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The twisted de Rham cohomology for basic constructions of hyperplane arrangements and its applications

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Abstract. In this paper we study the twisted de Rham cohomology for an arrangement of hyperplanes. We prove formulas for basic constructions of arrangements: the twisted Künneth formula for products or decompositions, the long exact sequence for triple constructions, and the relation for coning and deconing constructions. Moreover, we obtain the vanishing, dimension, and basis of the twisted de Rham cohomology for an arrangement in general position and a generic arrangement with arbitrary weight.

Key words: twisted de Rham cohomology, hyperplane arrangement.

1. Introduction

Fundamental results of the twisted de Rham theory are founded in [De]. Let M be a complex affine manifold of dimension n. Denote the sheaves of holomorphic p-forms on M by Ω_M^p with $\Omega_M^0 = \mathcal{O}_M$ and denote $\Omega^p(M) = \Gamma(M, \Omega_M^p)$. Let ω be a closed holomorphic 1-form on M. The covariant derivation with respect to ω is defined by

$$\nabla_\omega := d + \omega\wedge$$

and then we have the (smooth) twisted de Rham complex $(\Omega^\bullet(M), \nabla_\omega)$ of M. Denote the cohomology of $(\Omega^\bullet(M), \nabla_\omega)$ by $H^*(M, \nabla_\omega)$. Deligne gives the twisted de Rham theorem in [De, II; 6.3]: there exists an isomorphism

$$H^p(M, \mathcal{L}_\omega) \simeq H^p(M, \nabla_\omega),$$

where $\mathcal{L}_\omega$ is the locally constant sheaf of solutions of $\nabla_\omega h = 0$, $h \in \mathcal{O}_M$, which is a complex local system of rank one on M. We note that $H^p(M, \mathcal{L}_\omega) = 0$ for $p > n$.

The twisted de Rham cohomology on complements of hyperplanes is an important subject in the Aomoto-Gelfand multivariable theory of hypergeometric functions [DM, AK, OT2].

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An *ℓ-arrangement $\mathcal{A}$ of hyperplanes* is a finite set of distinct hyperplanes in the ℓ-dimensional complex space $V = \mathbb{C}^\ell$. Denote the complement of $\mathcal{A}$ by $M(\mathcal{A}) = \mathbb{C}^\ell \backslash \cup_{H\in\mathcal{A}} H$. We define a *logarithmic* 1*-form* ω_H of a hyperplane H by $\omega_H = d\log\alpha_H$, where α_H is a degree one polynomial defining H.

A *weight* $\lambda = \lambda(\mathcal{A})$ of $\mathcal{A}$ is defined by $\lambda = (\lambda_H; H \in \mathcal{A})$; $\lambda_H \in \mathbb{C}$. We call λ_H the *weight* of H. We define the logarithmic 1-form $\omega(\mathcal{A}, \lambda)$ by

$$\omega(\mathcal{A}, \lambda) = \sum_{H\in\mathcal{A}} \lambda_H \omega_H.$$

We obtain the twisted de Rham complex $(\Omega^\bullet(M(\mathcal{A})), \nabla_\lambda)$, where we denote $\nabla_\lambda = \nabla_{\omega(\mathcal{A},\lambda)}$, for simplicity. In the Section 2, for basic constructions of arrangements we shall obtain twisted versions of formulas in [OT], which are the twisted Künneth formula for products or decompositions, the long exact sequence for triple constructions, and the relation for coning and deconing constructions.

By the way, we review combinatorics of arrangements. The *intersection set* $L = L(\mathcal{A})$ of $\mathcal{A}$ is the set of nonempty intersections of elements of $\mathcal{A}$. By convention L includes V. We call an element of L an *edge* of $\mathcal{A}$. The *rank* $r(X)$ of $X \in L$ is defined by $r(X) = \operatorname{codim}(X)$. Note that $r(V) = 0$. The *rank* $r(\mathcal{A})$ of $\mathcal{A}$ is the maximum rank of any edges of $\mathcal{A}$, which is the maximum number of linearly independent hyperplanes of $\mathcal{A}$ with nonempty intersection. For $X \in L(\mathcal{A})$, define $\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subset H\}$ and $\lambda_X := \sum_{H\in\mathcal{A},\, X\subset H} \lambda_H$. An edge $X \in L(\mathcal{A})$ is called *dense* in $\mathcal{A}$, if the central arrangement $\mathcal{A}_X$ is not decomposable (the decomposability will be defined in the Section 2.2). Denote the set of all dense edges in $\mathcal{A}$ by $\mathrm{D}(\mathcal{A})$. We note that all hyperplanes of $\mathcal{A}$ are dense edges in $\mathcal{A}$. Let $\mathcal{A}_\infty$ be the projective closure of $\mathcal{A}$ which is defined in the Section 2.5.

Theorem 1.1 ([ESV, STV]) *Let $\mathcal{A}$ be an arrangement of rank ℓ. If a weight λ of $\mathcal{A}$ satisfies the condition*:

$$\lambda_X \notin \mathbb{Z}_{\geq 0} \quad \textit{for every} \quad X \in \mathrm{D}(\mathcal{A}_\infty), \tag{Mon}$$

then we have

$$H^p(M(\mathcal{A}), \nabla_\lambda) = 0, \quad \textit{for } p \neq \ell.$$

In the Section 3, using Theorem 1.1 and a long exact sequence for triple of arrangements obtained in the Section 2, we shall obtain the vanishing, dimension, decomposition, and basis of the twisted de Rham cohomology for arrangements in general position with arbitrary weight, which does not necessarily satisfy the condition (Mon). In the Section 4, we shall obtain the vanishing, dimension, and basis of the twisted de Rham cohomology for a generic arrangement. This is another example not satisfying the condition (Mon).

2. Basic constructions

In this section we study the twisted de Rham cohomology for basic constructions of arrangements. We refer to [OT, OT2] for definitions and fundamental properties.

2.1. Preliminaries

In this section, we work in the category of complex affine manifolds. The twisted Mayer-Vietoris sequence and the twisted Künneth formula are formulated in [IK]. In general, we can prove a twisted version of Leray-Hirsch Theorem.

Theorem 2.1 (Twisted Leray-Hirsch Theorem) *Let $\pi\colon E \to M$ be a fiber bundle with fiber F. Suppose that the cohomology of F is finite dimensional and that there exist global cohomology classes on E of which restrictions to each fiber freely generate the cohomology of the fiber. Let ω be a closed holomorphic 1-form on M. Then we have an isomorphism*

$$\bigoplus_{p+q=k} H^p(M,\,\nabla_\omega)\otimes H^q(F) \xrightarrow{\sim} H^k(E,\,\nabla_{\pi^*\omega}).$$

Proof. We have only to follow the proof of Leray-Hirsch Theorem in [BT, 5.11] by replacing d to ∇_ω over M. □

Let $\mathcal{A}$ be an arrangement and λ its weight. A weight λ is said to be *trivial*, if all λ_H's are integers.

Lemma 2.2 *Let λ and λ' be two weights of an arrangement $\mathcal{A}$. If $\lambda-\lambda'$ is trivial, then there exists an isomorphism*:

$$\begin{aligned} H^k(M(\mathcal{A}),\,\nabla_\lambda) &\xrightarrow{\sim} H^k(M(\mathcal{A}),\,\nabla_{\lambda'}) \\ \varphi &\mapsto f\cdot\varphi, \end{aligned}$$

where $f=\prod_{H\in\mathcal{A}}\alpha_H^{\lambda_H-\lambda_{H'}}$.

Proof. Since $\lambda - \lambda'$ is trivial, f is a nowhere vanishing single valued analytic function on $M = M(\mathcal{A})$. Let $\omega = \omega(\mathcal{A}, \lambda')$. Then we have $\omega(\mathcal{A}, \lambda) = \omega + d \log f$. The multiplication by f induces an isomorphism of twisted de Rham complexes: $(\Omega^\bullet(M), \nabla_{\omega + d\log f}) \xrightarrow{\sim} (\Omega^\bullet(M), \nabla_\omega)$, since we have

$$\begin{aligned}\nabla_\omega(f\varphi) &= (d+\omega)(f\varphi) = df \wedge \varphi + f d\varphi + f\omega \wedge \varphi \\ &= f(d + \omega + d\log f)\varphi = f\nabla_{\omega + d\log f}(\varphi).\end{aligned}$$

□

Write $\lambda_{\mathcal{A}} := \sum_{H\in\mathcal{A}} \lambda_H$. The weight λ is said to be *standard*, if λ_H's and $\lambda_{\mathcal{A}}$ consist of zeros and non-integers: $\lambda_H, \lambda_{\mathcal{A}} \in \{\mathbb{C}\setminus\mathbb{Z}\} \cup \{0\}$ for $H \in \mathcal{A}$.

An ℓ-arrangement $\mathcal{A}$ is called *essential* if $r(\mathcal{A}) = \ell$. For an ℓ-arrangement $\mathcal{A}$ with rank $r = r(\mathcal{A})$, there exists the essential r-arrangement $\mathbf{ess}\mathcal{A}$ such that $L(\mathcal{A}) = L(\mathbf{ess}\mathcal{A})$ and $M(\mathcal{A}) = M(\mathbf{ess}\mathcal{A}) \times \mathbb{C}^{\ell - r}$. By the twisted Künneth formula, we have the following.

Lemma 2.3 *Let $\mathcal{A}$ be an ℓ-arrangement with rank $r = r(\mathcal{A})$ and λ be a weight. We have an isomorphism*

$$H^k(M(\mathcal{A}), \nabla_\lambda) \simeq H^k(M(\mathbf{ess}\mathcal{A}), \nabla_\lambda) \quad \textit{for all} \quad k,$$

and thus we have $H^k(M(\mathcal{A}), \nabla_\lambda) = 0$ for $k > r$.

In this paper, we work under the assumption that arrangements are essential and that weights are standard.

2.2. Product and decomposition

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be arrangements in V_1 and V_2, respectively. The *product* arrangement $\mathcal{A}_1 \times \mathcal{A}_2$ is an arrangement in $V_1 \oplus V_2$ defined by

$$\mathcal{A}_1 \times \mathcal{A}_2 = \{H_1 \oplus V_2; H_1 \in \mathcal{A}_1\} \cup \{V_1 \oplus H_2; H_2 \in \mathcal{A}_2\}.$$

Note that $M(\mathcal{A}_1 \times \mathcal{A}_2) = M(\mathcal{A}_1) \times M(\mathcal{A}_2)$. Let λ^1 and λ^2 be weights of $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. We can define the weight λ of $\mathcal{A}_1 \times \mathcal{A}_2$ by (λ^1, λ^2). By the twisted Künneth formula, we have the following

Lemma 2.4

$$H^k(M(\mathcal{A}_1 \times \mathcal{A}_2), \nabla_\lambda) = \bigoplus_{p+q=k} H^p(M(\mathcal{A}_1), \nabla_{\lambda^1}) \otimes H^q(M(\mathcal{A}_2), \nabla_{\lambda^2}).$$

Conversely, an arrangement $\mathcal{A}$ is said to be *decomposable*, if there exist nonempty arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ after some linear coordinate change. A weight $\lambda = \lambda(\mathcal{A})$ of $\mathcal{A}$ induces weights of $\mathcal{A}_1$ and $\mathcal{A}_2$ as subarrangements.

2.3. Deletion and restriction

Let $\mathcal{A}$ be an arrangement of hyperplanes and λ a weight of $\mathcal{A}$. Let $\mathcal{B}$ be a subarrangement $\mathcal{B}$ of $\mathcal{A}$. A weight λ of $\mathcal{A}$ induces a weight $\lambda(\mathcal{B})$ of $\mathcal{B}$ defined by $(\lambda_H)_{H\in\mathcal{B}}$. We denote $\lambda(\mathcal{B})$ by the same λ. For $X \in L(\mathcal{A})$, define an arrangement $\mathcal{A}^X$ in X by

$$\mathcal{A}^X := \{X \cap H \mid H \in \mathcal{A} \setminus \mathcal{A}_X,\, X \cap H \neq \emptyset\},$$

which is called the *restriction* of $\mathcal{A}$ to X. A weight λ of $\mathcal{A}$ induces a weight $(\lambda_Y)_{Y\in A^X}$ of $\mathcal{A}^X$, which is denoted by $\lambda = \lambda(\mathcal{A}^X)$, where

$$\lambda_Y = \sum_{H\in\mathcal{A}_Y\setminus\mathcal{A}_X} \lambda_H.$$

Note that the induced weight of $\mathcal{A}^X$ depends only on the induced weight of $\mathcal{A} \setminus \mathcal{A}_X$. It is easy to see the following.

Lemma 2.5 *For $X \in L(\mathcal{A})$, we have $M(\mathcal{A}^X) = M(\mathcal{A} \setminus \mathcal{A}_X) \cap X$ and*

$$H^k(M(\mathcal{A}^X),\, \nabla_\lambda) = H^k(M(\mathcal{A} \setminus \mathcal{A}_X) \cap X,\, \nabla_{\lambda(\mathcal{A}\setminus\mathcal{A}_X)}).$$

For a subarrangement $\mathcal{B}$ of $\mathcal{A}$ and $X \in L(\mathcal{A})$ such that $\mathcal{B} \cap \mathcal{A}_X \neq \emptyset$, we have $M(\mathcal{B}) \cap X = M((\mathcal{B} \cup \mathcal{A}_X)^X)$ and

$$H^k(M(\mathcal{B}) \cap X,\, \nabla_{\lambda(\mathcal{B})}) = H^k(M((\mathcal{B} \cup \mathcal{A}_X)^X),\, \nabla_\lambda).$$

2.4. Triple

Let $\mathcal{A}$ be an arrangement of hyperplanes. For $H_0 \in \mathcal{A}$, a *triple* $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ of arrangements with distinguished hyperplane H_0 consists of $\mathcal{A}$, a subarrangement $\mathcal{A}' = \mathcal{A} \setminus \{H_0\}$, and a restriction $\mathcal{A}'' = \mathcal{A}^{H_0}$. Let λ be a weight of $\mathcal{A}$ with $\lambda_{H_0} = 0$. The weight λ induces weights of $\mathcal{A}'$ and $\mathcal{A}''$ as a subarrangement of $\mathcal{A}$ and a restriction of $\mathcal{A}$, respectively. There exist inclusions

$$i = i_{H_0} \colon M(\mathcal{A}) \hookrightarrow M(\mathcal{A}') \quad \text{and } j = j_{H_0} \colon M(\mathcal{A}'') \hookrightarrow M(\mathcal{A}').$$

We clearly have $\omega(\mathcal{A}, \lambda) = i^*\omega(\mathcal{A}', \lambda)$ and $\omega(\mathcal{A}'', \lambda) = j^*\omega(\mathcal{A}', \lambda)$.

A tubular neighborhood T_H of a hyperplane H in $\mathbb{C}^\ell$ can be regarded as a trivial bundle over H with fiber $F = \mathbb{C}$. The complement of the zero section is denoted by $T_H^* = T_H \setminus H$. Since H is contractible, for any fiber F_0 of T_H^*, we have

$$H^1(F_0) = H^1(T_H^*) = H^1(M(\{H\})) = \mathbb{C}\{\omega_H\}.$$

Theorem 2.6 *Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple of arrangements with distinguished hyperplane $H_0 \in \mathcal{A}$. Let λ be a weight of $\mathcal{A}$ with $\lambda_{H_0} = 0$. Then we have a long exact sequence*:

$$\begin{aligned}
\cdots \to & H^k(M(\mathcal{A}'), \nabla_\lambda) \to H^k(M(\mathcal{A}), \nabla_\lambda) \\
& \to H^{k-1}(M(\mathcal{A}''), \nabla_\lambda) \otimes H^1(T_{H_0}^*) \xrightarrow{\delta} H^{k+1}(M(\mathcal{A}'), \nabla_\lambda) \to \cdots .
\end{aligned}$$

Proof. Denote $M = M(\mathcal{A})$, $M' = M(\mathcal{A}')$ and $M'' = M(\mathcal{A}'')$. By [OT, lemma 5.79], there exists a tubular neighborhood $E = E_{H_0} \subset M'$ such that (E, E_0) is a trivial bundle over M'' with fiber $(\mathbb{C}, \mathbb{C}^*)$, where $E_0 = E \cap M$ is the complement of the zero section in E. Then we can assume that E is a restriction of T_{H_0} to M'. We have $H^k(F) = H^k(T_{H_0})$ for any fiber $F = \mathbb{C}$ of E. The cohomology of any fiber $F_0 = \mathbb{C}^*$ of E_0 is generated by the logarithmic 1-form ω_{H_0} and $H^1(F_0) = H^1(T_{H_0}^*)$. By the twisted Leray-Hirsch formula and the Künneth formula we have isomorphisms

$$\begin{aligned}
H^k(E, \nabla) & \simeq \bigoplus_{p+q=k} H^p(M'', \nabla) \otimes H^q(T_{H_0}) \\
& \simeq H^k(M'', \nabla)
\end{aligned}$$

and

$$\begin{aligned}
H^k(E_0, \nabla) & \simeq \bigoplus_{p+q=k} H^p(M'', \nabla) \otimes H^q(T_{H_0}^*) \\
& \simeq H^k(M'', \nabla) \oplus \left[H^{k-1}(M'', \nabla) \otimes H^1(T_{H_0}^*) \right],
\end{aligned}$$

where $\nabla = \nabla_\lambda$. On the other hand, since

$$E \cap M = E_0 \quad \text{and} \quad E \cup M = M',$$

we get a twisted Mayer-Vietoris sequence:

$$\begin{aligned}
& \to H^k(M', \nabla) \to H^k(E, \nabla) \oplus H^k(M, \nabla) \to H^k(E_0, \nabla) \\
& \xrightarrow{\delta} H^{k+1}(M', \nabla) \to \cdots .
\end{aligned}$$

Therefore we have the following exact sequence:

$$\to H^k(M', \nabla) \to \begin{bmatrix} H^k(M'', \nabla) \\ \oplus \\ H^k(M, \nabla) \end{bmatrix}$$
$$\to \begin{bmatrix} H^k(M'', \nabla) \\ \oplus \\ H^{k-1}(M'', \nabla) \otimes H^1(T^*_{H_0}) \end{bmatrix} \xrightarrow{\delta} \cdots .$$

The third map induces an isomorphism $H^k(M'', \nabla) \xrightarrow{\sim} H^k(M'', \nabla)$. So we can delete this isomorphism from the above sequence and hence we obtain the exact sequence in Theorem. □

Remark This is a generalization of Corollary 5.81 in [OT]. Daniel C. Cohen [Co] obtains this, independently.

2.5. Coning and deconing

An arrangement is called *central* if the intersection of all its hyperplanes is nonempty. This intersection is called the *center*. Let $\mathcal{C}$ be a nonempty central arrangement in $\mathbb{C}^{\ell+1}$ with center $T = T(\mathcal{C}) := \cap_{H\in\mathcal{C}} H \neq \emptyset$. We may assume that $0 \in T(\mathcal{C})$. Recall the Hopf bundle $p\colon \mathbb{C}^{\ell+1} \setminus \{0\} \to \mathbb{P}^\ell$ with fiber $\mathbb{C}^*$, which identifies $x \in \mathbb{C}^{\ell+1} \setminus \{0\}$ with tx for $t \in \mathbb{C}^*$. For a hyperplane H in $\mathbb{C}^{\ell+1}$ containing 0, we get a projective hyperplane $\mathbb{P}H$ in $\mathbb{P}^\ell$.

Projective quotient: The *projective quotient* $\mathbb{P}\mathcal{C}$ of $\mathcal{C}$ is a projective ℓ-arrangement given by $\mathbb{P}\mathcal{C} := \{\mathbb{P}H \ ; \ H \in \mathcal{C}\}$.

Deconing: Fix a hyperplane $H_0 \in \mathcal{C}$. A *decone* $\mathbf{d}_{H_0}\mathcal{C}$ of $\mathcal{C}$ with respect to H_0 is an arrangement of hyperplanes in $\mathbb{C}^\ell = \mathbb{P}^\ell \setminus \mathbb{P}H_0$ defined by

$$\mathbf{d}_{H_0}\mathcal{C} = \mathbb{P}\mathcal{C} \setminus \mathbb{P}H_0 = \{\mathbf{d}_{H_0}H := \mathbb{P}H \setminus \mathbb{P}H_0 \ ; \ H \in \mathcal{C} \setminus \{H_0\}\}.$$

Note that $M(\mathbb{P}\mathcal{C}) = M(\mathbf{d}_{H_0}\mathcal{C})$. A weight $\lambda = (\lambda_H; H \in \mathcal{C})$ of $\mathcal{C}$ with $\lambda_\mathcal{C} = 0$ induces a weight $\lambda = \lambda(\mathbf{d}_{H_0}\mathcal{C})$ of $\mathbf{d}_{H_0}\mathcal{C}$ defined by $\lambda_{\mathbf{d}_{H_0}H} := \lambda_H$ for $H \in \mathcal{C} \setminus \{H_0\}$.

Theorem 2.7 *Let $\mathcal{C}$ be a central arrangement of hyperplanes and λ a weight of $\mathcal{C}$. If $\lambda_\mathcal{C} = 0$ then we have an isomorphism*:

$$H^k(M(\mathcal{C}), \nabla_\lambda) \simeq \bigoplus_{p+q=k} H^p(M(\mathbf{d}_{H_0}\mathcal{C}), \nabla_{\lambda^0}) \otimes H^q(\mathbb{C}^*).$$

Remark We can choose any one of ω_H, $H \in \mathcal{C}$ as a basis of the one-dimensional vector space $H^1(\mathbb{C}^*)$. Through the isomorphism above, $H^\bullet(M(\mathcal{C}), \nabla_\lambda)$ is generated by exterior products of the forms $\omega_H - \omega_{H_0}$, $H \in \mathcal{C}$.

Proof. By [OT, Proposition 5.1], the restriction of the Hopf bundle $p\colon M(\mathcal{C}) \to M(\mathbf{d}\mathcal{C})$ is trivial bundle with fiber $\mathbb{C}^*$. By simple computations, we have

$$\begin{array}{rclrcl} p^*\omega_{\mathbf{d}H} &=& \omega_H - \omega_{H_0}, & p^*\omega(\mathbf{d}\mathcal{C}, \lambda) &=& \omega(\mathcal{C}, \lambda), \\ \omega_H|_{\mathbb{C}^*} &=& \omega_{H_0}|_{\mathbb{C}^*} = \dfrac{dt}{t}, & \omega(\mathcal{C}, \lambda)|_{\mathbb{C}^*} &=& 0, \end{array}$$

where $H \in \mathcal{C} \setminus \{H_0\}$ and $\mathbb{C}^*(t)$ is each fiber. Then the (twisted) de Rham cohomology of the fiber is

$$H^0 = \mathbb{C},\ H^1 = \mathbb{C} \simeq \mathbb{C}\{\omega_{H_0}\},\ H^k = 0,\ k \neq 0, 1.$$

The restriction to each fiber $\mathbb{C}^*$ of the logarithmic form ω_{H_0} for H_0 is a basis for the de Rham cohomology of the fiber. Moreover, by applying to the twisted Leray-Hirsch Theorem, we obtain this theorem. □

Coning: An affine ℓ-arrangement $\mathcal{A}$ give rise to a central $(\ell+1)$-arrangement $\mathbf{c}\mathcal{A}$, called the *cone* over $\mathcal{A}$. The cone $\mathbf{c}\mathcal{A}$ has the additional hyperplane H_0 so that $\mathbf{d}_{H_0}(\mathbf{c}\mathcal{A}) = \mathcal{A}$. A weight λ of $\mathcal{A}$ induces a weight λ of $\mathbf{c}\mathcal{A}$ defined by $\lambda_{H_0} = -\lambda_{\mathcal{A}}$. Note that $\lambda_{\mathbf{c}\mathcal{A}} = 0$.

Projective closure: Let $\mathbb{P}^\ell$ be the complex projective space, which is a compactification of $\mathbb{C}^\ell$. Denote by $\bar{H}$ the projective closure of an affine hyperplane H and by $\bar{H}_\infty$ the infinite hyperplane. For an arrangement $\mathcal{A}$ in $\mathbb{C}^\ell$, we define an arrangement $\mathcal{A}_\infty$ in $\mathbb{P}^\ell$ by $\mathcal{A}_\infty := \{\bar{H}\ ;\ H \in \mathcal{A}\} \cup \{\bar{H}_\infty\}$, which is called the *projective closure* of $\mathcal{A}$. Note that $\mathcal{A}_\infty = \mathbb{P}(\mathbf{c}\mathcal{A})$.

Corollary 2.8 (Yuzvinsky [Yu]) *Let $\mathbf{c}\mathcal{A}$ be the cone over $\mathcal{A}$ and λ a weight of $\mathcal{A}$. Then we have*

$$H^k(M(\mathbf{c}\mathcal{A}), \nabla_\lambda) \simeq H^k(M(\mathcal{A}), \nabla_\lambda) \oplus \Big[H^{k-1}(M(\mathcal{A}), \nabla_\lambda) \otimes H^1(\mathbb{C}^*)\Big].$$

In other words, there exists a short exact sequence:

$$0 \to H^*(M(\mathcal{A}), \nabla_\lambda) \to H^*(M(\mathbf{c}\mathcal{A}), \nabla_\lambda) \to H^{*-1}(M(\mathcal{A}), \nabla_\lambda) \to 0.$$

Remark An algebraic proof is given by [Yu]. This is a generalization of Corollary 3.57 in [OT] which gives a short exact sequence of Orlik-Solomon algebras.

3. An arrangement of hyperplanes in general position

Definition 3.1 For an arrangement $\mathcal{A}$ and a weight λ of $\mathcal{A}$, we define a decomposition $\mathcal{A} = \mathcal{G}(\mathcal{A}, \lambda) \cup \mathcal{N}(\mathcal{A}, \lambda)$ of $\mathcal{A}$ by

$$\mathcal{G} = \mathcal{G}(\mathcal{A}, \lambda) := \{H \in \mathcal{A}; \lambda_H \in \mathbb{C} \setminus \mathbb{Z}\}$$

and

$$\mathcal{N} = \mathcal{N}(\mathcal{A}, \lambda) := \{H \in \mathcal{A}; \lambda_H \in \mathbb{Z}\}.$$

Moreover, subarrangements $\mathcal{G}$ and $\mathcal{N}$ has weights $\lambda(\mathcal{G})$ and $\lambda(\mathcal{N})$ induced by $\lambda(\mathcal{A})$, respectively. Since $\lambda(\mathcal{N})$ is trivial, by Lemma 2.2 we can assume that λ is standard, i.e., $\lambda_H = 0$ for $H \in \mathcal{N}$.

An ℓ-arrangement $\mathcal{A}$ is said to be *in general position* if $r(H_1 \cap \cdots \cap H_p) = p$ for every subset $\{H_1, \ldots, H_p\}$ of $\mathcal{A}$ with $p \leq \ell$ and $H_1 \cap \cdots \cap H_p = \emptyset$ when $p > \ell$. In the case where an arrangement is in general position, we obtain the the following:

Theorem 3.2 *Let $\mathcal{A}$ be an ℓ-arrangement of hyperplanes in general position and λ a non-trivial weight. Then we have*

$$H^k(M(\mathcal{A}), \nabla_\lambda) = 0, \quad \textit{for } k \neq \ell, \tag{3.1}$$

$$\dim H^\ell(M(\mathcal{A}), \nabla_\lambda) = \binom{|\mathcal{A}| - 1}{\ell}, \tag{3.2}$$

and $H^\ell(M(\mathcal{A}), \nabla_\lambda)$ is isomorphic to

$$\bigoplus_{X \in L(\mathcal{N})} H^{\dim(X)}(M(\mathcal{G}) \cap X, \nabla_\lambda) \otimes H^{\operatorname{codim}(X)}(M(\mathcal{N}_X)). \tag{3.3}$$

Moreover, for $H_0 \in \mathcal{A}$ such that λ_{H_0} is a non-integer, we write

$$\mathcal{A} := \{H_0, H_1, \ldots, H_n\}$$

and $\omega_i = \omega_{H_i}$. Then the set

$$\{\omega_{i_1} \wedge \cdots \wedge \omega_{i_\ell}; \quad 0 < i_1 < \cdots < i_\ell \leq n\} \tag{3.4}$$

is a basis of $H^\ell(M(\mathcal{A}), \nabla_\lambda)$.

Remark The vanishing (3.1) is the twisted de Rham cohomological version of the result [Ha]. When every weight λ_H of $H \in \mathcal{A}$ and $\lambda_{\mathcal{A}}$ are not integers ($\mathcal{N} = \emptyset$ and $\mathcal{A} = \mathcal{G}$), this theorem is obtained in [AK].

We note that the rank of $\mathcal{A}$ is equal to ℓ in our assumption. By (3.3), since $L(\mathcal{N})$ contains $V = \mathbb{C}^\ell$ as the zero-codimensional space, $H^\ell(M(\mathcal{G}), \nabla_\lambda)$ is a subspace of $H^\ell(M(\mathcal{A}), \nabla_\lambda)$. For $X \in L(\mathcal{N})$ with $\dim X = 0$, we have

$$H^{\dim(X)}(M(\mathcal{G}) \cap X, \nabla_\lambda) \simeq H^0(X) = \mathbb{C},$$

because $M(\mathcal{G}) \cap X = X$ is a point. Then

$$\bigoplus_{X \in L(\mathcal{N}), \dim(X)=0} H^\ell(M(\mathcal{N}_X))$$

is also a subspace of $H^\ell(M(\mathcal{A}), \nabla_\lambda)$.

3.1. One-dimensional case

Let $\mathcal{A}$ be a finite set of points in $\mathbb{C}$ and λ its weight. We use a coordinate z of $\mathbb{C}$. Note that $\omega(\mathcal{A}, \lambda) = \sum_{p \in \mathcal{A}} \lambda_p \big(dz/(z-p)\big)$. Recall $\mathcal{G} := \mathcal{G}(\mathcal{A}, \lambda)$ and $\mathcal{N} := \mathcal{N}(\mathcal{A}, \lambda)$ defined by Definition 3.1. We shall prove the following:

Proposition 3.3 *Let $\mathcal{A}$ be an arrangement of points in $\mathbb{C}$ and λ be a non-trivial weight. Then we have*

$$H^k(M(\mathcal{A}), \nabla_\lambda) = 0 \quad \textit{for } k \neq 1, \tag{3.5}$$

$$\dim H^1(M(\mathcal{A}), \nabla_\lambda) = |\mathcal{A}| - 1, \tag{3.6}$$

$$H^1(M(\mathcal{A}), \nabla_\lambda) \simeq H^1(M(\mathcal{G}), \nabla_\lambda) \oplus H^1(M(\mathcal{N})), \tag{3.7}$$

and, when we fix $p_0 \in \mathcal{G}$, the set

$$\left\{ \frac{dz}{z-p} \,;\, p \in \mathcal{A} \setminus \{p_0\} \right\} \tag{3.8}$$

is a basis of $H^1(M(\mathcal{A}), \nabla_\lambda)$.

Proof. Since λ is non-trivial, we have $\mathcal{G} \neq \emptyset$ and $H^k(M(\mathcal{A}), \nabla_\lambda) = 0$ for $k > 1$. A weight λ of $\mathcal{A}$ satisfies the condition (Mon), if and only if, $\lambda_p \notin \mathbb{Z}_{\geq 0}$ for every $p \in \mathcal{A}$ and $\lambda_{\mathcal{A}} \notin \mathbb{Z}_{\geq 0}$. In this case, this proposition is known (cf. [AK]).

For $p \in \mathcal{A}$, we take a sufficiently small open disc T_p around p and let $T_p^* := T_p \setminus \{p\}$. We fix $p \in \mathcal{N}$, namely $\lambda_p = 0 \in \mathbb{Z}$. Let $M = M(\mathcal{A})$, $\mathcal{A}' =$

$\mathcal{A} \setminus \{p\}$ and $M' = M(\mathcal{A}')$. Note that $M = M(\mathcal{A}) \subset M' = M(\mathcal{A}') \subset M(\mathcal{G})$. Since $\nabla_\lambda = d$ on T_p, we have

$$H^k(T_p, \nabla_\lambda) = H^k(T_p), \quad H^k(T_p^*, \nabla_\lambda) = H^k(T_p^*).$$

Since $T_p \cup M = M'$ and $T_p \cap M = T_p^*$, we obtain a twisted Mayer-Vietoris sequence for T_p and M. Since $0 \to H^0(T_p) \xrightarrow{\sim} H^0(T_p^*) \to 0$, we have two exact sequences:

$$0 \to H^0(M', \nabla_\lambda) \xrightarrow{\sim} H^0(M, \nabla_\lambda) \to 0$$

and

$$0 \to H^1(M', \nabla_\lambda) \to H^1(M, \nabla_\lambda) \to H^1(T_p^*) \to 0.$$

Therefore we have

$$H^1(M, \nabla_\lambda) = H^1(M', \nabla_\lambda) \oplus H^1(T_p^*).$$

We repeat this and eventually obtain

$$H^0(M, \nabla_\lambda) \simeq H^0(M', \nabla_\lambda) \simeq \cdots \simeq H^0(M(\mathcal{G}), \nabla_\lambda)$$

and

$$H^1(M, \nabla_\lambda) = H^1(M(\mathcal{G}), \nabla_\lambda) \oplus \bigoplus_{p \in \mathcal{N}} H^1(T_p^*).$$

Hence, we get (3.7). If $\lambda_{\mathcal{A}} = \lambda_{\mathcal{G}}$ is a non-integer, the weight λ of $\mathcal{G}$ satisfies the condition (Mon) and then (3.5), (3.6) and (3.8) clearly hold.

Now, assume that $\lambda_{\mathcal{A}}$ is an integer, namely $\lambda_{\mathcal{A}} = 0$. There exist at least two points in $\mathcal{A}$ whose weights are not integers. Then we fix such two points $p_0, q \in \mathcal{A}$ with $\lambda_{p_0}, \lambda_q \notin \mathbb{Z}$. We consider the affine line $\mathbb{C}^1(z)$ as in the projective line $\mathbb{P}^1$. We take another affine line $\mathbb{C}^1(w)$ such that $z = q$ is the point at infinity. Let φ be the transition function from $\mathbb{C}^1(z)$ to $\mathbb{C}^1(w)$: $w = \varphi(z) = 1/(z - q)$. Define the arrangement

$$\tilde{\mathcal{A}} := \{\varphi(p)\,;\, p \in \mathcal{A} \setminus \{q\}\} \cup \{w = 0\}$$

in $\mathbb{C}^1(w)$ and the weight λ of $\tilde{\mathcal{A}}$ is defined to be $\lambda_{\varphi(p)} = \lambda_p$ and $\lambda_{w=0} = \lambda_{\mathcal{A}} = 0$. Then we have $\varphi^*\omega_{\varphi(p)} = \omega_p - \omega_q$ for $p \in \mathcal{A} \setminus \{q\}$ and $\varphi^*\omega_{w=0} = -\omega_q$. Note that $M(\mathcal{A}) = M(\tilde{\mathcal{A}})$ and $\omega(\mathcal{A}, \lambda) = \varphi^*\omega(\tilde{\mathcal{A}}, \lambda)$. Since $H^1(T^*_{w=0})$ has a basis $\omega_{w=0}$ and the induced weight λ of $\tilde{\mathcal{G}} = \mathcal{G}(\tilde{\mathcal{A}}, \lambda)$ satisfies the condition

(Mon), then we have

$$H^0(M(\mathcal{A}),\, \nabla_\lambda) = H^0(M(\tilde{\mathcal{A}}),\, \nabla_\lambda) \simeq H^0(M(\tilde{\mathcal{G}}),\, \nabla_\lambda) = 0.$$

Moreover, the set $\{\omega_{w=0}\} \cup \{\omega_{\varphi(p)}\,;\, p \in \mathcal{A} \setminus \{p_0,\, q\}\}$ is a basis of $H^1(M(\tilde{\mathcal{A}}),\, \nabla_\lambda)$. Thus, the set $\{-\omega_q\} \cup \{\omega_p - \omega_q\,;\, p \in \mathcal{A} \setminus \{p_0,\, q\}\}$ is a basis of $H^1(M(\mathcal{A}),\, \nabla_\lambda)$ and then the set $\{\omega_p\,;\, p \in \mathcal{A} \setminus \{p_0\}\}$ is so. Hence, we obtain (3.8) and (3.6). □

3.2. Proof of Theorem 3.2

Proof of the vanishing (3.1). We may assume that $\lambda_{\mathcal{A}}$ is not a integer. Since $\mathcal{A}'$ and $\mathcal{A}''$ is in general position, we shall prove vanishings for $0 \le k < \ell$, using the double induction for dimensions ℓ and the cardinality s of $\mathcal{N}(\mathcal{A},\, \lambda)$. When $\ell = 1$, it is (3.5). When $s = 0$, it is well-known (see [AK, ESV, STV]). The long exact sequence in Theorem 2.6 implies that, if (3.1) holds for $(\ell,\, s-1)$ and $(\ell-1,\, s-1)$, then so does for $(\ell,\, s)$. □

Clearly, by (3.1), the long exact sequence in Theorem 2.6 induces the following.

Lemma 3.4 *Let $\mathcal{A}$ be an ℓ-arrangement in general position and let $(\mathcal{A},\, \mathcal{A}',\, \mathcal{A}'')$ be a triple of arrangements with distinguished hyperplane $H_0 \in \mathcal{A}$. Let λ be a weight of $\mathcal{A}$ with $\lambda_{H_0} = 0$. Then we have a short exact sequence*:

$$\begin{aligned} 0 \to H^\ell(M(\mathcal{A}'),\, \nabla_\lambda) &\to H^\ell(M(\mathcal{A}),\, \nabla_\lambda) \\ &\to H^{\ell-1}(M(\mathcal{A}''),\, \nabla_\lambda) \otimes H^1(T^*_{H_0}) \to 0. \end{aligned}$$

Remark This short exact sequence is a twisted version of the short exact sequence for a triple [OT, Theorem 5.87], in the case of general position.

Proof of the decomposition (3.3). We shall prove that $H^\ell(M(\mathcal{A}),\, \nabla_\lambda)$ is isomorphic to (3.3), using the double induction for the dimension ℓ and the cardinality s of $\mathcal{N}(\mathcal{A},\, \lambda)$. When $s = 0$ for each dimension ℓ, it obviously holds. When $\ell = 1$, it is just (3.7). Now we assume that it holds for $(\ell,\, s-1)$ and $(\ell-1,\, s-1)$. Let $\mathcal{A}$ be an arrangement in general position and λ a weight of $\mathcal{A}$. Assume that the cardinality s of $\mathcal{N} = \mathcal{N}(\mathcal{A},\, \lambda)$ is positive. We take a triple $(\mathcal{A},\, \mathcal{A}',\, \mathcal{A}'')$ with distinguished hyperplane $H_0 \in \mathcal{N}$. Then both of the cardinalities of $\mathcal{N}(\mathcal{A}',\, \lambda)$ and $\mathcal{N}(\mathcal{A}'',\, \lambda)$ are $s-1$. By the short exact sequence in Lemma 3.4 and the inductive assumption, we have

$$
\begin{aligned}
&H^k(M(\mathcal{A}),\nabla_\omega)\\
&\simeq H^k(M(\mathcal{A}'),\nabla_\omega)\oplus\left[H^{k-1}(M(\mathcal{A}''),\nabla_\omega)\otimes H^1(T^*_{H_0})\right]\\
&\simeq \bigoplus_{X\in L(\mathcal{N}\setminus\{H_0\})}\left[H^{\dim(X)}(M(\mathcal{G})\cap X,\nabla_{\omega_0})\otimes H^{\operatorname{codim}(X)}(M(\mathcal{N}_X))\right]\\
&\quad\oplus\bigoplus_{Y\in L(\mathcal{N}^{H_0})}\left[\begin{array}{c} H^{\dim_0(Y)}(M(\mathcal{G})\cap H_0\cap Y,\nabla_{\omega_0})\otimes\\ H^{\operatorname{codim}_0(Y)}(M((\mathcal{N}^{H_0})_Y))\otimes H^1(T^*_{H_0})\end{array}\right],
\end{aligned}
$$

where $\dim_0$ and codim_0 denote the dimension and the codimension in H_0, respectively. Since the arrangements $\mathcal{A}$ and $\mathcal{N}$ are in general position, $L(\mathcal{N})$ is a disjoint union

$$L(\mathcal{N})=L(\mathcal{N}\setminus\{H_0\})\cup L(\mathcal{N}^{H_0}),$$

where we consider $Y\in L(\mathcal{N}^{H_0})$ of H_0 to be an affine subspace of $\mathbb{C}^k$. Note that $\dim_0(Y)=\dim Y$, $\operatorname{codim}_0(Y)=\operatorname{codim}Y-1$, and $(\mathcal{N}^{H_0})_Y=\mathcal{N}_Y\setminus\{H_0\}$. Then we have

$$H^{\operatorname{codim}_0(Y)}(M((\mathcal{N}^{H_0})_Y))\otimes H^1(T^*_{H_0})\simeq H^{\operatorname{codim}(Y)}(M(\mathcal{N}_Y)).$$

This implies that it holds for $(\ell,\, s)$. □

Proof of a basis (3.4). We shall prove that (3.4) is a basis of $H^\ell(M(\mathcal{A}),\nabla_\lambda)$. Note that if it holds, then (3.2) holds.

Case 1: Assume that $\lambda_{\mathcal{A}}\notin\mathbb{Z}$. In this case $\lambda(\mathcal{G})$ satisfies the condition (Mon). For $X\in L(\mathcal{N})$ with $\dim X=p$, the p-arrangement $(\mathcal{G}\cup\mathcal{N}_X)^X$ is in general position with $M((\mathcal{G}\cup\mathcal{N}_X)^X)=M(\mathcal{G})\cap X$ and the induced weight $\lambda((\mathcal{G}\cup\mathcal{N}_X)^X)$ also satisfies the condition (Mon). Then, by [ESV, STV], $H^p(M(\mathcal{G})\cap X,\,\nabla_\lambda)$ has a basis:

$$\omega_{i_1}\wedge\cdots\wedge\omega_{i_p}\,;\,\{H_{i_1},\,\ldots,\,H_{i_p}\}\subseteq\mathcal{G}\setminus\{H_0\}.$$

On the other hand, since $\mathcal{N}_X$ is a boolean arrangement, $H^{n-p}(M(\mathcal{N}_X))$ is one-dimensional and generated by

$$\omega_{j_1}\wedge\cdots\wedge\omega_{j_{n-p}}\,;\,\{H_{j_1},\,\ldots,\,H_{j_{n-p}}\}=\mathcal{N}_X.$$

Thus $H^p(M(\mathcal{G})\cap X,\,\nabla_\lambda)\otimes H^{n-p}(M(\mathcal{N}_X))$ has a basis:

$$
\begin{aligned}
&\omega_{i_1}\wedge\cdots\wedge\omega_{i_p}\wedge\omega_{j_1}\wedge\cdots\wedge\omega_{j_{n-p}};\\
&\qquad\qquad\{H_{i_1},\,\ldots,\,H_{i_p}\}\subseteq\mathcal{G}\setminus\{H_0\},\ \{H_{j_1},\,\ldots,\,H_{j_{n-p}}\}=\mathcal{N}_X.
\end{aligned}
$$

By the decomposition (3.3), thus, (3.4) is a basis.

Case 2: Assume that $\lambda_{\mathcal{A}} = 0$. We fix $F \in \mathcal{A} \setminus \{H_0\}$ such that λ_F is not an integer. We take an affine coordinate $x = (x_1, x_2, \ldots, x_\ell)$ of $\mathbb{C}^\ell$ such that $F = \{x_\ell = 0\}$. We consider $\mathbb{C}^\ell(x)$ to be embedded in the projective space $\mathbb{P}^\ell$. We take another affine cover

$$\mathbb{C}^\ell(y)\,;\; y = (y_0, y_1, \ldots, y_{\ell-1})$$

such that F is the hyperplane at infinity. Let φ be the transition function from $\mathbb{C}^\ell(x)$ to $\mathbb{C}^\ell(y)$:

$$(y_0, y_1, \ldots, y_{\ell-1}) = \left(\frac{1}{x_\ell}, \frac{x_1}{x_\ell}, \ldots, \frac{x_{\ell-1}}{x_\ell}\right).$$

We set the hyperplane $G := \{y_0 = 0\}$ and the arrangement

$$\tilde{\mathcal{A}} := \{\varphi(H)\,;\, H \in \mathcal{A} \setminus \{F\}\} \cup \{G\}$$

in $\mathbb{C}^\ell(y)$ and define the weight λ of $\tilde{\mathcal{A}}$ by $\lambda_{\varphi(H)} = \lambda_H$ and $\lambda_G = -\lambda_{\mathcal{A}} = 0$. Then we have

$$\varphi^*\omega_{\varphi(H)} = \omega_H - \omega_F, \quad \text{for } H \in \mathcal{A} \setminus \{F\}, \quad \text{and } \varphi^*\omega_G = -\omega_F.$$

Note that $M(\mathcal{A}) = M(\tilde{\mathcal{A}})$ and $\omega(\mathcal{A}, \lambda) = \varphi^*\omega(\tilde{\mathcal{A}}, \lambda)$, by a simple computation. By applying to Case 1, we have the following basis of $H^\ell(M(\tilde{\mathcal{A}}), \nabla_\lambda)$:

$$\omega_{i_1} \wedge \cdots \wedge \omega_{i_\ell}\,; \quad \{H_{i_1}, \ldots, H_{i_\ell}\} \subseteq \tilde{\mathcal{A}} \setminus \{\varphi H_0\}.$$

So $H^\ell(M(\mathcal{A}), \nabla_\lambda)$ has a basis:

$$\omega_{i_1} \wedge \cdots \wedge \omega_{i_\ell} + \left[\sum_k (-1)^* \omega_{i_1} \wedge \cdots \widehat{\omega_{i_k}} \cdots \wedge \omega_{i_\ell}\right] \wedge \omega_F; \qquad \{H_{i_1}, \ldots, H_{i_\ell}\} \subseteq \mathcal{A} \setminus \{H_0, F\}$$

$$\omega_{j_1} \wedge \cdots \wedge \omega_{j_{\ell-1}} \wedge \omega_F\,; \quad \{H_{j_1}, \ldots, H_{j_{\ell-1}}\} \subseteq \mathcal{A} \setminus \{H_0, F\}.$$

Therefore this basis induces a basis (3.4). □

4. Generic arrangement

A central ℓ-arrangement $\mathcal{C}$ is said to be *generic* if the hyperplanes of every subarrengement $\mathcal{B} \subseteq \mathcal{A}$ with $|\mathcal{B}| = \ell$ are linearly independent. Note that a decone of a generic arrangement is an arrangement in general position. If the cardinality of a generic ℓ-arrangement $\mathcal{C}$ is greater than ℓ, then $\mathrm{D}(\mathcal{C}_\infty)$ consists of hyperplanes in $\mathcal{C}$ and the center $T(\mathcal{C}) = \cap_{H \in \mathcal{C}} H$ of $\mathcal{C}$. The following gives an example that does not satisfy (Mon).

Proposition 4.1 *Let $\mathcal{C}$ be a generic $(\ell+1)$-arrangement and λ a nontrivial weight of $\mathcal{C}$ with $\lambda_{\mathcal{C}} = 0$. Then we have*

$$H^k(M(\mathcal{C}), \nabla_\lambda) = 0, \quad \text{for } k \neq \ell,\, \ell+1 \tag{4.1}$$

and

$$\dim H^{\ell+1}(M(\mathcal{C}), \nabla_\lambda) = \dim H^{\ell}(M(\mathcal{C}), \nabla_\lambda) = \binom{|\mathcal{C}|-2}{\ell}. \tag{4.2}$$

Moreover, for $H_0 \in \mathcal{A}$ such that λ_{H_0} is a non-integer, we write

$$\mathcal{C} := \{H_0, H_1, \ldots, H_{n+1}\}$$

and $\omega_i = \omega_{H_i}$. Then the set

$$\{\omega_{i_1} \wedge \cdots \wedge \omega_{i_\ell} \wedge \omega_{n+1}\,;\quad 0 < i_1 < \cdots < i_\ell \leq n\} \tag{4.3}$$

is a basis of $H^{\ell+1}(M(\mathcal{C}), \nabla_\lambda)$ and the set

$$\{\partial(\omega_{i_1} \wedge \cdots \wedge \omega_{i_\ell} \wedge \omega_{n+1})\,;\quad 0 < i_1 < \cdots < i_\ell \leq n\} \tag{4.4}$$

is a basis of $H^{\ell}(M(\mathcal{C}), \nabla_\lambda)$, where we define

$$\partial(\omega_{i_1} \wedge \cdots \wedge \omega_{i_p}) = \sum_{k=1}^{p} (-1)^{k-1} \omega_{i_1} \wedge \cdots \wedge \widehat{\omega_{i_k}} \wedge \cdots \wedge \omega_{i_p}$$

for $p \geq 2$.

Proof. Assume that $\mathcal{C} = \{H_0, H_1, \ldots, H_{n+1}\}$ and that λ_{H_0} is a non-integer. Let $\mathcal{A} = \{\mathbf{d}H_i \mid i = 0, 1, \ldots, n\}$ be a decone $\mathbf{d}_{H_{n+1}}\mathcal{C}$ of $\mathcal{C}$ with respect to H_{n+1}. Then $\mathcal{A}$ is in general position. By (3.2) and Theorem 2.7, we get (4.1). Furthermore, since $\lambda_{\mathbf{d}H_0} = \lambda_{H_0}$ is non-integer, by (3.4), the set

$$\{\tilde{\omega}_{i_1} \wedge \cdots \wedge \tilde{\omega}_{i_\ell}\,;\quad 0 < i_1 < \cdots < i_\ell \leq n\}$$

is a basis of $H^{\ell}(M(\mathcal{A}), \nabla_\lambda)$, where $\tilde{\omega}_{i_k} = \omega_{\mathbf{d}H_{i_k}}$. Due to Theorem 2.7, the set

$$\{(\omega_{i_1} - \omega_{n+1}) \wedge \cdots \wedge (\omega_{i_\ell} - \omega_{n+1}) \wedge \omega_{n+1}\,;\quad 0 < i_1 < \cdots < i_\ell \leq n\}$$

is a basis of $H^{\ell+1}(M(\mathcal{C}),\, \nabla_\lambda)$ and the set

$$\{(\omega_{i_1} - \omega_{n+1}) \wedge \cdots \wedge (\omega_{i_\ell} - \omega_{n+1})\,; \quad 0 < i_1 < \cdots < i_\ell \leq n\}$$

is a basis of $H^{\ell}(M(\mathcal{C}),\, \nabla_\lambda)$. By direct computations, we make sure the following:

$$(\omega_{i_1} - \omega_{n+1}) \wedge \cdots \wedge (\omega_{i_\ell} - \omega_{n+1}) = (-1)^n \partial(\omega_{i_1} \wedge \cdots \wedge \omega_{i_\ell} \wedge \omega_{n+1}).$$

We have thus proved the proposition. □

By the same argument as the proof of Theorem 3.2 using Theorem 2.6, if $\lambda_{\mathcal{C}}$ is non-integer then we have $H^k(M(\mathcal{C}),\, \nabla_\lambda) = 0$ for all k.

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