Hokkaido Mathematical Journal Vol. 34 (2005) p. 513–531

Some results on the heat kernel asymptotics of the Laplace operator on Finsler spaces

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(Received July 22, 2003; Revised November 16, 2004)

Abstract. In this paper we consider Bao-Lackey's extension of the Laplace operator on a Finsler space. We prove that this operator is of Laplace type on scalars and on top degree forms, and compute the first heat coefficients. In exchange, the BL Laplacian on 1-forms is nonminimal and a study of its heat kernel asymptotics is more difficult. The results obtained in this paper for the 1-formed Laplacian concern Finsler surfaces and direct products of Finsler surfaces. We apply our computation of the heat coefficients to prove that, on Randers spaces, the scalar BL Laplacian and the scalar Laplacian of the metric a_{ij} have the same eigenvalues if and only if the Randers space is Riemann.

Key words: Finsler spaces, Laplace operator, asymptotic expansion.

1. Introduction

Throughout this paper (M, F) is a compact n-dimensional Finsler manifold. Thus, the Finsler function $F(x, y)$ defined on the slit tangent bundle $\widetilde{TM}$ is positive homogeneous of degree one in y and the Finsler metric tensor $g_{ij} := (1/2)\partial^2 F^2/\partial y^i \partial y^j$ is positive definite.

In a known paper ([7]), Bao and Lackey introduced the Laplacian on the base manifold M and obtained a Hodge decomposition theorem.

We will denote this extension by Δ_{BL} and, when it is restricted to $A^p(M)$, by $(\Delta_{BL})_p$. Let us recall below the Bao-Lackey method ([7], [8]).

Consider the indicatrix bundle SM with base manifold M and fibers $S_xM := \{y \in T_xM : F(x, y) = 1\}$. The volume form of S_xM is well-known:

$$\theta_x := \sqrt{g}\sum_{i=1}^{n}(-1)^{i-1} y^i dy^1 \wedge \ldots \wedge dy^{i-1} \wedge dy^{i+1} \wedge \ldots \wedge dy^n.$$

It is a fact that $\mathrm{vol}(x)$, the volume of S_xM, generally depends on x. For this reason one considers the normalized volume form $\zeta_x := (1/\mathrm{vol}(x))\theta_x$.

There are several ways to introduce a volume form on M. For instance, $\sqrt{G}dx := (\int_{S_xM}\sqrt{g}(x, y)\zeta_x)dx$ is a volume form. Then, the metric on

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2000 Mathematics Subject Classification : 53C60, 53C21, 58C40.

$A^p(M)$ is defined by $G(\omega, \theta) = (1/p!)\omega_I \theta_J G^{IJ}$, where $\omega = (1/p!)\omega_I dx^I$, $\theta = (1/p!)\theta_J dx^J$, $I = (i_1, \ldots, i_p)$, $J = (j_1, \ldots, j_p)$ and

$$G^{IJ}(x) := G^{i_1 j_1 \cdots i_p j_p}(x) = \frac{1}{\sqrt{G}} \int_{S_x M} \sqrt{g} g^{i_1 j_1} \cdots g^{i_p j_p} \zeta_x.$$

One can see that $G^{i_1 j_1 \cdots i_p j_p}(x)$ does not reduce to $G^{i_1 j_1}(x) \cdots G^{i_p j_p}(x)$, so that we have different metrics on each space $A^p(M)$. If

$$\langle \omega, \theta \rangle_M := \int_M G(\omega, \theta) \sqrt{G} dx$$

denotes the global scalar product on $A^p(M)$ and $\langle\ ,\ \rangle_{SM}$ is the standard global scalar product on SM given by the Sasaki lift of g, then it can be checked the fundamental property: $\langle \pi^* \omega, \pi^* \theta \rangle_{SM} = \langle \omega, \theta \rangle_M$

Consider d, the exterior differential on M and d^*_{BL}, the adjoint of d with respect to $\langle\ ,\ \rangle_M$. The BL Laplacian is defined by

$$\Delta_{BL} := d d^*_{BL} + d^*_{BL} d.$$

It is clear that Δ_{BL} reduces to the standard Laplacian if the Finsler structure is Riemannian.

At this point, it is natural to investigate other ways to extend the Laplacian on Finsler spaces. Certainly, we expect that such an extension reduce to the standard Laplacian for Riemannian structures. Such is the case of the Laplacian introduced by Antonelli and Zastawniak from the viewpoint of diffusion theory ([3]), or of the so-called Mean-Value Laplacian of Centore ([9]) and of the non-linear Laplacian defined by Shen ([18]). The excellent book [2] gathers some interesting results concerning these non-equivalent extensions.

In a study of the heat kernel asymptotics, the BL Laplacian has the advantage of being defined in a way that enables us to make connections with Laplace type operators and with the so-called nonminimal operators.

2. Heat kernel asymptotics of Δ_{BL}

Our objective in this section is to obtain information on the asymptotic expansion of the heat kernel of Δ_{BL}. This makes sense because the BL-Laplacian is an elliptic operator. There are several methods for effective calculation of the heat kernel invariants ([4]). We will use the so-called invariant method of P. Gilkey.

We begin with two useful settings. Let us consider the strictly positive functions $K,\, \mu \in C^\infty(M)$,

$$K := \sqrt{G} \cdot \sqrt{\det \int g^\sharp} \quad \text{and} \quad \mu := \frac{\int \det g^\sharp}{\det \int g^\sharp},$$

where $\det \int g^\sharp := \det(G^{ij})$ and $\int \det g^\sharp := (1/\sqrt{G}) \int_{S_x M} (1/\sqrt{g}) \zeta_x$.

It is a consequence of Jensen's inequality that K and μ satisfy $K^2 \mu \geq 1$. Equality holds iff $\sqrt{g}$ depends only on x, that is iff g_{ij} depends only on x.

These two functions play an important role in the theory of Finslerian Laplacians. For instance, for Finsler surfaces, $(\Delta_{BL})_1$ admits a Weitzenböck formula if and only if the 'geometric ratio' μ satisfies: $1/3 \leq \mu \leq 3$ (see [8]). Both functions become constant equal to 1 for Riemannian Finsler structure, i.e., $g_{ij} = g_{ij}(x)$.

Let $\langle\ ,\ \rangle$ be the Riemannian global scalar product on M, defined by the metric G^{ij}. We stress that for $\omega,\, \theta \in A^1(M)$ and for $\varphi,\, \psi \in A^n(M)$ we have

$$\langle \omega,\, \theta \rangle_M = \langle \omega,\, K\theta \rangle, \quad \langle \varphi,\, \psi \rangle_M = \langle \varphi,\, K\mu\psi \rangle.$$

These two formulas will be used when we discuss on the heat kernel asymptotic expansion of $(\Delta_{BL})_0$ and $(\Delta_{BL})_n$.

2.1. Heat kernel asymptotic expansion of the scalar Laplacian

In order to obtain the formula of the scalar Laplacian $(\Delta_{BL})_0$, observe that for $\omega \in A^1(M)$ and $f \in C^\infty(M)$ it holds:

$$\langle d^*_{BL}\omega,\, f \rangle_M = \langle \omega,\, df \rangle_M = \langle d^*(K\omega),\, f \rangle = \left\langle \frac{1}{K} d^*(K\omega),\, f \right\rangle_M.$$

Now, using the formula $d^*(K\omega) = Kd^*\omega - G(\omega,\, dK)$, it follows that

$$(\Delta_{BL})_0 f = \Delta_0 f - G(df,\, d\log K),$$

where Δ_0 is the standard Laplacian on M, defined by G_{ij}.

It is useful to regard $A^0(M) = C^\infty(M)$ as a vector bundle with the (conformal) metric $K(\ ,\ )$ and the base manifold M with the metric G_{ij}, the inverse of G^{ij}. Then the scalar Laplacian is of Laplace type and one can verify that

$$(\Delta_{BL})_0 = \tilde{\nabla}^* \tilde{\nabla} - \mathcal{E},$$

where the connection $\tilde{\nabla}$ and the endomorphism $\mathcal{E}$ on $C^\infty(M)$ are given by:

$$\tilde{\nabla}_X f = Xf + \frac{1}{2}(X \log K)f;$$
$$\mathcal{E} = \frac{1}{2}\left(\Delta \log K - \frac{1}{2}G(d\log K,\, d\log K)\right).$$

Such a Weitzenböck formula was investigated in [15] for 1-forms. In the quoted paper, the Laplace-type operator acts on a fiber bundle (for instance SM) and the function K normalizes the volumes of the fibers (the function $\mathrm{vol}(x)$ on SM).

Now let us return to our framework. As a direct consequence of Theorem 4.8.16 from [11], for the heat kernel asymptotics of the scalar Laplacian

$$\mathrm{Tr}(e^{-t(\Delta_{BL})_0}) \sim \sum_{k\geq 0}(4\pi)^{-n/2}t^{(k-n)/2}a_k\big((\Delta_{BL})_0\big)$$

one can compute the following coefficients

$$a_0\big((\Delta_{BL})_0\big) = \int_M dv$$
$$a_2\big((\Delta_{BL})_0\big) = \frac{1}{6}\int_M (\tau + 6\mathcal{E})dv$$
$$a_4\big((\Delta_{BL})_0\big) = \frac{1}{360}\int_M \big(5(\tau + 6\mathcal{E})^2 + 2|R|^2 - 2|\mathrm{Ric}|^2\big)dv,$$

where R, Ric and τ are the curvature tensor, the Ricci tensor and the scalar curvature of the metric G_{ij}, respectively.

Now let $\mathrm{spec}(M,F)$ denote the spectrum of the scalar Laplacian $(\Delta_{BL})_0$. Consider another compact Riemannian manifold $(M^{'},\, g^{'})$. At this point we ask whether some geometric properties of $(M^{'},\, g^{'})$ are reflected by spectrality.

Proposition 1 *Suppose that* $\dim M = 3$, $(M^{'},\, g^{'})$ *has constant curvature and* $\mathrm{spec}(M,\, F) = \mathrm{spec}(M^{'},\, g^{'})$. *Then K is constant and M is of the same constant curvature as $M^{'}$.*

Proof. We use that $|R|^2 \geq |\mathrm{Ric}|^2$, where equality holds if and only if M has constant curvature. Therefore,

$$\int_M (\tau + 6\mathcal{E})dv = \int_{M'} \tau^{'} dv^{'}, \quad \int_M (\tau + 6\mathcal{E})^2 dv \leq \int_{M'} (\tau^{'})^2 dv'$$

and by the Schwartz inequality we get

$$\mathrm{vol}(M)\int_M(\tau+6\mathcal{E})^2dv\le\Big(\int_{M'}\tau' dv'\Big)^2=\Big(\int_M(\tau+6\mathcal{E})dv\Big)^2.$$

Again, the Schwartz inequality implies that $\tau+6\mathcal{E}$ is constant on M. Therefore, $\tau+6\mathcal{E}=\tau'$ and $|R|^2=|\mathrm{Ric}|^2$. Clearly, M has constant curvature and $\mathcal{E}=\mathrm{const}$. Moreover, since $\int_M\mathcal{E}dv\le 0$ we obtain that $\mathcal{E}\le 0$. The proof is now obvious using the maximum principle of E. Hopf. □

It is a fact that generally K is not a constant function on M. Then the Finsler space cannot be isospectral to a Riemannian space of constant curvature. The two-dimensional analogue of Proposition 1 is given by

Proposition 2 *If* $\dim M=2$, (M',g') *has constant positive curvature and* $\mathrm{spec}(M,F)=\mathrm{spec}(M',g')$, *then* K *is constant and* M *is of the same constant positive curvature as* M'.

Proof. As a whole, this proof is similar to the one given above. We have:

$$\int_M(\tau+6\mathcal{E})dv=\tau'\,\mathrm{vol}(M'),$$
$$\int_M\big(5(\tau+6\mathcal{E})^2+\tau^2\big)dv=6(\tau')^2\,\mathrm{vol}(M').$$

Next, use that

$$\begin{aligned}6\Big(\int_M(\tau+6\mathcal{E})dv\Big)^2&=6(\tau')^2\,\mathrm{vol}(M')^2\\&=\mathrm{vol}(M)\int_M\big(5(\tau+6\mathcal{E})^2+\tau^2\big)dv\\&\ge 5\Big(\int_M(\tau+6\mathcal{E})dv\Big)^2+\mathrm{vol}(M)\int_M\tau^2dv\end{aligned}$$

to obtain the following inequality: $\mathrm{vol}(M)\int_M\tau^2dv\le(\int_M(\tau+6\mathcal{E})dv)^2$.

We also have $0\le\tau'\,\mathrm{vol}(M')=\int_M(\tau+6\mathcal{E})dv\le\int_M\tau dv$, so that $(\int_M(\tau+6\mathcal{E})dv)^2\le(\int_M\tau dv)^2$. This implies $\tau=\mathrm{const.}$ and, the same as above, we get $K=\mathrm{const.}$ □

2.2. Heat kernel asymptotic expansion of the Laplacian on forms

At this moment, our purpose is to determine the leading symbol of $(\Delta_{BL})_n$. It is important to notice that for any $\omega,\theta\in A^n(M)$, $G(\omega,\theta)=$

$\mu(\omega,\,\theta)$, where $(\omega,\,\theta)$ is the usual Riemannian scalar product defined by G^{ij}. Further, we wish to express simpler the metrics on $A^{n-1}(M)$. This is necessary since the Laplacian involves the metrics on both $A^n(M)$ and $A^{n-1}(M)$. Let us denote by $N := (1,\,\ldots,\,n)$ and for $r \in \{1,\,\ldots,\,n\}$ by $N_r := (1,\,\ldots,\,r-1,\,r+1,\,\ldots,\,n)$. Then any $n-1$ forms φ and ψ have local expressions

$$\varphi = \sum_r \varphi_{N_r} dx^{N_r}, \quad \psi = \sum_s \psi_{N_s} dx^{N_s}$$

and the metrics act on $A^{n-1}(M)$ by

$$G(\varphi,\,\psi) = \sum_{r,\,s} (-1)^{r+s} \bar{G}_{rs} \varphi_{N_r} \psi_{N_s},\ (\varphi,\,\psi) = \sum_{r,\,s} (-1)^{r+s} G^*_{rs} \varphi_{N_r} \psi_{N_s},$$

where $\bar{G}_{rs}$ and G^*_{rs} are given by

$$\bar{G}_{rs} := \frac{1}{\sqrt{G}} \int_{S_xM} \frac{g_{rs}}{\sqrt{g}} \zeta_x \quad \text{and} \quad G^*_{rs} := \frac{G_{rs}}{\det G}.$$

Now we are ready to compute the leading symbol of $(\Delta_{BL})_n$. First, note that we have

$$\langle d^*_{BL}\omega,\,\theta\rangle_M = \langle \omega,\,d\theta\rangle_M = \langle \mu K\omega,\,\theta\rangle = \langle d^*(\mu K\omega),\,\theta\rangle.$$

Next, it follows that

$$\sum_i (-1)^i \bar{G}_{ri} (d^*_{BL}\omega)_{N_i} = \sum_j (-1)^j G^*_{rj} \left(\frac{1}{K} d^*(\mu K\omega)\right)_{N_j}.$$

Let $(\bar{G}^{ri})$ denote the inverse matrix of $(\bar{G}_{ri})$; then the following formula holds:

$$(d^*_{BL}\omega)_{N_i} = \sum_j (-1)^{i+j} G^*_{rj} \bar{G}^{ri} \left(\frac{1}{K} d^*(\mu K\omega)\right)_{N_j}.$$

If we want to determine only the leading symbol, we may neglect the 0-order terms of $d^*(\mu K\omega)$. Then, modulo 0-order terms, by direct calculus is found that for $\omega = f dv$, $f \in C^\infty(M)$,

$$(d^*_{BL}\omega)_{N_i} = \sum_h (-1)^i \mu \frac{\bar{G}^{hi}}{\det G} \left(\frac{\partial f}{\partial x^h}\right) \sqrt{\det G}.$$

Therefore, the leading symbol of $(\Delta_{BL})_n$ is simply

$$(\Delta_{BL})_n = -\sigma_n^{ih}\partial_i\partial_h + \cdots, \quad \text{where} \quad \sigma_n^{ih} := \mu\frac{\bar{G}^{hi}}{\det G}.$$

Notice that one can naturally define the metric

$$H_{ij} := \sqrt{G}\int_{S_xM}\frac{g_{ij}}{\sqrt{g}}\zeta_x.$$

Then σ_n^{ih} and H_{ij} are tied by $\sigma_n^{ih} = \mu K^2 H^{ij}$, where (H^{ij}) is the inverse of (H_{ij}). However, we see that $(\Delta_{BL})_n$ is of Laplace type: choose on M the conformal Riemannian metric $(1/(\mu K^2))H_{ij}$; then the leading symbol is given precisely by this metric tensor. Clearly, $(\Delta_{BL})_n$ is self-adjoint with respect to $\mu K^2(\sqrt{G}/\sqrt{H})G(\ ,\ )$, where $\sqrt{H} := \sqrt{\det(H_{ij})}$.

Albeit the Weitzenböck formula of $(\Delta_{BL})_n$ can be obtained taking into account the 0-order terms of $d^*(\mu K\omega)$, we restrict below to Finsler surfaces, i.e. $\dim M = 2$. The reason is that for Finsler surfaces $K^2H^{ij} = G^{ij}$.

The Weitzenböck formula can be obtained by direct calculus:

$$(\Delta_{BL})_2 = \tilde{\nabla}^*\tilde{\nabla} - \Sigma,$$

where the following settings were made

$$\begin{aligned} \tilde{\nabla}_X &:= \bar{\nabla}_X + \frac{1}{2}X\log K;\\ \Sigma &:= -\frac{1}{2}\mu\bigg(\Delta\log K + \frac{1}{2}G(d\log K,\, d\log K)\bigg). \end{aligned}$$

Here $\bar{\nabla}$ is the Levi-Civita connection of the metric $\bar{G}^{ij} := \mu G^{ij}$.

As a consequence, we get the heat kernel asymptotic expansion of $(\Delta_{BL})_2$

$$\operatorname{Tr}\big(e^{-t(\Delta_{BL})_2}\big) \sim \sum_{k\geq 0}\frac{1}{4\pi}t^{k/2-1}a_k\big((\Delta_{BL})_2\big),$$

where the coefficients are given by

$$\begin{aligned} a_0\big((\Delta_{BL})_2\big) &= \int_M \frac{1}{\mu}dv\\ a_2\big((\Delta_{BL})_2\big) &= \frac{1}{6}\int_M\frac{1}{\mu}(\bar{\tau}+6\Sigma)dv = a_2\big((\Delta_{BL})_0\big)\\ a_4\big((\Delta_{BL})_2\big) &= \frac{1}{360}\int_M\frac{1}{\mu}\big(5(\bar{\tau}+6\Sigma)^2 + 2|\bar{R}|^2 - 2|\overline{\mathrm{Ric}}|^2\big)dv. \end{aligned}$$

Here $\bar{R}$, $\overline{\mathrm{Ric}}$ and $\bar{\tau}$ are the curvature tensor, the Ricci tensor and the scalar curvature of the metric $\bar{G}_{ij}$, respectively.

Different from the cases studied above, the leading symbol of $(\Delta_{BL})_1$ is generally not given by a metric tensor on M. For example, for Finsler surfaces $(\Delta_{BL})_1$ is of Laplace-type if and only if $\mu = 1$.

The leading symbol of $(\Delta_{BL})_p$ is computed in [8] for an arbitrary manifold. For Finsler surfaces and $p = 1$, it has the simple formula

$$(\sigma_1^{ij})_k^s = \mu G^{ij}\delta_k^s + (1-\mu)G^{sj}\delta_k^i.$$

Notice that if μ is constant on M, then σ_1 is the leading symbol of a weighted Laplacian: $\sigma\big((\Delta_{BL})_1\big) = \sigma(dd^* + \mu d^* d)$.

Let us return to the case $d\mu \neq 0$. Without using the Weitzenböck formula (which, however, is complicated enough even for Finsler surfaces) we can obtain the heat kernel invariants of the Laplace operator.

Theorem 1 *The heat kernel of the BL-Laplacian on* 1*-forms has the following asymptotics*

$$\mathrm{Tr}\big(e^{-t(\Delta_{BL})_1}\big) \sim \sum_{k\geq 0} \frac{1}{4\pi} t^{k/2-1} a_k\big((\Delta_{BL})_1\big),$$

where the coefficients $a_k\big((\Delta_{BL})_1\big)$ are given by:

$$a_0\big((\Delta_{BL})_1\big) = \int_M \left(1 + \frac{1}{\mu}\right) dv$$
$$a_2\big((\Delta_{BL})_1\big) = -\frac{1}{6}\int_M \big(4\tau + 3G(d\log K,\, d\log K)\big) dv.$$

Proof. First, let us write down the eigenvalues of the leading symbol $(\sigma_1^{ij}\xi_i\xi_j)_k^s = \mu G(\xi,\,\xi)\delta_k^s + (1-\mu)\xi^s\xi_k$, where $\xi = \xi_k dx^k \in A^1(M)$. By direct calculus the eigenvalues are found to be $\lambda_1(\xi) = G(\xi,\,\xi)$ and $\lambda_2(\xi) = (1/\mu)G(\xi,\,\xi)$.

The first coefficient a_0 can be computed directly, for example using Lemma 1.7.4 of [11].

To prove the formula for the second coefficient, recall that $\chi(M) = (1/4\pi)\int_M \tau dv = \mathrm{index}(d + d^*)$. It is easy to prove that $\dim \mathrm{Ker}\big((\Delta_{BL})_p\big) = \dim \mathrm{Ker}(\Delta_p)$, thus $\mathrm{index}(d + d^*) = \mathrm{index}(d + d^*_{BL})$. The index of an elliptic operator can be computed in terms of its heat kernel invariants, in our context $\int_M \tau dv = a_2\big((\Delta_{BL})_0\big) - a_2\big((\Delta_{BL})_1\big) + a_2\big((\Delta_{BL})_2\big)$.

Now, $a_2\big((\Delta_{BL})_1\big)$ follows from the formula for the conformal scalar curvature: $\bar{\tau} = \mu(\tau + \Delta \log \mu)$. It is worthwhile to mention that $a_0\big((\Delta_{BL})_1\big)$ and $a_4\big((\Delta_{BL})_1\big)$ can be computed too as a consequence of the index formula. $\square$

Nonminimal operators have been extensively studied by many authors, we mention here only [5] and [12]. Our viewpoint is that the nonminimal operators introduced by Bao and Lackey are interesting too. We consider an open problem to determine the heat coefficients of the 1-formed Laplacian on an arbitrary Finsler manifold of dimension > 2. First, one must compute the eigenvalues of the leading symbol. What we know for sure is that on a Finsler space of arbitrary dimension the symbol $(\sigma_1^{ij}\xi_i\xi_j)_k^s$ has always the eigenvalue $\lambda_1 = G(\xi,\, \xi)$. Indeed, observe that $(\sigma_1^{ij}\xi_i\xi_j)_k^s = G_{rk}\big(\int_{S_xM}(g^{ij}g^{rs} - g^{ir}g^{js})\sqrt{g}\zeta_x\big)\xi_i\xi_j + G^{sj}\xi_j\xi_k$ and therefore $(\sigma_1^{ij}\xi_i\xi_j)_k^s\xi_s = G(\xi,\, \xi)\xi_k$. Yet, generally the eigenvalues of the symbol are not even of the form $fG(\xi,\, \xi)$, for $f \in C^\infty(M)$. This can be seen by taking its trace: $(\sigma_1^{ij}\xi_i\xi_j)_k^k = G_{rk}(G^{ijrk} - G^{irjk})\xi_i\xi_j + G(\xi,\, \xi)$.

It results that the sum of the other $n-1$ eigenvalues is precisely the first term in the trace and generally $G_{rk}(G^{ijrk} - G^{irjk}) \neq fG^{ij}$. Before we study a particular case, namely product manifolds, let us notice that although eigenvalues may have an intricate form, the corresponding eigenvectors remain orthogonal to ξ. Indeed, suppose that η is an eigenvector of the symbol, $(\sigma_1^{ij}\xi_i\xi_j)_k^s\eta_s = \lambda(\xi)\eta_k$. Then $\lambda(\xi)G(\xi,\, \eta) = (\sigma_1^{ij}\xi_i\xi_j)^{sh}\eta_s\xi_h = G(\xi,\, \xi)G(\xi,\, \eta)$. We assumed that $\lambda \neq G(\xi,\, \xi)$, therefore $G(\xi,\, \eta) = 0$.

2.2.1. Product Manifolds Consider two arbitrary Finsler manifolds, $(M_i,\, F_i)$, $i \in \{1,\, 2\}$ and the product Finsler manifold $(M,\, F)$, where $M = M_1 \times M_2$ and $F = \sqrt{(F_1)^2 \oplus (F_2)^2}$.

To distinguish various objects on M_i and M, to any object that lives on M_i it will be assigned the subscript index i, while objects on M will have no subscript index. For instance,

$$F(x,\, y) = F(x_1,\, x_2,\, y_1,\, y_2) = \sqrt{F_1^2(x_1,\, y_1) + F_2^2(x_2,\, y_2)},$$

where $x = (x_1,\, x_2)$ and $y = y_1 + y_2$. It is easy to check that $g = g_1 \oplus g_2$, but as far as the indicatrix is concerned things are not so simple. In fact, $\dim(S_xM) = m_1 + m_2 - 1$, while $\dim(S_{x_1}M_1 \times S_{x_2}M_2) = m_1 + m_2 - 2$. This is the main reason that makes us work with $\{y \in T_xM \colon F^2(x,\, y) \leq 1\}$ rather than with the indicatrix.

Consider the indicatrix in r, $S(r)=\{y\in R^m\colon F(y)=r\}$. Here we omit the point x for simplicity. $r=F(y)$ is usually called the radial variable. The volume form of $S(r)$ is $(1/r)\theta$, where θ is the volume form of $S(1)$, while the volume form of $\{y\in R^m\colon F(y)\le 1\}$ is $\sqrt{g}dy=dr\wedge(1/r)\theta$. Consequently, if $f(y)$ is a 0-homogenous function, then

$$\begin{aligned}\int_{F^2(y)\le 1} f(y)\sqrt{g}dy &= \int_0^1 dr\int_{S(r)} f(y)\frac{1}{r}\theta\\ &= \int_0^1 \frac{1}{r}r^m dr\cdot\int_{S(1)} f(y)\theta\\ &= m\int_{F(y)=1} f(y)\theta\end{aligned}$$

Notice that the integral in r is improper in 0 and $\sqrt{g}=\sqrt{g_1}\sqrt{g_2}$ usually has mild singularities on $y_i=0$. However, all functions involved are integrable.

One can alternatively use a careful manipulation of Stokes theorem or spherical coordinates to obtain the above formula.

The same reasoning for two 0-homogenous functions, $f_1(y_1)$ and $f_2(y_2)$, on M_i yields:

$$\int_{S_{x_i}M_i} f_i\theta_i = m_i\int_{F_i^2(x_i,y_i)\le 1} f_i\sqrt{g_i}dy_i$$

$$\int_{S_xM} f_1f_2\theta = m\int_{F^2(x,y)\le 1} f_1f_2\sqrt{g_1}\sqrt{g_2}dy_1\wedge dy_2$$

In the following Lemma, $\mathrm{vol}(S^{d-1})$ denotes the euclidean volume of the standard unit sphere in R^d.

Lemma 1 *We have the following formula*

$$\begin{aligned}\int_{S_xM} f_1(y_1)f_2(y_2)\theta(y) = \alpha\left(\int_{S_{x_1}M_1} f_1(y_1)\theta_1(y_1)\right)\\ \cdot\left(\int_{S_{x_2}M_2} f_2(y_2)\theta_2(y_2)\right),\end{aligned}$$

$$\textit{where}\quad \alpha=\frac{\mathrm{vol}(S^{m-1})}{\mathrm{vol}(S^{m_1-1})\cdot\mathrm{vol}(S^{m_2-1})}.$$

Proof. We use the above observations and the Fubini theorem:

$$\begin{aligned}
&\int_{S_xM} f_1 f_2 \theta(y)\\
&= m\int_{F^2(y)\le 1} f_1(y_1) f_2(y_2)\sqrt{g_1}\sqrt{g_2}dy_1dy_2\\
&= m\int_{F_2^2(y_2)\le 1} f_2(y_2)\Big(\int_{F_1^2(y_1)\le 1-F_2^2(y_2)} f_1(y_1)\sqrt{g_1}dy_1\Big)\sqrt{g_2}dy_2\\
&= m\int_{F_1^2(y_1)\le 1} f_1\sqrt{g_1}dy_1\cdot\int_{F_2^2(y_2)\le 1}\big(1-F_2^2(y_2)\big)^{m_1/2} f_2\sqrt{g_2}dy_2.
\end{aligned}$$

We now use again the radial variable to compute

$$\begin{aligned}
&\int_{F_2^2\le 1}\big(1-F_2^2(y_2)\big)^{m_1/2} f_2\sqrt{g_2}dy_2\\
&= \int_0^1 (1-r^2)^{m_1/2} r^{m_2-1}dr\cdot\int_{S_{x_2}M_2} f_2\theta_2,
\end{aligned}$$

thus, there exists a constant C such that:

$$\int_{F_2^2\le 1}(1-F_2^2)^{m_1/2} f_2\sqrt{g_2}dy_2 = C\cdot\int_{F_2^2\le 1} f_2\sqrt{g_2}dy_2.$$

Returning now to the first formula, we see that

$$\begin{aligned}
\int_{S_xM} f_1f_2\theta &= mC\Big(\int_{F_1^2\le 1} f_1\sqrt{g_1}dy_1\Big)\cdot\Big(\int_{F_2^2\le 1} f_2\sqrt{g_2}dy_2\Big)\\
&= \frac{mC}{m_1m_2}\Big(\int_{S_{x_1}M_1} f_1\theta_1\Big)\cdot\Big(\int_{S_{x_2}M_2} f_2\theta_2\Big).
\end{aligned}$$

One may select $f_1 = f_2 = 1$ and F_1, F_2 given by Riemannian metrics to obtain that $mC/(m_1m_2) = \alpha$. □

Remark 1 This Lemma could also be justified working directly on the indicatrix. One can see that

$$\theta(y) = \big(dF_1(y_1) - dF_2(y_2)\big)\wedge\frac{1}{F_1(y_1)}\theta_1(y_1)\wedge\frac{1}{F_2(y_2)}\theta_2(y_2)$$

Now, if $r = F_1(y_1)$ is the radial variable, Lemma 1 can be proved by a similar argument, taking into account that $F_2(y_2) = \sqrt{1-r^2}$.

Let us use Lemma 1 to prove the following fundamental result.

Theorem 2 *We have the identities*:
1. $\mathrm{vol}(S_xM) = \alpha \cdot \mathrm{vol}(S_{x_1}M_1) \cdot \mathrm{vol}(S_{x_2}M_2)$.
2. $\mathrm{vol}(M) = \mathrm{vol}(M_1) \cdot \mathrm{vol}(M_2)$.
3. $G(\xi, \eta) = G_1^{ij}\xi_i\eta_j + G_2^{ab}\xi_a\eta_b$, *where* $\xi_1 = \xi_i dx_1^i$, $\xi_2 = \xi_a dx_2^a$, *and the same is for* η.
4. $G(\varphi, \psi) = 1/2! G_1^{ijkl}\varphi_{ik}\psi_{jl} + G_1^{ij}G_2^{ab}\varphi_{ia}\psi_{jb} + 1/2! G_2^{abcd}\varphi_{ac}\psi_{bd}$, *where*

$$\varphi = \frac{1}{2!}\varphi_{ik}dx_1^i \wedge dx_1^k + \varphi_{ia}dx_1^i \wedge dx_2^a + \frac{1}{2!}\varphi_{ab}dx_2^a \wedge dx_2^b.$$

Proof. 1. It follows directly for $f_1 = 1$ and $f_2 = 1$.
2. Recall that the geometrical objects on M are defined using the normalized volume form. Clearly,

$$\int_{S_xM} f(y)\zeta(y) = \left(\int_{S_{x_1}M_1} f_1(y_1)\zeta_1(y_1)\right) \cdot \left(\int_{S_{x_2}M_2} f_2(y_2)\zeta_2(y_2)\right)$$

Now, take $f_1 = \sqrt{g_1}$ and $f_2 = \sqrt{g_2}$ to get that $\sqrt{G} = \sqrt{G_1} \cdot \sqrt{G_2}$.
3. It follows for $f_1 = g_1^{ij}\sqrt{g_1}$, $f_2 = \sqrt{g_2}$ and $f_1 = \sqrt{g_1}$, $f_2 = g_2^{ab}\sqrt{g_2}$.
4. It can be found for $f_1 = g_1^{ij}g_1^{kl}\sqrt{g_1}$, $f_2 = \sqrt{g_2}$; $f_1 = g_1^{ij}\sqrt{g_1}$, $f_2 = \sqrt{g_2}$ and $f_1 = \sqrt{g_1}$, $f_2 = g_2^{ab}g_2^{cd}\sqrt{g_2}$. □

This Theorem not only shows that for product manifolds we have similar results to those in Riemannian geometry, but also allows us to find the eigenvalues of the 1-formed Laplacian.

In the remainder of this subsection, $m_1 = m_2 = 2$, and we restrict to Bao-Lackey Laplacian on 1-forms. Let σ, σ_1 and σ_2 denote the symbols of Δ_{BL} on M, M_1 and M_2, respectively. We continue to denote by i, j, k, l the indices on M_1, by a, b, c, d the indices on M_2, and by α, β, γ, δ the indices on M, $\alpha, \beta, \gamma, \delta, \ldots \in \{i, j, k, l, \ldots, a, b, c, d, \ldots\}$. It is a direct consequence of Theorem 2 that

$$\begin{aligned}
(\sigma^{\alpha\beta}\xi_\alpha\xi_\beta)^i_j &= (\sigma_1^{kh}\xi_k\xi_h)^i_j + (G_2^{ab}\xi_a\xi_b)\delta^i_j \\
(\sigma^{\alpha\beta}\xi_\alpha\xi_\beta)^i_a &= (\sigma^{\alpha\beta}\xi_\alpha\xi_\beta)^a_j = 0 \\
(\sigma^{\alpha\beta}\xi_\alpha\xi_\beta)^a_b &= (\sigma_1^{cd}\xi_c\xi_d)^a_b + (G_1^{ij}\xi_i\xi_j)\delta^a_b
\end{aligned}$$

Clearly, finding the eigenvalues of $(\sigma^{\alpha\beta}\xi_\alpha\xi_\beta)^\gamma_\delta$ reduces to finding the eigenvalues of $(\sigma^{\alpha\beta}\xi_\alpha\xi_\beta)^i_j$ and $(\sigma^{\alpha\beta}\xi_\alpha\xi_\beta)^a_b$. This can be done easily and the eigenvalues are found to be

$$\lambda_1{=}G(\xi,\,\xi),\quad \lambda_2=\mu_1 G_1(\xi_1,\,\xi_1)+G_2(\xi_2,\,\xi_2)$$
$$\lambda_3{=}G(\xi,\,\xi),\quad \lambda_4=G_1(\xi_1,\,\xi_1)+\mu_2 G_2(\xi_2,\,\xi_2)$$

The calculus of $a_0\big((\Delta_{BL})_1\big)$ on M, denoted here simply by $a_0(M)$, reduces to an integral of the form $\int\sum_{i=1,\,4}\exp\big(-\lambda_i(\xi)\big)d\xi$. Changing the variables first with respect to ξ_1 and then with respect to ξ_2, this integral yields $1+1+1/\mu_1+1/\mu_2=(1+1/\mu_1)(1+1/\mu_2)+1-1/\mu$. Forasmuch $\int_{M_i}(1+1/\mu_i)dv_i=a_0(M_i)$, the first heat invariant on M has the formula:

$$a_0(M)=a_0(M_1)\cdot a_0(M_2)+\mathrm{vol}(M)-\int_M\frac{1}{\mu}dv$$

We end this section with an emphasis on the fact that generally eigenvalues might have cumbersome formulas and only in such particular cases we were able to find them. Even so, only $a_0((\Delta_{BL})_1)$ can be computed directly, while $a_2\big((\Delta_{BL})_1\big)$ would probably need tremendous calculations.

2.3. Spectrum of Randers spaces

In this section we study on Randers spaces some of the invariants which appeared in the previous sections. On the flat torus we give a partial spectral resolution of the scalar Laplacian. Since the Randers spaces involve another Riemannian metric a_{ij} on M, it is natural to ask when $\mathrm{spec}(M,\,F)=\mathrm{spec}(M,\,a)$. It happens if and only if the Finsler structure is Riemannian.

Consider $(M,\,a)$ a Riemannian manifold and $b=b_idx^i\in A^1(M)$ satisfying $|b|^2:=a^{ij}b_ib_j<1$. Then a and b define a Finsler structure on M in the simply way: $F(x,\,y):=\alpha(x,\,y)+\beta(x,\,y)$, where $\alpha=\sqrt{a_{ij}y^iy^j}$ and $\beta=b_iy^i$. We recall the following classical formulas ([13], p. 206; [6], p. 289)

$$g^{ij}(x,\,y)=\frac{\alpha}{F}a^{ij}+\left(\frac{\beta}{F}+\frac{\alpha}{F}|b|^2\right)\frac{y^i}{F}\frac{y^j}{F}-\frac{\alpha}{F}\left(a^{ik}\frac{y^j}{F}b_k+a^{jk}\frac{y^i}{F}b_k\right);$$
$$\sqrt{g}(x,\,y)=\left(\frac{F}{\alpha}\right)^{(n+1)/2}\sqrt{a}.$$

Even on Randers spaces, finding the eigenvalues of the BL-Laplacian is an intricate problem. This can be tried on the flat torus.

2.4. Eigenvalues of the flat torus

Although for $K=$ const. a large class of manifolds have known spectrum, finding the eigenvalues of $(\Delta_{BL})_0$ for $dK\neq 0$ is more difficult. The Laplace equation $(\Delta_{BL})_0f=\lambda f$ is a Sturm-Liouville equation that generally cannot be integrated. The fact that the function K is not computable on Randers spaces is an impediment in studying this Sturm-Liouville equation.

Consider the flat torus T^2 with periodic coordinates $(t,\, s)$. The metric a is the flat metric and we choose the 1-form b to depend only on t, that is $b = b_1(t)dt$. The Finsler function is simply $F = \sqrt{(y^1)^2 + (y^2)^2} + b_1 y^1$. Yet, the volume function *vol* is given by an elliptic integral, therefore K cannot be computed either. We find easier to study the eigenvalues of a conformal Finsler function $\bar{F}$. Then, the eigenvalues of $(M,\, F)$ and those of $(M,\, \bar{F})$ can be compared using a known result of J. Dodziuk ([10]).

First, notice that the metric G^{ij} depends only on t and so does K. It is easy to check that $G^{12} = 0$.

Now consider the conformal Finsler structure $\bar{F} := (\sqrt{G^{11}}/K)F$. Clearly, the structure $(M,\, \bar{F})$ is not Riemannian. The conformal factor $\sqrt{G^{11}}/K$ yields a Sturm-Liouville equation which partially can be integrated.

We wish to study the spectrum of $(M,\, \bar{F})$. First let us mark out that the function K is a conformal invariant, therefore $\bar{K} = K$. Now use that $\bar{G}^{ij} = (K^2/G^{11})G^{ij}$ and $\bar{\Delta}_0 = (K^2/G^{11})\Delta_0$ (here Δ_0 is the standard Laplacian of G_{ij} and $\bar{\Delta}_0$ is the standard Laplacian of the conformal metric $\bar{G}_{ij}$) to obtain that $(\bar{\Delta}_{BL})_0 = (K^2/G^{11})(\Delta_{BL})_0$.

The eigenvalues of $(\bar{\Delta}_{BL})_0$ are given by $(\Delta_{BL})_0 f = (G^{11}/K^2)\lambda f$, that is

$$-\frac{\partial^2 f}{\partial t^2} - \frac{G^{22}}{G^{11}}\frac{\partial^2 f}{\partial s^2} - \frac{\partial f}{\partial t}\frac{\partial \log K}{\partial t} = \frac{\lambda}{K^2} f.$$

Now K depends only on t, thus we may infer that the eigenfunction f is of the form $f(t,\, s) = A(t)e^{ils}$, where $l \in \mathbf{Z}$. Our Sturm-Liouville equation is

$$-A''(t) + l^2\frac{G^{22}}{G^{11}}A(t) - A'(t)\frac{1}{K}K' = \frac{\lambda}{K^2}A.$$

For $l = 0$ this equation is simply $-(KA')' = (\lambda/K)A$. We use a method like in [1] and obtain that $A = \exp\{i\sqrt{\lambda}\int_0^t (1/K(u))du\}$. Now, the function A must satisfy $A(2\pi) = A(0)$, for we have used periodic coordinates. Thus, for $l = 0$ the eigenvalues are $\left\{4\pi^2 n^2/\left(\int_0^{2\pi}(1/K(t))dt\right)^2 \mid n \in \mathbf{Z}\right\}$.

Of course, we found the eigenfunctions (and their eigenvalues) depending only on t. We do not know for sure how the other eigenvalues look like.

However, one can determine an upper bound for the first positive eigenvalue of $(\Delta_{BL})_0$, in terms of $\max b_1(t)$. Recall that $\bar{G}^{ij} = (K^2/G^{11})G^{ij}$, so that using the Rayleigh quotient in our context, we get the estimates

$$\lambda_1 \min_{T^2} \frac{K^2}{G^{11}} \leq \overline{\lambda}_1 \leq \lambda_1 \max_{T^2} \frac{K^2}{G^{11}},$$

where λ_1 and $\bar{\lambda}_1$ denote the first positive eigenvalue of $(\Delta_{BL})_0$ and $(\bar{\Delta}_{BL})_0$, respectively. Certainly, $\bar{\lambda}_1 \leq 4\pi^2/\big(\int_0^{2\pi}(1/K(t))dt\big)^2$, therefore

$$\lambda_1 \leq \frac{4\pi^2}{\Big(\int_0^{2\pi}\big(1/K(t)\big)dt\Big)^2} \max_{T^2} \frac{G^{11}}{K^2}.$$

This upper bound can be expressed in terms of $\max b_1(t)$, using the formulas for K and G^{ij}, and the sharp inequalities $K \geq 1$, $\mathrm{vol} \geq 2\pi$.

2.5. Randers surfaces

Let (M, F) be a compact Randers surface. Recall that the volume form of S_xM is $\theta_x = \sqrt{g}(y^1dy^2 - y^2dy^1)$ or, in polar coordinates $y^1 = r\cos t$, $y^2 = r\sin t$, it is $\theta_x = \sqrt{g}r^2dt$.

In a point $x \in M$ we choose geodesic polar coordinates with respect to the metric a_{ij}. It follows that $a_{ij}(x) = \delta_{ij}$ and $(\partial a_{ij}/\partial x^k)(x) = 0$. Then in the point x the Finsler function is $F(x, y) = r(1 + b_1(x)\cos t + b_2(x)\sin t)$, where $b_1^2 + b_2^2 < 1$ and $\sqrt{g}(x) = 1/\alpha^{3/2} = 1/r^{3/2}$. Thus, the volume in x is precisely $\mathrm{vol}(x) = \int_0^{2\pi} 1/\sqrt{1 + b_1(x)\cos t + b_2(x)\sin t}\, dt$.

Proposition 3 *For any $x \in M$ we have the following sharp inequalities*

$$\mathrm{vol}(x) \geq 2\pi, \quad K(x) \geq 1$$

and either equality takes place if and only if $b_1(x) = b_2(x) = 0$.

Proof. By Jensen's inequality we get

$$\frac{1}{2\pi}\,\mathrm{vol}(x) = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{r}\right)^{-1/2} dt \geq \left(\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{r}dt\right)^{-1/2} = 1.$$

The proof of $K(x) \geq 1$ uses the integral Minkowski inequality (see [8] or Proposition 5 in this paper). □

An interesting problem is to find necessary and sufficient conditions for K and vol respectively to be constant. Certainly, if K is constant on M, then $(\Delta_{BL})_0 = \Delta_0$. If K is not constant on M, then (M, F) is not isospectral to a Riemannian surface of positive scalar curvature.

Proposition 4 below gives such conditions for the volume function. It is possible that they work for K too.

It is known that vol is constant on Landsberg surfaces ([6], p. 102). Recall that a Randers surface is Landsberg iff it is locally Minkowski ([16]). The Randers space (T^2, F), with $F = \sqrt{(y^1)^2 + (y^2)^2} + b_1(t, s)y^1 + b_2(t, s)y^2$ and $b_1 = A\cos(t+s)$, $b_2 = -A\sin(t+s)$, $0 < A < 1$, has constant volume function, but it is not locally Minkowski. Therefore, the condition M be Landsberg is only sufficient.

Proposition 4 *The function* vol *is constant on M if and only if $|b|$ is constant on M.*

Proof. First, consider the elliptic integral

$$I(b_1, b_2) = \int_0^{2\pi} \frac{dt}{\sqrt{1 + b_1\cos t + b_2\sin t}}.$$

Suppose that $b_1^2 + b_2^2 = b^2$, where $b < 1$. In polar coordinates, $b_1 = b\cos\theta$ and $b_2 = b\sin\theta$, the function I depends on b and θ

$$I\colon [0, 1) \times [0, 2\pi) \to \mathbf{R}, \quad I(b, \theta) = \int_0^{2\pi} \frac{1}{\sqrt{1 + b\cos(\theta + t)}} dt.$$

In fact, I depends only on b, because

$$\begin{aligned}\frac{\partial I}{\partial \theta} &= -\frac{1}{2}\int_0^{2\pi} \frac{b\sin(\theta+t)}{\big(1 + b\cos(\theta+t)\big)^{3/2}} dt \\ &= \big(1 + b\cos(\theta+t)\big)^{-1/2}\Big|_0^{2\pi} = 0.\end{aligned}$$

Therefore, $I(b, \theta) = I(b, 0) = \int_0^{2\pi} 1/\sqrt{1 + b\cos t}\, dt$ and I is a strictly increasing function in the argument $b > 0$.

Now, $\mathrm{vol}(x) = c$ if and only if $b(x) = b$, where $I(b, 0) = c$. This completes the proof of the Proposition. □

We are able to give the answer to the problem formulated at the beginning of this section: when $\mathrm{spec}(M, F) = \mathrm{spec}(M, a)$?

Theorem 3 $\mathrm{Spec}(M, F) = \mathrm{Spec}(M, a)$ *if and only if* $b = 0$.

Proof. Let $\mathrm{vol}(M,\,F)$, $\mathrm{vol}(M,\,G)$ and $\mathrm{vol}(M,\,a)$ denote the volumes of M with respect to the volume forms $\sqrt{G}dx$, $\sqrt{\det G}dx$ and $\sqrt{a}dx$, respectively. Then

$$\mathrm{vol}(M,\,a) = \max\bigl\{\mathrm{vol}(M,\,F),\,\mathrm{vol}(M,\,G),\,\mathrm{vol}(M,\,a)\bigr\}.$$

Indeed, from Proposition 3 we get $\mathrm{vol}(M,\,F) \geq \mathrm{vol}(M,\,G)$. In geodesic coordinates in x we obtain $\sqrt{G}(x) = 2\pi/\mathrm{vol}(x)$, thus $\sqrt{G}/\sqrt{a} \leq 1$. This proves that $\mathrm{vol}(M,\,a) \geq \mathrm{vol}(M,\,F) \geq \mathrm{vol}(M,\,G)$. In each of these inequalities, the equality holds if and only if the Finsler structure is Riemannian.

□

2.6. Randers spaces of dimension three

In this section we generalize the previous results on Randers surfaces. Consider $(M,\,F)$ a Randers space, $\dim M = 3$. In a point $x \in M$ use geodesic coordinates and spherical coordinates on $S_xM\colon y^1 = r\cos u\sin v$, $y^2 = r\sin u\sin v$, $y^3 = r\cos v$, $0 \leq v \leq \pi$ and $0 \leq u \leq 2\pi$. Then $\theta_x = r\sin v dv\wedge du$. The following result generalizes Proposition 3.

Proposition 5 *For any x we have the following sharp inequalities*:

$$\mathrm{vol}(x) \geq 4\pi, \quad K(x)\,\mathrm{vol}(x) \geq 4\pi$$

and each equality holds if and only if $b = 0$.

Proof. Let $D = [0,\,2\pi]\times[0,\,\pi]$. Then $\mathrm{vol}(x) = \int_D r\sin v du dv = 4\pi\int_D r d\sigma$, where $d\sigma := \sin v/(4\pi)du dv$. That $\mathrm{vol}(x) \geq 4\pi$ follows from Jensen's inequality: $\int_D r d\sigma = \int_D (1/r)^{-1}d\sigma \geq (\int_D (1/r)d\sigma)^{-1} = 1$. Equality holds iff r is constant, i.e., $b = 0$. For the second inequality, notice that $K^2 = 1/\sqrt{G}\det\int g^\sharp = (\mathrm{vol}(x)/(4\pi))\det\int g^\sharp$ and by the inequality of Minkowski

$$\begin{aligned}\left(\det\int g^{ij}\sqrt{g}\zeta_x\right)^{1/3} \geq& \int\bigl(\det(g^{ij}\sqrt{g})\bigr)^{1/3}\zeta_x\\ =&\frac{1}{\mathrm{vol}(x)}\int_D r^{1/3}\sin v du dv.\end{aligned}$$

The same as above, we have $\int_D r^{1/3}\sin v du dv \geq 4\pi$. □

Albeit we did not prove that $K \geq 1$, Proposition 5 is sufficient to prove that

$$\mathrm{vol}(M,\,a) = \max\bigl\{\mathrm{vol}(M,\,F),\,\mathrm{vol}(M,\,G),\,\mathrm{vol}(M,\,a)\bigr\}.$$

Proposition 6 Spec$(M,\, F) = $ Spec$(M,\, a)$ *if and only if* $F = \alpha$.

Acknowledgment The author would like to thank the referee for his valuable suggestions.

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