Hokkaido Mathematical Journal Vol. 34 (2005) p. 541–586

L^p-L^q estimate for wave equation with bounded time dependent coefficient

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(Received October 14, 2003; Revised September 1, 2004)

Abstract. The goal of the present paper is to derive L^p-L^q estimates for wave equations with a bounded time dependent coefficient. A classification of the oscillating behaviour of the coefficient is given. This classification determines in an essential way the possibility of deriving L^p-L^q decay estimates.

Key words: wave equation, oscillating time dependent coefficient, WKB-representation, Fourier multipliers, Floquet's theory.

1. Introduction and Main Results

Studying L^p-L^q decay estimates for solutions to hyperbolic problems goes back to [15], where, for the solution of

$$\left.\begin{aligned} &\partial_t^2 u(t, x) - \Delta_x u(t, x) = 0, \quad (t, x) \in [0, \infty) \times \mathbb{R}^n, \\ &u(0, x) = \varphi(x), \ \partial_t u(0, x) = \psi(x), \quad x \in \mathbb{R}^n, \ \varphi, \psi \in C_0^\infty, \end{aligned}\right\} \tag{1.1}$$

the following estimate was proved:

$$\|(u_t(t, \cdot), \nabla_x u(t, \cdot))\|_{L^q} \le C(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|(\nabla_x \varphi, \psi)\|_{W^{N_p, p}}.$$

Here $n \ge 2$, $1/p + 1/q = 1$, $1 < p \le 2$ and $N_p \ge n(1/p - 1/q)$. We shall use C throughout to denote a positive constant which may differ at each occurrence and $W^{k,p} = W^{k,p}(\mathbb{R}^n)$ to denote the standard Sobolev spaces.

More recently, the influence of a time-dependent coefficient on such decay estimates was studied in a series of papers [11], [12], [13]. A classification for decay estimates of solutions to the Cauchy problem

$$\left.\begin{aligned} &\partial_t^2 u(t, x) - b(t)^2 \lambda(t)^2 \Delta_x u(t, x) = 0, \ (t, x) \in [0, \infty) \times \mathbb{R}^n, \\ &u(0, x) = \varphi(x), \ \partial_t u(0, x) = \psi(x), \ x \in \mathbb{R}^n, \ \varphi, \psi \in C_0^\infty, \end{aligned}\right\} \tag{1.2}$$

is given, where $b(t)$ is a bounded function and $\lambda(t)$ is a strictly increasing function which satisfy, for some positive constants C_0, C_1, C, c, c_k ($k =$

2000 Mathematics Subject Classification : 35L05, 35L15.

0, 1, 2, ...),

$$
\begin{aligned}
&0 < C_0 \le b(t)^2 \le C_1, \quad \text{for large } t;\\
&\Lambda(t) := \int_0^t \lambda(s)ds \to \infty \quad \text{as } t \to \infty;\\
&c(\log \Lambda(t))^{-c} \le c_0 \frac{\lambda(t)}{\Lambda(t)} \le \frac{\lambda'(t)}{\lambda(t)} \le c_1 \frac{\lambda(t)}{\Lambda(t)} \le C(\log \Lambda(t))^C,\\
&\hfill \text{for large } t;\\
&|D_t^k \lambda(t)| \le c_k \lambda(t) \left(\frac{\lambda(t)}{\Lambda(t)}\right)^k, \quad \text{for } k = 2,\, 3,\, \dots \text{ and } t \text{ large.}
\end{aligned}
$$

This classification is based on the interplay between $b(t)$ and $\lambda(t)$, that is the so-called *speed of oscillations*; more precisely, the condition

$$
|D_t^k b(t)| \le C_{b,k} \left(\frac{\lambda(t)}{\Lambda(t)}\big(\log \Lambda(t)\big)^\beta\right)^k, \text{ for all } k \in \mathbb{N} \text{ and large } t, \tag{1.3}
$$

plays the fundamental rôle: if (1.3) holds for $\beta \in [0,\, 1]$ then estimates of the form

$$
\begin{aligned}
&\|u_t(t,\,\cdot)\|_{L^q} + \|\lambda(t)\nabla_x u(t,\,\cdot)\|_{L^q}\\
&\le C(1+\Lambda(t))^{-\big((n-1)/2\big)(1/p-1/q)+\beta_0}(\|\varphi\|_{W_p^{L+1}} + \|\psi\|_{W_p^L})
\end{aligned} \tag{1.4}
$$

hold for the solution $u = u(t,\, x)$ to (1.2) for some constant β_0 which depends on β; here $L = n(1/p-1/q)+1$, $1 < p < 2$ and $1/p+1/q = 1$. Furthermore, if (1.3) does not hold for $\beta = 1$ then no such estimate can be found: a counterexample is constructed in [11].

In [14] the Cauchy problem for general second order strictly hyperbolic operators with increasing time-dependent coefficients is studied. That is, the problem

$$
\begin{aligned}
&\partial_t^2 u(t,\, x) + \sum_{i=1}^n b_i(t)\partial_{x_i t}^2 u(t,\, x) - \sum_{i,j=1}^n a_{ij}(t)\partial_{x_i x_j}^2 u(t,\, x) = 0,\\
&u(0,\, x) = \varphi(x),\ u_t(0,\, x) = \psi(x),
\end{aligned}
$$

where the quadratic form $\sum_{i,j=1}^n a_{ij}(t)\xi_i\xi_j$ satisfies

$$
d_0\nu(t)^2|\xi|^2 \le \sum_{i,j=1}^n a_{ij}(t)\xi_i\xi_j \le d_1\nu(t)^2|\xi|^2,
$$

for some positive function $\nu \in C^\infty(0, \infty)$ and positive constants d_0, d_1. Also, the following conditions, which are analogous to those above for the operator in (1.2), are assumed to hold:

$$c_0 \frac{\nu(t)}{N(t)} \leq \frac{\nu'(t)}{\nu(t)} \leq c_1 \frac{\nu(t)}{N(t)} \text{ and}$$
$$|D_t^k \nu(t)| \leq c_k \left(\frac{\nu(t)}{N(t)}\right)^k \nu(t) \text{ for } k = 2,\, 3,\, \ldots \text{ and } t \text{ large,}$$
$$\text{where } N(t) := \int_0^t \nu(s)ds \to \infty \text{ as } t \to \infty.$$

For this problem, only the case which corresponds to that of $\beta = 0$ in (1.3) is studied; that is, if the following conditions are assumed for the coefficients for $\xi \in \mathbb{R}^n$, large t, and each $k = 0, 1, 2, \ldots$:

$$\left| D_t^k \sum_{i=1}^n b_i(t)\xi_i \right| \leq C_k \nu(t) \left(\frac{\nu(t)}{N(t)}\right)^k |\xi|,$$
$$\left| D_t^k \sum_{i,j=1}^n a_{ij}(t)\xi_i\xi_j \right| \leq C_k \nu(t)^2 \left(\frac{\nu(t)}{N(t)}\right)^k |\xi|^2,$$
$$\text{stabilization conditions: } \lim_{t\to\infty} \frac{b_i(t)}{\nu(t)},\ \lim_{t\to\infty} \frac{a_{ij}(t)}{\nu(t)^2} \text{ exist,}$$

then an estimate of the form (1.4) with a, in general, nonnegative β_0 holds. However, in contrast to the problem (1.2), no classification involving the "log-effect" (i.e. an analogue to the condition (1.3) for $\beta \in (0, 1)$) is currently known. A detailed representation of all these results with proofs can be found in [10].

In this paper, we study the limiting case of (1.2) where $\lambda(t) \equiv 1$, which is not covered by the above results. For this limiting case we will give a more precise classification of oscillations and describe the corresponding more precise classification of decay estimates. This case under consideration corresponds in some sense (see Remark 4.5) to the case of strictly hyperbolic equations with non-regular coefficients, a topic which has developed in the last few years in an astonishing way (see [8] and references therein).

Indeed, we consider the following Cauchy problem for $u = u(t, x)$:

$$\left.\begin{aligned} &\partial_t^2 u - a(t)\Delta u = 0, \quad (t,\, x) \in [0,\, \infty) \times \mathbb{R}^n, \\ &u(0,\, x) = \varphi(x),\ \partial_t u(0,\, x) = \psi(x),\ \varphi,\, \psi \in C_0^\infty(\mathbb{R}^n),\, x \in \mathbb{R}^n, \end{aligned}\right\} \tag{1.5}$$

where $a = a(t)$ is a bounded, smooth function which satisfies $a(t) \geq C > 0$ for all $t \geq 0$, so the equation from (1.5) is strictly hyperbolic.

Definition 1.1 Classification of Oscillations
Let $a = a(t)$ be a smooth function satisfying

$$|D_t^k a(t)| \leq C_k \left(\frac{1}{t+e^3} \Big(\log(t+e^3) \Big)^{\gamma} \right)^k, \quad k \in \mathbb{N}. \tag{1.6}$$

The parameter γ controls the oscillations of a. We say that the oscillations of a are **very slow, slow** or **fast** if $\gamma = 0$, $0 < \gamma < 1$ or $\gamma = 1$ respectively. If (1.6) is not satisfied for $\gamma = 1$, then we say a has **very fast oscillations**.

We show that if we have very slow, slow or fast oscillations, then L^p-L^q decay estimates can be proved for the solutions of (1.5):

Theorem 1.1 *Consider the strictly hyperbolic Cauchy problem* (1.5) *where the coefficient* $a = a(t)$ *satisfies* (1.6) *with* $\gamma \in [0, 1]$. *Then there exists a constant* C *such that the following* L^p-L^q *estimate holds for the solution* $u = u(t, x)$:

$$\|(u_t(t, \cdot), \nabla_x u(t, \cdot))\|_{L^q} \leq C(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)+s_0} \|(\nabla_x \varphi, \psi)\|_{W^{N_p,p}},$$

where $1/p + 1/q = 1$, $1 < p \leq 2$, $N_p \geq n(1/p - 1/q)$ *and*

- $s_0 = 0$ *if* $\gamma = 0$; *in this case* C *only depends on* p, n;
- $s_0 = \varepsilon$ *if* $\gamma \in (0, 1)$ *for all* $\varepsilon > 0$; *in this case* C *depends on* p, n *and* ε;
- s_0 *is a fixed constant* (*which can be determined*) *if* $\gamma = 1$; *in this case* C *is independent of* φ, ψ.

Remark 1.1 We may interpret the non-negative constant s_0 as "loss of derivatives", that is, a number which explains how the decay rate differs from the classical Strichartz' decay rate $-\{(n-1)/2\}(1/p - 1/q)$. If the oscillations are very slow, slow, fast, then no loss, arbitrary small loss, finite loss of derivatives, respectively, appears. The same influence of such oscillations on a loss of derivatives which really exists can be proved, e.g. in [2]. Thus, the questions of H^s *well-posedness for strictly hyperbolic Cauchy problems of type* (1.5) and L^p-L^q *decay estimates for solutions of a strictly hyperbolic wave equation with a bounded coefficient* correspond to each other from the point of view of the theory of degenerate hyperbolic problems [7].

Example Let us consider the Cauchy problem

$$\left.\begin{aligned} &\partial_t^2 u - (2+\sin(2\pi(\log(t+e^3))^\alpha))\Delta u = 0, \\ &u(0,\,x) = \varphi(x),\ \partial_t u(0,\,x) = \psi(x). \end{aligned}\right\} \tag{1.7}$$

The coefficient is smooth and bounded. The oscillations are very slow, slow, fast if $\alpha = 1$, $\alpha \in (1,\,2)$, $\alpha = 2$, respectively. Consequently, Theorem 1.1 can be applied to (1.7).

If $\alpha > 2$, then the oscillations in (1.7) are very fast. In Section 3 we show that, from the point of view of L^p-L^q decay estimates, the behaviour of solutions for (1.7) changes in a rigorous way from $\alpha = 2$ to $\alpha > 2$. The main point is to understand how to describe such a change of behaviour. For the moment, we restrict ourselves to formulating the main result.

Theorem 1.2 *Let us consider the Cauchy problem*

$$\left.\begin{aligned} &\partial_t^2 u - (2+\sin(2\pi(\log(t+e^3))^\alpha))^2\Delta u = 0, \\ &u(t_0,\,x) = \varphi(x),\ \partial_t u(t_0,\,x) = \psi(x), \end{aligned}\right\} \tag{1.8}$$

with $\alpha > 2$. There do not exist constants p, q, M, C_1, C_2 such that, for all initial times t_0 and for all initial data φ, $\psi \in C_0^\infty(\mathbb{R}^n)$, the following L^p-L^q estimate holds for all $t \geq t_0$:

$$E(u)(t)\,\big|_{L^q} \leq C_1 \exp(C_2(\log(t+e^3))^r)E(u)(t_0)\,\big|_{W^{M,p}}, \tag{1.9}$$

where $r < \alpha - 1$. Here the (non-standard) energy $E(u)(t)\big|_{W^{M,p}}$ is defined by

$$E(u)(t)\big|_{W^{M,p}} := \|\sigma(t)\nabla_x u(t,\,\cdot)\|_{W^{M,p}} + \left\|\frac{1}{\sigma(t)^2}\partial_t(u(t,\,\cdot)\sigma(t))\right\|_{W^{M,p}}$$

with $\sigma(t) := \sqrt{\{\alpha(\log(t+e^3))^{\alpha-1}\}/(t+e^3)}$.

Remark 1.2 The heart of the proof of Theorem 1.2 is the use of Floquet's theory which is applied to, amongst other things, Hill's equation $w_{tt} + \lambda b(t)^2 w = 0$ (see [5]). The function $b = b(t)$ is periodic; λ is a constant. In the proof we show that there is a relation between the equation from (1.8) and Hill's equation. For this reason we use the square in the coefficient of (1.8).

2. Proof of Theorem 1.1

In order to prove this, we first derive a WKB representation for the solution to the auxiliary problem

$$\partial_t^2 v + a(t)|\xi|^2 v = 0,\ v(0,\,\xi) = \widehat{\varphi}(\xi),\ \partial_t v(0,\,\xi) = \widehat{\psi}(\xi), \tag{2.1}$$

which is obtained from (1.5) by partial Fourier transformation with respect to x. Then we use standard techniques from the theory of Fourier multipliers to obtain L^p-L^q estimates.

2.1. WKB representation of solution

2.1.1. Division of phase space into zones To find a WKB representation for the solution of (2.1) we divide the phase space $[0,\,\infty) \times \mathbb{R}^n_\xi$ into two zones, the *hyperbolic zone* and the *pseudodifferential zone*, denoted $Z_{\mathrm{hyp}}(N)$, $Z_{\mathrm{pd}}(N)$ respectively. These enable us to use the hyperbolicity of our starting problem (1.5) and tools from hyperbolic theory only in the hyperbolic zone.

Definition 2.1 Several Zones
For a given $N > 0$, define the zones $Z_{\mathrm{hyp}}(N)$ and $Z_{\mathrm{pd}}(N)$ of the phase space $[0,\,\infty) \times \mathbb{R}^n$ by

$$Z_{\mathrm{hyp}}(N) := \{(t,\,\xi) \in [0,\,\infty) \times \mathbb{R}^n \colon |\xi|(t+e^3) \geq N(\log(t+e^3))^\gamma\},$$
$$Z_{\mathrm{pd}}(N) := \{(t,\,\xi) \in [0,\,\infty) \times \mathbb{R}^n \colon |\xi|(t+e^3) \leq N(\log(t+e^3))^\gamma\}.$$

Here γ is the parameter from (1.6).

We shall denote the line that separates these zones by $t_\xi = t(|\xi|)$ which is defined for $\{\xi \colon |\xi| \leq p_0\}$, $p_0 := Ne^{-3}3^\gamma$, implicitly by the formula

$$|\xi|(t_\xi + e^3) = N(\log(t_\xi + e^3))^\gamma.$$

Lemma 2.1 *For t_ξ as defined above we have, for all multi-indices α with $|\alpha| \geq 1$, the inequality*

$$|\partial_\xi^\alpha t_\xi| \leq C_{\alpha,N}|\xi|^{-1-|\alpha|}(\log(t_\xi + e^3))^\gamma.$$

We also subdivide $Z_{\mathrm{hyp}}(N)$ into two smaller zones, the *oscillations subzone* $Z_{\mathrm{osc}}(N)$ and the *regular subzone* $Z_{\mathrm{reg}}(N)$.

Definition 2.2 Several Subzones

For a given $N > 0$ define the subzones $Z_{\mathrm{osc}}(N)$ and $Z_{\mathrm{reg}}(N)$ of $Z_{\mathrm{hyp}}(N)$ by

$$Z_{\mathrm{osc}}(N) := \left\{ (t,\, \xi) \colon \begin{array}{r} N(\log(t+e^3))^{\gamma} \le |\xi|(t+e^3) \\ \le 2N(\log(t+e^3))^{2\gamma} \end{array} \right\},$$

$$Z_{\mathrm{reg}}(N) := \{(t,\, \xi) \colon |\xi|(t+e^3) \ge 2N(\log(t+e^3))^{2\gamma}\}.$$

We denote the separating line by $\tilde{t}_\xi = \tilde{t}(|\xi|)$ which is defined for $\{\xi \colon |\xi| \le p_1\}$, $p_1 := 2Ne^{-3}3^{2\gamma}$, implicitly by the formula

$$|\xi|(\tilde{t}_\xi + e^3) = 2N(\log(\tilde{t}_\xi + e^3))^{2\gamma}.$$

Lemma 2.2 *For $\tilde{t}_\xi$ as defined above we have, for all multi-indices α with $|\alpha| \ge 1$, the inequalities*

$$|\partial_\xi^\alpha \tilde{t}_\xi| \le C_{\alpha,N} |\xi|^{-1-|\alpha|} (\log(\tilde{t}_\xi + e^3))^{2\gamma}.$$

2.1.2. Representation in the pseudodifferential zone In $Z_{\mathrm{pd}}(N)$ it is straightforward to get a representation for the solution; observe that (2.1) can be written as the first order system

$$D_t U = \begin{pmatrix} 0 & |\xi| \\ a(t)|\xi| & 0 \end{pmatrix} U, \quad U(0,\, \xi) = U_0(\xi) := \begin{pmatrix} |\xi|\widehat{\varphi}(\xi) \\ \widehat{\psi}(\xi) \end{pmatrix},$$

where $D_t = (1/\sqrt{-1})\partial_t$ and

$$U = U(t,\, \xi) := \begin{pmatrix} |\xi| v(t,\, \xi) \\ D_t v(t,\, \xi) \end{pmatrix}.$$

Hence, we can write $U(t,\, \xi) = E(t,\, 0,\, \xi)U_0(\xi)$ where $E = E(t,\, s,\, \xi)$, $0 \le s \le t$, solves

$$D_t E = \begin{pmatrix} 0 & |\xi| \\ a(t)|\xi| & 0 \end{pmatrix} E, \quad E(s,\, s,\, \xi) = I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Naturally, this can be written as an infinite sum via the matrizant representation:

$$E(t,\, s,\, \xi) = I + \sum_{j=1}^{\infty} \int_s^t A(t_1,\, \xi) \int_s^{t_1} A(t_2,\, \xi) \cdots \int_s^{t_{j-1}} A(t_j,\, \xi)\, dt_j \cdots dt_1, \quad (2.2)$$

where $A(t, \xi) = \sqrt{-1}\begin{pmatrix} 0 & |\xi| \\ a(t)|\xi| & 0 \end{pmatrix}$. Observing that $\|A(t, \xi)\| \leq Ca(t)|\xi|$, we see

$$\|E(t, s, \xi)\| \leq \exp\left(\int_s^t \|A(r, \xi)\| dr\right) \leq e^{C_a N (\log(t+e^3))^\gamma},$$

with $C_a := C \sup_t a(t)$, for $0 \leq s \leq t \leq t_\xi$. Later we need, for special representations of solutions to (2.1) in subzones of the phase space, the behaviour of $\|\partial_\xi^\alpha E(t_\xi, 0, \xi)\|$.

Lemma 2.3 *The following estimates hold for all multi-indices* α:

$$\|\partial_\xi^\alpha E(t_\xi, 0, \xi)\| \leq C_{\alpha,N} |\xi|^{-|\alpha|} (\log(t_\xi + e^3))^{|\alpha|\gamma} e^{C_a N (\log(t_\xi + e^3))^\gamma}.$$

Proof. The proof follows from the representation (2.2) and from the statement of Lemma 2.1. □

Summarising all the above information we have

Proposition 2.4 *For* $0 \leq t \leq t_\xi$ *the following representation holds*:

$$|\xi| v(t, \xi) = E_{11}(t, 0, \xi)|\xi|\widehat{\varphi}(\xi) + E_{12}(t, 0, \xi)\widehat{\psi}(\xi),$$
$$D_t v(t, \xi) = E_{21}(t, 0, \xi)|\xi|\widehat{\varphi}(\xi) + E_{22}(t, 0, \xi)\widehat{\psi}(\xi),$$

and

$$|\partial_\xi^\alpha E_{kl}(t_\xi, 0, \xi)| \leq C_{\alpha,N} |\xi|^{-|\alpha|} (\log(t_\xi + e^3))^{|\alpha|\gamma} e^{C_a N (\log(t_\xi + e^3))^\gamma},$$

for each multi-index α, *for all* $0 \leq t \leq t_\xi$ *and for all* $k, l = 1, 2$.

2.1.3. Symbol classes in the hyperbolic zone The hyperbolic zone $Z_{\mathrm{hyp}}(N)$ consists of two parts $Z_{\mathrm{hyp}}^{(1)}(N) := \{(t, \xi) \in [t_\xi, \infty) \times \{\xi \colon |\xi| \leq p_0\}\}$ and $Z_{\mathrm{hyp}}^{(2)}(N) := \{(t, \xi) \in [0, \infty) \times \{\xi \colon |\xi| \geq p_0\}\}$. In what follows, we shall restrict our considerations to $Z_{\mathrm{hyp}}^{(1)}(N)$. In order to give a representation for the solution to (2.1) in $Z_{\mathrm{hyp}}(N)$, we carry out a diagonalisation procedure with suitable remainder at each step. The following definition of symbol classes exactly characterises the necessary properties of the remainders.

Definition 2.3 For each $m_1 \in \mathbb{R}$ and $m_2, N \geq 0$ we define $S_N\{m_1, m_2\}$ to be the set of functions $\sigma = \sigma(t, \xi) \in C^\infty(Z_{\mathrm{hyp}}^{(1)}(N))$ such that for all $(t, \xi) \in Z_{\mathrm{hyp}}^{(1)}(N)$, multi-indices α and $k \in \mathbb{N}$

$$|D_t^k D_\xi^\alpha \sigma(t,\,\xi)| \le C_{k,\alpha}|\xi|^{m_1-|\alpha|}\left(\frac{1}{t+e^3}\big(\log(t+e^3)\big)^{\gamma}\right)^{m_2+k},$$

with nonnegative constants $C_{k,\alpha}$ depending only on k and α.

Lemma 2.5 *The classes $S_N\{m_1,\,m_2\}$ have the following properties*:

i) *if $\sigma \in S_N\{m_1,\,m_2\}$ then $D_\xi^\alpha\sigma \in S_N\{m_1-|\alpha|,\,m_2\}$ and $D_t^k\sigma \in S_N\{m_1,\,m_2+k\}$;*

ii) *if $\sigma_1 \in S_N\{m_1,\,m_2\}$, $\sigma_2 \in S_N\{p_1,\,p_2\}$ then $\sigma_1\sigma_2 \in S_N\{m_1+p_1,\,m_2+p_2\}$;*

iii) *for all $r \ge 0$ we have $S_N\{m_1,\,m_2\} \subset S_N\{m_1+r,\,m_2-r\}$.*

Proof. Properties i) and ii) are clear by the definition of $S_N\{m_1,\,m_2\}$. To show iii), simply observe that, by the definition of $Z_{\rm hyp}(N)$,

$$\frac{(\log(t+e^3))^{\gamma}}{|\xi|(t+e^3)} \le \frac{1}{N}.$$

Hence, if $\sigma \in S_N\{m_1,\,m_2\}$ then

$$\begin{aligned}|D_t^k D_\xi^\alpha \sigma(t,\,\xi)| &\le C_{k,\alpha}|\xi|^{m_1+r-|\alpha|}\left(\frac{1}{t+e^3}(\log(t+e^3))^{\gamma}\right)^{m_2-r+k}\\ &\qquad\times\frac{(\log(t+e^3))^{\gamma r}}{(|\xi|(t+e^3))^{r}}\\ &\le C_{k,\alpha}N^{-r}|\xi|^{m_1+r-|\alpha|}\left(\frac{1}{t+e^3}(\log(t+e^3))^{\gamma}\right)^{m_2-r+k},\end{aligned}$$

so $\sigma \in S_N\{m_1+r,\,m_2-r\}$. □

2.1.4. Diagonalisation modulo $S_N\{0,1\}$ The equation from (2.1) is equivalent to the first order system

$$D_tV = \begin{pmatrix} 0 & \sqrt{a(t)}|\xi| \\ \sqrt{a(t)}|\xi| & 0\end{pmatrix}V + \frac{D_ta(t)}{2a(t)}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \tag{2.3}$$

where

$$V = V(t,\,\xi) = \begin{pmatrix}\sqrt{a(t)}|\xi|v(t,\,\xi)\\ D_tv(t,\,\xi)\end{pmatrix} \quad \text{for } t \ge t_\xi.$$

Note that the leading matrix is in $S_N\{1,\,0\}$, and the remainder lies in $S_N\{0,\,1\}$. In order to have a useful representation for the solution to (2.1) in $Z_{\rm hyp}^{(1)}(N)$, we must diagonalise this system. Since the eigenvalues of the first matrix are $\tau_1 = \tau_1(\xi) = -\sqrt{a(t)}|\xi|$ and $\tau_2 = \tau_2(\xi) = \sqrt{a(t)}|\xi|$, it is simple to show that

$$M \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} M^{-1} = \begin{pmatrix} 0 & \sqrt{a(t)}|\xi| \\ \sqrt{a(t)}|\xi| & 0 \end{pmatrix},$$

where

$$M = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad \text{and } M^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Setting $V_0 = V_0(t, \xi) := M^{-1}V(t, \xi)$, we obtain the following system for V_0:

$$\begin{aligned} D_t V_0 = &\begin{pmatrix} \tau_1(\xi) + D_t a(t)/4a(t) & 0 \\ 0 & \tau_2(\xi) + D_t a(t)/4a(t) \end{pmatrix} V_0 \\ &+ \frac{D_t a(t)}{4a(t)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} V_0. \end{aligned} \tag{2.4}$$

We shall use the notation:

$$\begin{aligned} \mathcal{D} &:= \begin{pmatrix} \tau_1(\xi) + D_t a(t)/4a(t) & 0 \\ 0 & \tau_2(\xi) + D_t a(t)/4a(t) \end{pmatrix}, \\ \mathcal{R}_0 &:= \frac{D_t a(t)}{4a(t)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{2.5}$$

The system (2.4) has diagonal leading part $\mathcal{D} \in S_N\{1, 0\}$ with remainder $\mathcal{R}_0 \in S_N\{0, 1\}$. Thus, we have obtained in $Z_{\text{hyp}}^{(1)}(N)$ the *diagonalisation of the system* (2.3) *modulo remainder* $\mathcal{R}_0 \in S_N\{0, 1\}$.

2.1.5. Further considerations in the oscillations subzone: Diagonalisation modulo $S_N\{-1, 2\}$ The oscillations subzone $Z_{\text{osc}}(N)$ consists of two parts

$$Z_{\text{osc}}^{(1)}(N) := \{(t, \xi) \in [t_\xi, \tilde{t}_\xi] \times \{\xi \colon |\xi| \le p_0\}\}$$

and

$$Z_{\text{osc}}^{(2)}(N) := \{(t, \xi) \in [0, \tilde{t}_\xi] \times \{\xi \colon p_0 \le |\xi| \le p_1\}\}.$$

In what follows, we restrict our considerations to $Z_{\text{osc}}^{(1)}(N)$ if we have in mind the oscillations subzone. In $Z_{\text{hyp}}^{(1)}(N)$ we carry out one more step of the diagonalisation procedure. Let

$$\begin{aligned}\mathcal{N}^{(1)} &= \mathcal{N}^{(1)}(t,\,\xi)\\ &:= -\frac{D_t a(t)}{4a(t)}\begin{pmatrix}0 & 1/(\tau_1-\tau_2)\\ 1/(\tau_2-\tau_1) & 0\end{pmatrix} \in S_N\{-1,\,1\}.\end{aligned}$$

Now set $N_1 = N_1(t,\,\xi) := I + \mathcal{N}^{(1)}$; this is invertible since

$$\|\mathcal{N}^{(1)}\| \le C\frac{(\log(t+e^3))^\gamma}{|\xi|(t+e^3)} \le C/N \quad \text{by definition of } Z_{\mathrm{hyp}}(N),$$

and we choose N in the definition of $Z_{\mathrm{hyp}}(N)$, $Z_{\mathrm{pd}}(N)$ large enough so that $C/N < 1/2$ here. Let $V_1 = V_1(t,\,\xi) := N_1^{-1}M^{-1}V$; then we obtain the following equivalent problem to (2.3) for V_1 for $t \ge t_\xi$:

$$\begin{aligned}&(D_t - \mathcal{D} - \mathcal{R}_1)V_1 = 0,\\ &V_1(t_\xi,\,\xi) = V_{1,0}(\xi) := N_1^{-1}(t_\xi,\,\xi)M^{-1}V(t_\xi,\,\xi),\end{aligned} \tag{2.6}$$

where $\mathcal{R}_1 \in S_N\{-1,\,2\}$. This is a consequence of

Lemma 2.6 *Let* $\mathcal{R}_1 := -N_1^{-1}(D_t\mathcal{N}^{(1)} - \mathcal{R}_0\mathcal{N}^{(1)})$. *Then in* $Z_{\mathrm{hyp}}^{(1)}(N)$ *the following identity holds*:

$$(D_t - \mathcal{D} - \mathcal{R}_0)N_1 = N_1(D_t - \mathcal{D} - \mathcal{R}_1),$$

where $\mathcal{D}$, $\mathcal{R}_0$ *are as in* (2.5).

Proof. Follows immediately from the observation that $[\mathcal{N}^{(1)},\,\mathcal{D}] = \mathcal{R}_0$. □

Thus, we have obtained in $Z_{\mathrm{hyp}}^{(1)}(N)$ the *diagonalisation of the system* (2.3) *modulo remainder* $\mathcal{R}_1 \in S_N\{-1, 2\}$.

2.1.6. Further considerations in regular subzone: Diagonalisation modulo $S_N\{-m, m+1\}$ In $Z_{\mathrm{osc}}^{(1)}(N)$ there is no point in carrying out any more steps of diagonalisation since there is no useful improvement of regularity between amplitudes from the classes $S_N\{-(l-1),\,l\}$ and $S_N\{-l,\,l+1\}$ when $l \ge 2$. However, we get such an improvement of regularity in the regular subzone $Z_{\mathrm{reg}}(N)$. There we carry out m steps of the diagonalisation procedure. The number m is chosen and motivated in Section 2.2. The step depends on the next result, which generalises Lemma 2.6.

Proposition 2.7 *For each* $m \in \mathbb{N}$ *there exist matrix-valued functions*

$$N_m = N_m(t,\,\xi) \in S_N\{0,\,0\}, \quad F_m = F_m(t,\,\xi) \in S_N\{-1,\,2\},$$
and $\mathcal{R}_m = \mathcal{R}_m(t,\,\xi) \in S_N\{-m,\,m+1\}$

such that F_m is diagonal, N_m is invertible and its inverse also lies in $S_N\{0, 0\}$, and the following identity holds in $Z^{(1)}_{\mathrm{hyp}}(N)$:

$$(D_t - \mathcal{D} - \mathcal{R}_0)N_m = N_m(D_t - \mathcal{D} + F_m - \mathcal{R}_m),$$

where $\mathcal{D}$, $\mathcal{R}_0$ are as in (2.5). *Also, for N (in the definition of $Z_{\mathrm{hyp}}(N)$) chosen large enough, N_m is invertible and its inverse also lies in $S_N\{0, 0\}$.*

Proof. We seek representations for N_m, F_m in the form

$$N_m = \sum_{r=0}^{m} \mathcal{N}^{(r)}, \quad F_m = \sum_{r=0}^{m-1} \mathcal{F}^{(r)}.$$

To do this, define inductively the matrix-valued functions $B^{(r)} = \begin{pmatrix} B^{(r)}_{11} & B^{(r)}_{12} \\ B^{(r)}_{21} & B^{(r)}_{22} \end{pmatrix}$, $\mathcal{N}^{(r)}$ and $\mathcal{F}^{(r)}$ in the manner below:

$$\begin{aligned} &\mathcal{N}^{(0)} := I,\ B^{(0)} := \mathcal{R}_0,\ \mathcal{F}^{(r)} := \operatorname{diag} B^{(r)}, \\ &\mathcal{N}^{(r+1)} := \begin{pmatrix} 0 & B^{(r)}_{12}/(\tau_1 - \tau_2) \\ B^{(r)}_{21}/(\tau_2 - \tau_1) & 0 \end{pmatrix}, \\ &B^{(r+1)} := (D_1 - \mathcal{D} - \mathcal{R}_0)\Bigg(\sum_{\rho=0}^{r+1} \mathcal{N}^{(\rho)}\Bigg) \\ &\qquad\qquad - \Bigg(\sum_{\rho=0}^{r+1} \mathcal{N}^{(\rho)}\Bigg)\Bigg(D_t - \mathcal{D} + \sum_{\rho=0}^{r} \mathcal{F}^{(\rho)}\Bigg). \end{aligned}$$

We claim that $\mathcal{N}^{(r)} \in S_N\{-r, r\}$ and $B^{(r)} \in S_N\{-r, r+1\}$. For $r = 1$ this is clear from Lemma 2.6. Assume it holds for $r = k$; then, observing that $\tau_2 - \tau_1 = 2\sqrt{a(t)}|\xi| \in S_N\{1, 0\}$ and noting by the induction hypothesis that $B^{(k)}_{12}$, $B^{(k)}_{21} \in S_N\{-k, k+1\}$, we see that $\mathcal{N}^{(k+1)} \in S_N\{-(k+1), k+1\}$. Also,

$$\begin{aligned} B^{(k+1)} &= B^{(k)} + (D_t - D - \mathcal{R}_0)\mathcal{N}^{(k+1)} - \sum_{\rho=0}^{k} \mathcal{N}^{(\rho)}\mathcal{F}^{(k)} \\ &\quad - \mathcal{N}^{(k+1)}\Bigg(D_t - \mathcal{D} + \sum_{\rho=0}^{k} \mathcal{F}^{(\rho)}\Bigg) \\ &= B^{(k)} - \mathcal{F}^{(k)} - [\mathcal{D}, \mathcal{N}^{(k+1)}] + \mathcal{S}, \end{aligned}$$

where $\mathcal{S} = D_t\mathcal{N}^{(k+1)} - \mathcal{R}_0\mathcal{N}^{(k+1)} - \mathcal{N}^{(k+1)}\sum_{\rho=0}^{k}\mathcal{F}^{(\rho)} - \sum_{\rho=1}^{k}\mathcal{N}^{(\rho)}\mathcal{F}^{(k)}$, which lies in $S_N\{-(k+1),\, k+2\}$ by the induction hypothesis and the rules of the symbolic calculus of Lemma 2.5. Furthermore, by definition of $\mathcal{F}^{(k)}$ and $\mathcal{N}^{(k+1)}$,

$$B^{(k)} - \mathcal{F}^{(k)} - [\mathcal{D},\, \mathcal{N}^{(k+1)}] = 0.$$

Therefore, $B^{(k+1)} = \mathcal{S}$ proving the induction step. So the claim is proved. Now we claim that $N_m := \sum_{r=0}^{m}\mathcal{N}^{(r)}$ is invertible; this is true because

$$\|\mathcal{N}^{(r)}\| \le C_r\left(\frac{\big(\log(1/(t+e^3))\big)^{\gamma}}{(t+e^3)|\xi|}\right)^r \le \frac{C_r}{N^r},$$

by the definition of $Z_{\mathrm{hyp}}(N)$. Choose N in the definition of $Z_{\mathrm{hyp}}(N)$ so that

$$\frac{C_r}{N^r} \le \frac{1}{2^{r+1}} \quad \text{for } r = 1,\, \ldots,\, m.$$

The value of m shall be chosen later, but since it is fixed, this fixes N. Hence,

$$\|N_m - I\| \le \sum_{r=1}^{m}\|\mathcal{N}^{(r)}\| \le \sum_{r=1}^{m}\frac{1}{2^{r+1}} < \frac{1}{2},$$

thus proving the invertibility of N_m. Finally, noting that $\mathcal{F}^{(0)} = 0$, so $\mathcal{F}^{(m)} \in S_N\{-1,\, 2\}$, and setting $R_m := -N_m^{-1}B^{(m)} \in S_N\{-m,\, m+1\}$ completes the proof of the proposition. □

The regular subzone $Z_{\mathrm{reg}}(N)$ consists of three parts $Z_{\mathrm{reg}}^{(1)}(N) := \{(t,\,\xi) \in [\tilde{t}_\xi,\, \infty) \times \{\xi\colon |\xi| \le p_0\}\}$, $Z_{\mathrm{reg}}^{(2)}(N) := \{(t,\,\xi) \in [\tilde{t}_\xi,\, \infty) \times \{\xi\colon p_0 \le |\xi| \le p_1\}\}$ and $Z_{\mathrm{reg}}^{(3)}(N) := \{(t,\,\xi) \in [0,\, \infty) \times \{\xi\colon |\xi| \ge p_1\}\}$. In what follows we restrict our considerations to $Z_{\mathrm{reg}}^{(1)}(N)$ if we have in mind the regular subzone. Now we set $V_m = V_m(t,\,\xi) := N_m^{-1}V_0$ for $t \ge \tilde{t}_\xi$ and see that the system (2.4) for V_0 is equivalent in $Z_{\mathrm{reg}}^{(1)}(N)$ to

$$\begin{aligned} &(D_t - \mathcal{D} + F_m - \mathcal{R}_m)V_m = 0,\\ &V_m(\tilde{t}_\xi,\, \xi) = V_{m,\,0}(\xi) := N_m^{-1}(\tilde{t}_\xi,\, \xi)N_1(\tilde{t}_\xi,\, \xi)V_1(\tilde{t}_\xi,\, \xi). \end{aligned} \tag{2.7}$$

Thus, we have obtained in $Z_{\mathrm{reg}}^{(1)}(N)$ the *diagonalisation of the system* (2.3) *modulo remainder* $\mathcal{R}_m \in S_N\{-m,\, m+1\}$, while in $Z_{\mathrm{osc}}^{(1)}(N)$ it is sufficient to carry out the *diagonalisation of the system* (2.3) *modulo remainder* $\mathcal{R}_1 \in$

$S_N\{-1,\, 2\}$.

2.1.7. Fundamental solutions and their properties Now let us construct representations for the fundamental solutions to the matrix-valued operators appearing in (2.6) and (2.7). First, let $E_2 = E_2(t,\, s,\, \xi)$ solve

$$D_t E_2 - \mathcal{D} E_2 = 0, \quad E_2(s,\, s,\, \xi) = I,$$

where $t,\, s \geq t_\xi$ and the matrix $\mathcal{D}$ is as in (2.5). We see that

$$E_2(t,\, s,\, \xi) = \left(\frac{a(t)}{a(s)}\right)^{1/4} \begin{pmatrix} e^{i\int_s^t \tau_1(r,\xi)dr} & 0 \\ 0 & e^{i\int_s^t \tau_2(r,\xi)dr} \end{pmatrix}.$$

Hence, by the strict hyperbolicity of our starting Cauchy problem (1.5) we get

$$\|E_2(t, s, \xi)\| \leq \left(\frac{a(t)}{a(s)}\right)^{1/4} \leq C_a \quad \text{for all } t,\, s \geq t_\xi. \tag{2.8}$$

Later we need, for special representations of solutions to (2.1) in $Z_{\mathrm{osc}}^{(1)}(N)$ and in $Z_{\mathrm{reg}}^{(1)}(N)$, the behaviour of $\|\partial_\xi^\alpha E_2(t_\xi,\, 0,\, \xi)\|$ and of $\|\partial_\xi^\alpha E_2(\tilde{t}_\xi,\, t_\xi,\, \xi)\|$.

Lemma 2.8 *The following estimates hold for all multi-indices α:*

$$\|\partial_\xi^\alpha E_2(t_\xi,\, 0,\, \xi)\| \leq C_{\alpha,N} |\xi|^{-|\alpha|} (\log(t_\xi + e^3))^{|\alpha|\gamma},$$
$$\|\partial_\xi^\alpha E_2(\tilde{t}_\xi,\, t_\xi,\, \xi)\| \leq C_{\alpha,N} |\xi|^{-|\alpha|} (\log(\tilde{t}_\xi + e^3))^{2|\alpha|\gamma}.$$

Proof. Follows immediately from Lemmas 2.1 and 2.2 together with assumption (1.6) and estimate (2.8). □

Now we define $E_{\mathrm{osc}} = E_{\mathrm{osc}}(t,\, s,\, \xi)$, $t_\xi \leq s \leq t \leq \tilde{t}_\xi$ to be the fundamental solution to (2.6) in $Z_{\mathrm{osc}}^{(1)}(N)$. This can be written in the form $E_{\mathrm{osc}} = E_2(t,\, s,\, \xi) Q_1(t,\, s,\, \xi)$ where $Q_1 = Q_1(t,\, s,\, \xi)$ solves

$$D_t Q_1 = E_2(s,\, t,\, \xi) \mathcal{R}_1(t,\, \xi) E_2(t,\, s,\, \xi) Q_1, \quad Q_1(s,\, s,\, \xi) = I.$$

Letting $P_1 = P_1(t,\, s,\, \xi) := \sqrt{-1} E_2(s,\, t,\, \xi) \mathcal{R}_1(t,\, \xi) E_2(t,\, s,\, \xi)$, we have the matrizant representation for Q_1:

$$Q_1(t,\, s,\, \xi) = I + \sum_{j=1}^{\infty} \int_s^t P_1(t_1,\, s,\, \xi) \cdots \int_s^{t_{j-1}} P_1(t_j,\, s,\, \xi) dt_j \cdots dt_1.$$

Now $\mathcal{R}_1 \in S_N\{-1, 2\}$; therefore, using (2.8), we see that

$$\|P_1(t, s, \xi)\| \le C|\xi|^{-1}\left(\frac{(\log(t+e^3))^\gamma}{t+e^3}\right)^2.$$

Hence,

$$\|Q_1(t, s, \xi)\| \le \exp\left(\int_s^t \|P_1(r, s, \xi)\|\, dr\right) \le e^{C_{N,\gamma}(\log(t_\xi+e^3))^\gamma}$$

for all $t_\xi \le s,\, t \le \tilde{t}_\xi$.

We need, for a special representation of solution to (2.1) in $Z^{(1)}_{\mathrm{osc}}(N)$ and in $Z^{(1)}_{\mathrm{reg}}(N)$, the behaviour of $\|\partial_\xi^\alpha Q_1(t, t_\xi, \xi)\|$ for $t \in [t_\xi, \tilde{t}_\xi]$.

Lemma 2.9 *The following estimate holds for all multi-indices α and for all $t \in [t_\xi, \tilde{t}_\xi]$:*

$$\|\partial_\xi^\alpha Q_1(t, t_\xi, \xi)\| \le C_{\alpha,N}|\xi|^{-|\alpha|}(\log(t+e^3))^{2|\alpha|\gamma} e^{C_{N,\gamma}(\log(t_\xi+e^3))^\gamma}.$$

Proof. Follows immediately from the above representation and estimate for $Q_1 = Q_1(t, s, \xi)$, and from Lemma 2.1 together with assumption (1.6) and estimate (2.8). □

Similarly, in $Z^{(1)}_{\mathrm{reg}}(N)$ we define $E_{\mathrm{reg}} = E_{\mathrm{reg}}(t, s, \xi)$, $\tilde{t}_\xi \le s,\, t$, to be the fundamental solution to (2.7). We write this in the form $E_{\mathrm{reg}}(t, s, \xi) = \tilde{E}_2(t, s, \xi)Q_m(t, s, \xi)$, where $Q_m = Q_m(t, s, \xi)$ solves

$$D_t Q_m = \tilde{E}_2(s, t, \xi)\mathcal{R}_m(t, \xi)\tilde{E}_2(t, s, \xi)Q_m, \quad Q_m(s, s, \xi) = I.$$

Here we define for $\tilde{t}_\xi \le s,\, t$,

$$\begin{aligned}\tilde{E}_2(t, s, \xi) = &\left(\frac{a(t)}{a(s)}\right)^{1/4}\\ &\times \begin{pmatrix} e^{i\int_s^t \tau_1(r,\xi)dr - \int_s^t f_m^{(1)}(r,\xi)dr} & 0 \\ 0 & e^{i\int_s^t \tau_2(r,\xi)dr - \int_s^t f_m^{(2)}(r,\xi)dr}\end{pmatrix},\end{aligned}$$

where $F_m := \begin{pmatrix} f_m^{(1)} & 0 \\ 0 & f_m^{(2)}\end{pmatrix}$. Using $F_m \in S_N\{-1, 2\}$ we have

$$\left|\int_s^t f_m^{(l)}(r, \xi)dr\right| \le C_m \quad \text{for all } \tilde{t}_\xi \le s,\, t \text{ and for } l = 1, 2.$$

Then, observing that $\mathcal{R}_m \in S_N\{-m, m+1\}$, we see that the following estimate holds:

$$\|Q_m(t,\, s,\, \xi)\| \le C_m \quad \text{for all } \tilde{t}_\xi \le s,\, t.$$

We also need, for a special representation of solution to (2.1) in $Z^{(1)}_{\mathrm{reg}}(N)$, the behaviour of $\|\partial^\alpha_\xi Q_m(t,\, \tilde{t}_\xi,\, \xi)\|$ for $t \ge \tilde{t}_\xi$.

Lemma 2.10 *The following estimate holds for all multi-indices α with $|\alpha| \le (m-1)/2$ and for all $\tilde{t}_\xi \le t$:*

$$\|\partial^\alpha_\xi Q_m(t,\, \tilde{t}_\xi,\, \xi)\| \le C_{\alpha,m}|\xi|^{-|\alpha|} \quad \textit{for all } \tilde{t}_\xi \le t.$$

Proof. It is sufficient to discuss the derivatives with respect to ξ of the term

$$g(t,\, \xi) := a(\tilde{t}_\xi)^{-1/4} \exp\left\{ i \int_{\tilde{t}_\xi}^{t} \sqrt{a(r)}dr|\xi| - \int_{\tilde{t}_\xi}^{t} f^{(1)}_m(r,\, \xi)dr \right\} r_m(t,\, \xi)$$

with $r_m \in S_N\{-m,\, m+1\}$. Such terms appear in the matrizant representation for Q_m. We have

$$|g(t,\, \xi)| \le C_m|\xi|^{-m}\left(\frac{1}{t+e^3}(\log(t+e^3))^{2\gamma}\right)^{m+1}.$$

Derivatives of r_m with respect to ξ generate $|\xi|^{-|\alpha|}$. By Lemma 2.2, assumption (1.6) and the definition of $Z_{\mathrm{reg}}(N)$ we conclude that

$$|\partial_{\xi_1} a(\tilde{t}_\xi)^{-1/4}| \le C\frac{(\log(\tilde{t}_\xi + e^3))^{3\gamma}}{|\xi|^2(\tilde{t}_\xi + e^3)} \le C_N(\log(\tilde{t}_\xi + e^3))^{\gamma}.$$

In the same way one can show

$$|\partial^\alpha_\xi a(\tilde{t}_\xi)^{-1/4}| \le C_{\alpha,N}(\log(\tilde{t}_\xi + e^3))^{2|\alpha|\gamma}.$$

Differentiating $\int_{\tilde{t}_\xi}^{t} f^{(1)}_m(r,\, \xi)dr$ with respect to ξ_1 gives

$$\int_{\tilde{t}_\xi}^{t} \partial_{\xi_1} f^{(1)}_m(r,\, \xi)dr - f^{(1)}_m(\tilde{t}_\xi,\, \xi)\frac{\partial \tilde{t}_\xi}{\partial \xi_1}.$$

The integral can be estimated by $C_m|\xi|^{-1}$. Taking account of $f^{(1)}_m \in S_N\{-1,\, 2\}$ gives the estimate

$$\left| f_m^{(1)}(\tilde{t}_\xi,\, \xi) \frac{\partial \tilde{t}_\xi}{\partial \xi_1} \right| \le C_m \frac{(\log(\tilde{t}_\xi + e^3))^{4\gamma}}{|\xi|^3(\tilde{t}_\xi + e^3)^2} \le C_{m,N}|\xi|^{-1}.$$

Higher derivatives of $\tilde{t}_\xi$ give rise to log terms. Thus, we get

$$\left| \partial_\xi^\alpha \exp\left\{ - \int\limits_{\tilde{t}_\xi}^{t} f_m^{(1)}(r,\, \xi) dr \right\} \right| \le C_{\alpha,N}(\log(\tilde{t}_\xi + e^3))^{2|\alpha|\gamma}.$$

The main problem arises from $\int_{\tilde{t}_\xi}^t \sqrt{a(r)}dr|\xi|$. Differentiation ∂_{ξ_1} allows only an estimate like

$$\left| \partial_{\xi_1} \left(\int_{\tilde{t}_\xi}^t \sqrt{a(r)}dr|\xi| \right) \right| \le C_a(t + e^3).$$

But now we can use that $r_m \in S_N\{-m,\, m+1\}$. If we differentiate α times, then for all $t \ge \tilde{t}_\xi$ we have

$$\begin{aligned}
|(t+e^3)^{|\alpha|} r_m(t,\, \xi)| &\le C_m \frac{(\log(t+e^3))^{\gamma(m+1)}}{|\xi|^m(t+e^3)^{m+1-|\alpha|}} \\
&\le \frac{C_m}{|\xi|^{|\alpha|}} \frac{(\log(t+e^3))^{\gamma(m+1)}}{|\xi|^{m-|\alpha|}(t+e^3)^{m+1-|\alpha|}} \\
&\le \frac{C_m}{|\xi|^{|\alpha|}} \frac{(\log(t+e^3))^{\gamma(m-1)}(\log(t+e^3))^{2\gamma}}{(|\xi|(t+e^3))^{m-1-|\alpha|}|\xi|(t+e^3)^2} \\
&\le \frac{C_{m,N}}{|\xi|^{|\alpha|}} \frac{(\log(t+e^3))^{2\gamma}}{|\xi|(t+e^3)^2}
\end{aligned}$$

if $|\alpha| \le (m-1)/2$. Consequently we earn $|\xi|^{-|\alpha|}$ and a term which is integrable over $[\tilde{t}_\xi,\, t]$ for all t. It remains to explain how we proceed with the terms $(\log(\tilde{t}_\xi + e^3))^{2|\alpha|\gamma}$ arising in the above estimates. These terms we couple with r_m also and get, for $|\alpha| \le (m-1)/2$,

$$\begin{aligned}
&|(\log(\tilde{t}_\xi + e^3))^{2|\alpha|\gamma} r_m(t,\, \xi)| \\
&\le C_m \left| (\log(t+e^3))^{(m-1)\gamma} \frac{(\log(t+e^3))^{(m+1)\gamma}}{|\xi|^m(t+e^3)^{m+1}} \right| \\
&\le C_m \frac{(\log(t+e^3))^{2(m-1)\gamma}}{(|\xi|(t+e^3))^{m-1}} \frac{(\log(t+e^3))^{2\gamma}}{|\xi|(t+e^3)^2}.
\end{aligned}$$

Using the definition of $Z_{\text{reg}}(N)$, the first factor is uniformly bounded. The second one is integrable over $[\tilde{t}_\xi, t]$. This completes the proof of our lemma. □

2.1.8. Representation of solutions in subzones Now we are in position to give representations for the solution to (2.1) in $Z_{\text{osc}}^{(1)}$ and $Z_{\text{reg}}^{(1)}$. The vector-function $V = V(t, \xi)$ is a solution of (2.3).

In $Z_{\text{osc}}^{(1)}(N)$: For $t_\xi \le t \le \tilde{t}_\xi$ we have

$$\begin{aligned} V(t, \xi) = {} & M N_1(t, \xi) E_2(t, 0, \xi) E_2(0, t_\xi, \xi) \\ & \cdot Q_1(t, t_\xi, \xi) N_1(t_\xi, \xi)^{-1} M^{-1} \\ & \cdot \begin{pmatrix} \sqrt{a(t_\xi)} & 0 \\ 0 & 1 \end{pmatrix} E(t_\xi, 0, \xi) U_0(\xi), \end{aligned} \tag{2.9}$$

where we recall $E(t_\xi, 0, \xi)$ is obtained in the representation of the solution in the pseudodifferential zone.

In $Z_{\text{reg}}(N)$: For $t \ge \tilde{t}_\xi$ we have

$$\begin{aligned} V(t, \xi) = {} & M N_m(t, \xi) E_2(t, 0, \xi) E_2(0, \tilde{t}_\xi, \xi) \tilde{F}_m(t, \tilde{t}_\xi, \xi) \\ & \cdot Q_m(t, \tilde{t}_\xi, \xi) N_m(\tilde{t}_\xi, \xi)^{-1} N_1(\tilde{t}_\xi, \xi) \\ & \cdot E_2(\tilde{t}_\xi, t_\xi, \xi) Q_1(\tilde{t}_\xi, t_\xi, \xi) N_1(t_\xi, \xi)^{-1} M^{-1} \\ & \cdot \begin{pmatrix} \sqrt{a(t_\xi)} & 0 \\ 0 & 1 \end{pmatrix} E(t_\xi, 0, \xi) U_0(\xi), \end{aligned}$$

with

$$\tilde{F}_m(t, \tilde{t}_\xi, \xi) = \begin{pmatrix} e^{-\int_{\tilde{t}_\xi}^t f_m^{(1)}(r, \xi) dr} & 0 \\ 0 & e^{-\int_{\tilde{t}_\xi}^t f_m^{(2)}(r, \xi) dr} \end{pmatrix},$$

where we have used the representation (2.9) at $t = \tilde{t}_\xi$.

Before we discuss the representation of the solution to (2.1) we collect together some useful estimates.

Lemma 2.11 *The following estimates hold for all multi-indices α:*

$$\|\partial_\xi^\alpha N_1(t_\xi, \xi)\| \le C_\alpha |\xi|^{-|\alpha|} (\log(t_\xi + e^3))^{|\alpha|\gamma},$$
$$\|\partial_\xi^\alpha N_m(\tilde{t}_\xi, \xi)\| \le C_\alpha |\xi|^{-|\alpha|} (\log(\tilde{t}_\xi + e^3))^{2|\alpha|\gamma},$$

$$\|\partial_\xi^\alpha \tilde{F}_m(t,\, \tilde{t}_\xi,\, \xi)\| \le C_\alpha |\xi|^{-|\alpha|}(\log(\tilde{t}_\xi + e^3))^{2|\alpha|\gamma}, \quad \textit{for all } t \ge \tilde{t}_\xi.$$

Proof. Follows immediately from the above representation for $\tilde{F}_m = \tilde{F}_m(t,\, \tilde{t}_\xi,\, \xi)$, from Lemmas 2.1 and 2.2 together with assumption (1.6) and the definition of zones. □

Summarising all the above results we have

Proposition 2.12 *The following WKB representations hold for the solution to* (2.1):

$$\begin{aligned}
|\xi|v(t,\xi) = {} & b_{11}^{(1)}(t,\xi)\exp\Big\{-i|\xi|\int_0^t \sqrt{a(s)}ds\Big\}|\xi|\widehat{\varphi}(\xi)\\
& + b_{12}^{(1)}(t,\xi)\exp\Big\{-i|\xi|\int_0^t \sqrt{a(s)}ds\Big\}\widehat{\psi}(\xi)\\
& + b_{21}^{(1)}(t,\xi)\exp\Big\{i|\xi|\int_0^t \sqrt{a(s)}ds\Big\}|\xi|\widehat{\varphi}(\xi)\\
& + b_{22}^{(1)}(t,\xi)\exp\Big\{i|\xi|\int_0^t \sqrt{a(s)}ds\Big\}\widehat{\psi}(\xi),
\end{aligned}$$

$$\begin{aligned}
D_t v(t,\xi) = {} & b_{11}^{(2)}(t,\xi)\exp\Big\{-i|\xi|\int_0^t \sqrt{a(s)}ds\Big\}|\xi|\widehat{\varphi}(\xi)\\
& + b_{12}^{(2)}(t,\xi)\exp\Big\{-i|\xi|\int_0^t \sqrt{a(s)}ds\Big\}\widehat{\psi}(\xi)\\
& + b_{21}^{(2)}(t,\xi)\exp\Big\{i|\xi|\int_0^t \sqrt{a(s)}ds\Big\}|\xi|\widehat{\varphi}(\xi)\\
& + b_{22}^{(2)}(t,\xi)\exp\Big\{i|\xi|\int_0^t \sqrt{a(s)}ds\Big\}\widehat{\psi}(\xi).
\end{aligned}$$

Here the amplitudes $b_{kl}^{(p)}(t,\,\xi)$, $p,\, k,\, l = 1,\, 2$, *satisfy the following estimates*:

- *in* $Z_{\mathrm{pd}}(N) \cup Z_{\mathrm{osc}}^{(1)}(N)$: $|b_{kl}^{(p)}(t,\,\xi)| \le C_{N,a,\gamma} e^{C_{N,a,\gamma}(\log(t+e^3))^\gamma}$;

- *in* $Z^{(1)}_{\mathrm{reg}}(N)$: $|\partial^\alpha_\xi b^{(p)}_{kl}(t,\,\xi)| \le C_{N,a,\gamma,\alpha}|\xi|^{-|\alpha|}(\log(t+e^3))^{2|\alpha|\gamma}$ $\times \exp\{C_{N,a,\gamma}(\log(t+e^3))^\gamma\}$ *for all* $|\alpha| \le (m-1)/2$, *where* m *is the number of steps of diagonalisation in* $Z^{(1)}_{\mathrm{reg}}(N)$.

We obtain similar representations in the other parts of the phase space. The amplitudes satisfy at worst the estimates above.

2.2. L^p-L^q estimates for Fourier multipliers

Using Proposition 2.12 we can write down the following Fourier multiplier representation for the solution $u = u(t,\,x)$ to (1.5):

$$\begin{aligned}|D_x|u(t,\,x) = \mathcal{F}^{-1}\Bigg(& b^{(1)}_{11}(t,\,\xi)\exp\Big\{-i|\xi|\int_0^t \sqrt{a(s)}ds\Big\}|\xi|\widehat{\varphi}(\xi)\\ &+ b^{(1)}_{12}(t,\,\xi)\exp\Big\{-i|\xi|\int_0^t \sqrt{a(s)}ds\Big\}\widehat{\psi}(\xi)\\ &+ b^{(1)}_{21}(t,\,\xi)\exp\Big\{i|\xi|\int_0^t \sqrt{a(s)}ds\Big\}|\xi|\widehat{\varphi}(\xi)\\ &+ b^{(1)}_{22}(t,\,\xi)\exp\Big\{i|\xi|\int_0^t \sqrt{a(s)}ds\Big\}\widehat{\psi}(\xi)\Bigg),\end{aligned} \tag{2.10}$$

$$\begin{aligned}D_t u(t,\,x) = \mathcal{F}^{-1}\Bigg(& b^{(2)}_{11}(t,\,\xi)\exp\Big\{-i|\xi|\int_0^t \sqrt{a(s)}ds\Big\}|\xi|\widehat{\varphi}(\xi)\\ &+ b^{(2)}_{12}(t,\,\xi)\exp\Big\{-i|\xi|\int_0^t \sqrt{a(s)}ds\Big\}\widehat{\psi}(\xi)\\ &+ b^{(2)}_{21}(t,\,\xi)\exp\Big\{i|\xi|\int_0^t \sqrt{a(s)}ds\Big\}|\xi|\widehat{\varphi}(\xi)\\ &+ b^{(2)}_{22}(t,\,\xi)\exp\Big\{i|\xi|\int_0^t \sqrt{a(s)}ds\Big\}\widehat{\psi}(\xi)\Bigg),\end{aligned} \tag{2.11}$$

where $\mathcal{F}^{-1}$ denotes the inverse to the partial Fourier transform with respect to x. Here the amplitudes $b_{kl}^{(p)}(t,\xi)$, $p, k, l = 1, 2$, satisfy the following estimates:

- in $Z_{\mathrm{pd}}(N) \cup Z_{\mathrm{osc}}(N)$: $|b_{kl}^{(p)}(t,\xi)| \le C_{N,a,\gamma} e^{C_{N,a,\gamma}(\log(t+e^3))^\gamma}$;
- in $Z_{\mathrm{reg}}(N)$: $|\partial_\xi^\alpha b_{kl}^{(p)}(t,\xi)| \le C_{N,a,\gamma,\alpha}|\xi|^{-|\alpha|}(\log(t+e^3))^{2|\alpha|\gamma}$ $\times \exp\{C_{N,a,\gamma}(\log(t+e^3))^\gamma\}$ for all $|\alpha| \le (m-1)/2$, where m is the number of steps of diagonalisation in $Z_{\mathrm{reg}}^{(1)}(N)$.

Our next goal is to estimate these Fourier multipliers. We use the approach from [10] and from [13].

2.2.1. L^p-L^q estimates for Fourier multipliers with amplitudes vanishing in regular subzone Let us choose a function $\psi \in C^\infty(\mathbb{R}^n)$ satisfying $\psi(\xi) \equiv 0$ for $|\xi| \le 1$, $\psi(\xi) \equiv 1$ for $|\xi| \ge 2$ and $0 \le \psi(\xi) \le 1$. Further, denote $K(t) := N(\log(t+e^3))^{2\gamma}/(t+e^3)$.

Theorem 2.13 *Let us consider Fourier multipliers which are defined by*

$$\mathcal{F}^{-1}\Big(e^{i|\xi|\int_0^t \sqrt{a(s)}ds}\big(1-\psi(\xi/K(t))\big)|\xi|^{-2r}a(t,\xi)\mathcal{F}(\varphi)(\xi)\Big).$$

Suppose that $a = a(t,\xi)$ satisfies the following assumption:

- *in $Z_{\mathrm{pd}}(N) \cup Z_{\mathrm{osc}}(N)$: $|a(t,\xi)| \le C_{N,a,\gamma}\exp\{C_{N,a,\gamma}(\log(t+e^3))^\gamma\}$.*

Then we have the L^p-L^q estimate

$$\Big\|\mathcal{F}^{-1}\Big(e^{i|\xi|\int_0^t \sqrt{a(s)}ds}\big(1-\psi(\xi/K(t))\big)|\xi|^{-2r}a(t,\xi)\mathcal{F}(\varphi)(\xi)\Big)\Big\|_{L^q} \le C_{N,a,\gamma}K(t)^{-2r+n(1/p-1/q)}\exp\big\{C_{N,a,\gamma}(\log(t+e^3))^\gamma\big\}\|\varphi\|_{L^p},$$

provided that $0 \le 2r \le n(1/p-1/q)$, $1 < p \le 2$, $1/p+1/q = 1$.

Proof. Let us consider

$$I_0 := \Big\|\mathcal{F}^{-1}\Big(e^{i|\xi|\int_0^t \sqrt{a(s)}ds}\big(1-\psi(\xi/K(t))\big)|\xi|^{-2r}a(t,\xi)\mathcal{F}(\varphi)(\xi)\Big)\Big\|_{L^q}^q.$$

Using the transformations $\xi = K(t)\eta$ and $z = K(t)x$ we conclude

$$I_0 = K(t)^{nq-2rq-n}\Big\|\mathcal{F}^{-1}\Big(e^{iK(t)|\eta|\int_0^t \sqrt{a(s)}ds}(1-\psi(\eta))|\eta|^{-2r} \cdot a(t,K(t)\eta)\mathcal{F}(\varphi)(K(t)\eta)\Big)\Big\|_{L^q}^q.$$

The point $(t,\, K(t)\eta)$ with $|\eta| \leq 2$ (support of $1-\psi$) belongs to $Z_{\mathrm{pd}}(N) \cup Z_{\mathrm{osc}}(N)$. Therefore $|a(t,\, K(t)\eta)| \leq C_{N,a,\gamma} \exp(C_{N,a,\gamma}(\log(t+e^3))^\gamma)$. For I_0 we obtain

$$I_0^{1/q} \leq K(t)^{n-2r-n/q} \cdot \Big\| \mathcal{F}^{-1}\Big(e^{iK(t)|\eta| \int_0^t \sqrt{a(s)}ds} (1-\psi(\eta))|\eta|^{-2r} a(t,\, K(t)\eta) \Big) * \mathcal{F}^{-1}(\mathcal{F}(\varphi)(K(t)\eta)) \Big\|_{L^q}.$$

Now let us denote

$$T_t := \mathcal{F}^{-1}\Big(e^{iK(t)|\eta| \int_0^t \sqrt{a(s)}ds} (1-\psi(\eta))|\eta|^{-2r} a(t,\, K(t)\eta) \Big) \cdot \exp(-C_{N,a,\gamma}(\log(t+e^3))^\gamma).$$

We have, together with the estimate for $a(t,\, K(t)\eta)$,

$$\mathrm{meas}\{\eta\colon |\mathcal{F}(T_t)| \geq l\} \leq \mathrm{meas}\{\eta\colon |\eta| \leq C l^{-1/(2r)}\} \leq C l^{-n/(2r)}.$$

Due to Theorem 1.11 from [3] we have $\mathcal{F}(T_t) \in M_p^q$ for all $2r \leq n(1/p - 1/q)$. Here M_p^q denotes the set of Fourier transforms $\mathcal{F}(T)$ of distributions $T \in L_p^q$, where L_p^q denotes the space of tempered distributions such that $\|T * u\|_{L^q} \leq C\|u\|_{L^p}$ with a constant C independent of u. Hence $T_t \in L_p^q$ and

$$\|T_t * \mathcal{F}^{-1}(\mathcal{F}(\varphi)(K(t)\eta))\|_{L^q} \leq C_p K(t)^{-n+n/p} \|\varphi\|_{L^p}.$$

Thus, we have proved

$$I_0^{1/q} \leq C K(t)^{-2r+n(1/p-1/q)} e^{C_{N,a,\gamma}(\log(t+e^3))^\gamma} \|\varphi\|_{L^p},$$

and

$$\Big\| \mathcal{F}^{-1}\Big(e^{i|\xi| \int_0^t \sqrt{a(s)}ds} (1-\psi(\xi/K(t)))|\xi|^{-2r} a(t,\, \xi)\mathcal{F}(\varphi)(\xi) \Big) \Big\|_{L^q} \leq C K(t)^{-2r+n(1/p-1/q)} e^{C_{N,a,\gamma}(\log(t+e^3))^\gamma} \|\varphi\|_{L^p},$$

respectively. Thus, we have derived the statement of our theorem. □

2.2.2. L^p-L^q estimates for Fourier multipliers with amplitudes supported in regular subzone

Theorem 2.14 *Let us consider Fourier multipliers which are defined by*

$$\mathcal{F}^{-1}\Big(e^{i|\xi|\int_0^t \sqrt{a(s)}ds}\psi(\xi/(2K(t)))|\xi|^{-2r}a(t,\xi)\mathcal{F}(\varphi)(\xi)\Big).$$

Suppose that $a = a(t,\xi)$ satisfies the following assumption:

- *in $Z_{\mathrm{reg}}(N)$: $|\partial_\xi^\alpha a(t,\xi)| \le C_{N,a,\gamma,\alpha}|\xi|^{-|\alpha|}(\log(t+e^3))^{2|\alpha|\gamma}$*
$$\times \exp(C_{N,a,\gamma}(\log(t+e^3))^\gamma)$$
for all $|\alpha| \le (m-1)/2$.

Then we have the L^p-L^q estimate

$$\begin{aligned}&\Big\|\mathcal{F}^{-1}\Big(e^{i|\xi|\int_0^t \sqrt{a(s)}ds}\psi(\xi/(2K(t)))|\xi|^{-2r}a(t,\xi)\mathcal{F}(\varphi)(\xi)\Big)\Big\|_{L^q}\\ &\quad\le CK(t)^{-2r+n(1/p-1/q)}(\log(t+e^3))^{2M\gamma}e^{C_{N,a,\gamma}(\log(t+e^3))^\gamma}\|\varphi\|_{L^p}\end{aligned}$$

provided that $1 < p \le 2$, $1/p + 1/q = 1$, $\big((n+1)/2\big)(1/p-1/q) \le 2r \le n(1/p-1/q)$ and with a suitable positive constant M.

Proof. We generalize the proof of [6] to Fourier multipliers depending on a parameter. If $(t,\xi) \in \operatorname{supp}\psi(\xi/(2K(t)))$, then $(t,\xi) \in Z_{\mathrm{reg}}(N)$. We choose a nonnegative function $\phi = \phi(\xi)$ having compact support in $\{\xi \in \mathbb{R}^n : 1/2 \le |\xi| \le 2\}$. We set

$$\phi_k(\xi) := \phi(2^{-k}\xi) \quad \text{for } k \in \mathbb{N} \text{ while } \phi_0(\xi) := 1 - \sum_{k=1}^\infty \phi_k(\xi).$$

The function ϕ_0 has its support in $\{\xi \in \mathbb{R}^n : |\xi| \le 2\}$.
The L^q-norm of

$$\mathcal{F}^{-1}\Big(e^{i|\xi|\int_0^t \sqrt{a(s)}ds}\psi(\xi/(2K(t)))\phi_0(\xi/(2K(t)))|\xi|^{-2r}a(t,\xi)\mathcal{F}(\varphi)(\xi)\Big)$$

can be estimated as in Theorem 2.13. Thus we can restrict ourselves to considering the integral

$$\begin{aligned}&\Big\|\mathcal{F}^{-1}\Big(e^{i|\xi|\int_0^t \sqrt{a(s)}ds}\psi(\xi/(2K(t)))\\ &\qquad\cdot\phi_k(\xi/(2K(t)))|\xi|^{-2r}a(t,\xi)\mathcal{F}(\varphi)(\xi)\Big)\Big\|_{L^q}\end{aligned}$$

with $k \in \mathbb{N}$. First we study this integral for $t \geq t_0$, t_0 large.

a) L^1-L^∞ *continuity*: To estimate

$$I_k := \Big\| \mathcal{F}^{-1}\Big(e^{i|\xi| \int_0^t \sqrt{a(s)}ds} \psi(\xi/(2K(t))) \cdot \phi_k(\xi/(2K(t)))|\xi|^{-2r} a(t,\, \xi)\mathcal{F}(\varphi)(\xi)\Big)\Big\|_{L^q}$$

we set $\xi/(2K(t)) = 2^k\eta$. We are going to apply Lemma 3 from [1]. For this reason we use the inequality

$$\Big\| \mathcal{F}^{-1}\Big(e^{i|\xi| \int_0^t \sqrt{a(s)}ds} \psi(\xi/(2K(t)))\phi_k(\xi/(2K(t)))|\xi|^{-2r} a(t,\, \xi)\Big)\Big\|_{L^\infty}$$
$$\leq C2^{k(n-2r)}(2K(t))^{n-2r} \Big\| \mathcal{F}^{-1}\Big(e^{i2^k 2K(t)|\eta| \int_0^t \sqrt{a(s)}ds} \psi(2^k\eta)\phi(\eta) \cdot |\eta|^{-2r} a(t,\, 2^{k+1}K(t)\eta)\Big)\Big\|_{L^\infty}.$$

Let us denote $v_k(t,\, \eta) := \phi(\eta)\psi(2^k\eta)a(t,\, 2^{k+1}K(t)\eta)$. These function have their supports in $\{\eta \in \mathbb{R}^n\colon 1/2 \leq |\eta| \leq 2\}$. According to [4] (see also [1] or [6]) we have (here we need t_0 large)

$$\Big\| \mathcal{F}^{-1}\Big(e^{i2^{k+1}K(t)|\eta| \int_0^t \sqrt{a(s)}ds} |\eta|^{-2r} v_k(t,\, \eta)\Big)\Big\|_{L^\infty}$$
$$\leq C\Big(2^{k+1}K(t)\int_0^t \sqrt{a(s)}ds\Big)^{-(n-1)/2} \sum_{|\alpha|\leq M} \|D_\eta^\alpha(|\eta|^{-2r}v_k(t,\, \eta))\|_{L^\infty}.$$

According to the assumption for $a = a(t,\, \xi)$ we have

$$\|D_\eta^\alpha(|\eta|^{-2r}v_k(t,\, \eta))\|_{L^\infty} \leq C_{N,a,\gamma,M}(\log(t+e^3))^{2|\alpha|\gamma} e^{C_{N,a,\gamma}(\log(t+e^3))^\gamma}$$

for all $|\alpha| \leq (m-1)/2$. If we use $m = 2M+1$ steps in our diagonalisation procedure in $Z_{\text{reg}}(N)$, then the last inequality holds for all $|\alpha| \leq M$. Hence,

$$\Big\| \mathcal{F}^{-1}\Big(e^{i2^{k+1}K(t)|\eta| \int_0^t \sqrt{a(s)}ds} |\eta|^{-2r} v_k(t,\, \eta)\Big)\Big\|_{L^\infty}$$
$$\leq C_{N,a,\gamma,M} 2^{-(k/2)(n-1)} \Big(K(t)\int_0^t \sqrt{a(s)}ds\Big)^{-(n-1)/2} \cdot (\log(t+e^3))^{2M\gamma} e^{C_{N,a,\gamma}(\log(t+e^3))^\gamma}.$$

All together we have shown

$$\begin{aligned}&\Big\|\mathcal{F}^{-1}\Big(e^{i|\xi|\int_0^t\sqrt{a(s)}ds}\psi(\xi/(2K(t)))\\&\qquad\cdot\phi_k(\xi/(2K(t)))|\xi|^{-2r}a(t,\,\xi)\mathcal{F}(\varphi)(\xi)\Big)\Big\|_{L^\infty}\\&\le C_{N,a,\gamma,M}2^{k\left((n+1)/2-2r\right)}K(t)^{n-2r}\Big(K(t)\int_0^t\sqrt{a(s)}ds\Big)^{-(n-1)/2}\\&\qquad\cdot(\log(t+e^3))^{2M\gamma}e^{C_{N,a,\gamma}(\log(t+e^3))^\gamma}\|\varphi\|_{L^1}.\end{aligned}$$

b) L^2-L^2 *continuity*: To estimate L^2–norms we apply Lemma 3 from [1]. To this end we take into consideration

$$\begin{aligned}&\Big\|e^{i|\xi|\int_0^t\sqrt{a(s)}ds}\psi(\xi/(2K(t)))\phi_k(\xi/(2K(t)))|\xi|^{-2r}a(t,\,\xi)\Big\|_{L^\infty}\\&\le\sup_{2^{k-1}\le|\xi|/(2K(t))\le 2^{k+1}}|\xi|^{-2r}|a(t,\,\xi)|\\&\le C_{N,a,\gamma,0}2^{-2kr}K(t)^{-2r}e^{C_{N,a,\gamma}(\log(t+e^3))^\gamma}.\end{aligned}$$

Hence,

$$\begin{aligned}&\Big\|\mathcal{F}^{-1}\Big(e^{i|\xi|\int_0^t\sqrt{a(s)}ds}\psi(\xi/(2K(t)))\\&\qquad\cdot\phi_k(\xi/(2K(t)))|\xi|^{-2r}a(t,\,\xi)\mathcal{F}(\varphi)(\xi)\Big)\Big\|_{L^2}\\&\le C_{N,a,\gamma,0}2^{-2kr}K(t)^{-2r}e^{C_{N,a,\gamma}(\log(t+e^3))^\gamma}\|\varphi\|_{L^2}.\end{aligned}$$

c) *Interpolation argument*: An interpolation argument between L^1-L^∞ and L^2-L^2 estimates from a) and b) yields

$$\begin{aligned}&\Big\|\mathcal{F}^{-1}\Big(e^{i|\xi|\int_0^t\sqrt{a(s)}ds}\psi(\xi/(2K(t)))\\&\qquad\cdot\phi_k(\xi/(2K(t)))|\xi|^{-2r}a(t,\,\xi)\mathcal{F}(\varphi)(\xi)\Big)\Big\|_{L^q}\\&\le C_{N,a,\gamma,M}2^{k\left(((n+1)/2)(1/p-1/q)-2r\right)}K(t)^{n(1/p-1/q)-2r}\\&\qquad\cdot(\log(t+e^3))^{2M\gamma}e^{C_{N,a,\gamma}(\log(t+e^3))^\gamma}\|\varphi\|_{L^p}\end{aligned}$$

for $t\ge t_0$, t_0 large, provided $1<p\le 2$, $1/p+1/q=1$, $\big((n+1)/2\big)(1/p-1/q)\le 2r\le n(1/p-1/q)$. Applying Lemma 2 from [1] proves the statement

of the theorem for $t \geq t_0$, t_0 large.

d) *Estimates for small* t: It remains to estimate the L^q–norms of

$$\mathcal{F}^{-1}\Big(e^{i|\xi|\int\limits_0^t \sqrt{a(s)}ds}\psi(\xi/(2K(t)))\phi_k(\xi/(2K(t)))|\xi|^{-2r}a(t,\xi)\mathcal{F}(\varphi)(\xi)\Big)$$

for $t \in [0, t_0]$. Here we will not use the stationary phase method, the key tool to get the above estimates for $t \geq t_0$. Instead we apply the Hardy–Littlewood inequality as we did to get the estimates in Theorem 2.13. Let us sketch the differences in the proof. Using the transformations $\xi = 2^{k+1}K(t)\eta$ and $z = 2^{k+1}K(t)x$ we conclude for $k \in \mathbb{N}$

$$\begin{aligned} I_k := &(2^{k+1}K(t))^{nq-2rq-n} \\ &\cdot \Big\|\mathcal{F}^{-1}\Big(e^{i\,K(t)|\eta|\int\limits_0^t \sqrt{a(s)}ds}\psi(2^k\eta)\phi(\eta)|\eta|^{-2r} \\ &\qquad \cdot a(t,\, 2^{k+1}K(t)\eta)\mathcal{F}(\varphi)(2^{k+1}K(t)\eta)\Big)\Big\|^q_{L^q}. \end{aligned}$$

The point $(t,\, 2^{k+1}K(t)\eta)$ with $|\eta| \in [1/2,\, 2]$ (support of ϕ) belongs to $Z_{\text{reg}}(N)$. Therefore $|a(t,\, 2^{k+1}K(t)\eta)| \leq C_{N,a,\gamma,0}\exp(C_{N,a,\gamma}(\log(t+e^3))^\gamma)$. For I_k we obtain

$$\begin{aligned} I_k^{1/q} \leq &(2^{k+1}K(t))^{n-2r-n/q} \\ &\cdot \Big\|\mathcal{F}^{-1}\Big(e^{iK(t)|\eta|\int\limits_0^t \sqrt{a(s)}ds}\psi(2^k\eta)\phi(\eta)|\eta|^{-2r}a(t,\, 2^{k+1}K(t)\eta)\Big) \\ &\qquad * \mathcal{F}^{-1}(\mathcal{F}(\varphi)(2^{k+1}K(t)\eta))\Big\|_{L^q}. \end{aligned}$$

Now let us denote

$$\begin{aligned} T_{t,k} := &\mathcal{F}^{-1}\Big(e^{iK(t)|\eta|\int\limits_0^t \sqrt{a(s)}ds}\psi(2^k\eta))\phi(\eta)|\eta|^{-2r}a(t,\, 2^{k+1}K(t)\eta)\Big) \\ &\cdot \exp(-C_{N,a,\gamma}(\log(t+e^3))^\gamma). \end{aligned}$$

Then $T_{t,k}$ has the same properties as described for T_t in the proof to Theorem 2.13. Thus we can derive

$$\begin{aligned} &\Big\|\mathcal{F}^{-1}\Big(e^{i|\xi|\int\limits_0^t \sqrt{a(s)}ds}\psi(\xi/(2K(t))) \\ &\qquad \cdot \phi_k(\xi/(2K(t)))|\xi|^{-2r}a(t,\, \xi)\mathcal{F}(\varphi)(\xi)\Big)\Big\|_{L^q} \end{aligned}$$

$$\leq C(2^k K(t))^{-2r+n(1/p-1/q)} e^{C_{N,a,\gamma}(\log(t+e^3))^\gamma} \|\varphi\|_{L^p}.$$

Lemma 2 from [1] proves with $2r \geq n(1/p - 1/q)$ the statement of the theorem for $t \in [0, t_0]$. This completes the proof. $\square$

2.3. End of the proof

Proof. The statements of Theorems 2.13 and 2.14 applied to the representations (2.10) and (2.11) enable us to derive the estimates from Theorem 1.1. If $t \in (0, t_0]$, then we choose in the estimates from Theorems 2.13 and 2.14 the parameter $2r = n(1/p - 1/q)$. This fixes the necessary regularity $N_p = n(1/p - 1/q)$. If $t \in [t_0, \infty)$, then we choose in Theorem 2.14 the parameter $2r = \big((n+1)/2\big)(1/p - 1/q)$. Now let us distinguish the different cases for γ. If $\gamma = 0$, then we directly obtain the classical Strichartz' L^p-L^q decay estimate from Theorem 1.1 with $s_0 = 0$. If $\gamma \in (0, 1)$, then the main influence on changes to the classical Strichartz' decay rate comes from the term $\exp(C_{N,a,\gamma}(\log(t + e^3))^\gamma$ in Theorems 2.13 and 2.14. For each ε this term can be estimated by $C_\varepsilon(1+t)^\varepsilon$. Thus $s_0 = \varepsilon$ for all $\varepsilon > 0$. Finally, if $\gamma = 1$, then this term produces, together with the log terms, a factor like $(1+t)^{s_0}$, where s_0 eventually becomes a large positive constant. $\square$

Let us formulate a corollary of Theorem 1.1. At first sight, the statement of this corollary does not seem to be very surprising, but its meaning lies in a comparison of the cases $\gamma \in [0, 1]$ and $\gamma > 1$ in (1.6).

Corollary 2.15 *Consider the strictly hyperbolic Cauchy problem*

$$\left.\begin{aligned} &\partial_t^2 u - a(t)\Delta u = 0, \quad (t, x) \in [0, \infty) \times \mathbb{R}^n, \\ &u(t_0, x) = \varphi(x),\ \partial_t u(t_0, x) = \psi(x),\ t_0 \geq T, \end{aligned}\right\} \tag{2.12}$$

where T is large and the coefficient $a = a(t)$ satisfies (1.6) *with $\gamma \in [0, 1]$. Then there exists a constant C which is independent of $t_0 \geq T$ and $t \geq t_0$ such that the following L^p-L^q estimate holds for the solution $u = u(t, x)$:*

$$\|(u_t(t, \cdot), \nabla_x u(t, \cdot))\|_{L^q} \leq C(1+t)^{s_0} \|(\nabla_x \varphi, \psi)\|_{W^{N_p,p}},$$

where $1/p + 1/q = 1$, $1 < p \leq 2$, $N_p \geq n(1/p - 1/q)$ and

- *$s_0 = 0$ if $\gamma = 0$; in this case C only depends on p, n;*
- *$s_0 = \varepsilon$ if $\gamma \in (0, 1)$ for all $\varepsilon > 0$; in this case C depends on p, n and ε;*
- *s_0 is a fixed constant* (*which can be determined*) *if $\gamma = 1$; in this case C is independent of φ, ψ.*

Proof. The transformation $t := t_0 + \tau$ transfers the above Cauchy problem to

$$\partial_\tau^2 u - a_{t_0}(\tau)\Delta u = 0, \quad u(0,\,x) = \varphi(x),\ \partial_\tau u(0,\,x) = \psi(x),$$

where $a_{t_0}(\tau) := a(t_0+\tau)$. The coefficients $a_{t_0}(\tau)$ satisfy, for all $t_0 \geq T$, the estimates (1.6) with the same constants C_k. Thus we can follow the proof of Theorem 1.1 and obtain the L^p-L^q estimate

$$\begin{aligned}&\|(u_\tau(\tau,\,\cdot),\,\nabla_x u(\tau,\,\cdot))\|_{L^q}\\ &\leq C(1+\tau)^{-\big((n-1)/2\big)(1/p-1/q)+s_0}\|(\nabla_x\varphi,\,\psi)\|_{W^{N_p,p}}.\end{aligned}$$

Setting $\tau = t - t_0$ in the last inequality gives the statement of the corollary. □

3. Proof of Theorem 1.2

The proof is based on an application of Floquet's theory, an idea used in [16] to show that the Cauchy problem for $\partial_t^2 - \exp(-2t^{-\alpha})b(t^{-1})\partial_x^2$ is not C^∞ well-posed when $0 < \alpha < 1/2$, where $b = b(t)$ is a positive, smooth, 1-periodic function. A similar idea is used in [11] to study L^p-L^q estimates for hyperbolic equations with increasing coefficients.

Proof. In order to apply Floquet's theory, it is necessary to first transform (1.8) so that the coefficient is periodic. This idea is used in [2] when studying the C^∞ well-posedness of strictly hyperbolic equations with non-Lipschitz coefficients. Then, a lower bound is found for a suitable energy of the solution of the transformed problem via estimates for an auxiliary problem. Finally, we derive a contradiction to (1.7) by obtaining a lower bound for the non-standard energy of Theorem 1.2 of the solution to (1.8).

3.1. Transformation of the Cauchy problem (1.8)

Let

$$\begin{aligned}&s = s(t) := (\log(t+e^3))^\alpha, \quad (\text{with inverse } t(s) := e^{s^{1/\alpha}} - e^3),\\ &w = w(s,x) := \sqrt{\tau(s)}\,u(t(s),\,x),\end{aligned}$$

where $\tau(s) := (ds/dt)(t(s)) = \alpha s^{1-(1/\alpha)}e^{-s^{1/\alpha}}$, and, instead of (1.8), consider the Cauchy problem obtained after this transformation. By simple calculations, we have

$$\begin{aligned} w_s(s,\,x) &= \frac{1}{2}\frac{\tau'(s)}{\tau(s)}w(s,\,x) + \frac{1}{\sqrt{\tau(s)}}u_t(t(s),\,x); \\ w_{ss}(s,\,x) &= \frac{1}{4}\left(\frac{2\tau''(s)\tau(s) - \tau'(s)^2}{\tau(s)^2}\right)w(s,\,x) \\ &\quad + \frac{1}{\tau(s)^2}(2+\sin(2\pi s))^2\Delta w(s,\,x) \\ &= \frac{1}{4\alpha^2 s^2}(s^{2/\alpha} - \alpha^2 + 1)w \\ &\quad + \frac{1}{\alpha^2}e^{2s^{1/\alpha}}s^{(2/\alpha)-2}(2+\sin(2\pi s))^2\Delta w, \end{aligned}$$

since $t'(s) = 1/\tau(s)$. Transforming the initial data, we obtain the following conditions for $w(s,\,x)$ at $s = s_0$:

$$\left.\begin{aligned} w(s_0,\,x) &= \sqrt{\tau(s_0)}\varphi(x) =: \widetilde{\varphi}(x), \\ w_s(s_0,\,x) &= \frac{1}{2}\frac{\tau'(s_0)}{\sqrt{\tau(s_0)}}\varphi(x) + \frac{1}{\sqrt{\tau(s_0)}}\psi(x) =: \widetilde{\psi}(x). \end{aligned}\right\} \tag{3.1}$$

The problem is now in the form

$$\left.\begin{aligned} &w_{ss} - \nu(s)^2 b(s)^2\Delta w + \mu(s)w = 0, \\ &w(s_0,\,x) = \widetilde{\varphi}(x),\ w_s(s_0,\,x) = \widetilde{\psi}(x), \end{aligned}\right\} \tag{3.2}$$

where

$$\begin{aligned} &\nu(s) := e^{s^{1/\alpha}}s^{(1/\alpha)-1},\ b(s) := (2+\sin(2\pi s))/\alpha, \\ &\mu(s) := \frac{1}{4\alpha^2 s^2}(\alpha^2 - 1 - s^{2/\alpha}) = O(s^{(2/\alpha)-2}) \quad \text{as } s \to \infty. \end{aligned}$$

Note that $b(s)$ is a non-constant, smooth, positive, periodic function with period 1. This is now in a form where the application of Floquet's theory is possible.

3.2. Application of Floquet's theory

Consider the second order ordinary differential equation for $v = v(s)$,

$$v_{ss} + \lambda b(s)^2 v = 0.$$

Let X be the fundamental matrix corresponding to this problem. That is, $X = X(s,\,s_0)$ solves the first order system of ordinary differential equations

$$d_s X = \begin{pmatrix} 0 & -\lambda b(s)^2 \\ 1 & 0 \end{pmatrix} X, \quad X(s_0, s_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.3}$$

We recall the following lemma from Floquet's theory (see for example [5], [16]):

Lemma 3.1 *Suppose $b(s)$ is a 1-periodic, non-constant, positive and smooth function on $\mathbb{R}$ and $s_0 \in \mathbb{N} \cup \{0\}$. Then there exists $\lambda_0 > 0$ such that the fundamental matrix $X(s, s_0)$ corresponding to $v_{ss} + \lambda_0 b(s)^2 v = 0$ evaluated at $s = s_0 + 1$ (i.e. $X(s_0 + 1, s_0)$) has eigenvalues μ_0, μ_0^{-1} and $|\mu_0| > 1$.*

We use this to approximate the solution to the ordinary differential equation

$$v_{ss} + \lambda(s, \xi) b(s)^2 v = 0, \tag{3.4}$$

with suitable Cauchy data, where $\lambda(s, \xi) = \lambda_1(s, \xi) + \lambda_2(s)$ and

$$\lambda_1(s, \xi) := |\xi|^2 \nu(s)^2 = |\xi|^2 s^{(2/\alpha)-2} e^{2s^{1/\alpha}},$$
$$\lambda_2(s) := \frac{\mu(s)}{b(s)^2} = \frac{\alpha^2 - 1 - s^{2/\alpha}}{4\alpha^2 s^2 b(s)^2}.$$

Observe that $\partial_s \lambda_1(s, \xi) = |\xi|^2 e^{2s^{1/\alpha}} \big(((2/\alpha)-2) s^{(2/\alpha)-3} + (2/\alpha) s^{(3/\alpha)-3}\big) > 0$ for $s > T_0$ for large enough T_0. Henceforth, we shall always assume $s > T_0$. So, for each $\xi \in \mathbb{R}^n$, $\lambda_1(s, \xi)$ is a monotonically increasing function in s on its domain $[s_0, \infty)$. Also, it is clear that $\lambda_2(s) \to 0$ as $s \to \infty$.
We also define $s_\xi \in \mathbb{N}$ implicitly by the formula $\lambda(s_\xi, \xi) = \lambda_0$, where λ_0 is from Lemma 3.1. In addition, we require that $s_\xi > T$ where T is large enough to ensure that $s_\xi \to \infty$ as $|\xi| \to 0$:

Lemma 3.2 *There exists $T > 0$ such that for s_ξ as defined*

$$s_\xi \to \infty \quad \textit{as } |\xi| \to 0.$$

Proof. Since $\lambda_2(s) \to 0$ as $s \to \infty$, we can choose $T_1 > 0$ such that $s > T_1$ implies that $|\lambda_2(s)| < \lambda_0/2$. Then, by definition, as we insist $s_\xi > T_1$,

$$\frac{\lambda_1(s_\xi, \xi)}{|\xi|^2} = \frac{\lambda_0}{|\xi|^2} - \frac{\lambda_2(s_\xi)}{|\xi|^2} \geq \frac{\lambda_0}{2|\xi|^2} \to \infty \quad \text{as } |\xi| \to 0.$$

Now, since $\lambda_1(s, \xi)/|\xi|^2$ is monotonically increasing for $s > T_0$, by setting $T := \max\{T_0, T_1\}$ it follows that $\lim_{|\xi| \to 0} s_\xi = \infty$. $\square$

We remark that this result allows us to take $s_\xi \in \mathbb{N}$ for *any* (large) integer—simply choose $|\xi|$ appropriately small enough.

3.3. Properties of $\boldsymbol{\lambda(s, \xi)}$ and $\boldsymbol{X(s_\xi + 1, s_\xi)}$

For the function $\lambda(s,\, \xi)$ we have the following result.

Lemma 3.3 *There exist constants* $0 < \rho < 1$ *and* $K > 0$ *such that if* $0 \le \delta \le \rho s - K$ *then*

$$\begin{aligned}
&|\lambda_1(s, \xi) - \lambda_1(s - \delta,\, \xi)| \le C\delta\lambda_1(s,\, \xi)s^{(1/\alpha)-1}\\
&\textit{and}\quad |\lambda_2(s) - \lambda_2(s - \delta)| \le Cs^{(1/\alpha)-1},
\end{aligned}$$

for some positive constant C.

Proof. For the first part we apply the mean value theorem; this implies that there exists a constant $\tilde{s} \in (s - \delta,\, s)$ such that

$$\begin{aligned}
&|\lambda_1(s,\, \xi) - \lambda_1(s - \delta,\, \xi)|\\
&= |\xi|^2 \big| s^{(2/\alpha)-2} e^{2s^{1/\alpha}} - (s - \delta)^{(2/\alpha)-2} e^{2(s-\delta)^{1/\alpha}} \big|\\
&\le \frac{2}{\alpha} |\xi|^2 \delta e^{2\tilde{s}^{1/\alpha}} \tilde{s}^{3((1/\alpha)-1)} \big| 1 + (1 - \alpha)\tilde{s}^{-1/\alpha} \big|\\
&\le C\delta s^{(1/\alpha)-1} \lambda_1(s,\, \xi) e^{2(\tilde{s}^{1/\alpha} - s^{1/\alpha})} (s/\tilde{s})^{3(1-(1/\alpha))}\\
&\le C\delta s^{(1/\alpha)-1} \lambda_1(s,\, \xi),
\end{aligned}$$

since $e^{2s^{1/\alpha}} s^{-3(1-(1/\alpha))}$ is monotonically increasing for large s (we define K so that this is for $s > K$) and $s - \delta > K$ by hypothesis.
For the second part, simply observe that, with $b_0 := \min_s b(s)$,

$$\begin{aligned}
&|\lambda_2(s) - \lambda_2(s - \delta)|\\
&\le \frac{1}{4\alpha^2 b_0^2} \big((\alpha^2 - 1) |s^{-2} - (s - \delta)^{-2}| + |s^{(2/\alpha)-2} - (s - \delta)^{(2/\alpha)-2}| \big)\\
&\le C s^{(2/\alpha)-2} \Big((\alpha^2 - 1) s^{-2/\alpha} \Big| 1 - \Big(\frac{s}{s - \delta} \Big)^2 \Big| + \Big| 1 - \Big(\frac{s}{s - \delta} \Big)^{2-(2/\alpha)} \Big| \Big)\\
&\le C s^{(1/\alpha)-1},
\end{aligned}$$

when $0 \le \delta \le \rho s - K$ for some $0 < \rho < 1$. □

Now consider the fundamental matrix $X(s,\, s_0)$, which was defined as the solution of the system of ordinary differential equations (3.3), evaluated at the point $s = s_\xi + 1$, $s_0 = s_\xi$. We write this matrix as

$$X(s_\xi + 1,\, s_\xi) = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}.$$

By Lemma 3.1 this matrix has eigenvalues μ_0, μ_0^{-1} where $|\mu_0| > 1$. Observe that

$$\mu_0 + \mu_0^{-1} = \operatorname{tr} X(s_\xi + 1,\, s_\xi) = x_{11} + x_{22},$$

and so,

$$|\mu_0^{-1} - \mu_0| \le |x_{11} - \mu_0| + |x_{22} - \mu_0|.$$

Hence,

$$\max\{|x_{11} - \mu_0|,\, |x_{22} - \mu_0|\} \ge \frac{1}{2}|\mu_0^{-1} - \mu_0| > 0.$$

The last inequality follows from $|\mu_0| > 1$. We can assume, without loss of generality, that

$$|x_{11} - \mu_0| \ge \frac{1}{2}|\mu_0^{-1} - \mu_0|, \tag{3.5}$$

and then we also have

$$|x_{22} - \mu_0^{-1}| \ge \frac{1}{2}|\mu_0^{-1} - \mu_0|. \tag{3.6}$$

3.4. Auxiliary family of ODEs

Consider the family of ODEs

$$v_{ss} + \lambda(s_\xi - k + s,\, \xi)b(s_\xi + s)^2 v = 0, \quad k \in \mathbb{N} \cup \{0\},$$

where $s_\xi \in \mathbb{N}$ is as in Section 3.2 and λ_0 is as given in Lemma 3.1. Here we are using the 1-periodicity of $b(s)$.

To each problem associate the fundamental matrix $X_k(s,\, s_1)$ which satisfies

$$d_s X_k = \begin{pmatrix} 0 & -\lambda(s_\xi - k + s)b(s_\xi + s)^2 \\ 1 & 0 \end{pmatrix} X_k,\;\; X_k(s_1,\, s_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We study these matrices evaluated at $(s,\, s_1) = (1,\, 0)$; write

$$X_k(1,\, 0) = \begin{pmatrix} x_{11}(k) & x_{12}(k) \\ x_{21}(k) & x_{22}(k) \end{pmatrix}. \tag{3.7}$$

Denote the eigenvalues of this matrix by μ_k, μ_k^{-1} where $|\mu_k| \geq 1$ (in fact, later we see that $|\mu_k| > 1$ for all suitable k). That this matrix has determinant 1 is an immediate consequence of the formula for the derivative of the determinant of a matrix and the fact that $\operatorname{tr}(A_k(s,\,\xi)) = 0$ where

$$A_k(s,\,\xi) = \begin{pmatrix} 0 & -\lambda(s_\xi - k + s,\,\xi)b(s_\xi + s)^2 \\ 1 & 0 \end{pmatrix}. \tag{3.8}$$

The matrices $X_k(1,\,0)$ are uniformly bounded for suitably large k:

Lemma 3.4 *Let $F(s)$ be a function satisfying*

$$\lim_{s\to\infty} s^{(1/\alpha)-1}F(s) = 0. \tag{3.9}$$

Then we have

$$\max_{s,\,s_1\in[0,\,1]} \|X_k(s,\,s_1)\| \leq e^{C\lambda_0}$$

for $1 \leq k \leq cF(s_\xi)$ and some positive constants C, c.

Remark 3.1 Note that $F(s) := s^\beta$, where $\beta < 1-(1/\alpha)$, satisfies requirement (3.9).

Proof. We have the following representation for $X_k(s,s_1)$:

$$X_k(s,\,s_1) = I + \sum_{j=1}^{\infty}\int_{s_1}^{s} A_k(r_1,\,\xi)\int_{s_1}^{r_1} A_k(r_2,\,\xi) \cdots \int_{s_1}^{r_{j-1}} A_k(r_j,\,\xi)dr_j\cdots dr_2dr_1,$$

where $A_k(s,\,\xi)$ is as in (3.8). Now, by Lemma 3.3,

$$\begin{aligned}
\|A_k(s,\,\xi)\| &\leq 1 + b_1^2 \sup_{s\in[0,\,1]} |\lambda(s_\xi - k + s,\xi)| \\
&\leq 1 + b_1^2\Big|\lambda_1(s_\xi - k + 1,\,\xi) + \sup_{s>s_0}\lambda_2(s)\Big| \\
&= 1 + b_1^2\Big|\lambda_1(s_\xi - k + 1,\,\xi) - \lambda_1(s_\xi,\,\xi) - \lambda_2(s_\xi) + \lambda_0 + \sup_{s>s_0}\lambda_2(s)\Big| \\
&\leq 1 + b_1^2\Big(C(k-1)\lambda_1(s_\xi,\,\xi)s_\xi^{(1/\alpha)-1} + \lambda_0 + 2\sup_{s>s_0}|\lambda_2(s)|\Big)
\end{aligned}$$

provided $0 \leq k-1 \leq \rho s_\xi - K$; here $b_1 = \max_s b(s)$. So, by (3.9), $\|A_k(s,\,\xi)\| \leq 1 + C_1 b_1^2\lambda_0$ for large s_ξ when $1 \leq k \leq cF(s_\xi)$; here c is chosen to ensure that $k-1 \le \rho s_\xi - K$ is satisfied when $k \le cF(s_\xi)$. Therefore,

$$\max_{s,\, s_1 \in [0,\,1]} \|X_k(s,\, s_1)\| \le \exp\Big(\int_{s_1}^{s} \|A_k(r,\, \xi)\| dr\Big) \le C_0 e^{C_1 b_1^2 \lambda_0} = e^{C\lambda_0},$$

provided $1 \le k \le cF(s_\xi)$. The lemma is proved. $\square$

The next lemma shows that in some sense $X(s_\xi + 1,\, s_\xi)$ is "near" to the $X_k(1,\, 0)$ for suitable k.

Lemma 3.5 *Under the assumptions of* Lemma 3.4 *we have*

$$\|X_k(1,\, 0) - X(s_\xi + 1,\, s_\xi)\| \le C\lambda_0 s_\xi^{(1/\alpha)-1} F(s_\xi)$$

for $1 \le k \le cF(s_\xi)$ *and some positive constants* C, c.

Proof. First, note that $X(s_\xi + s,\, s_\xi) = X(s,\, 0)$, since $s_\xi \in \mathbb{N}$ and $b(s)$ is 1-periodic. Now, $X_k(s,\, 0)$ satisfies

$$\begin{aligned} d_s X_k(s,\, 0) &= \begin{pmatrix} 0 & -\lambda(s_\xi,\, \xi) b(s)^2 \\ 1 & 0 \end{pmatrix} X_k(s,\, 0) \\ &\quad + \begin{pmatrix} 0 & \big(\lambda(s_\xi,\, \xi) - \lambda(s_\xi - k + s,\, \xi)\big) b(s)^2 \\ 0 & 0 \end{pmatrix} X_k(s,\, 0), \end{aligned}$$

with $X_k(0,\, 0) = I$. Thus,

$$\begin{aligned} d_s\big(X_k(s,\, 0) - X(s,\, 0)\big) &= \begin{pmatrix} 0 & -\lambda(s_\xi,\, \xi) b(s)^2 \\ 1 & 0 \end{pmatrix} \big(X_k(s,\, 0) - X(s,\, 0)\big) \\ &\quad + \begin{pmatrix} 0 & \big(\lambda(s_\xi,\, \xi) - \lambda(s_\xi - k + s,\, \xi)\big) b(s)^2 \\ 0 & 0 \end{pmatrix} \\ &\quad \times X_k(s,\, 0), \end{aligned}$$

with initial data $X_k(0,\, 0) - X(0,\, 0) = 0$; here 0 denotes the zero matrix. Now, by Lemma 3.3,

$$\begin{aligned} &|\lambda_1(s_\xi,\, \xi) - \lambda_1(s_\xi - k + s,\, \xi)| \le C(k-s)\lambda_1(s_\xi,\, \xi) s_\xi^{(1/\alpha)-1} \\ &\text{and } |\lambda_2(s_\xi) - \lambda_2(s_\xi - k + s)| \le C s_\xi^{(1/\alpha)-1} \le C\lambda_0^{-1} k s_\xi^{(1/\alpha)-1} \lambda(s_\xi,\, \xi) \end{aligned}$$

for $0 \le k - s \le \rho s_\xi - K$. Therefore,

$$|\lambda(s_\xi,\, \xi) - \lambda(s_\xi - k + s,\, \xi)| \le Ck\lambda_0 s_\xi^{(1/\alpha)-1}$$

for $0 \le k - s \le \rho s_\xi - K$. Hence,

$$\|X_k(s,\,0) - X(s,\,0)\| \le \int_0^s C\lambda_0 \|X_k(r,\,0) - X(r,\,0)\| dr + \int_0^s Ck\lambda_0 s_\xi^{(1/\alpha)-1} \|X_k(r,\,0)\| dr.$$

So, by Lemma 3.4, Gronwall's inequality and the hypotheses on k,

$$\|X_k(1,\,0) - X(1,\,0)\| \le Ck\lambda_0 s_\xi^{(1/\alpha)-1} e^{C\lambda_0} \le C\lambda_0 s_\xi^{(1/\alpha)-1} F(s_\xi),$$

where $1 \le k \le cF(s_\xi)$; c here is chosen as in the proof of Lemma 3.4. This completes the proof of the lemma. □

Also, the $X_k(1,\,0)$ are, in a similar sense, "near" to each other.

Lemma 3.6 *The following inequality holds for all* $1 \le k \le cF(s_\xi)$*, with* c *as in* Lemma 3.4*,*

$$\|X_{k+1}(1,\,0) - X_k(1,\,0)\| \le C\lambda_0 s_\xi^{(1/\alpha)-1},$$

where C *is a positive constant and* $F(s)$ *satisfies* (3.9)*.*

Proof. Observe

$$\begin{aligned} &d_s\big(X_k(s,\,0) - X_{k+1}(s,\,0)\big) \\ &= \begin{pmatrix} 0 & -\lambda(s_\xi - (k+1) + s,\,\xi) b(s)^2 \\ 1 & 0 \end{pmatrix} \big(X_k(s,\,0) - X_{k+1}(s,\,0)\big) \\ &+ \begin{pmatrix} 0 & \big(\lambda(s_\xi - (k+1) + s,\,\xi) - \lambda(s_\xi - k + s,\,\xi)\big) b(s)^2 \\ 0 & 0 \end{pmatrix} X_k(s,\,0). \end{aligned}$$

By Lemma 3.3,

$$\begin{aligned} &|\lambda_1(s_\xi - (k+1) + s,\,\xi) - \lambda_1(s_\xi - k + s,\,\xi)| \\ &\le C\lambda_1(s_\xi - k + s,\,\xi)(s_\xi - k + s)^{(1/\alpha)-1} \\ &\text{and } |\lambda_2(s_\xi - (k+1) + s) - \lambda_2(s_\xi - k + s)| \le C(s_\xi - k + s)^{(1/\alpha)-1} \end{aligned}$$

for $1 \le \rho(s_\xi - k + s) - K$. This latter condition is satisfied when $1 \le k \le cF(s_\xi)$, where c is as in Lemma 3.4, for s_ξ chosen large enough. Now, $\lambda_1(t,\,\xi)t^{(1/\alpha)-1}$ is increasing for large t (see the proof of Lemma 3.3); in particular, it is increasing for $t \ge s_\xi - k + s$ when s_ξ is chosen large enough since

$$s_\xi - k + s \geq s_\xi - cF(\xi) \geq s_\xi - \frac{1}{2}s_\xi^{1-(1/\alpha)} \geq \frac{1}{2}s_\xi$$

by the hypotheses on k and $F(s)$. Also, using the argument above and the hypothesis on $F(s)$,

$$(s_\xi - k + s)^{(1/\alpha)-1} \leq \left(\frac{2}{s_\xi}\right)^{1-(1/\alpha)} \leq C\lambda_0 s_\xi^{(1/\alpha)-1}$$

when $k \leq cF(s_\xi)$. Thus,

$$|\lambda(s_\xi - (k+1) + s,\, \xi) - \lambda(s_\xi - k + s,\, \xi)| \leq C\lambda_0 s_\xi^{(1/\alpha)-1}$$

for $1 \leq k \leq cF(s_\xi)$. So, by a similar argument to that used in the proof of Lemma 3.5,

$$\|X_{k+1}(1,\, 0) - X_k(1,\, 0)\| \leq C\lambda_0 s_\xi^{(1/\alpha)-1} e^{C\lambda_0} \leq C\lambda_0 s_\xi^{(1/\alpha)-1}$$

for $1 \leq k \leq cF(s_\xi)$, as required. □

These "nearness" lemmas give information about the relations of the eigenvalues of these matrices.

Corollary 3.7 *For $1 \leq k \leq cF(s_\xi)$, where $F(s)$ satisfies* (3.9) *and $c > 0$ is as in* Lemma 3.4*, the following relations hold for suitably large s_ξ:*

1. *for each $\varepsilon > 0$, we can choose s_ξ large enough so that*

$$|\mu_k - \mu_0| < \varepsilon, \tag{3.10}$$

hence, for suitably chosen ε,

$$|\mu_k| \geq |\mu_0| - \varepsilon > 1; \tag{3.11}$$

2. *there exists $C > 0$ such that*

$$|\mu_k - \mu_{k-1}| \leq C\lambda_0 s_\xi^{(1/\alpha)-1}; \tag{3.12}$$

3. *there exists $C > 0$ such that*

$$|\mu_k - \mu_k^{-1}| \geq C; \tag{3.13}$$

4. *there exist constants C_1, $C_2 > 0$ such that*

$$|\mu_k - x_{11}(k)| \geq C_1, \tag{3.14}$$

$$|\mu_k^{-1} - x_{22}(k)| \geq C_2. \tag{3.15}$$

Proof. *to* 1. By Lemma 3.5,

$$\begin{aligned}|\mu_k + \mu_k^{-1} - \mu_0 - \mu_0^{-1}| &= |x_{11}(k) + x_{22}(k) - x_{11} - x_{22}|\\ &\le |x_{11}(k) - x_{11}| + |x_{22}(k) - x_{22}|\\ &\le C\lambda_0 s_\xi^{(1/\alpha)-1} F(s_\xi) \to 0 \quad \text{as } s_\xi \to \infty.\end{aligned}$$

On the other hand

$$|\mu_k + \mu_k^{-1} - \mu_0 - \mu_0^{-1}| = |(\mu_k - \mu_0)(1 - (\mu_k\mu_0)^{-1})| \ge C|\mu_k - \mu_0|,$$

where $C > 0$, since $|\mu_0| > 1$. Combining these two observations proves (3.10).
to 2. By Lemma 3.6

$$\begin{aligned}&|\mu_k - \mu_{k-1} + \mu_k^{-1} - \mu_{k-1}^{-1}|\\ &= |x_{11}(k) - x_{11}(k-1) + x_{22}(k) - x_{22}(k-1)| \le C\lambda_0 s_\xi^{(1/\alpha)-1}.\end{aligned}$$

Choosing s_ξ large enough so that (3.11) holds, we see, by a similar argument to 1., that (3.12) also holds.
to 3. This is clear since (3.11) holds for large enough s_ξ.
to 4. By (3.5), Lemma 3.5 and part 1. of this corollary we have the following for large enough s_ξ:

$$\begin{aligned}|\mu_k - x_{11}(k)| &= |\mu_k - \mu_0 + \mu_0 - x_{11} + x_{11} - x_{11}(k)|\\ &\ge |\mu_0 - x_{11}| - |\mu_k - \mu_0| - |x_{11} - x_{11}(k)|\\ &\ge \frac{1}{2}|\mu_0 - \mu_0^{-1}| - \frac{1}{8}|\mu_0 - \mu_0^{-1}| - \frac{1}{8}|\mu_0 - \mu_0^{-1}|\\ &= \frac{1}{4}|\mu_0 - \mu_0^{-1}| > 0.\end{aligned}$$

This proves (3.14). The proof of (3.15) is similar, but we use (3.6) in place of (3.5). □

Henceforth, we assume that s_ξ is chosen large enough so that all of the inequalities in Corollary 3.7 hold.

3.5. Lower bound for solution to auxiliary Cauchy problem

We are now in a position to give a lower bound for the solution to a Cauchy problem for (3.4).

Proposition 3.8 *Consider the Cauchy problem*

$$\left.\begin{aligned} &v_{ss} + \lambda(s,\,\xi)b(s)^2 v = 0,\\ &v(s_\xi - n_0,\,\xi) = 1,\ v_s(s_\xi - n_0,\,\xi) = \frac{x_{12}(n_0)}{\mu_{n_0} - x_{11}(n_0)}, \end{aligned}\right\} \tag{3.16}$$

where $cs_\xi^\beta - 1 \le n_0 \le cs_\xi^\beta$ *for* $1/\alpha < \beta < 1 - 1/\alpha$ *and* c *is some positive constant. Then the following estimate holds for the solution* $v = v(s,\,\xi)$ *at* $s = s_\xi$*:*

$$|v(s_\xi,\,\xi)| + |v_s(s_\xi,\,\xi)| \ge C\exp\biggl(a\Bigl(\log\Bigl|\frac{1}{\xi}\Bigr|\Bigr)^\gamma\biggr),$$

where $\gamma = \alpha\beta \in (1,\,\alpha - 1)$ *and* C, a *are positive constants.*

Proof. Throughout this proof we assume that $k \le n_0$ at each occurrence of k.

Observe that

$$\begin{pmatrix} v_s(s_\xi,\,\xi)\\ v(s_\xi,\,\xi)\end{pmatrix} = X_1(1,\,0)X_2(1,\,0) \cdots X_{n_0}(1,\,0)\begin{pmatrix} v_s(s_\xi - n_0,\,\xi)\\ v(s_\xi - n_0,\,\xi)\end{pmatrix}, \tag{3.17}$$

where $X_k(1,\,0)$ is as in (3.7). Now,

$$B_k = \begin{pmatrix} x_{12}(k)/\bigl(\mu_k - x_{11}(k)\bigr) & 1\\ 1 & x_{21}(k)/\bigl(\mu_k^{-1} - x_{22}(k)\bigr)\end{pmatrix},$$

is a diagonaliser for $X_k(1,\,0)$. This is a consequence of the facts that $\det X_k(1,\,0) = 1$ and $\operatorname{tr} X_k(1,\,0) = x_{11}(k) + x_{22}(k) = \mu_k + \mu_k^{-1}$.

Observe that

$$\|B_k\| \le C, \tag{3.18}$$

for some constant C independent of k; this follows from Lemma 3.4 and inequalities (3.14) and (3.15). Furthermore, B_k is invertible for each k since

$$\det B_k = \frac{\mu_k - \mu_k^{-1}}{\mu_k^{-1} - x_{22}(k)},$$

and (3.13) ensures that this is non-zero, together with Lemma 3.4 and $|\mu_k^{-1}| < 1$. From this and (3.18), it follows that, in addition, $\|B_k^{-1}\| \le C$ for some constant independent of k.

Also, by (3.14), (3.15), Lemma 3.4 and (3.12),

$$\begin{aligned}
&\|B_{k+1} - B_k\| \\
&\quad \le \max_{1\le k\le n_0} \|X_k(1,\, 0)\|(C_1|\mu_k - \mu_{k+1}| + C_2|\mu_k^{-1} - \mu_{k+1}^{-1}|) \\
&\quad \le C\lambda_0 s_\xi^{(1/\alpha)-1}. \qquad (3.19)
\end{aligned}$$

Hence, (3.17) can be rewritten as

$$\begin{aligned}
\begin{pmatrix} v_s(s_\xi,\, \xi) \\ v(s_\xi,\, \xi) \end{pmatrix} &= B_1 \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_1^{-1} \end{pmatrix} B_1^{-1} B_2 \begin{pmatrix} \mu_2 & 0 \\ 0 & \mu_2^{-1} \end{pmatrix} B_2^{-1} \\
&\qquad \cdots B_{n_0} \begin{pmatrix} \mu_{n_0} & 0 \\ 0 & \mu_{n_0}^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= B_1 \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\end{aligned}$$

Set $B_k^{-1} B_{k+1} = I + G_k$. So,

$$\begin{aligned}
\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} &= \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_1^{-1} \end{pmatrix} (I + G_1) \begin{pmatrix} \mu_2 & 0 \\ 0 & \mu_2^{-1} \end{pmatrix} \\
&\qquad \cdots (I + G_{n_0-1}) \begin{pmatrix} \mu_{n_0} & 0 \\ 0 & \mu_{n_0}^{-1} \end{pmatrix} \qquad (3.20) \\
&= \begin{pmatrix} \prod_{k=1}^{n_0} \mu_k & 0 \\ 0 & \prod_{k=1}^{n_0} \mu_k^{-1} \end{pmatrix} + M_1 + \cdots + M_{n_0-1},
\end{aligned}$$

where M_l is the matrix which is the sum of all the products of matrices from (3.20) containing exactly l of the G_k matrices; observe

$$\|M_l\| \le \left(\prod_{k=1}^{n_0} |\mu_k| \right) \left(\sum_{1\le i_1<\cdots<i_l\le n_0-1} \prod_{j=1}^{l} \|G_{i_j}\| \right).$$

By (3.18) and (3.19)

$$\begin{aligned}
\|G_k\| = \|B_k^{-1} B_{k+1} - I\| &= \|B_k^{-1}(B_{k+1} - B_k)\| \\
&\le \|B_k^{-1}\| \|B_{k+1} - B_k\| \le C\lambda_0 s_\xi^{(1/\alpha)-1}.
\end{aligned}$$

Therefore,

$$\|M_l\| \le \left(\prod_{k=1}^{n_0} |\mu_k| \right) \binom{n_0 - 1}{l} (C\lambda_0 s_\xi^{(1/\alpha)-1})^l.$$

Thus,

$$|y_{11}| \geq \Big(\prod_{k=1}^{n_0} |\mu_k|\Big)\Big(2 - \big(1 + C\lambda_0 s_\xi^{(1/\alpha)-1}\big)^{c s_\xi^\beta}\Big).$$

Taking account of $\beta < 1 - 1/\alpha$ gives immediately

$$|y_{11}| \geq \frac{1}{2}\Big(\prod_{k=1}^{n_0} |\mu_k|\Big).$$

On the other hand, $|y_{21}|$ is very small—it is less than $\nu \prod_{k=1}^{n_0} |\mu_k|$, where we can take ν as small as we like. Hence, using

$$\begin{pmatrix} v_s(s_\xi, \xi) \\ v(s_\xi, \xi) \end{pmatrix} = \begin{pmatrix} x_{12}(1)/\big(\mu_1 - x_{11}(1)\big) & 1 \\ 1 & x_{21}(1)/\big(\mu_1^{-1} - x_{22}(1)\big) \end{pmatrix} \begin{pmatrix} y_{11} \\ y_{21} \end{pmatrix}$$

and (3.11), it follows that

$$|v_s(s_\xi,\, \xi)| + |v(s_\xi,\, \xi)| \geq C(|\mu_0| - \varepsilon)^{n_0} \geq C e^{a s_\xi^\beta},$$

for some positive constants a, C. Finally, for large s_ξ,

$$s_\xi \sim \Big(\log \frac{1}{|\xi|}\Big)^\alpha, \tag{3.21}$$

and so we have the desired inequality. The proposition is proved. □

3.6. Lower bound for the energy of $w(s_\xi,\, x)$

We return to the transformed Cauchy problem (3.2) with initial time chosen as $s_0 = s_\xi - n_0$ and seek a representation for the solution at time $s = s_\xi$ in the unit ball $B_1(0)$. By the existence of a cone of dependence, this only depends on the initial data in the ball $B_R(0)$ at $s = s_\xi - n_0$, where $R = R(n_0,\, b) \leq C n_0 \min_s b(s)$. Set

$$\widetilde{\varphi}(x) = e^{ix\cdot\xi}\chi(x/R^2), \quad \widetilde{\psi}(x) = \frac{x_{12}(n_0)}{\mu_{n_0} - x_{11}(n_0)} e^{ix\cdot\xi}\chi(x/R^2) \tag{3.22}$$

to be the data at $s = s_0$, where $\chi(x)$ is a smooth cut-off function which is identically 1 on $|x| < 1$. By the uniqueness of solutions to strictly hyperbolic equations, the solution can be represented in the cone of dependence, and therefore in $B_1(0)$ at $s = s_\xi$, by

$$w = w(s,\, x) = e^{ix\cdot\xi} v(s,\, \xi);$$

here $v(s,\, x)$ is the solution to (3.16) at time s. Use $w(s_\xi,\, x,\, \xi) = e^{ix\cdot\xi}v(s_\xi,\, \xi)$ to denote this solution. Then the following lower bound holds for w:

$$\begin{aligned}
&\|\nabla_x w(s_\xi, \cdot)\|_{L^q} + \|w_s(s_\xi, \cdot)\|_{L^q} \\
&\quad \geq \|\nabla_x w(s_\xi, \cdot)\|_{L^q(B_1(0))} + \|w_s(s_\xi, \cdot)\|_{L^q(B_1(0))} \\
&\quad = (|\xi||v(s_\xi, \xi)| + |v_s(s_\xi, \xi)|)\,\mathrm{meas}(B_1(0))^{1/q} \\
&\quad \geq C|\xi| \exp\Big(a\Big(\log\frac{1}{|\xi|}\Big)^\gamma\Big),
\end{aligned} \tag{3.23}$$

where $L^q = L^q(\mathbb{R}^n)$.

3.7. Lower bound for the energy of $u(\tau_\xi,\, x)$

Finally, we return to the original problem (1.8).

Set $t_0 = t(s_0) = t(s_\xi - n_0) = e^{(s_\xi - n_0)^{1/\alpha}} - e^3$ and choose the following initial data:

$$\varphi(x) = \frac{1}{\sqrt{\tau(s(t_0))}} e^{ix\cdot\xi}\chi(x/R^2) = \frac{1}{\sigma(t_0)} e^{ix\cdot\xi}\chi(x/R^2), \tag{3.24}$$

$$\begin{aligned}
\psi(x) &= \Big(\sqrt{\tau(s(t_0))}\frac{x_{12}(n_0)}{\mu_{n_0} - x_{11}(n_0)} - \frac{\tau'(s(t_0))}{2\sqrt{\tau(s(t_0))}}\Big) e^{ix\cdot\xi}\chi(x/R^2) \\
&= \Big(\sigma(t_0)\frac{x_{12}(n_0)}{\mu_{n_0} - x_{11}(n_0)} - \frac{\sigma'(t_0)}{\sigma(t_0)^2}\Big) e^{ix\cdot\xi}\chi(x/R^2),
\end{aligned} \tag{3.25}$$

where $\sigma(t)$ is as in Theorem 1.2. Here we have taken into account (3.1) and (3.22). Now, by (3.23), the energy defined in Theorem 1.2 for $u = u(t,\, x)$ at $t = \tau_\xi := t(s_\xi) = e^{s_\xi^{1/\alpha}} - e^3$ can be estimated as follows:

$$\begin{aligned}
E(u)(\tau_\xi)\big|_{L^q} &= \Big\|\sigma(\tau_\xi)\nabla_x u(\tau_\xi, \cdot)\Big\|_{L^q} + \Big\|\frac{1}{\sigma(\tau_\xi)^2}\partial_t\big(u(t, \cdot)\sigma(t)\big)\big|_{t=\tau_\xi}\Big\|_{L^q} \\
&= \|\nabla_x w(s_\xi, \cdot)\|_{L^q} + \|w_s(s_\xi, \cdot)\|_{L^q} \\
&\geq C|\xi| \exp\Big(a\Big(\log\frac{1}{|\xi|}\Big)^\gamma\Big) \\
&\geq C\exp\big[-c_1 s_\xi^{1/\alpha} + ac_2 s_\xi^\beta\big] \\
&= C\exp\big[-c_1\log(\tau_\xi + e^3) + ac_2\big(\log(\tau_\xi + e^3)\big)^\gamma\big],
\end{aligned} \tag{3.26}$$

and $1 < \gamma < \alpha - 1$; here we have used (3.21) in the final equality. If we now assume (1.9) holds with the initial data (3.24), (3.25), then, for $1 < r < \gamma < \alpha - 1$,

$$
\begin{aligned}
E(u)(\tau_\xi)\big|_{L^q} &\le C_1 \exp\left(C_2(\log(\tau_\xi + e^3))^r\right)E(u)(t_0)\big|_{W^{M,p}}\\
&= C_1 e^{C_2(\log(\tau_\xi+e^3))^r}\left(1+\frac{x_{12}(n_0)}{\mu_{n_0}-x_{11}(n_0)}\right)\|e^{ix\cdot\xi}\chi(x/R^2)\|_{W^{M+1,p}},
\end{aligned}
$$

which contradicts (3.26) since $r < \gamma$. The proof of Theorem 1.2 is complete. $\square$

4. Concluding remarks

Remark 4.1 We observe that the L^p-L^q estimate from Corollary 2.15 derived for the cases very slow, slow, fast oscillations is of the form

$$E(u)(t)\big|_{L^q} \le C(1+t)^{s_0}E(u)(t_0)\big|_{W^{N_p,p}} \tag{4.1}$$

with a positive constant C independent of $t_0 \ge T$ and $t \ge t_0$. However, Theorem 1.2 states that in the case of very fast oscillations we cannot have an estimate of the form

$$E(u)(t)\big|_{L^q} \le C_1 \exp(C_2(\log(t+e^3))^r)E(u)(t_0)\big|_{W^{N_p,p}} \tag{4.2}$$

for $1 < r < \alpha - 1$ with positive constants C_1 and C_2 independent of $t_0 \ge T$ and $t \ge t_0$. Comparing (4.1) with (4.2) we have indeed an essential change in the behaviour of solutions to (1.5) from fast to very fast oscillations.

Remark 4.2 A special case of (4.2) is the L^2-L^2 estimate

$$E(u)(t)\big|_{L^2} \le C_1 \exp(C_2(\log(t+e^3))^r)E(u)(0)\big|_{L^2}$$

which cannot hold for $1 < r < \alpha - 1$. But the critical case $r = \alpha - 1$ cannot be excluded. In the case of very fast oscillations this critical L^2-L^2 estimate is much better than the L^2-L^2 estimate we obtain by applying Gronwall's inequality.

Remark 4.3 If we apply the change of variables $\tilde{t} = \Lambda(t) := \int_0^t \sqrt{a(\tau)}d\tau$ to (1.5), then we obtain the damped wave equation

$$\partial_{\tilde{t}}^2 u - \Delta u + b(\tilde{t})\partial_{\tilde{t}} u = 0 \quad \text{with } b(\tilde{t}) = \frac{(\sqrt{a})'(\Lambda^{-1}(\tilde{t}))}{a(\Lambda^{-1}(\tilde{t}))}.$$

Choosing $a(t) = 2 + \sin((\log(t+e^3))^\alpha)$ with $\alpha \in [1,\, 2]$ we get

$$\partial_{\tilde{t}}^2 u - \Delta u + b(\tilde{t})\partial_{\tilde{t}} u = 0$$

$$\text{with} \quad b(\tilde{t}) = \frac{\alpha \cos((\log(\Lambda^{-1}(\tilde{t}) + e^3))^\alpha)(\log(\Lambda^{-1}(\tilde{t}) + e^3))^{\alpha-1}}{a^{3/2}(\Lambda^{-1}(\tilde{t}))(\Lambda^{-1}(\tilde{t}) + e^3)}.$$

The term $b(\tilde{t})$ represents an oscillating dissipation with changing sign and with asymptotic behaviour $O((\log(\tilde{t} + e^3))^{\alpha-1}/\tilde{t})$ for $\tilde{t} \to \infty$. Here we have used $\tilde{t} \sim t$. The L^p-L^q decay estimate from Theorem 1.1 implies that

$$\|(u_{\tilde{t}}(\tilde{t}, \cdot), \nabla_x u(\tilde{t}, \cdot))\|_{L^q} \le C(1 + \tilde{t})^{-\frac{n-1}{2}(\frac{1}{p} - \frac{1}{q}) + s_0} \|(\nabla_x \varphi, \psi)\|_{W^{N_p, p}}$$

for the solution to the Cauchy problem

$$\partial_{\tilde{t}}^2 u - \Delta u + b(\tilde{t}) \partial_{\tilde{t}} u = 0, \ u(0, x) = \varphi(x), \ \partial_{\tilde{t}} u(0, x) = \psi(x).$$

Thus, we have an example for L^p-L^q decay estimates for solutions to the Cauchy problem for damped wave equations with non-monotone weak dissipation changing its sign. This result stimulates further considerations, generalizing the results of [17] and [9] for damped wave equations with monotone weak dissipation.

Remark 4.4 More general models which will be considered in forthcoming papers are $u_{tt} - a(t)\Delta u + m(t)u = 0$ or $u_{tt} - a(t)\Delta u + b(t)u_t = 0$, where $a = a(t)$ is a bounded coefficient. The following questions appear:

- How do the mass or dissipation change the classification of oscillations?
- How can we derive statements like those of Theorems 1.1 and 1.2 for solutions to more general models? How do we feel weak damping or overdamping?

Remark 4.5 The results of this paper complete the picture about the theory of degenerate hyperbolic problems [7]. This picture can be presented now in the following form:

strictly hyperbolic theory with non-Lipschitz continuous coefficients	$\longleftrightarrow$	L^p-L^q decay estimates for wave equations with bounded speed of propagation
$\updownarrow$		$\updownarrow$
weakly hyperbolic theory	$\longleftrightarrow$	L^p-L^q decay estimates for wave equations with increasing in time coefficients

Remark 4.6 The construction of counterexamples via Floquet's theory, as used in the proof of Theorem 1.2, can be adapted to constructing blow-up solutions for related nonlinear Cauchy problems. This is done in [18], where it is shown that, for carefully chosen initial data, no global solution exists to the Cauchy problem ($\alpha \in (-\infty, -1)$)

$$\left.\begin{aligned} &\partial_t^2 u - \exp(2t^\alpha) b(t)^2 \Delta u + (\partial_t u)^2 - \exp(2t^\alpha) b(t)^2 \sum_{j=1}^n (\partial_{x_j} u)^2 = 0, \\ &u(1, x) = \varphi(x), \quad u_t(1, x) = \psi(x). \end{aligned}\right\}$$

(Here $b(t)$ is as in the Introduction.)
This result shows us that it is reasonable to use the results of the present paper to study the *global existence of small data solutions*. We have the following two conjectures:

- *Conjecture* 1 (Floquet effect).
  In general, we have no global existence of small data solutions for the Cauchy problem

$$\left.\begin{aligned} &\partial_t^2 u - (2 + \sin(2\pi (\log(t + e^3))^\alpha))^2 \Delta u + (\partial_t u)^2 - \\ &\qquad\qquad (2 + \sin(2\pi (\log(t + e^3))^\alpha))^2 \sum_{j=1}^n (\partial_{x_j} u)^2 = 0, \\ &u(1, x) = \varphi(x), \quad u_t(1, x) = \psi(x), \end{aligned}\right\}$$

  when $\alpha > 2$.
- *Conjecture* 2 (L^p-L^q decay estimates).
  We have the global existence of small data solutions for the Cauchy problem

$$\left.\begin{aligned} &\partial_t^2 u - (2 + \sin(2\pi (\log(t + e^3))^\alpha))^2 \Delta u + (\partial_t u)^2 - \\ &\qquad\qquad (2 + \sin(2\pi (\log(t + e^3))^\alpha))^2 \sum_{j=1}^n (\partial_{x_j} u)^2 = 0, \\ &u(1, x) = \varphi(x), \quad u_t(1, x) = \psi(x), \end{aligned}\right\}$$

  when $\alpha \leq 2$; or, more generally, for the Cauchy problem

$$\left.\begin{aligned} &\partial_t^2 u - a(t) \Delta u + (\partial_t u)^2 - a(t) \sum_{j=1}^n (\partial_{x_j} u)^2 = 0, \\ &u(1, x) = \varphi(x), \quad u_t(1, x) = \psi(x), \end{aligned}\right\}$$

when $a(t)$ satisfies the assumption (1.6) for $\gamma \in [0, 1]$.

Acknowledgments The research of the first author was supported by the DFG grant A/03/11412. The second author would like to thank TU Bergakademie Freiberg for their warm hospitality and assistance during his stay in April–June 2003 and DAAD for the support in this period.

References

[1] Brenner P., *On L_p-$L_{p'}$ estimates for the wave-equation.* Math. Z. (3) **145** (1975), 251–254.

[2] Hirosawa F., *On the Cauchy problem for second order strictly hyperbolic equations with non–regular coefficients.* Math. Nachr. **256** (2003), 29–47.

[3] Hörmander L., *Estimates for translation invariant operators in L_p spaces.* Acta Math. **104** (1960), 93–140.

[4] Littman W., *Fourier transforms of surface carried measures and differentiability of surface averages.* Bull. Amer. Math. Soc. **69** (1963), 766–770.

[5] Magnus W. and Winkler S., *Hill's equation*, Interscience Tracts in Pure and Applied Mathematics, No. 20, Interscience Publishers John Wiley and Sons, New York-London-Sydney, 1966.

[6] Pecher H., *L_p-Abschätzungen und klassische Lösungen für nichtlineare Wellengleichungen.* I. Math. Z. (2) **150** (1976), 159–183.

[7] Reissig M., *A refined diagonalization procedure to handle fast oscillations in degenerate hyperbolic problems.* Hyperbolic problems and related topics (F. Colombini and T. Nishitani, eds.), International Press, Somerville, 2003, 295–318.

[8] Reissig M., *Hyperbolic equations with non-Lipschitz coefficients.* Rendiconti Del Seminario Matematico Torino (2) **61** (2003), 135–182.

[9] Reissig M. and Wirth J., *Wave equations with monotone weak dissipation.* Preprint 03-2, Fakultät für Mathematik und Informatik, Technische Universität Bergakademie Freiberg, ISSN 1433-9307, pp. 71.

[10] Reissig M. and Yagdjian K., *L_p-L_q estimates for the solutions of hyperbolic equations of second order with time dependent coefficients—oscillations via growth—.* Preprint 98-5, Fakultät für Mathematik und Informatik, Technische Universität Bergakademie Freiberg, ISSN 1433-9307, pp. 103.

[11] Reissig M. and Yagdjian K., *One application of Floquet's theory to L_p-L_q estimates for hyperbolic equations with very fast oscillations.* Math. Meth. in the Appl. Sci. **22** (1999), 937–951.

[12] Reissig M. and Yagdjian K., *About the influence of oscillations on Strichartz–type decay estimates.* Rendiconti Del Seminario Matematico Torino (3) **58** (2000), 375–388.

[13] Reissig M. and Yagdjian K., *L_p-L_q decay estimates for hyperbolic equations with oscillations in coefficients.* Chin. Ann. of Math. (2) **21B** (2000), 153–164.

[14] Reissig M. and Yagdjian K., *L_p-L_q decay estimates for the solutions of strictly hyperbolic equations of second order with increasing in time coefficients*. Math. Nachr. **214** (2000), 71–104.

[15] Strichartz R. S., *A priori estimates for the wave equation and some applications*. J. Functional Analysis **5** (1970), 218–235.

[16] Tarama S., *On the second order hyperbolic equations degenerating in the infinite order. Example*. Math. Japon. (3) **42** (1995), 523–533.

[17] Wirth J., *Solution representations for a wave equation with weak dissipation*. Math. Methods Appl. Sci. (1) **27** (2004), 101–124.

[18] Yagdjian K., *Parametric resonance and nonexistence of the global solution to nonlinear wave equations*. J. Math. Anal. Appl. (1) **260** (2001), 251–268.

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