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A weighted weak type estimate with power weights for sublinear operators

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Abstract. A weighted weak type estimate with power weights is established for sublinear operators which satisfy certain size condition.

Key words: sublinear operator, power weight, commutator.

1. Introduction

We will work on $\mathbb{R}^n$, $n \geq 1$. Let $K(x)$ be a function on $\mathbb{R}^n \backslash \{0\}$. A well known result of Stein [9] states that if K satisfies the size condition

$$|K(x)| \leq C|x|^{-n}, \ x \in \mathbb{R}^n \backslash \{0\} \tag{1}$$

and the operator

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x-y) f(y) dy$$

is bounded on $L^p(\mathbb{R}^n)$ for some p with $1 < p < \infty$, then T is also bounded on $L^p(\mathbb{R}^n, |x|^\alpha dx)$ provided that $-n < \alpha < n(p-1)$, where $L^p(\mathbb{R}^n, |x|^\alpha dx)$ denotes the weighted Lebesgue space defined by

$$L^p(\mathbb{R}^n, |x|^\alpha dx) = \Big\{ f \text{ is measurable on } \mathbb{R}^n \text{ and } \|f\|^p_{L^p(\mathbb{R}^n, |x|^\alpha dx)} = \int_{\mathbb{R}^n} |f(x)|^p |x|^\alpha dx < \infty \Big\}.$$

Soria and Weiss [8] gave some beautiful generalizations of Stein's result. In particular, they obtained the result of Stein in the case $p = 1$. However, either the result of Stein or the result of Soria and Weiss does not apply to the commutators of singular integral operators. Lu and Yang [4], Hu, Lu and Yang [3], and Hu [2] established some boundedness results for sublinear operators on weighted $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ spaces with power weights, and

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these boundedness results are very suitable for the commutators of singular integral operators. The main purpose of this paper is to give a weighted weak type estimate with power weights for certain sublinear operators. Our main result can be stated as follows.

Theorem 1 *Let m be a positive integer, $-n < \beta_1 < \beta_2 < \infty$, T be a sublinear operator which satisfies*

$$|Tf(x)| \le \int_{\mathbb{R}^n} |b(x) - b(y)|^m |K(x, y) f(y)| dy, \tag{2}$$

where $b \in \mathrm{BMO}(\mathbb{R}^n)$, $K(x, y)$ is a function on $\mathbb{R}^n \times \mathbb{R}^n \backslash \{(x, y)\colon x = y\}$. Suppose that

(i) *for some $\gamma \ge m$, T is bounded from $L^1(\log L)^\gamma(\mathbb{R}^n)$ to weak-$L^1(\mathbb{R}^n)$, namely*

$$\left|\left\{x \in \mathbb{R}^n\colon |Tf(x)| > \lambda\right\}\right| \le C \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log^\gamma \Big(2 + \frac{|f(x)|}{\lambda}\Big) dx;$$

(ii) *for any $0 < r < \infty$ and $\beta_1 < \beta < \beta_2$, the operator*

$$S_r f(x) = \int_{r<|x-y|\le 2r} |K(x, y)| \log^m (2 + r^n |K(x, y)|) |f(y)| dy \tag{3}$$

is bounded on $L^1(\mathbb{R}^n, |x|^\beta dx)$ with bound B and B is independent of r.

Then T is also bounded from $L^1(\log L)^\gamma(\mathbb{R}^n, |x|^\alpha dx)$ to weak-$L^1(\mathbb{R}^n, |x|^\alpha dx)$ provided that $\beta_1 < \alpha < \beta_2$, that is,

$$\int_{\left\{x\in\mathbb{R}^n\colon |Tf(x)|>\lambda\right\}} |x|^\alpha dx \le C \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \log^\gamma \Big(2 + \frac{|f(x)|}{\lambda}\Big) |x|^\alpha dx, \tag{4}$$

with the constant C independent of f and λ.

Throughout this paper, C denotes the constants that are independent of the main parameters involved but whose value may differ from line to line. For $r \ge 1$, r' is the dual exponent of r, i.e., $r' = r/(r-1)$. For any $\gamma \ge 1$, the function Φ_γ is defined by

$$\Phi_\gamma(t) = t \log^\gamma (2 + t), \quad t \ge 0.$$

2. Proof of Theorem 1

We begin with a preliminary lemma which will be important in the proof of Theorem 1 and has independent interest.

Lemma 1 *Let m, l and k be nonnegative integers. Then there is a positive constant $C = C_{m,l,k}$ such that for any $a > 0$ and $t_1,\ t_2,\ t_3 \geq 0$,*

$$\Phi_l(t_1 t_2^m)\Phi_k(t_3) \leq C\Phi_l(a^{-1})\big(\Phi_{l+m}(at_1)\Phi_{l+m+k}(t_3) + \exp t_2\big).$$

Proof. At first we claim that there is a positive constant C such that for any $t_1,\ t_2,\ t_3 \geq 0$,

$$\Phi_l(t_1 t_2^m)\Phi_k(t_3) \leq C\big(\Phi_{l+m}(t_1)\Phi_{l+m+k}(t_3) + \exp t_2\big).$$

In fact, if $0 < t_3 \leq 1$, then we have

$$\begin{aligned}\Phi_l(t_1 t_2^m)\Phi_k(t_3) \leq C\Phi_l(t_1 t_2^m) &\leq C\big(\Phi_{l+m}(t_1) + \exp t_2\big)\\ &\leq C\big(\Phi_{l+m}(t_1)\Phi_{l+m+k}(t_3) + \exp t_2\big),\end{aligned}$$

where in the second inequality we have employed the inequality

$$\Phi_l(t_1 t_2^m) \leq C\big(\Phi_{l+m}(t_1) + \exp t_2\big),$$

see [6] or [7] for details. On the other hand, if $t_3 > 1$, a trivial computation leads to that

$$\begin{aligned}\Phi_l(t_1 t_2^m) &\leq \Phi_l\big(\Phi_k(t_3)^{-1}\big)\Phi_l\big(\Phi_k(t_3)t_1 t_2^m\big)\\ &\leq C\Phi_k(t_3)^{-1}\big(\Phi_{l+m}\big(t_1\Phi_k(t_3)\big) + \exp t_2\big)\\ &\leq C\Phi_k(t_3)^{-1}\big(\Phi_{l+m}(t_1)\Phi_{l+m+k}(t_3) + \exp t_2\big).\end{aligned}$$

Lemma 1 now follows from the fact that

$$\Phi_l(t_1 t_2^m)\Phi_k(t_3) \leq \Phi_l(a^{-1})\Phi_l(at_1 t_2^m)\Phi_k(t_3).$$

□

Proof of Theorem 1. By the John-Nirenberg inequality, there are positive constants B_1 and B_2 such that

$$\sup_{Q\subset\mathbb{R}^n} \frac{1}{|Q|}\int_Q \exp\Big(\frac{|b(x) - m_Q(b)|}{B_2\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\Big)dx \leq B_1,$$

where Q is a ball, $m_Q(b)$ is the mean value of b on Q, i.e., $m_Q(b) =$

$|Q|^{-1}\int_Q b(y)dy$. By homogeneity, it suffices to prove that

$$\int_{\{x\in\mathbb{R}^n\colon |Tf(x)|>7DB_1\}}|x|^{\alpha}dx\le C\int_{\mathbb{R}^n}|f(x)|\log^{\gamma}\big(2+|f(x)|\big)|x|^{\alpha}dx,$$

where $D=D_1+D_2+D_3$ with D_1, D_2, D_3 positive constants which depend only on B_2, m and n, and will be chosen later. For each $k\in\mathbb{Z}$, set $C_k=\{x\in\mathbb{R}^n\colon 2^k<|x|\le 2^{k+1}\}$. Denote by χ_j the characteristic function of C_j. For each fixed $f\in L^1(\log L)^{\gamma}(\mathbb{R}^n,\,|x|^{\alpha}dx)$, write

$$f(x)=\sum_{j=-\infty}^{\infty}f(x)\chi_j(x)=\sum_{j=-\infty}^{\infty}f_j(x)$$

and

$$\begin{aligned}
&\int_{\{x\in\mathbb{R}^n\colon |Tf(x)|>7DB_1\}}|x|^{\alpha}dx\\
&\le C\sum_{k=-\infty}^{\infty}2^{k\alpha}\Big|\Big\{x\in C_k\colon T\Big(\sum_{j=k+3}^{\infty}f_j\Big)(x)>3DB_1\Big\}\Big|\\
&\quad+C\sum_{k=-\infty}^{\infty}2^{k\alpha}\Big|\Big\{x\in C_k\colon \Big|T\Big(\sum_{j=k-2}^{k+2}f_j\Big)(x)\Big|>DB_1\Big\}\Big|\\
&\quad+C\sum_{k=-\infty}^{\infty}2^{k\alpha}\Big|\Big\{x\in C_k\colon \Big|T\Big(\sum_{j=-\infty}^{k-3}f_j\Big)\Big|>3DB_1\Big\}\Big|\\
&=\mathrm{E}_1+\mathrm{E}_2+\mathrm{E}_3.
\end{aligned}$$

Since T is bounded from $L^1(\log L)^{\gamma}(\mathbb{R}^n)$ to weak-$L^1(\mathbb{R}^n)$, it follows that

$$\begin{aligned}
\mathrm{E}_2&\le C\sum_{k=-\infty}^{\infty}2^{k\alpha}\sum_{j=k-2}^{k+2}\int_{\mathbb{R}^n}|f_j(x)|\log^{\gamma}(2+|f_j(x)|)dx\\
&\le C\int_{\mathbb{R}^n}|f(x)|\log^{\gamma}(2+|f(x)|)|x|^{\alpha}dx.
\end{aligned}$$

To estimate the term E_1, we choose a number α_1 such that $\beta_1<\alpha_1<\alpha$ and write

$$\mathrm{E}_1\le C\sum_{k=-\infty}^{\infty}2^{k\alpha-k\alpha_1}\int_{\left\{x\in C_k\colon \left|T\left(\sum_{j=k+3}^{\infty}f_j\right)(x)\right|>3DB\right\}}|x|^{\alpha_1}dx.$$

Let $A_j = \{x \in \mathbb{R}^n : |x| \le 2^{j+1}\}$ and b_j be the mean value of b on A_j. By the John-Nirenberg inequality, we know that for any $k, j \in \mathbb{Z}$, $|b_j - b_k| \le C|k-j|$. Note that for $x \in C_k$,

$$\begin{aligned}
&T\Big(\sum_{j=k+3}^{\infty} f_j\Big)(x)\\
&\le \sum_{j=k+3}^{\infty} \int_{2^{j-2}\le|x-y|\le 2^{j+2}} |K(x, y)||b(x)-b(y)|^m|f_j(y)|dy\\
&\le \sum_{l=0}^{m}\binom{m}{l}\sum_{j=k+3}^{\infty}|b(x)-b_j|^l\int_{2^{j-2}\le|x-y|\le 2^{j+2}}|b(y)-b_j|^{m-l}|K(x, y)f_j(y)|dy\\
&\le C\sum_{l=0}^{m}\sum_{j=k+3}^{\infty}|b(x)-b_k|^l\int_{2^{j-2}\le|x-y|\le 2^{j+2}}|b(y)-b_j|^{m-l}|K(x, y)f_j(y)|dy\\
&\quad + C\sum_{l=0}^{m}\sum_{j=k+3}^{\infty}(j-k)^l\int_{2^{j-2}\le|x-y|\le 2^{j+2}}|b(y)-b_j|^{m-l}|K(x, y)f_j(y)|dy\\
&= \mathrm{U}_k f(x) + \mathrm{V}_k f(x).
\end{aligned}$$

By Lemma 1 (with $t_1 = |b(y)-b_j|^{m-l}|K(x, y)|/(B_2\|b\|_{\mathrm{BMO}(\mathbb{R}^n)})^{m-l}$, $t_2 = |b(x)-b_k|/(B_2\|b\|_{\mathrm{BMO}(\mathbb{R}^n)})$, $t_3 = 1$, and $a = 2^{(k-j)\alpha_1}|C_k|$) we have

$$\begin{aligned}
&\mathrm{U}_k f(x)\\
&\le C\sum_{l=0}^{m}\sum_{j=k+3}^{\infty}\frac{2^{(j-k)\alpha_1}}{|C_k|}\exp\Big(\frac{|b(x)-b_k|}{B_2\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\Big)\|f_j\|_1\\
&\quad + C\sum_{l=0}^{m}\sum_{j=k+3}^{\infty}\frac{2^{(j-k)\alpha_1}}{|C_k|}\\
&\quad \times\int_{2^{j-2}\le|x-y|\le 2^{j+2}}\Phi_l\Big(\frac{|b(y)-b_j|^{m-l}|C_k|}{2^{(j-k)\alpha_1}(B_2\|b\|_{\mathrm{BMO}(\mathbb{R}^n)})^{m-l}}\\
&\qquad\qquad \times|K(x, y)|\Big)|f_j(y)|dy\\
&\le C\sum_{j=k+3}^{\infty}\frac{2^{(j-k)\alpha_1}}{|C_k|}\exp\Big(\frac{|b(x)-b_k|}{B_2\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\Big)\|f_j\|_1\\
&\quad + C\sum_{l=0}^{m}\sum_{j=k+3}^{\infty}\frac{2^{(j-k)\alpha_1}}{|C_k|}
\end{aligned}$$

$$\times \int_{2^{j-2}\le|x-y|\le 2^{j+2}} \Phi_l\Big(\frac{|b(y)-b_j|^{m-l}|C_k|}{2^{(j-k)\alpha_1}(B_2\|b\|_{\mathrm{BMO}(\mathbb{R}^n)})^{m-l}} \times|K(x,\,y)|\Big)|f_j(y)|dy$$
$$= \mathrm{U}_k^{\mathrm{I}}f(x)+\mathrm{U}_k^{\mathrm{II}}f(x).$$

A standard computation leads to that

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} 2^{k\alpha-k\alpha_1}\int_{\{C_k:\ \mathrm{U}_k^{\mathrm{I}}f(x)>DB_1\}}|x|^{\alpha_1}dx\\
&\le C\sum_{k=-\infty}^{\infty}2^{k\alpha-k\alpha_1}\sum_{j=k+3}^{\infty}\|f_j\|_1\frac{2^{(j-k)\alpha_1}}{|C_k|}\times\int_{C_k}\exp\Big(\frac{|b(x)-b_k|}{B_2\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\Big)|x|^{\alpha_1}dx\\
&\le C\sum_{k=-\infty}^{\infty}2^{k\alpha-k\alpha_1}\sum_{j=k+3}^{\infty}\|f_j\|_1\frac{2^{-j\alpha_1}}{|A_k|}\int_{A_k}\exp\Big(\frac{|b(x)-b_k|}{B_2\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\Big)dx\\
&\le C\sum_{k=-\infty}^{\infty}2^{k\alpha-k\alpha_1}\sum_{j=k+3}^{\infty}\|f_j\|_{L^1(\mathbb{R}^n,\,|x|^\alpha dx)}2^{j(\alpha_1-\alpha)}\\
&\le C\sum_{j=-\infty}^{\infty}\|f_j\|_{L^1(\mathbb{R}^n,\,|x|^\alpha dx)}2^{j(\alpha_1-\alpha)}\sum_{k=-\infty}^{j-3}2^{k(\alpha-\alpha_1)}\\
&\le C\|f\|_{L^1(\mathbb{R}^n,\,|x|^\alpha dx)}.
\end{aligned}$$

Let $a_{k,\,j}=|C_j||C_k|^{-1}2^{j-k}2^{(j-k)\alpha_1}$. It is obvious that for $k,\,j\in\mathbb{Z}$ with $j\ge k+3$, $a_{k,\,j}>1$. Another application of Lemma 1 with $a=a_{k,\,j}$ yields

$$\begin{aligned}
&\mathrm{U}_k^{\mathrm{II}}f(x)\\
&\le C\sum_{j=k+3}^{\infty}\frac{2^{(j-k)\alpha_1}}{|C_k|a_{k,\,j}}\int_{C_j}\exp\Big(\frac{|b(y)-b_j|}{B_2\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\Big)dy\\
&\quad+C\sum_{j=k+3}^{\infty}\frac{2^{(j-k)\alpha_1}}{|C_k|a_{k,\,j}}\\
&\quad\times\int_{2^{j-2}\le|x-y|\le 2^{j+2}}\Phi_m\Big(\frac{a_{k,\,j}|C_k|}{2^{(j-k)\alpha_1}}|K(x,\,y)|\Big)\Phi_m(|f_j(y)|)dy
\end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{j=k+3}^{\infty}\frac{2^{k-j}}{|A_j|}\int_{A_j}\exp\Big(\frac{|b(y)-b_j|}{B_2\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\Big)dy\\
&\quad + C\sum_{j=k+3}^{\infty}\frac{2^{k-j}}{|C_j|}\\
&\quad \times\int_{2^{j-2}\le|x-y|\le 2^{j+2}}\Phi_m\Big(|C_j|2^{j-k}|K(x,\,y)|\Big)\Phi_m(|f_j(y)|)dy\\
&\le D_1B_1 + C\sum_{j=k+3}^{\infty}\frac{(j-k)^m}{|C_j|}\\
&\quad \times\int_{2^{j-2}\le|x-y|\le 2^{j+2}}\Phi_m\Big(|C_j||K(x,\,y)|\Big)\Phi_m(|f_j(y)|)dy.
\end{aligned}
$$

This in turn implies that

$$
\begin{aligned}
&\sum_{k=-\infty}^{\infty}2^{k\alpha-k\alpha_1}\int_{\{x\in C_k:\,\mathrm{U}_k^{\mathrm{II}}f(x)>2DB_1\}}|x|^{\alpha_1}dx\\
&\le\sum_{k=-\infty}^{\infty}2^{k\alpha-k\alpha_1}\int_{\left\{\begin{array}{c} \sum_{j=k+3}^{\infty}\frac{(j-k)^m}{|C_j|}\\ x\in C_k:\times\int_{2^{j-2}\le|x-y|\le 2^{j+2}}\Phi_m\big(|C_j||K(x,\,y)|\big)\\ \times\Phi_m(|f_j(y)|)dy>CDB_1\end{array}\right\}}|x|^{\alpha_1}dx\\
&\le C\sum_{k=-\infty}^{\infty}2^{k\alpha-k\alpha_1}\sum_{j=k+3}^{\infty}(j-k)^m\Big\|S_{2^j}\Big(\Phi_m(|f_j|)\Big)\Big\|_{L^1(\mathbb{R}^n,\,|x|^{\alpha_1}dx)}\\
&\le C\sum_{j=-\infty}^{\infty}\|\Phi_m(|f_j|)\|_{L^1(\mathbb{R}^n,\,|x|^{\alpha}dx)}\sum_{k=-\infty}^{j-3}(j-k)^m2^{(k-j)(\alpha-\alpha_1)}\\
&\le C\|\Phi_m(|f|)\|_{L^1(\mathbb{R}^n,\,|x|^{\alpha}dx)}.
\end{aligned}
$$

Similarly, for each fixed $k\in\mathbb{Z}$ and $x\in C_k$, we can easily deduce that

$$
\begin{aligned}
&\mathrm{V}_kf(x)\\
&\le C\sum_{l=0}^{m}\sum_{j=k+3}^{\infty}(j-k)^l\frac{2^{k-j}}{|C_j|}\int_{C_j}\exp\Big(\frac{|b(y)-b_j|}{B_2\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\Big)dy\\
&\quad + C\sum_{l=0}^{m}\sum_{j=k+3}^{\infty}(j-k)^l\frac{2^{k-j}}{|C_j|}
\end{aligned}
$$

$$\times \int_{2^{j-2}\le|x-y|\le 2^{j+2}} \Phi_{m-l}\big(|C_j|2^{(j-k)}|K(x,\,y)||f_j(y)|\big)dy$$
$$\le D_2B_1 + C\sum_{j=k+3}^{\infty}(j-k)^{2m}\frac{1}{|C_j|}$$
$$\times \int_{2^{j-2}\le|x-y|\le 2^{j+2}} \Phi_m(|C_j||K(x,\,y)|)\Phi_m(|f_j(y)|)dy.$$

Therefore,

$$\sum_{k=-\infty}^{\infty} 2^{k(\alpha-\alpha_1)}\int_{\{x\in C_k:\, \mathrm{V}_k f(x)>2DB_1\}}|x|^{\alpha_1}dx$$
$$\le C\sum_{k=-\infty}^{\infty}2^{k(\alpha-\alpha_1)}\sum_{j=k+3}^{\infty}(j-k)^{2m}\|S_{2^j}(\Phi_m(|f_j|)\|_{L^1(\mathbb{R}^n,\,|x|^{\alpha_1}dx)}$$
$$\le C\|\Phi_m(|f|)\|_{L^1(\mathbb{R}^n,\,|x|^{\alpha}dx)}.$$

Now we turn our attention to the term E_3. Take α_2 such that $\alpha<\alpha_2<\beta_2$ and write

$$\mathrm{E}_3 \le \sum_{k=-\infty}^{\infty} 2^{k(\alpha-\alpha_2)}\int_{\left\{x\in C_k:\, \left|T\left(\sum_{j=-\infty}^{k-3} f_j\right)(x)\right|>3DB_1\right\}}|x|^{\alpha_2}dx.$$

A familiar argument involving Lemma 1 tells us that for each fixed $k\in\mathbb{Z}$ and $x\in C_k$,

$$\Big|T\Big(\sum_{j=-\infty}^{k-3} f_j\Big)(x)\Big|$$
$$\le \sum_{j=-\infty}^{k-3}\int_{2^{k-2}\le|x-y|\le 2^{k+2}}|K(x,\,y)||b(x)-b(y)|^m|f_j(y)|dy$$
$$\le C\sum_{l=0}^{m}\sum_{j=-\infty}^{k-3}|b(x)-b_k|^l$$
$$\times\int_{2^{k-2}\le|x-y|\le 2^{k+2}}|b(y)-b_k|^{m-l}|K(x,\,y)f_j(y)|dy$$
$$\le C\sum_{l=0}^{m}\sum_{j=-\infty}^{k-3}\frac{2^{(j-k)\alpha_2}}{|C_k|}\exp\Big(\frac{|b(x)-b_k|}{B_2\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\Big)\|f_j\|_1$$

$$
\begin{aligned}
&+C\sum_{l=0}^{m}\sum_{j=-\infty}^{k-3}\frac{2^{(j-k)\alpha_2}}{|C_k|}\\
&\times\int_{2^{k-2}\le|x-y|\le 2^{k+2}}\Phi_l\Big(\frac{|b(y)-b_k|^{m-l}|C_k|}{2^{(j-k)\alpha_2}(B_2\|b\|_{\mathrm{BMO}(\mathbb{R}^n)})^{m-l}}|K(x,y)|\Big)\\
&\qquad\qquad\times|f_j(y)|dy\\
\le{}& C\sum_{j=-\infty}^{k-3}\frac{2^{(j-k)\alpha_2}}{|C_k|}\exp\Big(\frac{|b(x)-b_k|}{B_2\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\Big)\|f_j\|_1\\
&+C\sum_{l=0}^{m}\sum_{j=-\infty}^{k-3}\frac{2^{(j-k)\alpha_2}}{|C_k|}\\
&\times\int_{2^{k-2}\le|x-y|\le 2^{k+2}}\Phi_l\Big(\frac{|b(y)-b_k|^{m-l}|C_k|}{2^{(j-k)\alpha_2}(B_2\|b\|_{\mathrm{BMO}(\mathbb{R}^n)})^{m-l}}|K(x,y)|\Big)\\
&\qquad\qquad\times|f_j(y)|dy\\
={}&\mathrm{W}_k^{\mathrm{I}}f(x)+\mathrm{W}_k^{\mathrm{II}}f(x).
\end{aligned}
$$

For $\mathrm{W}_k^{\mathrm{I}}$, we have

$$
\begin{aligned}
&\sum_{k=-\infty}^{\infty}2^{k(\alpha-\alpha_2)}\int_{\{x\in C_k:\,|\mathrm{W}_k^{\mathrm{I}}f(x)|>DB_1\}}|x|^{\alpha_2}dx\\
\le{}& C\sum_{k=-\infty}^{\infty}2^{k(\alpha-\alpha_2)}\sum_{j=-\infty}^{k-3}2^{j\alpha_2}\|f_j\|_{L^1(\mathbb{R}^n)}\\
\le{}& C\sum_{j=-\infty}^{\infty}\|f_j\|_{L^1(\mathbb{R}^n,\,|x|^\alpha dx)}\sum_{k=j+3}^{\infty}2^{(k-j)(\alpha-\alpha_2)}\\
\le{}& C\|f\|_{L^1(\mathbb{R}^n,\,|x|^\alpha dx)}.
\end{aligned}
$$

Applying Lemma 1 with $\widetilde{a_{k,j}}=2^{k-j}2^{(j-k)\alpha_2}$, we have that

$$
\begin{aligned}
&\mathrm{W}_k^{\mathrm{II}}f(x)\\
\le{}& C\sum_{l=0}^{m}\sum_{j=-\infty}^{k-3}\frac{\Phi_l(\widetilde{a_{k,j}}^{-1})2^{(j-k)\alpha_2}}{|C_k|}\int_{C_k}\exp\Big(\frac{|b(y)-b_k|}{B_2\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\Big)dy\\
&+C\sum_{l=0}^{m}\sum_{j=-\infty}^{k-3}\frac{\Phi_l(\widetilde{a_{k,j}}^{-1})2^{(j-k)\alpha_2}}{|C_k|}
\end{aligned}
$$

$$\times \int_{C_k} \Phi_m\Big(\frac{\widetilde{a_{k,j}}|C_k|}{2^{(j-k)\alpha_2}}|K(x, y)|\Big)\Phi_m(|f_j(y)|)dy$$

$$\le D_3B_1 + C\sum_{j=-\infty}^{k-3}\frac{(k-j)^{m(1+\max(0,\,\alpha_2-1))}}{|C_k|} \times \int_{C_k}\Phi_m\big(|C_k||K(x, y)|\big)\Phi_m(|f_j(y)|)dy.$$

This together with our hypothesis states that

$$\sum_{k=-\infty}^{\infty} 2^{k(\alpha-\alpha_2)}\int_{\{x\in C_k:\, \mathrm{W}_k^{\mathrm{II}}f(x)>2DB_1\}}|x|^{\alpha_2}dx$$
$$\le C\sum_{k=-\infty}^{\infty}2^{k(\alpha-\alpha_2)}\sum_{j=-\infty}^{k-3}(k-j)^{2m}\Big\|S_{2^k}\big(\Phi_m(|f_j|)\big)\Big\|_{L^1(\mathbb{R}^n,\,|x|^{\alpha_2}dx)}$$
$$\le C\sum_{j=-\infty}^{\infty}\|\Phi_m(|f_j|)\|_{L^1(\mathbb{R}^n,\,|x|^{\alpha}dx)}\sum_{k=j+3}^{\infty}(k-j)^{2m}2^{(j-k)(\alpha_2-\alpha)}$$
$$\le C\|\Phi_m(f)\|_{L^1(\mathbb{R}^n,\,|x|^{\alpha}dx)}.$$

□

Remark Repeating the proof of Theorem 1, we can obtain

Theorem 2 *Let $-n < \beta_1 < \beta_2 < \infty$, T be a sublinear operator which satisfies*

$$|Tf(x)| \le \int_{\mathbb{R}^n}|K(x, y)|\frac{R_{m+1}(A;\, x, y)}{|x-y|^m}|f(y)|dy$$

where $K(x, y)$ is a function on $\mathbb{R}^n\times\mathbb{R}^n\backslash\{(x, y)\colon x\ne y\}$, m is a positive integer, $R_{m+1}(A; x, y) = A(x) - \sum_{|\alpha|\le m}\{\partial^\alpha A(y)/\alpha!\}(x-y)^\alpha$, and A has derivatives of order m in $\mathrm{BMO}(\mathbb{R}^n)$. Suppose that

(i) *for some $\gamma\ge 1$, T is bounded from $L^1(\log L)^\gamma(\mathbb{R}^n)$ to weak-$L^1(\mathbb{R}^n)$;*

(ii) *for any $0<r<\infty$ and $\beta_1<\beta<\beta_2$, the operator*

$$S_rf(x) = \int_{r<|x-y|\le 2r}\Phi(r^n|K(x, y)|)|f(y)|dy$$

is bounded on $L^1(\mathbb{R}^n, |x|^\beta dx)$ with bound Br^n and B is independent of r.

Then T is also bounded from $L^1(\log L)^\gamma(\mathbb{R}^n, |x|^\alpha dx)$ to weak-$L^1(\mathbb{R}^n, |x|^\alpha dx)$ provided that $\beta_1 < \alpha < \beta_2$.

3. Application

This section is devoted to an application of Theorem 1. We consider the sublinear operator which satisfies the size condition

$$|Tf(x)| \leq \int_{\mathbb{R}^n} |b(x) - b(y)|^m \frac{|\Omega(x-y)|}{|x-y|^n} |f(y)|dy, \tag{5}$$

where m is a positive integer and $b \in \mathrm{BMO}(\mathbb{R}^n)$, Ω is homogeneous of degree zero. Note that if $\Omega \in L^q(S^{n-1})$ for some $q > 1$, then $\Phi_m(|\Omega|) \in L^{q-\epsilon}(S^{n-1})$ for any positive integer m and ϵ with $0 < \epsilon < q$, and in this case, the operator

$$R_r f(x) = \frac{1}{r^n} \int_{r<|x-y|\leq 2r} \Phi_m(|\Omega(x-y)|)|f(y)|dy$$

is bounded on $L^1(\mathbb{R}^n, |x|^\alpha dx)$ with bound independent of r provided that $-1-(n-1)/q' < \alpha < 0$ (this can be proved, using the estimate

$$\left(\int_{S^{n-1}} \left(r^{-1} \int_r^{2r} |y - su|^\alpha ds\right)^{q'} d\sigma(u)\right)^{1/q'} \leq C|y|^\alpha,$$

cf. [1, page 874]). On the other hand, if $\Omega \in L(\log L)^m(S^{n-1})$, i.e.

$$\int_{S^{n-1}} |\Omega(x)| \log^m(2 + |\Omega(x)|)dx < \infty,$$

then the operator R_r is bounded on $L^1(\mathbb{R}^n, |x|^\alpha dx)$ provided that $-1 < \alpha \leq 0$ (this can be proved, using the estimate $r^{-1}\int_r^{2r} |y - su|^\alpha ds \leq C|y|^\alpha$, cf. [1, page 874]). Thus by Theorem 1, we have

Corollary 1 *Let T be a sublinear operator which satisfies the size condition (5). Suppose when $q > 1$, $\Omega \in L^q(S^{n-1})$; or when $q = 1$, $\Omega \in L(\log L)^m(S^{n-1})$. If T is bounded from $L^1(\log L)^m(\mathbb{R}^n)$ to weak-$L^1(\mathbb{R}^n)$ and $-1-(n-1)/q' < \alpha < 0$, then T is also bounded from $L^1(\log L)^m(\mathbb{R}^n, |x|^\alpha dx)$ to weak-$L^1(\mathbb{R}^n, |x|^\alpha dx)$.*

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