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The heat equation for the Hermite operator on the Heisenberg group

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Abstract. We give a formula for the one-parameter strongly continuous semigroup e^{-tL}, $t > 0$, generated by the Hermite operator L on the Heisenberg group $\mathbb{H}^1$ in terms of Weyl transforms, and use it to obtain an L^2 estimate for the solution of the initial value problem for the heat equation governed by L in terms of the L^p norm of the initial data for $1 \le p \le \infty$.

Key words: Hermite functions, Heisenberg groups, Hermite operators, Wigner transforms, Weyl transforms, Hermite semigroups, heat equations, Weyl-Heisenberg groups, localization operators, L^p–L^2 estimates.

1. The Hermite semigroup on $\mathbb{R}$

As a prologue to the Hermite semigroup on the Heisenberg group $\mathbb{H}^1$, we give an analysis of the Hermite semigroup on $\mathbb{R}$.

For $k = 0, 1, 2, \ldots$, the Hermite function of order k is the function e_k on $\mathbb{R}$ defined by

$$e_k(x) = \frac{1}{(2^k k! \sqrt{\pi})^{1/2}} e^{-x^2/2} H_k(x), \quad x \in \mathbb{R},$$

where H_k is the Hermite polynomial of degree k given by

$$H_k(x) = (-1)^k e^{x^2} \left(\frac{d}{dx}\right)^k (e^{-x^2}), \quad x \in \mathbb{R}.$$

It is well-known that $\{e_k \colon k = 0, 1, 2, \ldots\}$ is an orthonormal basis for $L^2(\mathbb{R})$.

Let A and $\overline{A}$ be differential operators on $\mathbb{R}$ defined by

$$A = \frac{d}{dx} + x$$

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and

$$\overline{A} = -\frac{d}{dx} + x.$$

In fact, $\overline{A}$ is the formal adjoint of A. The Hermite operator H is the ordinary differential operator on $\mathbb{R}$ given by

$$H = -\frac{1}{2}(A\overline{A} + \overline{A}A).$$

A simple calculation shows that

$$H = -\frac{d^2}{dx^2} + x^2.$$

The spectral analysis of the Hermite operator H is based on the following result, which is easy to prove.

Theorem 1.1 *For all x in $\mathbb{R}$,*

$$(Ae_k)(x) = 2ke_{k-1}(x), \quad k = 1, 2, \ldots,$$

and

$$(\overline{A}e_k)(x) = e_{k+1}(x), \quad k = 0, 1, 2, \ldots.$$

Remark 1.2 In view of Theorem 1.1, we call A and $\overline{A}$ the annihilation operator and the creation operator, respectively, for the Hermite functions e_k, $k = 0, 1, 2, \ldots$, on $\mathbb{R}$.

An immediate consequence of Theorem 1.1 is the following theorem.

Theorem 1.3 $He_k = (2k+1)e_k, \quad k = 0, 1, 2, \ldots.$

Remark 1.4 Theorem 1.3 says that for $k = 0, 1, 2, \ldots$, the number $2k+1$ is an eigenvalue of the Hermite operator H, and the Hermite function e_k on $\mathbb{R}$ is an eigenfunction of H corresponding to the eigenvalue $2k+1$.

We can now give a formula for the Hermite semigroup e^{-tH}, $t > 0$.

Theorem 1.5 *Let f be a function in the Schwartz space $\mathcal{S}(\mathbb{R})$. Then for $t > 0$,*

$$e^{-tH}f = \sum_{k=0}^{\infty} e^{-(2k+1)t}(f, e_k)e_k,$$

where the convergence is uniform and absolute on $\mathbb{R}$.

Theorem 1.6 *For* $t > 0$, *the Hermite semigroup* e^{-tH}, *initially defined on* $\mathcal{S}(\mathbb{R})$, *can be extended to a unique bounded linear operator from* $L^p(\mathbb{R})$ *into* $L^2(\mathbb{R})$, *which we again denote by* e^{-tH}, *and there exisis a positive constant* C *such that*

$$\|e^{-tH} f\|_{L^2(\mathbb{R})} \leq C^{2/p-1} \frac{1}{2\sinh t} \|f\|_{L^p(\mathbb{R})}$$

for all f *in* $L^p(\mathbb{R})$, $1 \leq p \leq 2$.

Remark 1.7 In fact, by a well-known asymptotic formula for Hermite functions,

$$\sup\{\|e_k\|_{L^\infty(\mathbb{R})} \colon k = 0, 1, 2, \ldots\} < \infty$$

and hence C can be any positive constant such that

$$C \geq \sup\{\|e_k\|_{L^\infty(\mathbb{R})} \colon k = 0, 1, 2, \ldots\}.$$

Proof of Theorem 1.6. Let $f \in \mathcal{S}(\mathbb{R})$. Then, by Theorem 1.5 and Minkowski's inequality,

$$\|e^{-tH} f\|_{L^2(\mathbb{R})} \leq \sum_{k=0}^{\infty} e^{-(2k+1)t} |(f, e_k)|. \tag{1.1}$$

Now, for $k = 0, 1, 2, \ldots$, by Schwarz' inequality,

$$|(f, e_k)| \leq \|f\|_{L^2(\mathbb{R})} \tag{1.2}$$

and

$$|(f, e_k)| \leq \|f\|_{L^1(\mathbb{R})} \|e_k\|_{L^\infty(\mathbb{R})}. \tag{1.3}$$

But, using an asymptotic formula in the book [4] by Szegö for Hermite functions, we can find a positive constant C, which can actually be estimated, such that

$$\|e_k\|_{L^\infty(\mathbb{R})} \leq C \tag{1.4}$$

for $k = 0, 1, 2, \ldots$. So, by (1.3) and (1.4),

$$|(f, e_k)| \leq C \|f\|_{L^1(\mathbb{R})}. \tag{1.5}$$

Hence, by (1.1), (1.2) and (1.5), we get

$$\|e^{-tH}f\|_{L^2(\mathbb{R})} \leq \frac{1}{2\sinh t}\|f\|_{L^2(\mathbb{R})} \tag{1.6}$$

and

$$\|e^{-tH}f\|_{L^2(\mathbb{R})} \leq \frac{1}{2\sinh t}C\|f\|_{L^1(\mathbb{R})}. \tag{1.7}$$

Hence, by (1.6), (1.7) and the Riesz-Thorin theorem, we get

$$\|e^{-tH}f\|_{L^2(\mathbb{R})} \leq C^{2/p-1}\frac{1}{2\sinh t}\|f\|_{L^p(\mathbb{R})}$$

for $1 \leq p \leq 2$. □

2. The Hermite operator on the Heisenberg group

Let $\partial/\partial z$ and $\partial/\partial\overline{z}$ be linear partial differential operators on $\mathbb{R}^2$ given by

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$$

and

$$\frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}.$$

Then we define the linear partial differential operator L on $\mathbb{R}^2$ by

$$L = -\frac{1}{2}(Z\overline{Z} + \overline{Z}Z),$$

where

$$Z = \frac{\partial}{\partial z} + \frac{1}{2}\overline{z}, \quad \overline{z} = x - iy,$$

and

$$\overline{Z} = \frac{\partial}{\partial \overline{z}} - \frac{1}{2}z, \quad z = x + iy.$$

The vector fields Z and $\overline{Z}$, and the identity operator I form a basis for a Lie algebra in which the Lie bracket of two elements is their commutator. In fact, $-\overline{Z}$ is the formal adjoint of Z and L is an elliptic partial differential operator on $\mathbb{R}^2$ given by

$$L = -\Delta + \frac{1}{4}(x^2 + y^2) - i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right),$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Thus, L is the ordinary Hermite operator $-\Delta + (1/4)(x^2 + y^2)$ perturbed by the partial differential operator $-iN$, where

$$N = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

is the rotation operator. We can think of L as the Hermite operator on $\mathbb{H}^1$. The vector fields Z and $\overline{Z}$, and the Hermite operator L are studied in the books [5, 6] by Thangavelu and [7] by Wong. The connection of L with the sub-Laplacian on the Heisenberg group $\mathbb{H}^1$ can be found in the book [6] by Thangavelu. The heat equations for the sub-Laplacians on Heisenberg groups are first solved explicitly and independently in [1] by Gaveau and in [2] by Hulanicki.

In this paper, we compute the Hermite semigroup on $\mathbb{H}^1$, *i.e.*, the one-parameter strongly continuous semigroup e^{-tL}, $t > 0$, generated by L using an orthonormal basis for $L^2(\mathbb{R}^2)$ consisting of special Hermite functions on $\mathbb{R}^2$, which are eigenfunctions of L. We give a formula for the Hermite semigroup on $\mathbb{H}^1$ in terms of pseudo-differential operators of the Weyl type, *i.e.*, Weyl transforms. The Hermite semigroup on $\mathbb{H}^1$ is then used to obtain an L^2 estimate for the solution of the initial value problem of the heat equation governed by L in terms of the L^p norm of the initial data for $1 \leq p \leq \infty$.

The results in this paper are valid for the Hermite operator L on $\mathbb{H}^n$ given by

$$L = -\frac{1}{2}\sum_{j=1}^{n}(Z_j\overline{Z}_j + \overline{Z}_j Z_j),$$

where, for $j = 1, 2, \ldots, n$,

$$Z_j = \frac{\partial}{\partial z_j} + \frac{1}{2}\overline{z}_j, \quad \overline{z}_j = x_j - iy_j,$$

and

$$\overline{Z}_j = \frac{\partial}{\partial \overline{z}_j} - \frac{1}{2} z_j, \quad z_j = x_j + i y_j.$$

Of course, for $j = 1, 2, \ldots, n$,

$$\frac{\partial}{\partial z_j} = \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j}$$

and

$$\frac{\partial}{\partial \overline{z}_j} = \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j}.$$

Section 4.4 of the book [5] by Thangavelu contains some information on the L^p–L^2 estimates of the solutions of the wave equation governed by the Hermite operator L. The L^p norm of the solution of the wave equation for the special Hermite operator in terms of the initial data for values of p near 2 is studied in the paper [3] by Narayanan and Thangavelu.

3. Weyl transforms

Let f and g be functions in the Schwartz space $\mathcal{S}(\mathbb{R})$ on $\mathbb{R}$. Then the Fourier-Wigner transform $V(f, g)$ of f and g is defined by

$$V(f, g)(q, p) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{iqy} f\left(y + \frac{p}{2}\right) \overline{g\left(y - \frac{p}{2}\right)} dy \tag{3.1}$$

for all q and p in $\mathbb{R}$. It can be proved that $V(f, g)$ is a function in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ on $\mathbb{R}^2$. We define the Wigner transform $W(f, g)$ of f and g by

$$W(f, g) = V(f, g)^\wedge, \tag{3.2}$$

where $\hat{F}$ is the Fourier transform of F, which we choose to define by

$$\hat{F}(\zeta) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-iz\cdot\zeta} F(z) dz, \quad \zeta \in \mathbb{R}^n,$$

for all F in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ on $\mathbb{R}^n$. It can be shown that

$$W(f, g)(x, \xi) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-i\xi p} f\left(x + \frac{p}{2}\right) \overline{g\left(x - \frac{p}{2}\right)} dp$$

for all x and ξ in $\mathbb{R}$. It is obvious that

$$W(f,\, g) = \overline{W(g,\, f)}, \quad f,\, g \in \mathcal{S}(\mathbb{R}). \tag{3.3}$$

Now, let $\sigma \in L^p(\mathbb{R}^2)$, $1 \le p \le \infty$, and let $f \in \mathcal{S}(\mathbb{R})$. Then we define $W_\sigma f$ to be the tempered distribution on $\mathbb{R}$ by

$$(W_\sigma f,\, g) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(x,\, \xi) W(f,\, g)(x,\, \xi) dx\, d\xi \tag{3.4}$$

for all g in $\mathcal{S}(\mathbb{R})$, where $(F,\, G)$ is defined by

$$(F,\, G) = \int_{\mathbb{R}^n} F(z)\overline{G(z)} dz$$

for all measurable functions F and G on $\mathbb{R}^n$, provided that the integral exists. We call W_σ the Weyl transform associated to the symbol σ. It should be noted that if σ is a symbol in $\mathcal{S}(\mathbb{R}^2)$, then $W_\sigma f$ is a function in $\mathcal{S}(\mathbb{R})$ for all f in $\mathcal{S}(\mathbb{R})$.

We need the following result, which is an abridged version of Theorem 14.3 in the book [7] by Wong.

Theorem 3.1 *Let $\sigma \in L^p(\mathbb{R}^2)$, $1 \le p \le 2$. Then $W_{\hat{\sigma}}$ is a bounded linear operator from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ and*

$$\|W_{\hat{\sigma}}\|_* \le (2\pi)^{-1/p} \|\sigma\|_{L^p(\mathbb{R}^2)},$$

where $\|W_{\hat{\sigma}}\|_*$ is the operator norm of $W_{\hat{\sigma}} \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

4. Hermite functions on $\mathbb{R}^2$

For $j,\, k = 0,\, 1,\, 2,\, \ldots$, we define the Hermite function $e_{j,k}$ on $\mathbb{R}^2$ by

$$e_{j,\,k}(x,\, y) = V(e_j,\, e_k)(x,\, y)$$

for all x and y in $\mathbb{R}$. Then we have the following fact, which is Theorem 21.2 in the book [7] by Wong.

Theorem 4.1 *$\{e_{j,\,k} \colon j,\, k = 0,\, 1,\, 2,\, \ldots\}$ is an orthonormal basis for $L^2(\mathbb{R}^2)$.*

The spectral analysis of the Hermite operator L on $\mathbb{H}^1$ is based on the following result, which is Theorem 22.1 in the book [7] by Wong.

Theorem 4.2 *For all x and y in $\mathbb{R}$,*

$$(Ze_{j,\,k})(x,\, y) = i(2k)^{1/2} e_{j,\,k-1}(x,\, y), \quad j = 0,\, 1,\, 2,\, \ldots,\, k = 1,\, 2, \ldots,$$

and

$$(\overline{Z}e_{j,\,k})(x,\,y) = i(2k+2)^{1/2}e_{j,\,k+1}(x,\,y), \quad j,\,k = 0,\,1,\,2,\,\ldots.$$

Remark 4.3 In view of Theorem 4.2, we call Z and $\overline{Z}$ the annihilation operator and the creation operator, respectively, for the special Hermite functions $e_{j,k}$, j, $k = 0, 1, 2, \ldots$, on $\mathbb{R}^2$.

An immediate consequence of Theorem 4.2 is the following theorem.

Theorem 4.4 $Le_{j,\,k} = (2k+1)e_{j,\,k}, \quad j,\,k = 0,\,1,\,2,\,\ldots.$

Remark 4.5 Theorem 4.4 says that for $k = 0, 1, 2, \ldots$, the number $2k+1$ is an eigenvalue of the Hermite operator L on $\mathbb{H}^1$, and the Hermite functions $e_{j,\,k}$, $j = 0, 1, 2, \ldots$, on $\mathbb{R}^2$ are eigenfunctions of L corresponding to the eigenvalue $2k+1$.

5. The Hermite semigroup on $\mathbb{H}^1$

A formula for the Hermite semigroup e^{-tL}, $t > 0$, on $\mathbb{H}^1$ is given in the following theorem.

Theorem 5.1 *Let $f \in \mathcal{S}(\mathbb{R}^2)$. Then for $t > 0$,*

$$e^{-tL}f = (2\pi)^{1/2}\sum_{k=0}^{\infty} e^{-(2k+1)t}V(W_{\hat{f}}e_k,\, e_k),$$

where the convergence is uniform and absolute on $\mathbb{R}^2$.

Proof. Let f be any function in $\mathcal{S}(\mathbb{R}^2)$. Then for $t > 0$, we use Theorem 4.4 to get

$$e^{-tL}f = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} e^{-(2k+1)t}(f,\, e_{j,\,k})e_{j,\,k}, \tag{5.1}$$

where the series is convergent in $L^2(\mathbb{R}^2)$, and is also uniformly and absolutely convergent on $\mathbb{R}^2$. Now, by (3.1)–(3.4) and Plancherel's theorem,

$$\begin{aligned}(f,\, e_{j,k}) &= \int_{\mathbb{R}^2} f(z)\overline{V(e_j,\, e_k)(z)}dz \\ &= \int_{\mathbb{R}^2} \hat{f}(\zeta)\overline{V(e_j,\, e_k)^\wedge(\zeta)}d\zeta\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^2} \hat{f}(\zeta)\overline{W(e_j,\, e_k)(\zeta)}d\zeta \\
&= (2\pi)^{1/2}(W_{\hat{f}}e_k,\, e_j) \qquad (5.2)
\end{aligned}
$$

for $j,\, k = 0,\, 1,\, 2,\, \ldots$. Similarly, for $j,\, k = 0,\, 1,\, 2,\, \ldots$, and g in $\mathcal{S}(\mathbb{R}^2)$, we get

$$
(e_{j,k},\, g) = \overline{(g,\, e_{j,k})} = (2\pi)^{1/2}\overline{(W_{\hat{g}}e_k,\, e_j)} = (2\pi)^{1/2}(e_j,\, W_{\hat{g}}e_k). \qquad (5.3)
$$

So, by (5.1)–(5.3), Fubini's theorem and Parseval's identity,

$$
\begin{aligned}
(e^{-tL}f,\, g) &= 2\pi \sum_{k=0}^{\infty} e^{-(2k+1)t} \sum_{j=0}^{\infty} (W_{\hat{f}}e_k,\, e_j)(e_j,\, W_{\hat{g}}e_k) \\
&= 2\pi \sum_{k=0}^{\infty} e^{-(2k+1)t}(W_{\hat{f}}e_k,\, W_{\hat{g}}e_k) \qquad (5.4)
\end{aligned}
$$

for $t > 0$, where the series is absolutely convergent on $\mathbb{R}$. But, by (3.2)–(3.4) and Plancherel's theorem,

$$
\begin{aligned}
(W_{\hat{f}}e_k,\, W_{\hat{g}}e_k) &= (2\pi)^{1/2}\overline{\int_{\mathbb{R}^2} \hat{g}(z)W(e_k,\, W_{\hat{f}}e_k)(z)dz} \\
&= (2\pi)^{1/2} \int_{\mathbb{R}^2} W(W_{\hat{f}}e_k,\, e_k)(z)\overline{\hat{g}(z)}dz \\
&= (2\pi)^{1/2} \int_{\mathbb{R}^2} V(W_{\hat{f}}e_k,\, e_k)(z)\overline{g(z)}dz \qquad (5.5)
\end{aligned}
$$

for $k = 0,\, 1,\, 2,\, \ldots$. Thus, by (5.4), (5.5) and Fubini's theorem,

$$
\begin{aligned}
(e^{-tL}f,\, g) &= (2\pi)^{1/2} \sum_{k=0}^{\infty} e^{-(2k+1)t}(V(W_{\hat{f}}e_k,\, e_k),\, g) \\
&= (2\pi)^{1/2} \left(\sum_{k=0}^{\infty} e^{-(2k+1)t} V(W_{\hat{f}}e_k,\, e_k),\, g \right) \qquad (5.6)
\end{aligned}
$$

for all f and g in $\mathcal{S}(\mathbb{R}^2)$ and $t > 0$. Thus, by (5.6),

$$
e^{-tL}f = (2\pi)^{1/2} \sum_{k=0}^{\infty} e^{-(2k+1)t} V(W_{\hat{f}}e_k,\, e_k)
$$

for all f in $\mathcal{S}(\mathbb{R}^2)$ and $t > 0$, where the uniform and absolute convergence of the series follows from (3.1) and Theorem 3.1. □

6. An L^p–L^2 estimate, $1 \le p \le 2$

We begin with the following result, which is known as the Moyal identity and can be found in the book [7] by Wong.

Theorem 6.1 *For all f and g in $\mathcal{S}(\mathbb{R})$,*

$$\|V(f,\, g)\|_{L^2(\mathbb{R}^2)} = \|f\|_{L^2(\mathbb{R})}\|g\|_{L^2(\mathbb{R})}.$$

We can now prove the following theorem as an application of the formula for the Hermite semigroup on $\mathbb{H}^1$ given in Theorem 5.1.

Theorem 6.2 *For $t > 0$, the Hermite semigroup e^{-tL} on $\mathbb{H}^1$, initially defined on $\mathcal{S}(\mathbb{R}^2)$, can be extended to a unique bounded linear operator from $L^p(\mathbb{R}^2)$ into $L^2(\mathbb{R}^2)$, which we again denote by e^{-tL}, and*

$$\|e^{-tL}f\|_{L^2(\mathbb{R}^2)} \le (2\pi)^{1/2-1/p}\frac{1}{2\sinh t}\|f\|_{L^p(\mathbb{R}^2)}$$

for all f in $L^p(\mathbb{R}^2)$, $1 \le p \le 2$.

Proof. Let $f \in \mathcal{S}(\mathbb{R}^2)$. Then, by Theorems 5.1 and 6.1, and Minkowski's inequality

$$\begin{aligned}\|e^{-tL}f\|_{L^2(\mathbb{R}^2)} &\le (2\pi)^{1/2}\sum_{k=0}^{\infty} e^{-(2k+1)t}\|V(W_{\hat{f}}e_k, e_k)\|_{L^2(\mathbb{R}^2)} \\ &= (2\pi)^{1/2}\sum_{k=0}^{\infty} e^{-(2k+1)t}\|W_{\hat{f}}e_k\|_{L^2(\mathbb{R})}\|e_k\|_{L^2(\mathbb{R})} \\ &= (2\pi)^{1/2}\sum_{k=0}^{\infty} e^{-(2k+1)t}\|W_{\hat{f}}e_k\|_{L^2(\mathbb{R})} \qquad (6.1)\end{aligned}$$

for $t > 0$. So, by (6.1) and Theorem 3.1, we get for $t > 0$,

$$\begin{aligned}\|e^{-tL}f\|_{L^2(\mathbb{R}^2)} &\le (2\pi)^{1/2}\sum_{k=0}^{\infty} e^{-(2k+1)t}(2\pi)^{-1/p}\|f\|_{L^p(\mathbb{R}^2)} \\ &= (2\pi)^{1/2-1/p}\frac{1}{2\sinh t}\|f\|_{L^p(\mathbb{R}^2)} \qquad (6.2)\end{aligned}$$

for all f in $\mathcal{S}(\mathbb{R}^2)$. Thus, by (6.2) and a density argument, the proof is complete. □

Remark 6.3 Theorem 6.2 gives an L^2 estimate for the solution of the initial value problem

$$\begin{cases} \frac{\partial u}{\partial t}(z, t) = (Lu)(z, t), & z \in \mathbb{R}^2, \ t > 0, \\ u(z, 0) = f(z), & z \in \mathbb{R}^2, \end{cases} \tag{6.3}$$

in terms of the L^p norm of the initial data f, $1 \leq p \leq 2$.

Remark 6.4 Instead of using Weyl transforms, Theorem 6.2 can be proved using an L^p–L^2 restriction theorem such as Theorem 2.5.4 in the book [5] by Thangavelu. To wit, we note that the formula (5.1) for the special Hermite semigroup gives

$$e^{-tL} f = \sum_{k=0}^{\infty} e^{-(2k+1)t} Q_k f, \quad f \in \mathcal{S}(\mathbb{R}^2),$$

where Q_k is the projection onto the eigenspace corresponding to the eigenvalue $2k + 1$. Thus, by Theorem 2.5.4 in [5], the estimate for $p = 1$ follows. The estimate for $p = 2$ is easy. Hence the estimate for $1 \leq p \leq 2$ follows if we interpolate.

7. An L^p–L^2 estimate, $1 \leq p \leq \infty$

Using the theory of localization operators on the Weyl-Heisenberg group in the paper [8] or Chapter 17 of the book [9] by Wong, we can give an L^p–L^2 estimate for $1 \leq p \leq \infty$. To this end, we need two results.

Theorem 7.1 *Let Λ be the function on $\mathbb{C}$ defined by*

$$\Lambda(z) = \pi^{-1} e^{-|z|^2}, \quad z \in \mathbb{C}.$$

Then for all $F \in L^p(\mathbb{C})$, $1 \leq p \leq \infty$,

$$W_{F*\Lambda} = L_F,$$

where L_F is the localization operator on the Weyl-Heisenberg group with symbol F.

Theorem 7.1 is Theorem 17.1 in the book [7] by Wong.

Theorem 7.2 *Let $F \in L^p(\mathbb{C})$, $1 \leq p \leq \infty$. Then*

$$\|L_F\|_* \leq (2\pi)^{-1/p} \|F\|_{L^p(\mathbb{C})}.$$

Theorem 7.2 is Theorem 17.11 in the book [9] by Wong.

The main result in this section is the following theorem.

Theorem 7.3 *Let $g \in L^p(\mathbb{C})$, $1 \le p \le \infty$, and let u be the solution of the initial value problem* (6.3) *with initial data $(g * \Lambda)^\vee$, where $\vee$ is the inverse Fourier transform. Then*

$$\|u\|_{L^2(\mathbb{R}^2)} \le (2\pi)^{1/2-1/p} \frac{1}{2\sinh t} \|g\|_{L^p(\mathbb{R}^2)}.$$

The proof is the same as that of Theorem 6.2 if we note that, by Theorem 7.1, $W_{\hat{f}} = W_{g*\Lambda} = L_g$ and hence the estimate follows from Theorem 7.2.

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