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Pluriharmonic maps in affine differential geometry and (1, 1)-geodesic affine immersions

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Abstract. We define a pluriharmonic map from a complex manifold with a complex affine connection to a manifold with an affine connection and obtain some fundamental results which generalize those for a pluriharmonic map from a Kähler manifold to a Riemannian manifold. Especially, by using an associated family, we find a sufficient condition for the product of two (1, 1)-geodesic affine immersions to an affine space to be a complex affine immersion from the manifold to the product of affine spaces with a certain complex structure.

Key words: pluriharmonic map, (1, 1)-geodesic affine immersion, complex affine immersion.

1. Introduction

An isometric immersion from a Kähler manifold to a Riemannian manifold is said to be (1, 1)-geodesic if the (1, 1)-part of the complexified second fundamental form vanishes. In [3], Dajczer and Gromoll showed that a (1, 1)-geodesic isometric immersion from a simply connected Kähler manifold to a Euclidean space has a distinguished deformation called an associated family which they used to construct a holomorphic isometric immersion. When the ambient space is a pseudo-Euclidean space, similar results are given in [7]. A (1, 1)-geodesic affine immersion is a special notion of a pluriharmonic map, that is, a map of which (1, 1)-part of the complexified Hessian vanishes. A pluriharmonic map can be considered as a generalization of a holomorphic map between complex manifolds. Recently, in [6], Eschenburg and Tribuzy characterize a pluriharmonic map from a Kähler manifold to a Riemannian symmetric space by the property of having an associated family.

In this paper, we define a pluriharmonic map from a complex manifold with a complex affine connection to a manifold with an affine connection and a (1, 1)-geodesic affine immersion as a special case and generalize some

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of the results in [3], [6], [7] and [11]. Let M be a complex manifold with a complex affine connection and $\widetilde{M}$ a manifold with an affine connection. In Section 2, we define a pluriharmonic map from M to $\widetilde{M}$ and obtain some fundamental results which include generalizations of some of the results in [6] and [11]. We also define an associated family for a map from M to $\widetilde{M}$ which is a generalization of that in [6] and obtain some results including a sufficient condition for the existence of the associated family when $\widetilde{M}$ is an affine space. In Section 3, we define a (1, 1)-geodesic affine immersion as a special case, apply the results for a pluriharmonic map to such an immersion and generalize some results in [3] and [7]. For an affine immersion from M to $\widetilde{M}$, we define an associated family as a special case and obtain generalized results in [3] and [7]. In Section 4, we prepare some results on the product of two affine immersions for the next section. In Section 5, we find a sufficient condition for the product of two maps from M to an affine space to be a holomorphic map from M to the product of affine spaces with a certain complex structure. Applying this result to the product of two affine immersions from M to an affine space, we find a sufficient condition for the product of two affine immersions to be a complex affine immersion. In particular, for a (1, 1)-geodesic affine immersion from M to an affine space, by using its associated family, we get a result which is a generalization of the corresponding result in [3] and [7].

2. Pluriharmonic maps

Throughout this paper, all objects and morphisms are assumed to be smooth. Let M be a manifold, TM its tangent bundle and T^*M its cotangent bundle. We use letters E, $\widetilde{E}$ to denote real vector bundles over M. The fibre of a vector bundle E at $x \in M$ is denoted by E_x, the dual bundle of E by E^*, the set of all connections on E by $\mathcal{C}(E)$ and the space of cross sections of E by $\Gamma(E)$. We denote by $A^p(E) = \Gamma(\wedge^p T^*M \otimes E)$ the space of E-valued p-forms over M. Let $\mathrm{Hom}(\widetilde{E}, E)$ be the vector bundle of which fibre $\mathrm{Hom}(\widetilde{E}, E)_x$ at $x \in M$ is the vector space $\mathrm{Hom}_{\mathbb{R}}(\widetilde{E}_x, E_x)$ of linear mapping from $\widetilde{E}_x$ to E_x. Let $\mathrm{HOM}(\widetilde{E}, E)$ be the space of vector bundle homomorphisms from $\widetilde{E}$ to E and $\mathrm{END}(E) := \mathrm{HOM}(E, E)$. We note that $\mathrm{HOM}(\widetilde{E}, E)$ can be identified with $\Gamma(\mathrm{Hom}(\widetilde{E}, E))$. For $\Phi \in \mathrm{HOM}(\widetilde{E}, E)$ and $x \in M$, put $\Phi_x := \Phi|_{E_x}$. The space of vector bundle isomorphisms from $\widetilde{E}$ to E is denoted by $\mathrm{ISO}(\widetilde{E}, E)$. Let M and $\widetilde{M}$ be manifolds, $f \colon M \to \widetilde{M}$ a map, $f^\sharp T\widetilde{M}$ and $f_\sharp\colon f^\sharp T\widetilde{M} \to T\widetilde{M}$ the induced bundle and its bundle map. We define $i^f\colon TM \to f^\sharp T\widetilde{M}$ by $i^f_x := (f_{\sharp x})^{-1} f_{*x}$ for each $x \in M$. We denote by $\mathcal{C}_0(TM)$ (resp. $\mathcal{C}_0(T\widetilde{M})$) the set of all torsion free affine connections on M (resp. $\widetilde{M}$). Let M, $\widetilde{M}$ be manifolds, $\nabla \in \mathcal{C}_0(TM)$ and $\widetilde{\nabla} \in \mathcal{C}_0(T\widetilde{M})$. For a map $f\colon M \to \widetilde{M}$, we denote by $f^\sharp\widetilde{\nabla}$ the pull-back of $\widetilde{\nabla} \in \mathcal{C}_0(T\widetilde{M})$. For a map $f\colon M \to \widetilde{M}$, we denote by H_f the Hessian of f defined by

$$H_f(X, Y) := f^\sharp\widetilde{\nabla}_X i^f Y - i^f \nabla_X Y$$

for $X, Y \in \Gamma(TM)$. Since both $\widetilde{\nabla}$ and ∇ are torsion free, H_f is a bilinear homomorphism and symmetric, that is, $H_f(X, Y) = H_f(Y, X)$ for each $x \in M$ and any $X, Y \in T_xM$. We denote by (M, J) a $2m$-dimensional manifold M with complex structure J and call it a complex manifold. For a complex manifold (M, J), we denote by $\mathcal{C}_0(TM, J)$ the set of all torsion free affine connections $\nabla \in \mathcal{C}_0(TM)$ such that $\nabla_X J = J\nabla_X$ for each $x \in M$ and any $X \in T_xM$. Such connections are called *complex affine connections*. Note that $\mathcal{C}_0(TM, J) \neq \{0\}$ since J is integrable. Hereafter in this paper, we always denote by $\widetilde{M}$ a manifold equipped with $\widetilde{\nabla} \in \mathcal{C}_0(T\widetilde{M})$.

Definition 2.1 For a complex manifold (M, J), $\nabla \in \mathcal{C}_0(TM, J)$ and a map $f\colon M \to \widetilde{M}$, we say that f is *pluriharmonic* if

$$H_f(JX, Y) = H_f(X, JY)$$

for each $x \in M$ and any $X, Y \in T_xM$.

Note that the equation above is equivalent to the condition that the (1, 1)-part of the complexified Hessian vanishes. We mention that any holomorphic or anti-holomorphic maps between complex manifolds with complex affine connections are pluriharmonic. The property that a map $f\colon M \to \widetilde{M}$ from a complex manifold (M, J) with $\nabla \in \mathcal{C}_0(TM, J)$ is pluriharmonic does not depend on the choice of $\nabla \in \mathcal{C}_0(TM, J)$. On the other hand, this property depends on the choice of $\widetilde{\nabla} \in \mathcal{C}_0(T\widetilde{M})$ as follows. For $\widetilde{\nabla}$ and $\widetilde{\nabla}' \in \mathcal{C}_0(T\widetilde{M})$, we define the difference tensor P by

$$P_U V := \widetilde{\nabla}_U V - \widetilde{\nabla}'_U V$$

for any $U, V \in \Gamma(T\widetilde{M})$ and we denote by $f^\sharp P$ the pull-back of P by f. If a map $f\colon M \to \widetilde{M}$ is pluriharmonic with respect to $\widetilde{\nabla} \in \mathcal{C}_0(T\widetilde{M})$, f is pluriharmonic with respect to $\widetilde{\nabla}'$ if and only if it holds that

$$(f^{\sharp}P)_X JY = (f^{\sharp}P)_{JX} Y$$

for each $x \in M$ and any $X, Y \in T_x M$. We have the following for a pluriharmonic map.

Proposition 2.2 *A map $f\colon M \to \widetilde{M}$ from a complex manifold (M, J) with $\nabla \in \mathcal{C}_0(TM, J)$ is pluriharmonic if and only if, for any holomorphic map $\phi\colon S \to M$ from a complex manifold (S, J^S) with complex affine connection to M, $f \circ \phi$ is also pluriharmonic.*

Proof. For simplicity, put $g = f \circ \phi$. Then we have

$$\begin{aligned} H_g(X, Y) = &(g_{\sharp x})^{-1} f_{\sharp\phi(x)} H_f(\phi_{*x}X, \phi_{*x}Y) \\ &+ (g_{\sharp x})^{-1} f_{*\phi(x)} \phi_{\sharp x} H_\phi(X, Y) \end{aligned}$$

for each $x \in S$ and any $X, Y \in T_x S$. If f is pluriharmonic, we get

$$\begin{aligned} &H_g(J^S X, Y) \\ &= (g_{\sharp x})^{-1} f_{\sharp\phi(x)} H_f(J\phi_{*x}X, \phi_{*x}Y) + (g_{\sharp x})^{-1} f_{*\phi(x)} \phi_{\sharp x} H_\phi(X, J^S Y) \\ &= (g_{\sharp x})^{-1} f_{\sharp\phi(x)} H_f(\phi_{*x}X, J\phi_{*x}Y) + (g_{\sharp x})^{-1} f_{*\phi(x)} \phi_{\sharp x} H_\phi(X, J^S Y) \\ &= H_g(X, J^S Y) \end{aligned}$$

for each $x \in S$ and any $X, Y \in T_x S$. The converse is trivial since the identity map of M is holomorphic. □

When $\widetilde{M}$ is a Riemannian manifold, a similar result as Proposition 2.2 is given in [11], where they use the Levi-Civita connections.

For $z \in \mathbb{C} \setminus \{0\}$, consider a (1, 1)-tensor field E^z on a complex manifold (M, J) defined by

$$E^z := \mathrm{Re}(z) I + \mathrm{Im}(z) J,$$

where I is the identity of TM, $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are the real part and the imaginary part of $z \in \mathbb{C}\setminus\{0\}$. It is clear that E^z has the following properties.

$$E^z E^{z^{-1}} = I, \tag{2.1}$$

$$\nabla_X E^z = E^z \nabla_X \tag{2.2}$$

for $\nabla \in \mathcal{C}_0(TM, J)$, each $x \in M$ and any $X \in T_x M$.

Definition 2.3 For a complex manifold (M, J) and $\nabla \in \mathcal{C}_0(TM, J)$, an *associated family* for a map $f\colon M \to \widetilde{M}$ is a family of maps $f_z\colon M \to \widetilde{M}$, $z \in \mathbb{C} \setminus \{0\}$, such that $f_1 = f$,

$$f_z^\sharp \widetilde{\nabla}_X \Psi_z = \Psi_z f^\sharp \widetilde{\nabla}_X, \tag{2.3}$$

$$\Psi_{zx} i_x^f E^z X = i_x^{f_z} X \tag{2.4}$$

for a bundle isomorphism $\Psi_z \in \mathrm{ISO}(f^\sharp T\widetilde{M}, f_z^\sharp T\widetilde{M})$, each $x \in M$ and any $X \in T_x M$.

An associated family is considered for a map from a Kähler manifold to a Riemannian symmetric space in [6], where they use a (1, 1)-tensor field

$$E^\theta := \cos\theta I + \sin\theta J \tag{2.5}$$

for $\theta \in [0, 2\pi)$.

From (2.3) and (2.4), we obtain

$$H_{f_z}(X, Y) = \Psi_{zx} H_f(X, E^z Y) \tag{2.6}$$

for each $x \in M$ and any $X, Y \in T_x M$.

Lemma 2.4 *For a complex manifold (M, J), $\nabla \in \mathcal{C}_0(TM, J)$ and a map $f\colon M \to \widetilde{M}$, if there is an associated family f_z, $z \in \mathbb{C} \setminus \{0\}$, for f, then f_z is pluriharmonic.*

Proof. From (2.6), we get

$$H_{f_z}(X, Y) = \mathrm{Re}(z)\Psi_{zx} H_f(X, Y) + \mathrm{Im}(z)\Psi_{zx} H_f(X, JY) \tag{2.7}$$

for each $x \in M$ and any $X, Y \in T_x M$. Since H_{f_z} is symmetric and $z \in \mathbb{C} \setminus \{0\}$ is taken arbitrary, f is pluriharmonic. Moreover, by (2.7), we have

$$\begin{aligned} H_{f_z}(JX, Y) &= \mathrm{Re}(z)\Psi_{zx} H_f(JX, Y) + \mathrm{Im}(z)\Psi_{zx} H_f(JX, JY) \\ &= \mathrm{Re}(z)\Psi_{zx} H_f(X, JY) + \mathrm{Im}(z)\Psi_{zx} H_f(X, J^2 Y) \\ &= H_{f_z}(X, JY) \end{aligned}$$

because f is pluriharmonic. This completes the proof. □

We prepare the following to show the existence of an associated family. For a complex manifold (M, J), $\nabla \in \mathcal{C}_0(TM, J)$, if a map $f\colon M \to \widetilde{M}$ is pluriharmonic, then it holds that

$$H_f(E^z X,\, Y) = H_f(X,\, E^z Y)$$

for each $z \in \mathbb{C} \setminus \{0\}$, $x \in M$ and any $X, Y \in T_xM$. Hence in a slight more general form, we consider the following. For a map $f\colon M \to \widetilde{M}$, consider a parallel (1, 1)-tensor field K on M such that

$$H_f(KX,\, Y) = H_f(X,\, KY)$$

for each $x \in M$ and any $X, Y \in T_xM$. For such a tensor field K, a map $f_K\colon M \to \widetilde{M}$ such that

$$f_K^\sharp \widetilde{\nabla}_X \Psi_K = \Psi_K f^\sharp \widetilde{\nabla}_X, \tag{2.8}$$

$$\Psi_{K_x} i_x^f KX = i_x^{f_K} X \tag{2.9}$$

for some $\Psi_K \in \mathrm{ISO}(f^\sharp T\widetilde{M},\, f_K^\sharp T\widetilde{M})$, each $x \in M$ and any $X \in T_xM$ is called an *associated map* with respect to K.

To state and prove the next proposition, we prepare the following. In this paper, we denote by $(\mathbb{R}^{n+p}, D)$ an $(n+p)$-dimensional affine space with the standard affine connection D. We denote by $(e_1, \ldots, e_{n+p})$ the standard basis of $\mathbb{R}^{n+p}$, $\bar{e}_\alpha$ the global parallel tangent vector field obtained from e_α and θ^α the dual of $\bar{e}_\alpha$, $\alpha = 1, \ldots, n+p$.

Proposition 2.5 *For a simply connected manifold M, $\nabla \in \mathcal{C}_0(TM)$, a parallel $(1, 1)$-tensor field K on M and a map $f\colon M \to \mathbb{R}^{n+p}$, if the Hessian satisfies $H_f(KX, Y) = H_f(X, KY)$ for each $x \in M$ and any $X, Y \in T_xM$, then there is an associated map f_K with respect to K.*

Proof. For the 1-forms $f^*\theta^\alpha \circ K$, $\alpha = 1, \ldots, n+p$, we have

$$f_*(KX) = \sum_\alpha (f^*\theta^\alpha)(KX)\bar{e}_\alpha \tag{2.10}$$

for any $X \in \Gamma(TM)$. For simplicity, we put $f^*\theta^\alpha \circ K = \omega_\alpha$. The condition that 1-forms ω_α, $\alpha = 1, \ldots, n+p$, are closed is equivalent to $\sum_\alpha d\omega_\alpha(f^\sharp \bar{e}_\alpha) = 0$, where $f^\sharp \bar{e}_\alpha \in \Gamma(f^\sharp T\mathbb{R}^{n+p})$ is defined by $(f^\sharp \bar{e}_\alpha)_x := (f_{\sharp x})^{-1}(\bar{e}_\alpha)_{f(x)}$ for each $x \in M$. On the other hand, by (2.10), we get

$$\begin{aligned}
&2\sum_\alpha (d\omega_\alpha)(X,\, Y)(f^\sharp \bar{e}_\alpha) \\
&= f^\sharp D_X i^f KY - f^\sharp D_Y i^f KX - i^f K(\nabla_X Y - \nabla_Y X) \\
&= H_f(X,\, KY) - H_f(Y,\, KX)
\end{aligned}$$

for any $X, Y \in \Gamma(TM)$. Therefore $(d\omega_\alpha)(X, Y) = 0$ for $\alpha = 1, \ldots, n+p$ if and only if $H_f(X, KY) = H_f(Y, KX)$ for each $x \in M$ and any $X, Y \in T_xM$. Then from the assumption, there exist φ^α such that

$$d\varphi^\alpha = f^*\theta^\alpha \circ K$$

by Poincaré's lemma for $\alpha = 1, \ldots, n+p$. Put $f_K(x) := \varphi^\alpha(x)e_\alpha$ and define Ψ_K by $\Psi_{K_x}(f^\sharp \bar{e}_\alpha)_x := (f_K^\sharp \bar{e}_\alpha)_x$ for each $x \in M$ and $\alpha = 1, \ldots, n+p$. Then the equations (2.8) and (2.9) hold. $\square$

From Lemma 2.4 and Proposition 2.5, we get

Proposition 2.6 *For a simply connected complex manifold (M, J), $\nabla \in \mathcal{C}_0(TM, J)$ and a map $f\colon M \to \mathbb{R}^{2m+p}$, there is an associated family f_z, $z \in \mathbb{C} \setminus \{0\}$, if and only if f is pluriharmonic.*

Proof. If f is pluriharmonic, then we obtain

$$H_f(E^zX, Y) = H_f(X, E^zY)$$

for each $x \in M$ and any $X, Y \in T_xM$. Hence from Proposition 2.5, there is an associated family f_z, $z \in \mathbb{C} \setminus \{0\}$, for f. Since f is pluriharmonic, f_z is also pluriharmonic by a direct calculation. The converse follows from Lemma 2.4. $\square$

For a pluriharmonic map from a Kähler manifold to a Riemannian symmetric space, Proposition 2.6 is proved in [6], where they use E^θ instead of E^z.

3. (1, 1)-geodesic affine immersions

Let M and $\widetilde{M}$ be manifolds and $f\colon M \to \widetilde{M}$ an immersion. For a subbundle N of $f^\sharp T\widetilde{M}$, if

$$f^\sharp T\widetilde{M} = i^f(TM) \oplus N,$$

then we call such an immersion an *immersion with transversal bundle N*. Let $\iota_{i^f(TM)}\colon i^f(TM) \to f^\sharp T\widetilde{M}$, $\iota_N\colon N \to f^\sharp T\widetilde{M}$ be the inclusions and $\pi_{i^f(TM)}\colon f^\sharp T\widetilde{M} \to i^f(TM)$, $\pi_N\colon f^\sharp T\widetilde{M} \to N$ the projections. We put $\widehat{i^f} := \pi_{i^f(TM)} i^f \in \mathrm{ISO}(TM, i^f(TM))$. Let $\nabla \in \mathcal{C}_0(TM)$ and $\widetilde{\nabla} \in \mathcal{C}_0(T\widetilde{M})$ be torsion free affine connections. For an immersion $f\colon M \to \widetilde{M}$ with transversal bundle N, if the induced connection $\pi_{i^f(TM)}(f^\sharp\widetilde{\nabla})\iota_{i^f(TM)}$ on $i^f(TM)$ for

$f^\sharp\widetilde{\nabla}$ coincides with $\widehat{i}^f\nabla(\widehat{i}^f)^{-1}$, we say such a morphism $(f, N)\colon (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$ an *affine immersion with transversal bundle* N and denote it by $f\colon (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$ if the transversal bundle is stated. When for an immersion $f\colon (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$, there is a subbundle N of $f^\sharp T\widetilde{M}$ such that $f\colon (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$ is an affine immersion with transversal bundle N, we call $f\colon (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$ an *affine immersion*. For an affine immersion $f\colon (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$ with transversal bundle N, we define the affine fundamental form $B \in A^1(\mathrm{Hom}(TM, N))$, the shape tensor $A \in A^1(\mathrm{Hom}(N, TM))$ and the transversal connection $\nabla^N \in \mathcal{C}(N)$ by

$$\begin{aligned} B&:=\pi_N(f^\sharp\widetilde{\nabla})\iota_{i^f(TM)}\widehat{i}^f,\\ A&:=-(\widehat{i}^f)^{-1}\pi_{i^f(TM)}(f^\sharp\widetilde{\nabla})\iota_N,\\ \nabla^N&:=\pi_N(f^\sharp\widetilde{\nabla})\iota_N. \end{aligned}$$

Since $\widetilde{\nabla}$ is torsion free, B is symmetric, that is, $B_XY = B_YX$ for each $x \in M$ and any $X, Y \in T_xM$. Note that B_XY (resp. $A_X\xi$) is usually denoted by $\alpha(X, Y)$ (resp. $A_\xi X$) for each $x \in M$, any $X, Y \in T_xM$ and $\xi \in N_x$. Then we can write the Gauss and Weingarten formulas as

$$\begin{aligned} (f^\sharp\widetilde{\nabla})_X i^f Y&=i^f\nabla_X Y + B_X Y,\\ (f^\sharp\widetilde{\nabla})_X \xi&=-i^f A_X\xi + \nabla^N_X\xi \end{aligned}$$

for each $x \in M$, any $X \in T_xM$, $Y \in \Gamma(TM)$ and $\xi \in \Gamma(N)$.

Next we consider another transversal bundle $\bar{N}$ and the decomposition

$$f^\sharp T\widetilde{M} = i^f(TM) \oplus \bar{N}. \tag{3.1}$$

According to the decomposition (3.1), let

$$\bar{\iota}_{i^f(TM)}\colon i^f(TM) \to f^\sharp T\widetilde{M}, \quad \iota_{\bar{N}}\colon \bar{N} \to f^\sharp T\widetilde{M}$$

be the inclusions and

$$\bar{\pi}_{i^f(TM)}\colon f^\sharp T\widetilde{M} \to i^f(TM), \quad \pi_{\bar{N}}\colon f^\sharp T\widetilde{M} \to \bar{N}$$

the projections. Note that $\iota_{i^f(TM)} = \bar{\iota}_{i^f(TM)}$. Let $\bar{\nabla} \in \mathcal{C}_0(TM)$ be a connection such that $f\colon (M, \bar{\nabla}) \to (\widetilde{M}, \widetilde{\nabla})$ is an affine immersion with transversal bundle $\bar{N}$ and $\bar{B}$, $\bar{A}$ and $\bar{\nabla}^{\bar{N}}$ the affine fundamental form, the shape tensor and the transversal connection of the affine immersion $f\colon (M, \bar{\nabla}) \to (\widetilde{M}, \widetilde{\nabla})$ with transversal bundle $\bar{N}$. Then we have

Lemma 3.1 [2], [10]

$$\begin{aligned}
\bar{\nabla}_X =& \nabla_X + (\widehat{i^f})^{-1}\bar{\pi}_{i^f(TM)}\iota_N B_X,\\
\bar{B}_X =& \pi_{\bar{N}}\iota_N B_X,\\
\bar{A}_X =& A_X\pi_N\iota_{\bar{N}} - \nabla_X(\widehat{i^f})^{-1}\pi_{i^f(TM)}\iota_{\bar{N}}\\
&- (\widehat{i^f})^{-1}\bar{\pi}_{i^f(TM)}\iota_N B_X(\widehat{i^f})^{-1}\pi_{i^f(TM)}\iota_{\bar{N}}\\
&- (\widehat{i^f})^{-1}\bar{\pi}_{i^f(TM)}\iota_N\nabla^N_X\pi_N\iota_{\bar{N}},\\
\bar{\nabla}^{\bar{N}}_X =& \pi_{\bar{N}}\iota_N\nabla^N_X\pi_N\iota_{\bar{N}} + \pi_{\bar{N}}\iota_N B_X(\widehat{i^f})^{-1}\pi_{i^f(TM)}\iota_{\bar{N}}
\end{aligned}$$

for each $x \in M$ and any $X \in T_xM$.

Note that the above equations in Lemma 3.1 uniquely determine the relations of induced objects, the connections on M, the affine fundamental forms, the shape tensors and the transversal connections, when we replace the transversal bundle $\bar{N}$ with N. From Lemma 3.1, both N and $\bar{N}$ induce the same connection ∇ on M if and only if $\bar{\pi}_{i^f(TM)}\iota_N B = 0$.

Definition 3.2 For a complex manifold (M, J), $\nabla \in \mathcal{C}_0(TM, J)$ and an affine immersion $f\colon (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$ with transversal bundle N, we say that f is (1, 1)-*geodesic* if

$$B_{JX}Y = B_X JY$$

for each $x \in M$ and any $X, Y \in T_xM$.

Note that the equation above is equivalent to the condition that (1, 1)-part of the complexified affine fundamental form vanishes. We note that an isometric immersion from a Kähler manifold to a pseudo-Riemannian manifold is (1, 1)-geodesic if and only if the shape tensor satisfies

$$A_{JX}\xi = -JA_X\xi \tag{3.2}$$

for each $x \in M$, any $X \in T_xM$ and $\xi \in T_x^{\perp}M$, where $T_x^{\perp}M$ is the normal space of the immersion at x. Hence any (1, 1)-geodesic isometric immersion from a Kähler manifold to a Riemannian manifold is minimal. For a (1, 1)-geodesic affine immersion from a complex manifold with complex affine connection to a manifold with affine connection, the equation (3.2) does not hold in general.

We mention that an isometric immersion from a Kähler manifold to a pseudo-Riemannian manifold is (1, 1)-geodesic if and only if the immersion is pluriharmonic as a map. If an affine immersion with transversal bundle N from a complex manifold with complex affine connection to a manifold with affine connection is pluriharmonic as a map, then the immersion is a (1, 1)-geodesic affine immersion with transversal bundle N. Conversely, a (1, 1)-geodesic affine immersion with transversal bundle from a complex manifold with complex affine connection to a manifold with affine connection is pluriharmonic as a map. From Lemma 3.1, we get

Proposition 3.3 *For a complex manifold* (M, J), $\nabla \in \mathcal{C}_0(TM, J)$ *and an affine immersion* $f\colon (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$ *with transversal bundle* N, *the property that* f *is* (1, 1)*-geodesic does not depend on the choice of transversal bundles.*

Proof. Let $\bar{N}$ be another transversal bundle of the affine immersion $f\colon (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$ and $\bar{B}$ the affine fundamental form for the affine immersion $f\colon (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$ with transversal bundle $\bar{N}$. Then by the second formula in Lemma 3.1, it holds that

$$\bar{B}_X = \pi_{\bar{N}} \iota_N B_X$$

for any $X \in T_xM$, $x \in M$. If the affine immersion $f\colon (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$ with transversal bundle N is (1, 1)-geodesic, we get

$$\bar{B}_X JY = \pi_{\bar{N}} \iota_N B_X JY = \pi_{\bar{N}} \iota_N B_{JX} Y = \bar{B}_{JX} Y$$

for any X, $Y \in T_xM$, $x \in M$ and the affine immersion $f\colon (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$ with transversal bundle $\bar{N}$ is (1, 1)-geodesic. $\square$

An affine immersion to a hyperquadric is considered in [9] and such an immersion can be considered as generalization of an isometric immersion to a space form. Let $(x^1, \ldots, x^{2m+p+1})$ be the standard coordinate of $\mathbb{R}^{2m+p+1}$ and D the standard connection of $\mathbb{R}^{2m+p+1}$. We define a hyperquadric Q in $\mathbb{R}^{2m+p+1}$ by

$$Q := \left\{ x \in \mathbf{R}^{2m+p+1} \;\middle|\; -\sum_{i=1}^{s} (x^i)^2 + \sum_{j=s+1}^{s+\bar{s}} (x^j)^2 = \varepsilon \right\},$$

where $0 \leqq s$, $0 \leqq \bar{s}$, $0 < s+\bar{s} \leqq 2m+p+1$ and $\varepsilon = \pm 1$. Let $\iota\colon Q \to \mathbb{R}^{2m+p+1}$

be the inclusion map. Define $\xi \in \Gamma(T\mathbb{R}^{2m+p+1})$ by

$$\xi := -\sum_{i=1}^{2m+p+1} x^i \frac{\partial}{\partial x^i}$$

and N^Q by

$$N_q^Q := \mathrm{Span}\{(\iota_{\sharp q})^{-1}\xi\}$$

for each $q \in Q$. Then N^Q is a transversal bundle of the immersion ι. Let ∇^Q be the connection determined by the above decomposition and D, $\iota\colon (Q, \nabla^Q) \to (\mathbb{R}^{2m+p+1}, D)$ is a centro-affine immersion with transversal bundle N^Q. We denote by B^Q the affine fundamental form of the affine immersion $\iota\colon (Q, \nabla^Q) \to (\mathbb{R}^{2m+p+1}, D)$ with transversal bundle N^Q. Then h^Q defined by $B_U^Q V = \varepsilon h^Q(U, V)(\iota_{\sharp q})^{-1}\xi$ for any $U, V \in T_qQ$, $q \in Q$ is a symmetric bilinear function on Q.

We mention that a non-degenerate hyperquadric which is immersed in an affine space as a centro-affine hypersurface corresponds to the space form of non-zero sectional curvature. It is given in [3] for an isometric immersion from a Kähler manifold to a space form of non-zero sectional curvature that if the immersion is (1, 1)-geodesic, then the dimension of the manifold equals to two. We can generalize this result to the case of an affine immersion, that is, we can prove the following proposition.

Proposition 3.4 *For a* $(1, 1)$-*geodesic affine immersion* $f\colon (M, \nabla) \to (Q, \nabla^Q)$ *with transversal bundle* N, *we assume that*

$$A_{JX} = -JA_X, \tag{3.3}$$
$$(f^*h^Q)_x(JX, JY) = (f^*h^Q)_x(X, Y) \tag{3.4}$$

for any $X, Y \in T_xM$, $x \in M$. *Then* $f^*h^Q = 0$ *on* M *or* $\dim M = 2$.

Proof. Fix a point $x \in M$. The equation of Gauss of the affine immersion $f\colon (M, \nabla) \to (Q, \nabla^Q)$ with transversal bundle N is

$$\begin{aligned} R_{X,Y}Z = {} & A_X B_Y Z - A_Y B_X Z \\ & + \varepsilon(f^*h^Q)_x(Y, Z)X - \varepsilon(f^*h^Q)_x(X, Z)Y \end{aligned} \tag{3.5}$$

for any $X, Y, Z \in T_xM$. By (3.3) and (3.5), we have

$$\mathrm{Ric}_{X,Y} = -\operatorname{tr}(A_X B_Y) + \varepsilon(2m-1)(f^*h^Q)_x(X, Y). \tag{3.6}$$

for any $X, Y \in T_xM$, $x \in M$. Since it holds that $R_{X,Y} = -JR_{X,Y}J$ for any $X, Y \in T_xM$, $x \in M$, (3.3), (3.4) hold and the immersion f is (1, 1)-geodesic, we obtain

$$\mathrm{Ric}_{X,Y} = -\operatorname{tr}(A_X B_Y) + \varepsilon(f^* h^Q)_x(X, Y). \tag{3.7}$$

From (3.6) and (3.7), it follows that $\varepsilon(2m-2)(f^*h^Q)_x(X, Y) = 0$ and we get $2m - 2 = 0$ or $(f^*h^Q)_x$ is identically zero. □

As a corollary of this proposition, we have

Corollary 3.5 *Under the same assumptions as in* Proposition 3.4*, if it holds that*

$$\max\left\{\bar{s} - p - \frac{1}{2}(1+\varepsilon), s - p - \frac{1}{2}(1-\varepsilon)\right\} > 0,$$

then $\dim M = 2$.

Proof. Fix a point $x \in M$. For any subspace W of $T_{f(x)}Q$ such that $h^Q|_{W\times W} = 0$, we have

$$\dim W \leqq \min\left\{2m + p - \bar{s} + \frac{1}{2}(1+\varepsilon),\ 2m + p - s + \frac{1}{2}(1-\varepsilon)\right\}.$$

From the assumption, we obtain

$$\min\left\{2m + p - \bar{s} + \frac{1}{2}(1+\varepsilon), 2m + p - s + \frac{1}{2}(1-\varepsilon)\right\} < 2m = \dim f_{*x}T_xM.$$

Thus $f^*h^Q \neq 0$. Hence, we get $\dim M = 2$ from Proposition 3.4. □

When Q is a Riemannian space form with non-zero sectional curvature, we have $s = (1-\varepsilon)/2$, $\bar{s} = 2m + p + (1+\varepsilon)/2$. If M is a Kähler manifold and the immersion $f\colon M \to Q$ is a (1, 1)-geodesic isometric immersion, assumptions (3.3) and (3.4) in Proposition 3.4 hold. Hence from Corollary 3.5, we see that $\dim M = 2$. This is one of the results given in [3].

As an analogue of Proposition 2.2, we obtain

Proposition 3.6 *For a complex manifold* (M, J)*,* $\nabla \in \mathcal{C}_0(TM, J)$ *and an affine immersion* $f\colon (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$ *with transversal bundle* N*,* f *is* (1, 1)*-geodesic if and only if, for any complex affine immersion* $\phi\colon (S, \nabla^S) \to (M, \nabla)$ *from a complex manifold* (S, J^S) *with* $\nabla^S \in \mathcal{C}_0(TS, J^S)$*,* $f \circ \phi$ *is a* (1, 1)*-geodesic affine immersion.*

Proof. We regard $g := f \circ \phi$ as an affine immersion with transversal bundle $\widehat{N}$ given by

$$\widehat{N}_x := (g_{\sharp x})^{-1} f_{\sharp\phi(x)} N_{\phi(x)} \oplus (g_{\sharp x})^{-1} f_{*\phi(x)} \phi_{\sharp x} \overline{N}_x$$

for each $x \in S$, where $\overline{N}$ is a transversal bundle of ϕ. Then the affine fundamental form B^g of g satisfies

$$B^g_X Y = (g_{\sharp x})^{-1} f_{\sharp\phi(x)} B_{\phi_{*x} X} \phi_{*x} Y + (g_{\sharp x})^{-1} f_{*\phi(x)} \phi_{\sharp x} B^{\phi}_X Y$$

for each $x \in S$ and any $X, Y \in T_x S$, where B^{ϕ} is the affine fundamental form of ϕ. Since ϕ is a complex affine immersion, ϕ is (1, 1)-geodesic and $\phi_* J^S = J\phi_*$. If f is (1, 1)-geodesic, we have

$$\begin{aligned} B^g_{J^S X} Y &= (g_{\sharp x})^{-1} f_{\sharp\phi(x)} B_{J\phi_{*x} X} \phi_{*x} Y + (g_{\sharp x})^{-1} f_{*\phi(x)} \phi_{\sharp x} B^{\phi}_X J^S Y \\ &= (g_{\sharp x})^{-1} f_{\sharp\phi(x)} B_{\phi_{*x} X} J\phi_{*x} Y + (g_{\sharp x})^{-1} f_{*\phi(x)} \phi_{\sharp x} B^{\phi}_X J^S Y \\ &= B^g_X J^S Y \end{aligned}$$

for each $x \in S$ and any $X, Y \in T_x S$. The converse is true since the identity map of M is regarded as a complex affine immersion. □

We define an associated family for an affine immersion from a complex manifold with complex affine connection to a manifold with affine connection, which is a special case of an associated family defined for a map in Section 2.

Definition 3.7 For a complex manifold (M, J), $\nabla \in \mathcal{C}_0(TM, J)$ and an affine immersion $f\colon (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$ with transversal bundle N, an *associated family* for f is a family of affine immersions $f_z\colon (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$, $z \in \mathbb{C} \setminus \{0\}$, with transversal bundle N_z such that f_z is an associated family for a map f and

$$\Psi_z(N) = N_z$$

for some $\Psi_z \in \mathrm{ISO}(f^{\sharp} T\widetilde{M}, f_z^{\sharp} T\widetilde{M})$.

We define $F_z \in \mathrm{ISO}(N, N_z)$ by $F_z := \pi_{N_z} \Psi_z \iota_N$, where $\pi_{N_z}\colon f_z^{\sharp} T\widetilde{M} \to N_z$ is the projection. Then from the definition, the affine fundamental form B^z, the shape tensor A^z and the transversal connection ∇^{N_z} of f_z satisfy

$$A^z_X F_z = E^{z^{-1}} A_X, \tag{3.8}$$

$$B^z_X = F_z B_X E^z, \tag{3.9}$$

$$F_z \nabla^N_X = \nabla^{N_z}_X F_z \tag{3.10}$$

for each $x \in M$, any $X \in T_xM$ and $z \in \mathbb{C} \setminus \{0\}$.

Remark For an isometric immersion from a Kähler manifold to a pseudo-Riemannian manifold, the equation (3.8) is equivalent to the equation (3.9).

An associated family is considered for an isometric immersion from a Kähler manifold to a Euclidean space in [3] and to a pseudo-Euclidean space in [7]. In these cases, they consider a (1, 1)-tensor field E^θ given by (2.5).

Remark For a complex manifold (M, J), $\nabla \in \mathcal{C}_0(TM, J)$ and an affine immersion $f\colon (M, \nabla) \to (\mathbb{R}^{2m+p}, D)$, assume that there is an associated family f_z, $z \in \mathbb{C} \setminus \{0\}$. If there exists $r \in \mathbb{R} \setminus \{0\}$ such that $z_1 = rz_2$ for $z_1, z_2 \in \mathbb{C} \setminus \{0\}$, then f_{z_1} and f_{z_2} are affine congruent of $\mathbb{R}^{2m+p}$.

The following result is similar to Lemma 2.4.

Lemma 3.8 *For a complex manifold* (M, J), $\nabla \in \mathcal{C}_0(TM, J)$ *and an affine immersion* $f\colon (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$ *with transversal bundle* N*, if there is an associated family* f_z, $z \in \mathbb{C} \setminus \{0\}$*, then* f *is* (1, 1)*-geodesic.*

Proof. From (3.9), we see that

$$B^z_X Y = F_{zx} B_X E^z Y = \mathrm{Re}(z) F_{zx} B_X Y + \mathrm{Im}(z) F_{zx} B_X JY$$

for each $x \in M$ and any $X, Y \in T_xM$. Since B is symmetric and $z \in \mathbb{C} \setminus \{0\}$ is taken arbitrary, we get

$$B_X JY = B_{JX} Y$$

for each $x \in M$ and any $X, Y \in T_xM$, that is, f is (1, 1)-geodesic. $\square$

By using the associated map defined in Section 2, we prepare the following. For an affine immersion $f\colon (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$ with transversal bundle N, consider a parallel invertible (1, 1)-tensor field K on M such that

$$B_{KX} Y = B_X KY$$

for each $x \in M$ and any $X, Y \in T_xM$. An affine immersion $f_K\colon (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$ with transversal bundle N_K is called an *associated immersion* with respect to K if f_K is an associated map for f with respect to K such that $\Psi_K(N) = N_K$. When we define $F_K \in \mathrm{ISO}(N, N_K)$ by $F_K := \pi_{N_K} \Psi_K \iota_N$, where $\pi_{N_K}\colon f_K^\sharp T\widetilde{M} \to N_K$ is the projection, the shape tensor A^K, the affine fundamental form B^K and the transversal connection ∇^{N_K} of the immersion f_K are characterized by

$$A^K_X F_K = K^{-1} A_X, \quad B^K_X = F_K B_X K, \quad \nabla^{N_K}_X F_K = F_K \nabla^N_X$$

for each $x \in M$ and any $X \in T_x M$. In [3], the authors consider an associated immersion for an isometric immersion from a Riemannian manifold to a Euclidean space by using a parallel orthogonal tensor field on the Riemannian manifold.

Proposition 3.9 *For a simply connected manifold M, $\nabla \in \mathcal{C}_0(TM)$, a parallel invertible* (1, 1)*-tensor field K on M and an affine immersion $f\colon (M, \nabla) \to (\mathbb{R}^{n+p}, D)$ with transversal bundle N, if the affine fundamental form B satisfies $B_{KX}Y = B_X KY$ for each $x \in M$ and any $X, Y \in T_x M$, then there is an associated immersion f_K with respect to K.*

Proof. For the 1-forms ω_α, $\alpha = 1, \ldots, n+p$, given in the proof of Proposition 2.5, we have

$$2 \sum_\alpha (d\omega_\alpha)(X, Y)(f^\sharp \bar{e}_\alpha) = B_X KY - B_Y KX$$

for any $X, Y \in \Gamma(TM)$ by (2.10). From the assumption, there are φ^α, $\alpha = 1, \ldots, n+p$, such that $d\varphi^\alpha = \omega_\alpha$ and $f_K\colon M \to \mathbb{R}^{n+p}$ given by $f_K(x) := \varphi^\alpha(x) e_\alpha$ for each $x \in M$ is an associated map with respect to K. Since f is an affine immersion, we see that f_K is an immersion and the image of the tangent space of f and f_K are parallel in $\mathbb{R}^{n+p}$ at each point of M. We may choose N_K such that $N_K := \Psi_K(N)$. The equations (2.8) and (2.9) imply

$$\begin{aligned} f^\sharp_K D_X i^{f_K} Y &= f^\sharp_K D_X \Psi_K i^f KY = \Psi_K f^\sharp D_X i^f KY \\ &= \Psi_K (i^f \nabla_X KY + B_X KY) = \Psi_K (i^f K \nabla_X Y + B_X KY) \\ &= i^{f_K} \nabla_X Y + \Psi_K B_X KY \end{aligned}$$

for each $x \in M$, any $X \in T_x M$ and $Y \in \Gamma(TM)$. Therefore the induced connection on M is ∇ and $f_K\colon (M, \nabla) \to (\mathbb{R}^{n+p}, D)$ is an affine immersion with transversal bundle N_K. Hence f_K is the associated immersion with respect to K for f. □

Note that Proposition 3.9 generalizes a result in [3], where they consider an isometric immersion from a Riemannian manifold to a Euclidean space and K is an orthogonal parallel tensor field on the Riemannian manifold.

Lemma 3.8 and Proposition 3.9 yield

Proposition 3.10 *For a simply connected complex manifold* (M, J), $\nabla \in \mathcal{C}_0(TM, J)$ *and an affine immersion* $f\colon (M, \nabla) \to (\mathbb{R}^{2m+p}, D)$ *with transversal bundle* N, *there exists an associated family* $f_z\colon (M, \nabla) \to (\mathbb{R}^{2m+p}, D)$, $z \in \mathbb{C} \setminus \{0\}$, *with transversal bundle* N_z *if and only if the immersion* f *is* $(1, 1)$*-geodesic.*

Proof. First we assume that f is $(1, 1)$-geodesic. Then we obtain

$$B_{E^z X} Y = B_X E^z Y$$

for each $x \in M$ and any $X, Y \in T_x M$. Therefore from Proposition 3.9, for each $z \in \mathbb{C} \setminus \{0\}$, there is an affine immersion $f_z\colon (M, \nabla) \to (\mathbb{R}^{2m+p}, D)$ with transversal bundle N_z, that is, there exists an associated family f_z for f. Since f is $(1, 1)$-geodesic, we get

$$B^z_{JX} Y = F_z B_{JX} E^z Y = F_z B_X J E^z Y = F_z B_X E^z J Y = B^z_X J Y$$

for each $x \in M$, any $X, Y \in T_x M$ and $z \in \mathbb{C} \setminus \{0\}$. The converse is true from Lemma 3.8. $\square$

We mention that an associated family for a $(1, 1)$-geodesic isometric immersion from a Kähler manifold to a Euclidean space is constructed by Dajczer and Gromoll in [3] and when the ambient space is a pseudo-Euclidean space, Furuhata constructed an associated family in [7].

4. Product of affine immersions

In this section, we study a product of two affine immersions for the next section.

Throughout this paper, we always assume that $i, j = 1, 2$ and $i \neq j$. Let M_i be a manifold. For $(x_1, x_2) \in M_1 \times M_2$, we define $q_{ix_j}\colon M_i \to M_1 \times M_2$ by

$$q_{ix_j}(x_i) := (x_1, x_2)$$

for each $x_j \in M_j$. For $Y \in T_{x_i} M_i$, we define the lift $\widetilde{Y}^i$ of Y to $M_1 \times M_2$ by

$$\widetilde{Y}^i := q_{ix_j * x_i} Y \in T_{(x_1, x_2)} M_1 \times M_2$$

for each $x_j \in M_j$. We often write $(\,\cdot\,)^{\sim i}$ instead of $\widetilde{(\,\cdot\,)}^i$. For $\nabla^i \in \mathcal{C}_0(TM_i)$, there is a unique connection $\widetilde{\nabla} \in \mathcal{C}(T(M_1 \times M_2))$ on a product manifold $M_1 \times M_2$ such that

$$\widetilde{\nabla}_{\widetilde{W_1}^1 + \widetilde{W_2}^2} \widetilde{X_1}^1 + \widetilde{X_2}^2 = (\nabla^1_{W_1} X_1)^{\sim 1} + (\nabla^2_{W_2} X_2)^{\sim 2}$$

for each $x_i \in M_i$, any $W_i \in T_{x_i} M_i$ and $X_i \in \Gamma(TM_i)$. By a direct calculation, we have $\widetilde{\nabla} \in \mathcal{C}_0(T(M_1 \times M_2))$ since $\nabla^i \in \mathcal{C}_0(TM_i)$. We call $\widetilde{\nabla}$ a *product connection* of ∇^1 and ∇^2. Note that when $(M_i, \nabla^i) = (\mathbb{R}^{n_i}, D^i)$, then the product connection $\bar{D}$ of D^1 and D^2 is affine diffeomorphic to the standard affine connection D of $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, where D^i is the standard affine connection of $\mathbb{R}^{n_i}$.

Let $\overline{M}_i$ be a manifold and $\overline{\nabla}^i \in \mathcal{C}_0(T\overline{M}_i)$. For an affine immersion $f_i \colon (M_i, \nabla^i) \to (\overline{M}_i, \overline{\nabla}^i)$ with transversal bundle N_i, we denote by B^i, A^i and ∇^{N_i} the affine fundamental form, the shape tensor and the transversal connection of f_i.

By a similar way, for each $y_i \in \overline{M}_i$, we denote the lift of $X \in T_{y_i} \overline{M}_i$ to $\overline{M}_1 \times \overline{M}_2$ by $\overline{X}^i$. We consider the immersion

$$\bar{f} := f_1 \times f_2 \colon M_1 \times M_2 \to \overline{M}_1 \times \overline{M}_2.$$

For each $x_i \in M_i$ and any $U_i \in (f_i^\sharp T\overline{M}_i)_{x_i}$, we define $\overline{U_i}^i \in (\bar{f}^\sharp T(\overline{M}_1 \times \overline{M}_2))_{(x_1, x_2)}$ by

$$\overline{U_i}^i := (\bar{f}_{\sharp(x_1,x_2)})^{-1} (\overline{f_{i\sharp x_i} U_i}^i)$$

for each $x_j \in M_j$. By a straightforward computation, we get

$$i^{\bar{f}}_{(x_1,x_2)} (\widetilde{Y_1}^1 + \widetilde{Y_2}^2) = \overline{i^{f_1}_{x_1} Y_1}^1 + \overline{i^{f_2}_{x_2} Y_2}^2$$

for each $x_i \in M_i$ and any $Y_i \in T_{x_i} M_i$. For the product connection $\overline{\nabla}$ of $\overline{\nabla}^1$ and $\overline{\nabla}^2$, we have

$$\bar{f}^\sharp \overline{\nabla}_{\widetilde{X_1}^1 + \widetilde{X_2}^2} (\overline{Z_1}^1 + \overline{Z_2}^2) = \overline{f_1^\sharp \overline{\nabla}^1_{X_1} Z_1}^1 + \overline{f_2^\sharp \overline{\nabla}^2_{X_2} Z_2}^2 \tag{4.1}$$

for each $x_i \in M_i$, any $X_i \in T_{x_i} M_i$ and $Z_i \in \Gamma(f_i^\sharp T\overline{M}_i)$. When we define

$$\overline{N}_{(x_1,x_2)} := \overline{N_{1x_1}}^1 \oplus \overline{N_{2x_2}}^2 \tag{4.2}$$

for each $(x_1, x_2) \in M_1 \times M_2$, it holds that

$$\bar{f}^\sharp T(\overline{M}_1 \times \overline{M}_2) = i^{\bar{f}}(T(M_1 \times M_2)) \oplus \overline{N}$$

and $\bar{f}$ is an immersion with transversal bundle $\overline{N}$. Since it follows that

$$
\begin{aligned}
\bar{f}^{\sharp}\overline{\nabla}_{\widetilde{X_1}^1+\widetilde{X_2}^2} i^{\bar{f}}(\widetilde{Y_1}^1+\widetilde{Y_2}^2) &= \bar{f}^{\sharp}\overline{\nabla}_{\widetilde{X_1}^1+\widetilde{X_2}^2}\overline{i^{f_1}Y_1}^1+\overline{i^{f_2}Y_2}^2 \\
&= \overline{f_1^{\sharp}\overline{\nabla}^1_{X_1} i^{f_1}Y_1}^1+\overline{f_2^{\sharp}\overline{\nabla}^2_{X_2} i^{f_2}Y_2}^2 \\
&= \overline{i^{f_1}\nabla^1_{X_1}Y_1}^1+\overline{i^{f_2}\nabla^2_{X_2}Y_2}^2 \\
&\quad +\overline{B^1_{X_1}Y_1}^1+\overline{B^2_{X_2}Y_2}^2, \\
i^{\bar{f}}(\widetilde{\nabla}_{\widetilde{X_1}^1+\widetilde{X_2}^2}\widetilde{Y_1}^1+\widetilde{Y_2}^2) &= i^{\bar{f}}((\nabla^1_{X_1}Y_1)^{\sim 1}+(\nabla^2_{X_2}Y_2)^{\sim 2}) \\
&= \overline{i^{f_1}\nabla^1_{X_1}Y_1}^1+\overline{i^{f_2}\nabla^2_{X_2}Y_2}^2
\end{aligned}
$$

for each $x_i \in M_i$, any $X_i \in T_{x_i}M_i$ and $Y_i \in \Gamma(TM_i)$, the induced connection on $M_1 \times M_2$ coincides with $\widetilde{\nabla}$ and $\bar{f}\colon (M_1 \times M_2, \widetilde{\nabla}) \to (\overline{M}_1 \times \overline{M}_2, \overline{\nabla})$ is an affine immersion with transversal bundle $\overline{N}$ given by (4.2). The equation (4.1) yields

$$
\begin{aligned}
\bar{f}^{\sharp}\overline{\nabla}_{\widetilde{X_1}^1+\widetilde{X_2}^2}(\overline{\xi_1}^1+\overline{\xi_2}^2) &= \overline{f_1^{\sharp}\overline{\nabla}^1_{X_1}\xi_1}^1+\overline{f_2^{\sharp}\overline{\nabla}^2_{X_2}\xi_2}^2 \\
&= \overline{-i^{f_1}A^1_{X_1}\xi_1}^1+\overline{-i^{f_2}A^1_{X_2}\xi_2}^2 \\
&\quad +\overline{\nabla^{N_1}_{X_1}\xi_1}^1+\overline{\nabla^{N_1}_{X_2}\xi_2}^2
\end{aligned}
$$

for each $x_i \in M_i$, any $X_i \in T_{x_i}M_i$ and $\xi_i \in \Gamma(N_i)$. Thus we obtain

Proposition 4.1 *For the affine immersion $\bar{f} = f_1 \times f_2\colon (M_1 \times M_2, \widetilde{\nabla}) \to (\overline{M}_1 \times \overline{M}_2, \overline{\nabla})$ with transversal bundle $\overline{N}$ given by* (4.2)*, the affine fundamental form $\bar{B}$, the shape tensor $\bar{A}$ and the transversal connection $\bar{\nabla}^{\bar{N}}$ are characterized by*

$$
\begin{aligned}
&\bar{B}_{\widetilde{X_1}^1+\widetilde{X_2}^2}(\widetilde{Y_1}^1+\widetilde{Y_2}^2) = \overline{B^1_{X_1}Y_1}^1+\overline{B^2_{X_2}Y_2}^2, \\
&\bar{A}_{\widetilde{X_1}^1+\widetilde{X_2}^2}(\overline{\xi_1}^1+\overline{\xi_2}^2) = (A^1_{X_1}\xi_1)^{\sim 1}+(A^2_{X_2}\xi_2)^{\sim 2}, \\
&\bar{\nabla}^{\bar{N}}_{\widetilde{X_1}^1+\widetilde{X_2}^2}(\overline{\xi_1}^1+\overline{\xi_2}^2) = \overline{\nabla^{N_1}_{X_1}\xi_1}^1+\overline{\nabla^{N_2}_{X_2}\xi_2}^2
\end{aligned}
$$

for each $x_i \in M_i$, any X_i, $Y_i \in T_{x_i}M_i$ and $\xi_i \in \Gamma(N_i)$.

From now on, we consider the case where $(M_1, \nabla^1) = (M_2, \nabla^2) = (M, \nabla)$. Let $\Delta\colon M \ni x \mapsto (x, x) \in M \times M$ be an immersion. We define

N^Δ by $N_x^\Delta := \mathrm{Span}\{\widetilde{X}^1 - \widetilde{X}^2 \mid X \in T_xM\}$ for each $x \in M$. Then we have

$$\Delta^\sharp T(M \times M) = i^\Delta(TM) \oplus \Delta^\sharp N^\Delta$$

and we see that Δ is an immersion with transversal bundle $\Delta^\sharp N^\Delta$. Since we get

$$\Delta^\sharp \widetilde{\nabla}_X i^\Delta Y = i^\Delta \nabla_X Y$$

for each $x \in M$, any $X \in T_xM$ and $Y \in \Gamma(TM)$, the induced connection on M is ∇ and the immersion $\Delta\colon (M, \nabla) \to (M \times M, \widetilde{\nabla})$ is an affine immersion with transversal bundle $\Delta^\sharp N^\Delta$. Moreover, the affine immersion Δ is totally geodesic and the shape tensor vanishes identically. Thus we can regard Δ as a natural immersion from M to $M \times M$.

We consider the immersion

$$\widetilde{f} := \bar{f} \circ \Delta = (f_1 \times f_2) \circ \Delta\colon M \to \overline{M}_1 \times \overline{M}_2.$$

For each $x \in M$ and any $U_i \in (f_i^\sharp T\overline{M}_i)_x$, we define $\overline{U_i}^i \in (\widetilde{f}^\sharp T(\overline{M}_1 \times \overline{M}_2))_x$ by

$$\overline{U_i}^i := (\widetilde{f}_{\sharp x})^{-1}(\overline{f_{i\sharp x} U_i}^i).$$

Then we obtain

$$i_x^{\widetilde{f}} X = \overline{i_x^{f_1} X}^1 + \overline{i_x^{f_2} X}^2, \tag{4.3}$$

$$\widetilde{f}^\sharp \overline{\nabla}_X (\overline{U_1}^1 + \overline{U_2}^2) = \overline{f_1^\sharp \overline{\nabla}^1_X U_1}^1 + \overline{f_2^\sharp \overline{\nabla}^2_X U_2}^2 \tag{4.4}$$

for each $x \in M$, any $X \in T_xM$ and $U_i \in \Gamma(f_i^\sharp T\overline{M}_i)$. Define $\widetilde{N}$ by

$$\widetilde{N}_x := \overline{N_{1x}}^1 \oplus \overline{N_{2x}}^2 \oplus (\widetilde{f}_{\sharp x})^{-1} \bar{f}_{*\Delta(x)} N_x^\Delta \tag{4.5}$$

for each $x \in M$. Then it holds that

$$\widetilde{f}^\sharp T(\overline{M}_1 \times \overline{M}_2) = i^{\widetilde{f}}(TM) \oplus \widetilde{N}$$

and $\widetilde{f}$ is an immersion with transversal bundle $\widetilde{N}$. We mention that there are various choices for a transversal bundle of the immersion $\widetilde{f}$. When M and $\overline{M}_i$ are Riemannian manifolds and both f_1 and f_2 are isometric immersions, $\widetilde{N}$ given by (4.5) corresponds to the normal bundle of the immersion $\widetilde{f}$, where N_i corresponds to the normal bundle of f_i. Therefore the immersion $\widetilde{f}$ with transversal bundle $\widetilde{N}$ given by (4.5) is a generalization of a product of isometric immersions. Since we have

$$\begin{aligned}\widetilde{f}^\sharp \nabla_X i^{\widetilde{f}} Y &= \overline{i^{f_1}\nabla_X Y}^1 + \overline{i^{f_2}\nabla_X Y}^2 + \overline{B^1_X Y}^1 + \overline{B^2_X Y}^2,\\ i^{\widetilde{f}}\nabla_X Y &= \overline{i^{f_1}\nabla_X Y}^1 + \overline{i^{f_2}\nabla_X Y}^2\end{aligned}$$

for each $x \in M$, any $X \in T_xM$ and $Y \in \Gamma(TM)$ from (4.3) and (4.4), the induced connection on M coincides with ∇ and $\widetilde{f}\colon (M, \nabla) \to (\overline{M}_1 \times \overline{M}_2, \overline{\nabla})$ is an affine immersion with transversal bundle $\widetilde{N}$. For each $x \in M$, any $X \in T_xM$, $Y \in \Gamma(TM)$ and $\xi_i \in \Gamma(N_i)$, it holds that

$$\begin{aligned}&\widetilde{f}^\sharp \nabla_X (\overline{\xi_1}^1 + \overline{\xi_2}^2 + \overline{i^{f_1}Y}^1 - \overline{i^{f_2}Y}^2)\\ &= \overline{f_1^\sharp \overline{\nabla}^1_X \xi_1}^1 + \overline{f_2^\sharp \overline{\nabla}^2_X \xi_2}^2 + \overline{f_1^\sharp \overline{\nabla}^1_X Y}^1 - \overline{f_1^\sharp \overline{\nabla}^2_X Y}^2\\ &= \overline{-i^{f_1}A^1_X\xi_1}^1 + \overline{-i^{f_2}A^2_X\xi_2}^2 + \overline{\nabla^{N_1}_X\xi_1}^1 + \overline{B^1_X Y}^1\\ &\quad + \overline{\nabla^{N_2}_X\xi_2}^2 - \overline{B^2_X Y}^2 + \overline{i^{f_1}\nabla_X Y}^1 - \overline{i^{f_2}\nabla_X Y}^2.\end{aligned}$$

Thus we recall

Proposition 4.2 *For the affine immersion $\widetilde{f}\colon (M, \nabla) \to (\overline{M}_1 \times \overline{M}_2, \overline{\nabla})$ with transversal bundle $\widetilde{N}$ given by (4.5), the affine fundamental form $\widetilde{B}$, the shape tensor $\widetilde{A}$ and the transversal connection $\widetilde{\nabla}^{\widetilde{N}}$ are characterized by*

$$\begin{aligned}\widetilde{B}_X Y &= \overline{B^1_X Y}^1 + \overline{B^2_X Y}^2,\\ \widetilde{A}_X(\overline{\xi_1}^1 + \overline{\xi_2}^2 + \overline{i^{f_1}Y}^1 - \overline{i^{f_2}Y}^2) &= \frac{1}{2}(A^1_X\xi_1 + A^2_X\xi_2),\\ \widetilde{\nabla}^{\widetilde{N}}_X(\overline{\xi_1}^1 + \overline{\xi_2}^2 + \overline{i^{f_1}Y}^1 - \overline{i^{f_2}Y}^2) &= \overline{\nabla^{N_1}_X\xi_1}^1 + \overline{\nabla^{N_2}_X\xi_2}^2 + \overline{B^1_X Y}^1 - \overline{B^2_X Y}^2\\ &\quad + \overline{i^{f_1}\nabla_X Y}^1 - \frac{1}{2}\overline{i^{f_1}(A^1_X\xi_1 - A^2_X\xi_2)}^1\\ &\quad - \overline{i^{f_2}\nabla_X Y}^2 + \frac{1}{2}\overline{i^{f_2}(A^1_X\xi_1 - A^2_X\xi_2)}^2\end{aligned}$$

for each $x \in M$, any $X \in T_xM$, $Y \in \Gamma(TM)$ and $\xi_i \in \Gamma(N_i)$.

5. Holomorphic maps and complex affine immersions

In this section, we investigate a complex affine immersion between complex manifolds with complex affine connections and prove our main theorem. Throughout this section, we always denote by (M, J) a real $2m$-dimensional complex manifold with complex structure J and assume that

$\nabla \in \mathcal{C}_0(TM,\, J)$.

We recall that for complex manifolds $(M,\, J)$ and $(\widetilde{M},\, \widetilde{J})$, a map $f\colon M \to \widetilde{M}$ is holomorphic if and only if $f_* J = \widetilde{J} f_*$ which is equivalent to $i^f J = (f^\sharp \widetilde{J}) i^f$.

Proposition 5.1 *For a complex manifold $(M,\, J)$ and a map $f_i\colon M \to \mathbb{R}^{2m+p}$, $i = 1,\, 2$, assume that there is*

$$\Psi \in \mathrm{ISO}(f_1^\sharp T\mathbb{R}^{2m+p},\ f_2^\sharp T\mathbb{R}^{2m+p})$$

such that

$$(f_2^\sharp D)_X \Psi = \Psi (f_1^\sharp D)_X, \tag{5.1}$$

$$-\Psi i^{f_1} J = i^{f_2} \tag{5.2}$$

for each $x \in M$ and any $X \in T_x M$. Then $\hat{J} \in \mathrm{END}(\widetilde{f}^\sharp T(\mathbb{R}^{2m+p} \times \mathbb{R}^{2m+p}))$ given by

$$\hat{J}(\overline{U_1}^1 + \overline{U_2}^2) := \overline{-\Psi^{-1} U_2}^1 + \overline{\Psi U_1}^2$$

for each $x \in M$ and any $U_i \in (f_i^\sharp T\mathbb{R}^{2m+p})_x$, $i = 1,\, 2$, can be extended to a parallel complex structure $\widetilde{J}$ on $\mathbb{R}^{2m+p} \times \mathbb{R}^{2m+p}$ such that $\widetilde{f} := (f_1 \times f_2) \circ \Delta$ is a holomorphic map with respect to J and $\widetilde{J}$.

Proof. It holds that $(\hat{J})^2 = -\,\mathrm{id}_{\widetilde{f}^\sharp T(\mathbb{R}^{2m+p} \times \mathbb{R}^{2m+p})}$ by a direct calculation. From (4.4) and (5.1), we have

$$\begin{aligned}
(\widetilde{f}^\sharp D)_X \hat{J}(\overline{U_1}^1 + \overline{U_2}^2) &= (\widetilde{f}^\sharp D)_X (\overline{-\Psi^{-1} U_2}^1 + \overline{\Psi U_1}^2) \\
&= \overline{(f_1^\sharp D)_X (-\Psi^{-1} U_2)}^1 + \overline{(f_2^\sharp D)_X (\Psi U_1)}^2 \\
&= \overline{-\Psi^{-1} (f_2^\sharp D)_X U_2}^1 + \overline{\Psi (f_1^\sharp D)_X U_1}^2 \\
&= \hat{J}(\overline{(f_1^\sharp D)_X U_1}^1 + \overline{(f_2^\sharp D)_X U_2}^2) \\
&= \hat{J}(\widetilde{f}^\sharp D)_X (\overline{U_1}^1 + \overline{U_2}^2)
\end{aligned}$$

for each $x \in M$ and any $X \in T_x M$ and $U_i \in \Gamma(f_i^\sharp T\mathbb{R}^{2m+p})$. Hence it holds that

$$(\widetilde{f}^\sharp D)_X \hat{J} = \hat{J} (\widetilde{f}^\sharp D)_X$$

for each $x \in M$ and any $X \in T_x M$ and there is a complex structure $\widetilde{J}$ of $\mathbb{R}^{2m+p} \times \mathbb{R}^{2m+p}$ such that $\widetilde{f}^\sharp \widetilde{J} = \hat{J}$. For each $x \in M$ and any $X \in T_x M$, we obtain

$$i^{\widetilde{f}}X = \overline{i^{f_1}X}^1 + \overline{i^{f_2}X}^2.$$

By (4.3) and (5.2), it follows that

$$\begin{aligned} i^{\widetilde{f}}JX &= \overline{i^{f_1}JX}^1 + \overline{i^{f_2}JX}^2 \\ &= \overline{-\Psi^{-1}i^{f_2}X}^1 + \overline{\Psi i^{f_1}X}^2 \\ &= (\widetilde{f}^\sharp\widetilde{J})(\overline{i^{f_1}X}^1 + \overline{i^{f_2}X}^2) \\ &= (\widetilde{f}^\sharp\widetilde{J})i^{\widetilde{f}}X \end{aligned}$$

for each $x \in M$ and any $X \in T_xM$. Therefore we obtain $(\widetilde{f}^\sharp\widetilde{J})i^{\widetilde{f}} = i^{\widetilde{f}}J$ and $\widetilde{f}$ is a holomorphic map with respect to J and $\widetilde{J}$. □

From Propositions 2.6 and 5.1, we get

Corollary 5.2 *For a complex manifold (M, J) and $\nabla \in \mathcal{C}_0(TM, J)$, we assume that M is simply connected and $f\colon M \to \mathbb{R}^{2m+p}$ is a pluriharmonic map. Then there is a parallel complex structure $\widetilde{J}$ of $\mathbb{R}^{2m+p} \times \mathbb{R}^{2m+p}$ such that $(f \times (-f_{\sqrt{-1}})) \circ \Delta$ is a holomorphic map with respect to J and $\widetilde{J}$.*

Proof. Since M is simply connected and f is pluriharmonic, there is an associated family f_z, $z \in \mathbb{C} \setminus \{0\}$ by Proposition 2.6. From the definition of an associated family, $-\Psi_{\sqrt{-1}} \in \mathrm{ISO}(f^\sharp T\mathbb{R}^{2m+p}, f^\sharp_{\sqrt{-1}}T\mathbb{R}^{2m+p})$ satisfies (5.1) and (5.2). Thus by virtue of Proposition 5.1, we obtain the result. □

When we choose a simply connected neighbourhood of each point and apply Corollary 5.2, we have

Corollary 5.3 *For a complex manifold (M, J) and $\nabla \in \mathcal{C}_0(TM, J)$, any pluriharmonic map $f\colon M \to \mathbb{R}^{2m+p}$ is real analytic.*

Next we prepare the definition of a complex affine immersion.

Definition 5.4 For a complex manifold (M, J), (resp. $(\widetilde{M}, \widetilde{J})$) with complex structure J (resp. $\widetilde{J}$), $\nabla \in \mathcal{C}_0(TM, J)$ (resp. $\widetilde{\nabla} \in \mathcal{C}_0(T\widetilde{M}, \widetilde{J})$) and an affine immersion $f\colon (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$ with transversal bundle N, if $f\colon M \to \widetilde{M}$ is a holomorphic map with respect to J and $\widetilde{J}$ and N is a complex subbundle of $f^\sharp T\widetilde{M}$, that is, $(f^\sharp\widetilde{J})(N) = N$, then such an affine immersion is said to be *complex* and we denote the induced complex structure of N by $J^N := \pi_N(f^\sharp\widetilde{J})\iota_N$.

A complex affine immersion is studied by many authors ([1], [4], [5], [8] and [12] for example). We note that if an isometric immersion between Kähler manifolds is holomorphic as a map, then the affine immersion is complex with respect to the Levi-Civita connections, where the transversal bundle of the affine immersion is the normal bundle of the isometric immersion. But an affine immersion between complex manifolds with complex affine connections which is a holomorphic map is not always a complex affine immersion. We find a sufficient condition for a product of two affine immersions from a complex manifold with complex affine connection to an affine space to be a complex affine immersion from the manifold to the product of affine spaces with a certain complex structure by using Proposition 5.1.

Theorem 5.5 *For a complex manifold* (M, J)*,* $\nabla \in \mathcal{C}_0(TM, J)$ *and an affine immersion* $f_i\colon (M, \nabla) \to (\mathbb{R}^{2m+p}, D)$ *with transversal bundle* N_i*, assume that there exists* $F \in \mathrm{ISO}(N_1, N_2)$ *such that*

$$-B^2_X = FB^1_X J, \quad A^2_X F = JA^1_X, \quad F\nabla^{N_1}_X = \nabla^{N_2}_X F$$

for each $x \in M$ *and any* $X \in T_xM$*. Then there is a parallel complex structure* $\widetilde{J}$ *of* $\mathbb{R}^{2m+p} \times \mathbb{R}^{2m+p}$ *such that an affine immersion*

$$\widetilde{f} = (f_1 \times f_2) \circ \Delta\colon (M, \nabla) \to (\mathbb{R}^{2m+p} \times \mathbb{R}^{2m+p}, D)$$

with transversal bundle $\widetilde{N}$ *given by* (4.5) *is a complex affine immersion with respect to* J *and* $\widetilde{J}$*.*

Proof. Define $\Psi\colon f_1^\sharp T\mathbb{R}^{2m+p} \to f_2^\sharp T\mathbb{R}^{2m+p}$ by

$$\Psi_x(i^{f_1}_x X + \xi) := i^{f_2}_x JX + F_x\xi$$

for each $x \in M$, any $X \in T_xM$ and $\xi \in N_{1_x}$. Then we see that $\Psi \in \mathrm{ISO}(f_1^\sharp T\mathbb{R}^{2m+p}, f_2^\sharp T\mathbb{R}^{2m+p})$. From the assumptions, we have

$$\begin{aligned}(f_2^\sharp D)_X \Psi(i^{f_1} Y + \xi) &= (f_2^\sharp D)_X (i^{f_2} JY + F\xi)\\ &= i^{f_2}\nabla_X JY + B^2_X JY - i^{f_2} A^2_X F\xi + \nabla^{N_2}_X F\xi\\ &= \Psi(f_1^\sharp D)_X (i^{f_1} Y + \xi)\end{aligned}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(N_1)$. On the other hand, from the definition of Ψ, it holds that $-\Psi i^{f_1} J = i^{f_2}$. Then from Proposition 5.1, there is a parallel complex structure $\widetilde{J}$ on $\mathbb{R}^{2m+p} \times \mathbb{R}^{2m+p}$ such that $\widetilde{f} := (f_1 \times$

$f_2) \circ \Delta$ is a holomorphic map with respect to J and $\widetilde{J}$. By a straightforward calculation, we get

$$\begin{aligned}
&(\widetilde{f}^\sharp \widetilde{J})(\overline{\xi}^1 + \overline{\eta}^2 + \overline{i^{f_1} Y}^1 - \overline{i^{f_2} Y}^2) \\
&\quad = \overline{-\Psi^{-1}\eta}^1 + \overline{\Psi\xi}^2 + \overline{\Psi^{-1} i^{f_2} Y}^1 + \overline{\Psi i^{f_1} Y}^2 \\
&\quad = \overline{-F^{-1}\eta}^1 + \overline{F\xi}^2 - (\overline{i^{f_1} JY}^1 - \overline{i^{f_2} JY}^2)
\end{aligned}$$

for any $Y \in \Gamma(TM)$, $\xi \in \Gamma(N_1)$ and $\eta \in \Gamma(N_2)$. Hence $\widetilde{N}$ is $\widetilde{f}^\sharp \widetilde{J}$-invariant. Therefore $\widetilde{f}$ is a complex affine immersion with transversal bundle $\widetilde{N}$. □

As a corollary, we have

Corollary 5.6 *For a simply connected complex manifold* (M, J), $\nabla \in \mathcal{C}_0(TM, J)$ *and a* $(1, 1)$*-geodesic affine immersion* $f\colon (M, \nabla) \to (\mathbb{R}^{2m+p}, D)$ *with transversal bundle* N*, there exists a parallel complex structure* $\widetilde{J}$ *of* $\mathbb{R}^{2m+p} \times \mathbb{R}^{2m+p}$ *such that an affine immersion* $(f \times (-f_{\sqrt{-1}})) \circ \Delta$ *with transversal bundle* $\widetilde{N}$ *given by* (4.5) *is a complex affine immersion with respect to* J *and* $\widetilde{J}$.

From Corollary 5.6, we obtain the following which is given in [7]. We mention that if the ambient space is a Euclidean space, the same result as next corollary in shown in [3].

Corollary 5.7 ([7]) *For a simply connected Kähler manifold* M *and a* $(1, 1)$*-geodesic isometric immersion* $f\colon M \to \mathbb{R}^{n+p}_N$ *to an* $(n+p)$*-dimensional pseudo-Euclidean space of index* N*, there exists a parallel complex structure of* $\mathbb{R}^{n+p}_N \times \mathbb{R}^{n+p}_N$ *such that* $(1/\sqrt{2})(f \times (-f_{\sqrt{-1}})) \circ \Delta$ *is a holomorphic isometric immersion.*

Proof. From Corollary 5.6, there is a parallel complex structure of $\mathbb{R}^{n+p}_N \times \mathbb{R}^{n+p}_N$ such that $(1/\sqrt{2})(f \times (-f_{\sqrt{-1}})) \circ \Delta$ is a complex affine immersion. By a direct calculation, we see that $(1/\sqrt{2})(f \times (-f_{\sqrt{-1}})) \circ \Delta$ is isometric. □

We will consider an example of a $(1, 1)$-geodesic affine immersion and construct a complex affine immersion which is a product of two $(1, 1)$-geodesic affine immersions by using its associated family. We denote by $(\mathbb{R}^2, J)$ a 2-dimensional affine space with the complex structure J which is induced from the standard complex structure J_0 of $\mathbb{R}^2$.

Let (x_1, x_2) be a coordinate of $\mathbb{R}^2$ such that $J\partial_1 = \partial_2$ and $J\partial_2 = -\partial_1$, where we denote by $\partial_i = \partial/\partial x_i$, $i = 1, 2$. Define $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ by

$$f(x_1, x_2) := (x_1, x_2, h),$$

where $h := x_1x_2 + (x_1^2/2) - (x_2^2/2)$. For $\xi := (0, 0, 1)$, define a transversal bundle N by

$$N_x := \mathrm{Span}\{(f_{\sharp x})^{-1}\xi\}$$

for each $x \in \mathbb{R}^2$. We regard f as an affine immersion with transversal bundle N by considering the standard affine connection D on $\mathbb{R}^3$ and denote by ∇ the induced connection on $\mathbb{R}^2$. Since f is a graph immersion, we have

$$(B_{\partial_\alpha}\partial_\beta)_x = (\partial_\alpha\partial_\beta h)_x(f_{\sharp x})^{-1}\xi$$

for each $x \in \mathbb{R}^2$, where α, $\beta = 1, 2$. By a direct calculation, we get

$$\begin{aligned}&(B_{\partial_1}\partial_1)_x{=}(f_{\sharp x})^{-1}\xi, \quad (B_{\partial_1}\partial_2)_x = (B_{\partial_2}\partial_1)_x = (f_{\sharp x})^{-1}\xi,\\ &(B_{\partial_2}\partial_2)_x{=}(B_{J\partial_1}J\partial_1)_x = -(B_{\partial_1}\partial_1)_x = -(f_{\sharp x})^{-1}\xi\end{aligned}$$

for each $x \in \mathbb{R}^2$ and we see that f is a (1, 1)-geodesic affine immersion with transversal bundle N.

Since $\mathbb{R}^2$ is simply connected, we can construct an associated family for f. For $z = a + b\sqrt{-1} \in \mathbb{C} \setminus \{0\}$, a, $b \in \mathbb{R}$, we define a map f_z by

$$f_z(x_1, x_2) = \Big(ax_1 - bx_2,\, bx_1 + ax_2,\, (a-b)x_1x_2 + (a+b)\frac{x_1^2 - x_2^2}{2}\Big).$$

Define Ψ_z by

$$\Psi_{zx} := (f_{z\sharp x})^{-1}(f_{\sharp x})$$

for each $x \in \mathbb{R}^2$. Then f_z is an associated family for the map $f\colon \mathbb{R}^2 \to \mathbb{R}^3$.

Define N_z by

$$N_{zx} := \mathrm{Span}\{(f_{z\sharp x})^{-1}\xi\}$$

for each $x \in \mathbb{R}^2$, then $f_z\colon (\mathbb{R}^2, \nabla) \to (\mathbb{R}^3, D)$ is an affine immersion with transversal bundle N_z and is an associated family for the affine immersion f. In this case, $\widetilde{f} := (f \times (-f_{\sqrt{-1}})) \circ \Delta\colon \mathbb{R}^2 \to \mathbb{R}^6$ given by

$$\widetilde{f}(x_1, x_2) = \Big(x_1,\, x_2,\, x_1x_2 + \frac{x_1^2 - x_2^2}{2},\, x_2,\, -x_1,\, x_1x_2 - \frac{x_1^2 - x_2^2}{2}\Big)$$

is an affine immersion with transversal bundle $\widetilde{N}$ given by

$$\widetilde{N}_x := \overline{N_x}^1 \oplus \overline{N_{\sqrt{-1}x}}^2 \oplus (\widetilde{f}_{\sharp x})^{-1}(f \times (-f_{\sqrt{-1}}))_{*\Delta(x)} N_x^\Delta$$

for each $x \in \mathbb{R}^2$. For the standard basis $e_1, \ldots, e_6$ of $\mathbb{R}^6$, we define a complex structure $\hat{J}_0$ of $\mathbb{R}^6$ by

$$\begin{aligned} &\hat{J}_0 e_1 = e_2, \quad \hat{J}_0 e_2 = -e_1, \quad \hat{J}_0 e_3 = e_6, \\ &\hat{J}_0 e_4 = e_5, \quad \hat{J}_0 e_5 = -e_4, \quad \hat{J}_0 e_6 = -e_3 \end{aligned}$$

and we denote by $\widetilde{J}$ the induced complex structure on $T\mathbb{R}^6$ from $\hat{J}_0$. From the definition of $\widetilde{J}$, we see that $\widetilde{N}$ is $\widetilde{f}^\sharp \widetilde{J}$-invariant and $\widetilde{f}$ is a complex affine immersion with respect to J and $\widetilde{J}$.

We put $z := x_1 + \sqrt{-1}x_2$ and let

$$e_1 + \sqrt{-1}e_2, \quad e_3 + \sqrt{-1}e_6, \quad e_4 + \sqrt{-1}e_5$$

be a complex basis of $\mathbb{C}^3$. Then we can write $\widetilde{f}$ as

$$\widetilde{f}(z) = \Big(z, -\sqrt{-1}z, \frac{(1-\sqrt{-1})z^2}{2}\Big).$$

Note that the real part of the right hand side is f and the imaginary part is $-f_{\sqrt{-1}}$.

For a product of two complex manifolds (M_i, J_i), $i = 1, 2$, the complex structure $J^{M_1\times M_2} \in \mathrm{END}(T(M_1 \times M_2))$ given by

$$J^{M_1\times M_2}(\widetilde{X_1}^1 + \widetilde{X_2}^2) = \widetilde{J_1X_1}^1 + \widetilde{J_2X_2}^2$$

for each $x_i \in M_i$ and $X_i \in T_{x_i}M_i$, $i = 1, 2$, equals the natural complex structure on $M_1 \times M_2$ induced from its complex analytic coordinate. We regard $M_1 \times M_2$ as a complex manifold with complex structure $J^{M_1\times M_2}$. We mention that the product connection $\widetilde{\nabla}$ of $\nabla^i \in \mathcal{C}_0(TM_i, J_i)$ satisfies $\widetilde{\nabla} \in \mathcal{C}_0(T(M_1 \times M_2), J^{M_1\times M_2})$, that is, $\widetilde{\nabla}$ is a complex affine connection with respect to $J^{M_1\times M_2}$. For a product of two maps from a complex manifold, we get

Proposition 5.8 *Let (M_i, J_i), $(M_1\times M_2, J^{M_1\times M_2})$ be complex manifolds, $\overline{M}_i$ a manifold, $f_i\colon M_i \to \overline{M}_i$ a map, $i = 1, 2$, $\widetilde{\nabla}$ the product connection of $\nabla^1 \in \mathcal{C}_0(TM_1, J_1)$ and $\nabla^2 \in \mathcal{C}_0(TM_2, J_2)$, $\overline{\nabla}$ the product connection of $\overline{\nabla}^1 \in \mathcal{C}_0(T\overline{M}_1)$ and $\overline{\nabla}^2 \in \mathcal{C}_0(T\overline{M}_2)$ and $\bar{f} := f_1 \times f_2\colon M_1 \times M_2 \to \overline{M}_1 \times \overline{M}_2$ a map. Then $\bar{f}$ is pluriharmonic with respect to $\widetilde{\nabla}$ and $\overline{\nabla}$ if and only if both*

f_1 and f_2 are pluriharmonic with respect to ∇^i and $\overline{\nabla}^i$, $i = 1, 2$.

Proof. The Hessian $H_{\bar{f}}$ of $\bar{f}$ is given by

$$\begin{aligned}
&H_{\bar{f}}(\widetilde{X_1}^1 + \widetilde{X_2}^2, \widetilde{Y_1}^1 + \widetilde{Y_2}^2) \qquad (5.3)\\
&= (\bar{f}^\sharp\overline{\nabla})_{\widetilde{X_1}^1 + \widetilde{X_2}^2} i^{\bar{f}}(\widetilde{Y_1}^1 + \widetilde{Y_2}^2) - i^{\bar{f}}\widetilde{\nabla}_{\widetilde{X_1}^1 + \widetilde{X_2}^2}(\widetilde{Y_1}^1 + \widetilde{Y_2}^2)\\
&= \sum_{k=1}^{2}\left(\overline{(f_k^\sharp\overline{\nabla}^k)_{X_k} i^{f_k} Y_k - i^{f_k}\nabla^k_{X_k} Y_k}^k\right)\\
&= \sum_{k=1}^{2}\overline{H_{f_k}(X_k, Y_k)}^k
\end{aligned}$$

for any $X_k, Y_k \in \Gamma(TM_k)$. If $\bar{f}$ is pluriharmonic, then it holds from (5.3) that

$$\begin{aligned}
\overline{H_{f_k}(J_k X_k, Y_k)}^k &= H_{\bar{f}}(J^{M_1\times M_2}\widetilde{X_k}^k, \widetilde{Y_k}^k)\\
&= H_{\bar{f}}(\widetilde{X_k}^k, J^{M_1\times M_2}\widetilde{Y_k}^k) = \overline{H_{f_k}(X_k, J_k Y_k)}^k
\end{aligned}$$

for each $x_k \in M_k$ and any $X_k, Y_k \in T_{x_k}M_k$ and we see that both f_1 and f_2 are pluriharmonic. The converse is trivial from (5.3). □

We have the following corollary.

Corollary 5.9 *Let (M, J) be a complex manifold, $\nabla \in \mathcal{C}_0(TM, J)$, $\overline{M}_i$ a manifold, $f_i \colon M \to \overline{M}_i$ a map, $i = 1, 2$, $\overline{\nabla}$ the product connection of $\overline{\nabla}^1 \in \mathcal{C}_0(T\overline{M}_1)$ and $\overline{\nabla}^2 \in \mathcal{C}_0(T\overline{M}_2)$ and $\widetilde{f} := (f_1 \times f_2) \circ \Delta \colon M \to \overline{M}_1 \times \overline{M}_2$ a map. Then $\widetilde{f}$ is pluriharmonic with respect to ∇ and $\overline{\nabla}$ if and only if both f_1 and f_2 are pluriharmonic with respect to ∇ and $\overline{\nabla}^i$, $i = 1, 2$.*

As a corollary of Proposition 5.8, for the product of two affine immersions, we obtain

Corollary 5.10 *Let (M, J) be a complex manifold, $\nabla \in \mathcal{C}_0(TM, J)$, $\overline{M}_i$ a manifold, $\overline{\nabla}$ the product connection of $\overline{\nabla}^1 \in \mathcal{C}_0(T\overline{M}_1)$ and $\overline{\nabla}^2 \in \mathcal{C}_0(T\overline{M}_2)$ and $f_i \colon (M, \nabla) \to (\overline{M}_i, \overline{\nabla}^i)$ an affine immersion with transversal bundle N_i. Then an affine immersion $\widetilde{f} = (f_1 \times f_2) \circ \Delta \colon (M, \nabla) \to (\overline{M}_1 \times \overline{M}_2, \overline{\nabla})$ with transversal bundle $\widetilde{N}$ given by (4.5) is a $(1, 1)$-geodesic affine immersion if and only if both f_1 and f_2 are $(1, 1)$-geodesic.*

Next we will construct an example for Corollary 5.10. Define $f_1\colon \mathbb{R}^2 \to \mathbb{R}^3$ by

$$f_1(x_1,\, x_2) := \Big(x_1,\, x_2,\, x_1x_2 + \frac{x_1^2 - x_2^2}{2}\Big)$$

and $f_2\colon \mathbb{R}^2 \to \mathbb{R}^5$ by

$$f_2(x_1,\, x_2) := \Big(x_1,\, x_2,\, -\frac{x_1^2 - x_2^2}{2},\, x_1x_2,\, x_1^3 - 3x_1x_2^2\Big).$$

For $\xi := (0,\, 0,\, 1)$, we define N_1 by

$$N_{1_x} := \mathrm{Span}\{(f_{1\sharp x})^{-1}\xi\}$$

for each $x \in \mathbb{R}^2$. Then f_1 is an affine immersion with transversal bundle N_1 by considering the standard affine connection D on $\mathbb{R}^3$ and denote by ∇ the induced connection on $\mathbb{R}^2$.

For $\eta_1 := (0,\, 0,\, 1,\, 0,\, 0)$, $\eta_2 := (0,\, 0,\, 0,\, 1,\, 0)$ and $\eta_3 := (0,\, 0,\, 0,\, 0,\, 1)$, we define a transversal bundle N_2 by

$$N_{2_x} := \mathrm{Span}\{(f_{2\sharp x})^{-1}\eta_1,\, (f_{2\sharp x})^{-1}\eta_2,\, (f_{2\sharp x})^{-1}\eta_3\}$$

for each $x \in \mathbb{R}^2$. Then f_2 is an affine immersion with transversal bundle N_2 by considering the standard affine connection D on $\mathbb{R}^5$ and denote by ∇ the induced connection on $\mathbb{R}^2$. We mention that the induced connection on M for f_1 and f_2 are the same connection and both f_1 and f_2 are full immersions. It is easy to show that both f_1 and f_2 are $(1,\, 1)$-geodesic.

On the other hand, $\widetilde{f} := (f_1 \times f_2) \circ \Delta\colon (\mathbb{R}^2,\, \nabla) \to (\mathbb{R}^8, D)$ given by

$$\begin{aligned}\widetilde{f}(x_1,\, x_2) := \Big(x_1,\, x_2,\, x_1x_2 + \frac{x_1^2 - x_2^2}{2},\, x_1,\, x_2,&\\ -\frac{x_1^2 - x_2^2}{2},\, x_1x_2,\, x_1^3 - 3x_1x_2^2\Big)&\end{aligned}$$

is an affine immersion with transversal bundle $\widetilde{N}$ given by (4.5). For an affine immersion $\widetilde{f}$, the affine fundamental form $\widetilde{B}$ satisfies

$$\begin{aligned}(\widetilde{B}_{\partial_1}\partial_1)_x &= \overline{(B^1_{\partial_1}\partial_1)_x}^{\,1} + \overline{(B^2_{\partial_1}\partial_1)_x}^{\,2}\\ &= \overline{((f_{1\sharp x})^{-1}\xi)_x}^{\,1} - \overline{((f_{2\sharp x})^{-1}(\eta_1 - 6x_1\eta_3))_x}^{\,2},\\ (\widetilde{B}_{\partial_1}\partial_2)_x &= (\widetilde{B}_{\partial_2}\partial_1)_x = \overline{(B^1_{\partial_2}\partial_1)_x}^{\,1} + \overline{(B^2_{\partial_2}\partial_1)_x}^{\,2}\\ &= \overline{((f_{1\sharp x})^{-1}\xi)_x}^{\,1} + \overline{((f_{2\sharp x})^{-1}(\eta_2 - 6x_2\eta_3))_x}^{\,2},\end{aligned}$$

$$\begin{aligned}(\widetilde{B}_{\partial_2}\partial_2)_x &= \overline{(B^1_{\partial_2}\partial_2)_x}^1 + \overline{(B^2_{\partial_2}\partial_2)_x}^2 \\ &= -\overline{((f_{1\sharp x})^{-1}\xi)_x}^1 + (\overline{((f_{2\sharp x})^{-1}(\eta_1 - 6x_1\eta_3))_x}^2) \\ &= -(\widetilde{B}_{\partial_1}\partial_1)_x\end{aligned}$$

for each $x \in \mathbb{R}^2$ and $\widetilde{f}$ is a (1, 1)-geodesic affine immersion. We note that the essential codimension of the affine immersion $\widetilde{f}\colon (\mathbb{R}^2, \nabla) \to (\mathbb{R}^8, D)$ with transversal bundle $\widetilde{N}$ given by (4.5) is four since the first normal space is contained in $\overline{N_{1x}}^1 \oplus \overline{N_{2x}}^2$ for each $x \in \mathbb{R}^2$ and the subbundle $\overline{N_1}^1 \oplus \overline{N_2}^2$ of $\widetilde{N}$ is parallel with respect to $\widetilde{\nabla}^{\widetilde{N}}$.

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