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An extension of the univalence criteria of Nehari and Ozaki and Nunokawa

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Abstract. In this paper, we obtain a sufficient condition for the univalence of analytic functions in the open unit disk $\mathbb{U}$. This condition involves two arbitrary functions $g(z)$ and $h(z)$ analytic in $\mathbb{U}$. Replacing $g(z)$ and $h(z)$ by some particular functions, we find the well-known conditions for univalency established by Z. Nehari (*Bull. Amer. Math. Soc.* **55** (1949)) and S. Ozaki and M. Nunokawa (*Proc. Amer. Math. Soc.* **33** (1972)). Likewise we find other new sufficient conditions.

Key words: univalent function, Löwner chain, Nehari criterion, Ozaki criterion.

1. Introduction

We denote by $\mathbb{U}_r = \{z \in \mathbb{C}\colon |z| < r\}$ the disk of z-plane, where $r \in (0, 1]$, $\mathbb{U}_1 = \mathbb{U}$ and $I = [0, \infty)$. Let $\mathcal{A}$ be the class of functions $f(z)$ which are analytic in $\mathbb{U}$ with the normalizations $f(0) = 0$ and $f'(0) = 1$. In the present paper, we consider the following conditions for univalency of functions $f(z)$ belonging to the class $\mathcal{A}$.

Theorem 1.1 ([1]) *Let $f(z) \in \mathcal{A}$. If, for all $z \in \mathbb{U}$, $f(z)$ satisfies*

$$\bigl|\{f\,;\,z\}\bigr| \leqq \frac{2}{(1-|z|^2)^2}, \tag{1.1}$$

where

$$\{f\,;\,z\} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2, \tag{1.2}$$

then the function $f(z)$ is univalent in $\mathbb{U}$.

Theorem 1.2 ([2]) *Let $f(z) \in \mathcal{A}$. If, for all $z \in \mathbb{U}$, $f(z)$ satisfies*

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| < 1, \tag{1.3}$$

then the function $f(z)$ is univalent in $\mathbb{U}$.

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Example 1.1 If we take Koebe functon $f(z) = z/(1-z)^2$ which is the extremal function for the class of starlike functions in $\mathbb{U}$, then

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| = |-z^2| < 1 \qquad (z \in \mathbb{U}).$$

2. Preliminaries

Our considerations are based on the theory of Löwner chains. We first recall here the following basic result of this theory by Pommerenke.

Theorem 2.1 ([4]) *Let $L(z, t) = a_1(t)z + a_2(t)z^2 + \cdots,\ a_1(t) \neq 0$ be analytic in $\mathbb{U}_r$ for all $t \in I$, locally absolutely continuous in I, and locally uniform with respect to $\mathbb{U}_r$. For almost all $t \in I$ suppose that*

$$z\frac{\partial L(z,\,t)}{\partial z} = p(z,\,t)\frac{\partial L(z,\,t)}{\partial t} \quad (\forall z \in \mathbb{U}_r),$$

where $p(z, t)$ is analytic in $\mathbb{U}$ and satisfies the condition $\operatorname{Re} p(z, t) > 0$ for all $z \in \mathbb{U},\ t \in I$. If $|a_1(t)| \to \infty$ for $t \to \infty$ and $\{L(z, t)/a_1(t)\}$ forms a normal family in $\mathbb{U}_r$, then, for each $t \in I$, the function $L(z, t)$ has an analytic and univalent extension to the whole disk $\mathbb{U}$.

3. Main results

Main theorem of our paper is contained in

Theorem 3.1 *Let $f(z) \in \mathcal{A}$. If, for $g(z) = 1 + b_1 z + \cdots$ and $h(z) = c_0 + c_1 z + \cdots$ which are analytic in $\mathbb{U}$, the following inequalities*

$$\left|\frac{f'(z)}{g(z)} - 1\right| < 1, \tag{3.1}$$

and

$$\begin{aligned}&\left|\left(\frac{f'(z)}{g(z)} - 1\right)|z|^4 + z(1-|z|^2)|z|^2\left(2\frac{f'(z)h(z)}{g(z)} + \frac{g'(z)}{g(z)}\right)\right.\\ &\qquad\left. + z^2(1-|z|^2)^2\left(\frac{f'(z)(h(z))^2}{g(z)} + \frac{g'(z)h(z)}{g(z)} - h'(z)\right)\right|\\ &\leqq |z|^2\end{aligned} \tag{3.2}$$

hold true for all $z \in \mathbb{U}$, then the function $f(z)$ is univalent in $\mathbb{U}$.

Proof. Let us consider the function $h_1(z, t)$ given by

$$h_1(z, t) = 1 + (e^t - e^{-t})zh(e^{-t}z).$$

For all $t \in I$ and $z \in \mathbb{U}$ we have $e^{-t}z \in \mathbb{U}$ and from the analyticity of $h(z)$ in $\mathbb{U}$ it follows that $h_1(z, t)$ is also analytic in $\mathbb{U}$. Since $h_1(0, t) = 1$, there exists a disk $\mathbb{U}_r,\ 0 < r < 1$ in which $h_1(z, t) \neq 0$ for all $t \in I$. Then the function $L(z, t)$ defined by

$$L(z, t) = f(e^{-t}z) + \frac{(e^t - e^{-t})zg(e^{-t}z)}{1 + (e^t - e^{-t})zh(e^{-t}z)}$$

is analytic in $\mathbb{U}_r$ for all $t \in I$ and has the following form

$$L(z, t) = a_1(t)z + a_2(t)z^2 + \cdots,$$

where $a_1(t) = e^t,\ a_1(t) \neq 0$ for all $t \in I$ and $\lim_{t\to\infty} |a_1(t)| = \infty$.
From the analyticity of $L(z, t)$ in $\mathbb{U}_r$, it follows that there exists a number r_1, $0 < r_1 < r$, and a constant $K = K(r_1)$ such that

$$\left|\frac{L(z, t)}{a_1(t)}\right| < K \quad (\forall z \in \mathbb{U}_{r_1},\ t \in I).$$

In consequence, the family $\{L(z, t)/a_1(t)\}$ is normal in $\mathbb{U}_{r_1}$. From the analyticity of $\partial L(z, t)/\partial t$, for all fixed numbers $T > 0$ and r_2, $0 < r_2 < r_1$, there exists a constant $K_1 > 0$ (that depends on T and r_2) such that

$$\left|\frac{\partial L(z, t)}{\partial t}\right| < K_1 \quad (\forall z \in \mathbb{U}_{r_2},\ t \in [0, T]).$$

It follows that the function $L(z, t)$ is locally absolutely continuous in I, locally uniform with respect to $\mathbb{U}_{r_2}$. Let us define the functions $p(z, t)$ and $w(z, t)$ by

$$p(z, t) = z\frac{\partial L(z, t)}{\partial z} \Big/ \frac{\partial L(z, t)}{\partial t}$$

and

$$w(z, t) = \frac{p(z, t) - 1}{p(z, t) + 1}.$$

Then the function $p(z, t)$ is analytic in $\mathbb{U}_{r_3}$, $0 < r_3 < r_2$, and the function $p(z, t)$ has an analytic extension with positive real part in $\mathbb{U}$, for all $t \in I$,

if the function $w(z,\, t)$ can be continued analytically in $\mathbb{U}$ and $|w(z,\, t)| < 1$ for all $z \in \mathbb{U}$ and $t \in I$.
After simple computation, we obtain that

$$\begin{aligned} w(z,\, t) &= \left(\frac{f'(e^{-t}z)}{g(e^{-t}z)} - 1\right) e^{-2t} \\ &+ (1-e^{-2t})e^{-t}z\left(\frac{2f'(e^{-t}z)h(e^{-t}z)}{g(e^{-t}z)} + \frac{g'(e^{-t}z)}{g(e^{-t}z)}\right) \\ &+ (1-e^{-2t})^2 z^2 \\ &\times \left(\frac{f'(e^{-t}z)(h(e^{-t}z))^2}{g(e^{-t}z)} + \frac{g'(e^{-t}z)h(e^{-t}z)}{g(e^{-t}z)} - h'(e^{-t}z)\right). \end{aligned} \tag{3.3}$$

From (3.1) and (3.2), we deduce that $g(z) \neq 0$ for all $z \in \mathbb{U}$ and then the function $w(z,\, t)$ is analytic in $\mathbb{U}$. In view of (3.1) and (3.3), we have

$$w(0,\, t) = 0 \quad \text{and} \quad |w(z,\, 0)| = \left|\frac{f'(z)}{g(z)} - 1\right| < 1. \tag{3.4}$$

If $t > 0$ is a fixed number and $z \in \mathbb{U}$, $z \neq 0$, then the function $w(z,\, t)$ is analytic in $\bar{\mathbb{U}}$ because $|e^{-t}z| \leq e^{-t} < 1$ for all $z \in \bar{\mathbb{U}}$, and it is known that

$$|w(z,\, t)| = \max_{|\zeta|=1} |w(\zeta,\, t)| = |w(e^{i\theta},\, t)|, \quad \theta = \theta(t) \in \mathcal{R}. \tag{3.5}$$

Let us denote by $u = e^{-t}e^{i\theta}$. Then $|u| = e^{-t}$ and, from (3.3), we get

$$\begin{aligned} |w(e^{i\theta},\, t)| = \Bigg| &\left(\frac{f'(u)}{g(u)} - 1\right)|u|^2 + (1-|u|^2)u\left(\frac{2f'(u)h(u)}{g(u)} + \frac{g'(u)}{g(u)}\right) \\ &+ (1-|u|^2)^2\frac{u^2}{|u|^2}\left(\frac{f'(u)(h(u))^2}{g(u)} + \frac{g'(u)h(u)}{g(u)} - h'(u)\right)\Bigg|. \end{aligned}$$

Since $u \in \mathbb{U}$, the relation (3.2) implies $|w(e^{i\theta},\, t)| \leq 1$ and, from (3.4) and (3.5), we conclude that $|w(z,\, t)| < 1$ for all $z \in \mathbb{U}$ and $t \in I$. This gives us that $L(z,\, t)$ is the Löwner chain and hence the function $L(z,\, 0) = f(z)$ is univalent in $\mathbb{U}$. □

We can get some corollaries for special cases of functions $g(z)$ and $h(z)$. So in the particular case $g(z) = f'(z)$ as a direct consequence of Theorem 3.1, we get

Theorem 3.2 *Let $f(z) \in \mathcal{A}$. If, for an analytic function $h(z) = c_0 + c_1 z + \cdots$ in $\mathbb{U}$, $f(z)$ satisfies*

$$\Bigg|(1-|z|^2)|z|^2\left(2h(z)+\frac{f''(z)}{f'(z)}\right) + z(1-|z|^2)^2\left((h(z))^2+\frac{f''(z)h(z)}{f'(z)}-h'(z)\right)\Bigg| \leqq |z| \tag{3.6}$$

for all $z \in \mathbb{U}$, then the function $f(z)$ is univalent in $\mathbb{U}$.

If we take

$$h(z) = -\frac{1}{2}\frac{f''(z)}{f'(z)} \tag{3.7}$$

in Theorem 3.2, then we have

Corollary 3.1 ([1]) *If $f(z) \in \mathcal{A}$ satisfies the inequality* (1.1) *for all $z \in \mathbb{U}$, then the function $f(z)$ is univalent in $\mathbb{U}$.*

Proof. For the function $h(z)$ defined by (3.7), the Schwartzian derivative (1.2) shows that

$$(h(z))^2+\frac{f''(z)h(z)}{f'(z)}-h'(z) = \frac{1}{2}\left[\frac{f'''(z)}{f'(z)}-\frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2\right] = \frac{1}{2}\{f\,;\,z\}.$$

and then the inequality (3.6) becomes (1.1). □

In the particular case $g(z) = (f(z)/z)^2$ in Theorem 3.1, we have

Theorem 3.3 *Let $f(z) \in \mathcal{A}$. If, for an analytic function $h(z) = c_0 + c_1 z + \cdots$ in $\mathbb{U}$, $f(z)$ satisfies*

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| < 1 \tag{3.8}$$

and

$$\Bigg|\left(\frac{z^2 f'(z)}{(f(z))^2}-1\right)|z|^4 + 2z(1-|z|^2)|z|^2\left(\frac{z^2 f'(z)h(z)}{(f(z))^2}+\frac{f'(z)}{f(z)}-\frac{1}{z}\right) + z^2(1-|z|^2)^2$$

$$\times \left[\frac{z^2 f'(z)(h(z))^2}{(f(z))^2} + 2h(z)\left(\frac{f'(z)}{f(z)} - \frac{1}{z} \right) - h'(z) \right] \Bigg| \leqq |z|^2 \qquad (3.9)$$

for all $z \in \mathbb{U}$, then the function $f(z)$ is univalent in $\mathbb{U}$.

We remark that the inequality (3.8) is just the inequality (1.3) and we will get the univalent criterion by Ozaki and Nunokawa [2] for a particular choise of the function $h(z)$. So, if we take in Theorem 3.3

$$h(z) = \frac{1}{z} - \frac{f(z)}{z^2}, \qquad (3.10)$$

then we obtain

Corollary 3.2 ([2]) *If $f(z) \in \mathcal{A}$ satisfies the inequality* (1.3) *for all $z \in \mathbb{U}$, then the function $f(z)$ is univalent in $\mathbb{U}$.*

Proof. For the function $h(z)$ defined by (3.10), we see that

$$\frac{z^2 f'(z)h(z)}{(f(z))^2} + \frac{f'(z)}{f(z)} - \frac{1}{z} = \frac{zf'(z)}{(f(z))^2} - \frac{1}{z}$$

and

$$\frac{z^2 f'(z)(h(z))^2}{(f(z))^2} + 2h(z)\left(\frac{f'(z)}{f(z)} - \frac{1}{z} \right) - h'(z) = \frac{f'(z)}{(f(z))^2} - \frac{1}{z^2}.$$

The inequality (3.9) becomes

$$\left| \left(\frac{z^2 f'(z)}{(f(z))^2} - 1 \right) \left(|z|^4 + 2|z|^2(1-|z|^2) + (1-|z|^2)^2 \right) \right| \leqq |z|^2,$$

and then

$$\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| \leqq |z|^2. \qquad (3.11)$$

It is easy to prove that if the inequality (1.3) is true, then the inequality (3.11) is also true. Indeed, if we put

$$w(z) = \frac{z^2 f'(z)}{(f(z))^2} - 1,$$

then the function $w(z)$ is analytic in $\mathbb{U}$ and, since $f(z) \in \mathcal{A}$, we observe that

$$w(z) = d_2 z^2 + d_3 z^3 + \cdots ,$$

which shows that $w(0) = w'(0) = 0$. By inequality (1.3), we have $|w(z)| < 1$. Thus the Schwartz's lemma gives us that $|w(z)| < |z|^2$. □

Finally, we give a example for Corollary 3.2.

Example 3.1 Let us consider the function $f(z)$ given by

$$f(z) = \frac{z}{1 + \sum_{n=2}^{\infty} \{2/\big(n(n^2-1)\big)\} z^n}.$$

Then we have that

$$\frac{z^2 f'(z)}{(f(z))^2} - 1 = -\sum_{n=2}^{\infty} \frac{2}{n(n+1)} z^n,$$

which gives that

$$\left| \frac{z^2 f'(z)}{f(z)^2} - 1 \right| < 2 \sum_{n=2}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1.$$

Therefore, the function $f(z)$ is univalent in $\mathbb{U}$.

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