|_{\mathbf{L}^\infty} < \epsilon_1, \tag{6.1}$$

where $\epsilon_1 > 0$ is sufficiently small. Let us prove the following estimate for all $t > 0$

$$(1+t)^{1/3-\gamma}(\|u(t)\|_{\mathbf{L}^2} + t^{1/3+\gamma}(1+t)^{1/3-\gamma}\|u_x(t)\|_{\mathbf{L}^2}) < \epsilon. \tag{6.2}$$

By Theorem 1 the norms $\|u(t)\|_{\mathbf{L}^2}$ and $\|u_x(t)\|_{\mathbf{L}^2}$ are continuous. Therefore via Remark 5 there exists a maximal time $T > 1$ such that the non strict estimate (6.2) is valid on $[0, T]$. We have by the formula (5.2)

$$\begin{aligned}\|u(t)\|_{\mathbf{L}^2} \le C\Big(&\int_0^a dy|u_0(y)|\|\mathcal{G}(\,\cdot\,, y, t)\|_{\mathbf{L}^2} \\ &+ \int_0^t d\tau \int_0^a |\mathbb{N}(u)(y, \tau)|\|\mathcal{G}(\,\cdot\,, y, t-\tau)\|_{\mathbf{L}^2}dy\Big).\end{aligned} \tag{6.3}$$

Since via Lemma 5 for all $t > 1$

$$t^{1/3-\gamma} \sup_{y\in[0,\, a]} \|\mathcal{G}(\,\cdot\,, y, t)\|_{\mathbf{L}^2} \le C,$$

and by virtue of (6.2)

$$\|\mathbb{N}(u)(t)\|_{\mathbf{L}^1} \le C\|u(t)\|_{\mathbf{L}^2}\|u_x(t)\|_{\mathbf{L}^2} \le \epsilon^2 t^{-1/3-\gamma}(1+t)^{-1+2\gamma}, \tag{6.4}$$

therefore we get

$$\begin{aligned}\|u(t)\|_{\mathbf{L}^2} \le\, & C\epsilon_1 t^{-1/3+\gamma} \\ &+ C\epsilon^2 \int_0^t (t-\tau)^{-1/3-\gamma}\tau^{-1/3-\gamma}(1+\tau)^{-1+2\gamma}d\tau \\ <\, & C(\epsilon_1+\epsilon^2)t^{-1/3+\gamma}.\end{aligned} \tag{6.5}$$

Now we estimate the norm $\|u_x(t)\|_{\mathbf{L}^2}$. We have by the formula (5.4)

$$\|u_x\|_{\mathbf{L}^2} \le C \int_0^{+\infty} d\tau|u_0|\|\mathcal{G}_x(\,\cdot\,, y, t)\|_{\mathbf{L}^2}$$

$$+ C\int_0^t d\tau \int_0^{+\infty} |\mathbb{N}(u)(y,\,\tau)| \|\mathcal{G}_x(\,\cdot\,,\,y,\,t-\tau)\|_{\mathbf{L}^2} dy. \quad (6.6)$$

Via Lemma 5 we obtain

$$\sup_{t\geq 0} t^{1+\gamma-\mu_1}(1+t)^{-2\gamma}\|\mathcal{G}_x(\,\cdot\,,\,y,\,t)\|_{\mathbf{L}^2} \leq Cy^{\mu_1}. \quad (6.7)$$

Therefore using (6.7) with $\mu_1 = 0$ we get for $t > 1$

$$\int_0^a d\tau |u_0| \|\mathcal{G}_x(\,\cdot\,,\,y,\,t)\|_{\mathbf{L}^2} \leq C\epsilon_1 t^{-1+\gamma}. \quad (6.8)$$

Also from (6.4) we obtain

$$\begin{aligned} &\int_0^t d\tau \int_0^a |\mathbb{N}(u)(y,\,\tau)| \|\mathcal{G}_x(\,\cdot\,,\,y,\,t-\tau)\|_{\mathbf{L}^2} dy \\ &\leq C\int_0^t d\tau (t-\tau)^{-1+\gamma} \|\mathbb{N}(u)(\tau)\|_{\mathbf{L}^1} \\ &\leq C\epsilon^2 \int_0^t d\tau (t-\tau)^{-1+\gamma} \tau^{-1/3-\gamma}(1+\tau)^{-1+2\gamma} \\ &\leq C\epsilon^2 t^{-1+\gamma}. \end{aligned} \quad (6.9)$$

We use (6.8) and (6.9) in the right-hand side of (6.6) to get

$$\sup_{t>1} t^{1-\gamma}\|u_x(t)\|_{\mathbf{L}^2} \leq C(\epsilon_1 + \epsilon^2). \quad (6.10)$$

From (6.5) and (6.10) we have estimate (6.2) for all $t \in [0,\,T]$. The contradiction obtained proves the estimate (6.2) for all $t > 0$. Moreover from Lemma 5 we have for $\delta \in (0,\,1)$, $\mu > 0$

$$\mathcal{G}(x,\,y,\,t) = t^{-2/3}\Lambda + O(t^{-2(1+\delta)/3}),$$

where

$$\Lambda = \frac{e^{-i(\pi/4)}\sqrt{2}}{2\pi i}\int_0^{+i\infty} e^{-K(z)} dz$$

Therefore we obtain from (5.2)

$$u(x,\,t) = t^{-1/\alpha} A\Lambda + R(x,\,t) + R_1(x,\,t),$$

where by virtue (6.1), (6.2)

$$A = \int_0^a u_0(y) dy + \int_0^{+\infty}\int_0^a \mathbb{N}(u)(y,\,t) dy dt < \infty$$

and

$$R(x,\, t) = O(t^{-2(1+\delta)/3}) \int_0^a u_0(y,\, t)dy + \int_0^\infty d\tau \int_0^a \mathbb{N}(u)(y,\, t)dy$$
$$+ \int_0^\infty d\tau \int_0^a \mathbb{N}(u)(y,\, t)(\mathcal{G}(x,\, y,\, t-\tau) - \mathcal{G}(x,\, y,\, t)).$$

By analogy with the proof of Lemma 5 we see that

$$\sup_{x,\, y \in [0,\, a]} |\mathcal{G}_t(x,\, y,\, t)| \le Ct^{-5/3}.$$

So we get for $\mu \in (0,\, 1)$

$$\sup_{x,\, y \in [0,\, a]} |\mathcal{G}(x,\, y,\, t-\tau) - \mathcal{G}(x,\, y,\, t)| \le C \sup_{x,\, y \in [0,\, a]} |\mathcal{G}_t(x,\, y,\, t)|^\mu \tau^{1-\mu}$$
$$\le Ct^{-5\mu/3}\tau^{1-\mu}.$$

Therefore choosing $\mu = 2/3 - 3\gamma$ from (6.4) we obtain

$$\left| \int_0^\infty d\tau \int_0^a \mathbb{N}(u)(y,\, \tau)(\mathcal{G}(x,\, y,\, t-\tau) - \mathcal{G}(x,\, y,\, t)) \right|$$
$$\le Ct^{-5\mu/3} \int_0^\infty \tau^{2/3-\gamma-\mu}(1+\tau)^{-1+2\gamma} d\tau = O(t^{-2(1+\delta)/3}),$$

where $\delta \in (0,\, 2/3)$. So we find

$$|R(x,\, t)| \le O(t^{-2(1+\delta)/3}) \left(\|u_0\|_{\mathbf{L}^1} + \int_0^t d\tau \|\mathbb{N}(\,\cdot\,,\, \tau)\|_{\mathbf{L}^1} \right)$$
$$+ O(t^{-2(1+\delta)/3})$$
$$= O(t^{-2(1+\delta)/3}),$$

where $\delta \in (0,\, 2/3)$. Theorem is proved.

Remark 7 By virtue of Remark 6 Theorem 2 is true for the following initial-boundary value problem (1.2).

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