Hokkaido Mathematical Journal Vol. 34 (2005) p. 375–391

Grassmann geometry on the 3-dimensional Heisenberg group

Jun-ichi Inoguchi, Kenji Kuwabara and Hiroo Naitoh

(Received July 24, 2003)

Abstract. In this paper the geometries of surfaces in the 3-dimensional Heisenberg group with a left invariant metric are classified from a stand point of the Grassmann geometry, and for each of them the existence or nonexistence of surfaces with constant mean curvature is clarified.

Key words: Heisenberg group, Grassmann geometry of surfaces, constant mean curvature.

1. Introduction

Let M be an m-dimensional connected Riemannian manifold and r be an integer such that $1 \leq r \leq m$. Given a nonempty subset Σ in the Grassman bundle $G^r(TM)$ over M, which consists of all r-dimensional linear subspaces of the tangent spaces of M, an r-dimensional connected submanifold S of M is called a Σ-submanifold if all tangent spaces of S belong to the set Σ, and the collection of such the submanifolds is called a Σ-geometry. "Grassmann geometry" is a collected name for such a Σ-geometry. When G is the identity component of the isometry group of M, it acts on $G^r(TM)$ as the differentials of isometries and then we have many G-orbits in $G^r(TM)$. If Σ is given by a G-orbit, Σ-geometry is in particular called of orbit type. If M is a Riemannian homogeneous manifold, such a Σ is a subbundle of $G^r(TM)$ over M.

In the study of Grassmann geometry, we should first consider whether a Σ-submanifold exists or not for an arbitrary Σ-geometry, and next consider weather the Σ-geometry has somewhat canonical Σ-submanifolds or not, *eg.*, minimal submanifolds, submanifolds with parallel mean curvature vectors, etc., and, if there do not exist such submanifolds, we would moreover like to find certain kinds of submanifolds suitable to the Σ-geometry.

In this paper, from this view of points, we will study a local theory of

2000 Mathematics Subject Classification : 53B25, 53C40, 53C30.
The first author was supported by JSPS. KAKENHI(14740053).
The third author was supported by JSPS. KAKENHI(13640077).

surfaces for the case where M is the 3-dimensional Heisenberg group with left invariant metrics parametrized by positive numbers c and Σ is a G-orbit in $G^2(TM)$. First of all, we shall show that the orbit space of the G-action on $G^2(TM)$ is parametrized by the values of the curvature function K, and K takes values in the closed interval $[-3c^2/4,\, c^2/4]$. The Grassmann geometry defined by the orbit determined by each $\alpha \in [-3c^2/4,\, c^2/4]$ will be called $\mathcal{O}(\alpha)$-geometry.

Our results are summarized as follows:

- There exist no surfaces in $\mathcal{O}(-3c^2/4)$-geometry.
- Surfaces in $\mathcal{O}(c^2/4)$-geometry are Hopf cylinders.
- For each α such that $-3c^2/4 < \alpha < c^2/4$, every surface in $\mathcal{O}(\alpha)$-geometry is of negative constant curvature $\alpha - c^2/4$ free of geodesic points. Moreover there exist no surfaces with constant mean curvature.

2. Heisenberg group and its Grassmann geometries

Let H be the 3-dimensional Heisenberg group, which is a 2-step nilpotent Lie group of all the 3×3 real matrices with the following form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x,\, y,\, z \in \mathbb{R},$$

and let $\mathfrak{h}$ be the Lie algebra of left invariant vector fields. Moreover take a left invariant metric $\langle\,\cdot\,,\,\cdot\,\rangle$ on H, which induces an inner product on $\mathfrak{h}$ since, for $X,\, Y \in \mathfrak{h}$, a function $\langle X,\, Y\rangle$ is constant on H. Then we can see that there exists an orthonormal basis $\{E_1,\, E_2,\, E_3\}$ of $\mathfrak{h}$ and a positive constant c such that the bracket relations on $\mathfrak{h}$ are represented in the following

$$[E_1,\, E_2] = c\, E_3, \quad [E_1,\, E_3] = [E_2,\, E_3] = 0,$$

where E_3 generates the center of $\mathfrak{h}$ and the constant c is determined by the left invariant metric $\langle\,\cdot\,,\,\cdot\,\rangle$, more precisely, the isometric classes of left invariant metrics on H are parametrized by all the positive constants c. (Refer [7] for the details.) In addition, using the orthonormal basis $\{E_1,\, E_2,\, E_3\}$ and the positive constant c, we can calculate the Levi-Civita connection ∇ and the curvature tensor R as follows:

$$\nabla_{E_1} E_2 = \frac{c}{2} E_3, \quad \nabla_{E_2} E_3 = \frac{c}{2} E_1, \quad \nabla_{E_3} E_1 = -\frac{c}{2} E_2, \tag{2.1}$$

$$\begin{aligned}
&\nabla_{E_2}E_1 = -\frac{c}{2}E_3, \quad \nabla_{E_3}E_2 = \frac{c}{2}E_1, \quad \nabla_{E_1}E_3 = -\frac{c}{2}E_2,\\
&R(E_1, E_2)E_1 = -R(E_2, E_1)E_1 = \frac{3}{4}c^2E_2,\\
&R(E_1, E_2)E_2 = -R(E_2, E_1)E_2 = -\frac{3}{4}c^2E_1,\\
&R(E_1, E_3)E_1 = -R(E_3, E_1)E_1 = -\frac{1}{4}c^2E_2,\\
&R(E_1, E_3)E_3 = -R(E_3, E_1)E_3 = \frac{1}{4}c^2E_1,\\
&R(E_2, E_3)E_2 = -R(E_3, E_2)E_2 = -\frac{1}{4}c^2E_3,\\
&R(E_2, E_3)E_3 = -R(E_3, E_2)E_3 = \frac{1}{4}c^2E_2,
\end{aligned}$$

and other $\nabla_{E_i}E_j$'s and $R(E_i, E_j)E_k$'s are zero.

Let exp be the exponential mapping of $\mathfrak{h}$ on H, which induces a diffeomorphism of $\mathbb{R}^3$ onto H by the correspondence $\mathfrak{h} = \mathbb{R}^3 \ni (u_1, u_2, u_3) \mapsto \exp(u_1E_1 + u_2E_2 + u_3E_3) \in H$. Hence $\exp^{-1}$ gives a global coordinates on H.

We have the following relations between vector fields (*cf.* [8]):

$$E_1 = \frac{\partial}{\partial u_1} - \frac{c}{2}u_2\frac{\partial}{\partial u_3}, \quad E_2 = \frac{\partial}{\partial u_2} + \frac{c}{2}u_1\frac{\partial}{\partial u_3}, \quad E_3 = \frac{\partial}{\partial u_3}.$$

Next let $S^2(\mathfrak{h})$ be the unit sphere in $\mathfrak{h}$ ($= \mathbb{R}^3$) centered at the origin. For an element $w = (w_1, w_2, w_3)$ in $S^2(\mathfrak{h})$, we can define its orthogonal plane $P(w)$ in $\mathfrak{h}$ as follows:

$$P(w) = \{(u_1, u_2, u_3) \in \mathbb{R}^3;\ w_1u_1 + w_2u_2 + w_3u_3 = 0\} \subset \mathbb{R}^3 = \mathfrak{h}.$$

Then, from the above calculations of R, the sectional curvature $K(P(w))$ of $P(w)$ is given in the following

$$K(P(w)) = \frac{c^2}{4}(-3w_3^2 + w_2^2 + w_1^2) = \frac{c^2}{4}(1 - 4\rho^2) \tag{2.2}$$

where $\rho = |w_3|$ and $0 \leq \rho \leq 1$. Here we note that $P(w)$ is a left invariant plane in $\mathfrak{h}$, and so $K(P(w))$ is well-defined. Now let r_θ, $\theta \in \mathbb{R}$, be the orthogonal transformation of $\mathfrak{h}$ which gives the θ-rotation on the u_1u_2-plane and fixes the u_3-axis. Then r_θ is an isometry of $\mathfrak{h}$ preserving the bracket product $[\,\cdot\,, \cdot\,]$, and so it also induces an isometric automorphism of H. Hence, for two planes P and P' of $\mathfrak{h}$, it holds that $K(P) = K(P')$ if and only if there exists an isometric automorphism φ of H such that $\varphi(P) = P'$. This fact immediately leads to the following proposition.

Proposition 2.1 *Let $G^2(TH)$ be the Grassmann bundle over H, and K be the curvature function on $G^2(TH)$ which assigns to a plane its sectional curvature. Moreover let G be the identity component of the isometry group of H. Then, it holds that $K(P) = K(P')$ for P, $P' \in G^2(TH)$ if and only if $G(P) = G(P')$. Namely, the orbit space of the G-action on $G^2(TH)$ is parametrized by the values of the curvature function K, where the values of K moves around on the interval $[-3c^2/4, c^2/4]$.*

Proof. Let P, P' be 2-planes in tangent spaces T_hH and $T_{h'}H$, respectively and assume that $K(P) = K(P')$. Next translate P and P' into the planes $L_{h^{-1}}P$ and $L_{h'^{-1}}P'$ in T_eH by the left translations $L_{h^{-1}}$ and $L_{h'^{-1}}$, respectively, where e is the unit element of H. Then, since left translations are isometries, it holds that $K(L_{h^{-1}}P) = K(L_{h'^{-1}}P')$. By the above fact, there exists a rotation isometry r_θ of H such that $r_\theta(L_{h^{-1}}P) = L_{h'^{-1}}P'$, and thus $(L_{h'} \circ r_\theta \circ L_{h^{-1}})P = P'$. Noting that $L_{h'}$, $L_{h^{-1}}$, and r_θ belong to the identity component G of isometries of H, we have that $G(P) = G(P')$. The converse is obvious since G acts isometrically on H. Also, the range of values of K is obvious by (2.2). □

Remark Using Lagrange's method of indeterminate coefficients, we can see that the critical values of the curvature function $K \circ P\colon S^2(\mathfrak{h}) \ni w \mapsto K(P(w)) \in \mathbb{R}$ are only the maximum $c^2/4$ and the minimum $-3c^2/4$.

Note that the identity component G of the isometry group of H is generated by left translations and rotation isometries r_θ on H. Hence H has 4-dimensional isometry group.

In the following sections, we will study the Grassmann geometries on H of orbit type. According to the parameterization of the orbit space $G\backslash G^2(TH)$ by the values α of K, we put

$$\Sigma = \{P \in G^2(TH);\, K(P) = \alpha\}$$

for each α such that $\alpha \in [-3c^2/4, c^2/4]$, and call the Grassmann geometry defined by this orbit the $\mathcal{O}(\alpha)$-*geometry*. Here $\alpha = (c^2/4)(1 - 4\rho^2)$. To study each $\mathcal{O}(\alpha)$-geometry, we prepare the following lemma, which plays important roles in the following arguments.

Lemma 2.2 *Let S be an $\mathcal{O}(\alpha)$-surface of H and p be a point in S. Then there exists a local involutive distribution D of H around p such that S is a local leaf and the leaves of D are all $\mathcal{O}(\alpha)$-surfaces.*

Proof. Since H is a Riemannian homogeneous space, we can locally and isometrically deform S around p for the direction normal to the tangent space T_pS. Then the collection of deformed surfaces defines a desired involutive distribution D around p. □

We first consider the $\mathcal{O}(-3c^2/4)$-geometry. Then we can easily see the following theorem.

Theorem 2.3 *There exists no $\mathcal{O}(-3c^2/4)$-surface.*

Proof. Assume that there exists an $\mathcal{O}(-3c^2/4)$-surface S. Then, by Lemma 2.2, there exists a local involutive distribution D whose leaves are all $\mathcal{O}(-3c^2/4)$-surfaces. In this case D is generated by the left invariant vector fields E_1 and E_2, since $\mathcal{O}(-3c^2/4) \cap G^2(T_eH)$ consists of only the plane generated by the vectors $(E_1)_e$ and $(E_2)_e$, where $G^2(T_eH)$ denotes the Grassman manifold over T_eH of 2-planes. Moreover, since D is involutive, it follows that $[E_1, E_2]$ belongs to D, but it holds that $[E_1, E_2] = cE_3$ by the bracket relation. This is a contradiction. □

Remark This result can be proved alternatively as follows: Denote by ω the left invariant 1-form dual to E_3. Then ω is a left invariant contact form on H. The distribution D is spanned by E_1 and E_2. Hence D is the contact distribution defined by $\omega = 0$ and hence D is non-integrable. The vector field E_3 is a unit Killing vector field. The flows of E_3 is called the *Reeb flows* of (H, ω). The formulas (2.1) imply that Reeb flows are geodesics.

Let S be a general surface of H. For a point q in S, there exists a unique left invariant 2-plane $P(q)$ in $\mathfrak{h}$ such that $(P(q))_q = T_qS$. Hence we can consider the Gauss mapping $\kappa\colon S \ni q \to P(q) \in G^2(\mathfrak{h})$ where $G^2(\mathfrak{h})$ denotes the Grassmann manifold over $\mathfrak{h}$ of 2-planes. As an application of Theorem 2.3, we then have the following.

Corollary 2.4 *Let S be a general surface of H and put*

$$\mathrm{Sing}(S) = \{p \in S;\ \kappa(p) = \text{the } u_1u_2\text{-plane}\}.$$

Then, the set $\mathrm{Sing}(S)$ *has no interior.*

Proof. If $\mathrm{Sing}(S)$ has an inner point, the interior of $\mathrm{Sing}(S)$ is an $\mathcal{O}(-3c^2/4)$ -surface. This contradicts to Theorem 2.3. $\square$

3. $\mathcal{O}(c^2/4)$-geometry

In this section we study the $\mathcal{O}(c^2/4)$-geomety. The orbit $\mathcal{O}(c^2/4)$ consits of all the planes P with the following form;

$$P = \mathbb{R} \cdot (E_3)_q + \mathbb{R} \cdot \big(\cos\theta (E_1)_q + \sin\theta (E_2)_q\big),$$

where $q \in H$ and $\theta \in \mathbb{R}$. For a local smooth function $\theta = \theta(u_1,\, u_2,\, u_3)$ on H, define a local distribution D^θ on H as follows:

$$(D^\theta)_q = \mathbb{R} \cdot (E_3)_q + \mathbb{R} \cdot \big(\cos\theta(q)(E_1)_q + \sin\theta(q)(E_2)_q\big),$$

where $q = (u_1,\, u_2,\, u_3)$. We will find local functions θ such that D^θ is involutive. The integrability condition $[D^\theta,\, D^\theta] \subset D^\theta$ induces the following:

$$[E_3,\, \cos\theta E_1 + \sin\theta E_2] = (E_3\theta)(-\sin\theta E_1 + \cos\theta E_2) \in D^\theta.$$

Since the smooth vector field $-\sin\theta E_1 + \cos\theta E_2$ is orthogonal to D^θ, the distribution D^θ is involutive if and only if $E_3\theta = 0$, and moreover, since $E_3 = \partial/\partial u_3$, this implies that $\theta(u_1,\, u_2,\, u_3)$ is independent on the variable u_3. In particular we can see that the $\mathcal{O}(c^2/4)$-geometry has many $\mathcal{O}(c^2/4)$-surfaces, and each $\mathcal{O}(c^2/4)$-surface is realized as a part of the inverse image $\pi^{-1}(\gamma)$ of a curve γ in the u_1u_2-plane by the projection $\pi\colon \mathbb{R}^3 \ni (u_1,\, u_2,\, u_3) \to (u_1,\, u_2) \in \mathbb{R}^2$. This inverse image is called a *Hopf cylinder* over γ. The Hopf cylinder over γ is flat and whose mean curvature is the half of the curvature of γ. See [5], p. 22.

Next we detail about geometric properties of $\mathcal{O}(c^2/4)$-surfaces. We set $\theta = \theta(u_1,\, u_2)$ and calculate the Riemannian connection ∇^θ, the curvature tensor R^θ, and the second fundamental form Π^θ of D^θ. The restrictions of them onto each leaf of D^θ give the Levi-Civita connection, the curvature tensor, and the second fundamental form of the leaf, respectively. The following lemma can be easily calculated by using (2.1) and the fact that $E_3\theta = 0$.

Lemma 3.1 *Let $(D^\theta)^\perp$ denote the orthogonal distribution of D^θ, and put $X = E_3$, $Y = \cos\theta E_1 + \sin\theta E_2 \in D^\theta$, and $N = \sin\theta E_1 - \cos\theta E_2 \in (D^\theta)^\perp$.*

Then it holds that

$$\nabla^\theta_X X = \nabla^\theta_Y Y = \nabla^\theta_X Y = \nabla^\theta_Y X = 0, \tag{3.3}$$

and thus $R^\theta = 0$. *Moreover*

$$\Pi^\theta(X, X) = 0, \quad \Pi^\theta(X, Y) = \frac{c}{2}N, \tag{3.4}$$
$$\Pi^\theta(Y, Y) = -\{\cos\theta(E_1\theta) + \sin\theta(E_2\theta)\}N.$$

Here we note that $\{X, Y, N\}$ is a local frame of orthonomal vector fields on H, and moreover it holds $E_1\theta = \partial\theta/\partial u_1$ and $E_2\theta = \partial\theta/\partial u_2$ since $\partial\theta/\partial u_3 = 0$. Then, together with Lemma 3.1, we have the following theorem.

Theorem 3.2 *Let S be an $\mathcal{O}(c^2/4)$-surface of H. Then, it satisfies the following geometric properties*:
(1) *S is a flat surface*;
(2) *S has no geodesic point*;
(3) *S is a minimal surface if and only if it is a part of a Hopf cylinder over a straight line in the u_1u_2-plane*;
(4) *S is a surface with nonzero constant mean curvature if and only if it is a part of a Hopf cylinder over a circle in the u_1u_2-plane.*

Proof. The claims (1) and (2) are obvious by (3.3) and (3.4), respectively. We show the claims (3) and (4). We consider a local involutive distribution D^θ as described in the above and then assume by virtue of Lemma 2.2 that the leaves of D^θ are all congruent and one of them locally contains S. Then, since X and Y are an orthonormal frame in D^θ, the mean curvature vector fields along the leaves of D^θ are given by $-\{\cos\theta(E_1\theta) + \sin\theta(E_2\theta)\}N$, and thus the leaves have the same constant mean curvature k with respect to the normal N if and only if $-\{\cos\theta(E_1\theta) + \sin\theta(E_2\theta)\} = k/2$, equivalently,

$$\cos\theta\frac{\partial\theta}{\partial u_1} + \sin\theta\frac{\partial\theta}{\partial u_2} = -\frac{k}{2} \tag{3.5}$$

since $E_1\theta = \partial\theta/\partial u_1$ and $E_2\theta = \partial\theta/\partial u_2$.

Now regard the $u_1u_2u_3$-space $\mathbb{R}^3$ as the commutative Lie group with flat metric such that $\partial/\partial u_i$'s are orthonormal left invariant vector fields and construct the distribution $(D')^\theta$ on the flat space $\mathbb{R}^3$ by using the same local function θ as given in the above, provided that the vector fields X, Y, and N read $\partial/\partial u_3$, $\cos\theta\partial/\partial u_1 + \sin\theta\partial/\partial u_2$, and $\sin\theta\partial/\partial u_1 - \cos\theta\partial/\partial u_2$, respectively. Denote by X', Y', and N' the latter vector fields, respectively. Then, we can observe that D^θ and $(D')^\theta$ define the same distribution since the planes generated by X and Y coincide with those generated by X' and Y'. Moreover, we can observe by the same way as done for the case of H that, in the flat space $\mathbb{R}^3$, the leaves of $(D')^\theta$ have the same constant mean curvature k if and only if the function θ satisfies the equation (3.5). By these observations, it follows that if S is a surface of H with constant mean curvature k, it is also such a surface of the flat space $\mathbb{R}^3$. Hence, by the theory of surfaces of $\mathbb{R}^3$, our claims (3) and (4) are proved. □

Remark Under the assumption that S is generally a Hopf cylinder of H, this theorem is already known by using another method (*cf.* [5], p. 22). Here we remark that the $\mathcal{O}(c^2/4)$-surfaces are nothing but the Hopf cylinders.

One can check that integral curves of the vector field $Y = \cos\theta E_1 + \sin\theta E_2$ are Legendre curves of constant torsion $c/2$. Thus every $\mathcal{O}(c^2/4)$-surface S is foliated by Legendre curves (curves orthogonal to E_3) of constant torsion and geodesics (Reeb flows). In particular, every $\mathcal{O}(c^2/4)$-surface with constant mean curvature is foliated by Legendre helices and Reeb flows. Compare with $\mathcal{O}(\alpha)$-surfaces with $-3c^2/4 < \alpha < c^2/4$ (Theorem 4.6).

4. $\mathcal{O}(\alpha)$-geometry $(-3c^2/4 < \alpha < c^2/4)$

Next we consider the $\mathcal{O}(\alpha)$-geometry such that $-3c^2/4 < \alpha < c^2/4$. Fix such an α and set $\rho = (1/c)\sqrt{(c^2/4) - \alpha}$, where $0 < \rho < 1$. Then the orbit $\mathcal{O}(\alpha)$ consists of all planes P with the following form:

$$\begin{aligned} P = &\mathbb{R} \cdot (\sin\theta(E_1)_q - \cos\theta(E_2)_q) \\ &+ \mathbb{R} \cdot \big(\rho\cos\theta(E_1)_q + \rho\sin\theta(E_2)_q - \sqrt{1-\rho^2}(E_3)_q\big), \end{aligned}$$

where $q \in H$ and $\theta \in \mathbb{R}$. By the same way as in the case of $\mathcal{O}(c^2/4)$-geometry, we define a local distribution D^θ on H for a local smooth function $\theta = \theta(u_1, u_2, u_3)$ on H as follows:

$$(D^\theta)_q = \mathbb{R} \cdot X_q + \mathbb{R} \cdot Y_q,$$

for $q = (u_1, u_2, u_3)$, where $X_q = \rho\cos\theta(q)(E_1)_q + \rho\sin\theta(q)(E_2)_q - \sqrt{1-\rho^2} \times (E_3)_q$ and $Y_q = \sin\theta(q)(E_1)_q - \cos\theta(q)(E_2)_q$. Then, the integrability condition of D^θ is given in the following form:

$$(1-\rho^2)(E_3\theta) - \rho\sqrt{1-\rho^2}\cos\theta(E_1\theta) - \rho\sqrt{1-\rho^2}\sin\theta(E_2\theta) + \rho^2 c = 0, \quad \text{i.e.,} \tag{4.6}$$

$$\rho\sqrt{1-\rho^2}\cos\theta\frac{\partial\theta}{\partial u_1} + \rho\sqrt{1-\rho^2}\sin\theta\frac{\partial\theta}{\partial u_2} + \left\{\frac{c}{2}\rho\sqrt{1-\rho^2}u_1\sin\theta - \frac{c}{2}\rho\sqrt{1-\rho^2}u_2\cos\theta - (1-\rho^2)\right\}\frac{\partial\theta}{\partial u_3} - \rho^2 c = 0. \tag{4.7}$$

We will give ourselves to study this 1st order PDE (4.7). The characteristic ODE's of (4.7) are given in the following:

$$\begin{aligned}
\frac{du_1}{dt} &= \rho\sqrt{1-\rho^2}\cos\theta,\\
\frac{du_2}{dt} &= \rho\sqrt{1-\rho^2}\sin\theta,\\
\frac{du_3}{dt} &= \frac{c}{2}\rho\sqrt{1-\rho^2}u_1\sin\theta - \frac{c}{2}\rho\sqrt{1-\rho^2}u_2\cos\theta - (1-\rho^2),\\
\frac{d\theta}{dt} &= \rho^2 c.
\end{aligned} \tag{4.8}$$

Take an initial surface $\mathcal{P}$ as $\mathcal{P} = \{(0,\, a,\, b);\ a, b \in \mathbb{R}\}$ and denote by $\varphi(a,\, b)$ an arbitrary initial function on $\mathcal{P}$. Moreover let $u_1(t,\, a,\, b)$, $u_2(t,\, a,\, b)$, $u_3(t,\, a,\, b)$, and $\theta(t,\, a,\, b)$ be the solution of (4.8) such that $u_1(0,\, a,\, b) = 0$, $u_2(0,\, a,\, b) = a$, $u_3(0,\, a,\, b) = b$, and $\theta(0,\, a,\, b) = \varphi(a,\, b)$. Then, it follows that $\theta(t,\, a,\, b) = \rho^2 ct + \varphi(a,\, b)$ and

$$\begin{aligned}
&\det\begin{pmatrix} \partial u_1/\partial t & \partial u_1/\partial a & \partial u_1/\partial b\\ \partial u_2/\partial t & \partial u_2/\partial a & \partial u_2/\partial b\\ \partial u_3/\partial t & \partial u_3/\partial a & \partial u_3/\partial b \end{pmatrix}(0,\, a,\, b)\\
&= \det\begin{pmatrix} \rho\sqrt{1-\rho^2}\cos\varphi(a,\, b) & 0 & 0\\ \rho\sqrt{1-\rho^2}\sin\varphi(a,\, b) & 1 & 0\\ -(c/2)\rho\sqrt{1-\rho^2}a\cos\varphi(a,\, b) - (1-\rho^2) & 0 & 1 \end{pmatrix}\\
&= \rho\sqrt{1-\rho^2}\cos\varphi(a,\, b).
\end{aligned}$$

If φ satisfies that $\cos\varphi(a_0,\, b_0) \neq 0$ at a point $(a_0,\, b_0)$, we can see by the inverse mapping theorem that the variables t, a, and b are solved by the variables u_i's around $(0,\, a_0,\, b_0)$ and consequently the PDE (4.7) has a local solution $\theta = \rho^2 ct(u_i) + \varphi(a(u_i),\, b(u_i))$ around $(0,\, a_0,\, b_0)$. For example, taking a constant function $\varphi = k$ where $\cos k \neq 0$, we can produce local

$\mathcal{O}(\alpha)$-surfaces and thus have the following.

Theorem 4.1 *For any α such that $-3c^2/4 < \alpha < c^2/4$, there exist $\mathcal{O}(\alpha)$-surfaces of H.*

Next we study geometric properties of $\mathcal{O}(\alpha)$-surfaces. Take a local function θ on H satisfying the equation (4.6) and consider the involutive distribution D^θ. There D^θ is spanned by the orthonormal vector fields X and Y given at the beginning of this section. Then, we have the following lemma.

Lemma 4.2 *Put $N = \sqrt{1-\rho^2}\cos\theta E_1 + \sqrt{1-\rho^2}\sin\theta E_2 + \rho E_3$ and $F^\theta = \sin\theta(E_1\theta) - \cos\theta(E_2\theta)$, where N is a unit vector field of D^θ. Then it holds that*

$$\begin{aligned}
\nabla_X X &= -\left(\rho\sqrt{1-\rho^2}(E_3\theta) - \frac{c\rho}{\sqrt{1-\rho^2}}\right)Y, \qquad (4.9)\\
\nabla_Y Y &= \sqrt{1-\rho^2}F^\theta N + \rho F^\theta X,\\
\nabla_Y X &= \frac{c}{2}N - \rho F^\theta Y,\\
\nabla_X Y &= \frac{c}{2}N + \left(\rho\sqrt{1-\rho^2}(E_3\theta) - \frac{c\rho}{\sqrt{1-\rho^2}}\right)X.
\end{aligned}$$

In paticular the second fundamental form Π^θ of D^θ is given as follows:

$$\Pi^\theta(X,\, X) = 0,\ \Pi^\theta(Y,\, Y) = \sqrt{1-\rho^2}F^\theta N,\ \Pi^\theta(X,\, Y) = \frac{c}{2}N. \qquad (4.10)$$

Moreover the sectional curvature K^θ of D^θ is a negative constant $-\rho^2c^2$.

Proof. The equations (4.9) follow by the same way as done for the $\mathcal{O}(c^2/4)$-geometry together with (2.1), and the equations (4.10) are obvious by (4.9). The sectional curvature K^θ is caluculated by using the Gauss equation of surfaces as follows:

$$\begin{aligned}
K^\theta &= \alpha + \langle \Pi^\theta(X,\, X), \Pi^\theta(Y,\, Y)\rangle - |\Pi^\theta(X,\, Y)|^2\\
&= \alpha - \frac{c^2}{4} = -\rho^2c^2.
\end{aligned}$$

□

By this lemma the following follows directly.

Proposition 4.3 *For α such that $-3c^2/4 < \alpha < c^2/4$, an $\mathcal{O}(\alpha)$-surface of H is always a surface of constant negative curvature $\alpha - c^2/4$ without geodesic points.*

Corollary 4.4 *The 3-dimensional Heisenberg group has no totally geodesic surface.*

Proof. Assume that the Heisenberg group H has a totally geodesic surface S. Then, since H is a Riemannian homogeneous manifold, S is locally Riemannian homogeneous. Hence, if S is connected, it has constant sectional curvature, denoted by λ. Moreover, since S is totally geodesic, the Gauss equation of S implies that sectional curvatures $K(T_pS)$, $p \in S$, of H equal to the constant λ. Then S is an $\mathcal{O}(\lambda)$-surface. Hence, by Theorems 2.3, 3.2, and the above proposition, S has no geodesic points. This is a contradiction. □

We here remark that the result of this corollary is well-known (*cf.* [3], [9]) but our proof stand on the Grassmann geometry.

Next, for any α such that $-3c^2/4 < \alpha < c^2/4$, we will show the non-existence of $\mathcal{O}(\alpha)$-surfaces with constant mean curvature. Our way is to solve the 1st order PDE (4.7) locally and to show that the distributions D^θ associated with the solutions θ don't have any leaves of constant mean curvature.

We return to the argument done in the above of Theorem 4.1. Then, for the initial surface $\mathcal{P}$ $(= (0, a, b);\, a, b \in \mathbb{R}\})$ and an arbitrary initial function φ on $\mathcal{P}$, the solution (u_1, u_2, u_3, θ) of the characteristic ODE's (4.8) is given in the following way:

$$u_1(t, a, b) = \frac{\sqrt{1-\rho^2}}{\rho c}\bigl\{\sin\bigl(\rho^2 ct + \varphi(a, b)\bigr) - \sin\varphi(a, b)\bigr\}, \tag{4.11}$$

$$u_2(t, a, b) = \frac{\sqrt{1-\rho^2}}{\rho c}\bigl\{-\cos\bigl(\rho^2 ct + \varphi(a, b)\bigr) + \cos\varphi(a, b)\bigr\} + a,$$

$$u_3(t, a, b) = -\frac{1}{2}\Bigl\{t + \frac{1}{(\rho^2 c)}\sin\rho^2 ct\Bigr\} - \frac{a\sqrt{1-\rho^2}}{2\rho}\sin\bigl(\rho^2 ct + \varphi(a, b)\bigr) + \frac{a\sqrt{1-\rho^2}}{2\rho}\sin\varphi(a, b) + b,$$

$$\theta(t, a, b) = \rho^2 ct + \varphi(a, b). \tag{4.12}$$

We here remark the following; if we fix a, b and consider the curve $\gamma(t)$ in the $u_1u_2u_3$-space defined by $\gamma(t) = \big(u_1(t,\,a,\,b),\,u_2(t,\,a,\,b),\,u_3(t,\,a,\,b) + t\big)$, it is the geodesic of H such that $\gamma(0) = (0,\,a,\,b)$ and $d\gamma/dt(0) = (\rho\sqrt{1-\rho^2}\cos\varphi(a,\,b),\,\rho\sqrt{1-\rho^2}\sin\varphi(a,\,b),\,\rho^2)$. (cf. See [8].)

We next calculate the Jacobian Δ of the transformation $(t,\,a,\,b) \to (u_1,\,u_2,\,u_3)$, and then the entries $\partial t/\partial u_i$, $\partial a/\partial u_i$, and $\partial b/\partial u_i$ in the inverse of the Jacobi matrix where $\Delta \neq 0$. At first the entries in the Jacobi matrix are given as follows:

$$
\begin{aligned}
\frac{\partial u_1}{\partial t} &= \rho\sqrt{1-\rho^2}\cos(\rho^2 ct + \varphi(a,\,b)),\\
\frac{\partial u_1}{\partial a} &= \frac{\sqrt{1-\rho^2}}{\rho c}\big\{\cos\big(\rho^2 ct + \varphi(a,\,b)\big) - \cos\varphi(a,\,b)\big\}\frac{\partial\varphi}{\partial a},\\
\frac{\partial u_1}{\partial b} &= \frac{\sqrt{1-\rho^2}}{\rho c}\big\{\cos\big(\rho^2 ct + \varphi(a,\,b)\big) - \cos\varphi(a,\,b)\big\}\frac{\partial\varphi}{\partial b},\\
\frac{\partial u_2}{\partial t} &= \rho\sqrt{1-\rho^2}\sin\big(\rho^2 ct + \varphi(a,\,b)\big),\\
\frac{\partial u_2}{\partial a} &= \frac{\sqrt{1-\rho^2}}{\rho c}\big\{\sin\big(\rho^2 ct + \varphi(a,\,b)\big) - \sin\varphi(a,\,b)\big\}\frac{\partial\varphi}{\partial a} + 1,\\
\frac{\partial u_2}{\partial b} &= \frac{\sqrt{1-\rho^2}}{\rho c}\big\{\sin\big(\rho^2 ct + \varphi(a,\,b)\big) - \sin\varphi(a,\,b)\big\}\frac{\partial\varphi}{\partial b},\\
\frac{\partial u_3}{\partial t} &= -\frac{1-\rho^2}{2}(1 + \cos\rho^2 ct) - \frac{ca}{2}\rho\sqrt{1-\rho^2}\cos\big(\rho^2 ct + \varphi(a,\,b)\big),\\
\frac{\partial u_3}{\partial a} &= -\frac{a\sqrt{1-\rho^2}}{2\rho}\big\{\cos\big(\rho^2 ct + \varphi(a,\,b)\big) - \cos\varphi(a,\,b)\big\}\frac{\partial\varphi}{\partial a},\\
\frac{\partial u_3}{\partial b} &= -\frac{a\sqrt{1-\rho^2}}{2\rho}\big\{\cos\big(\rho^2 ct + \varphi(a,\,b)\big) - \cos\varphi(a,\,b)\big\}\frac{\partial\varphi}{\partial b} + 1.
\end{aligned}
$$

Therefore the Jacobian Δ is given as follows:

$$
\begin{aligned}
\Delta = {}& \frac{1-\rho^2}{c}\sin\rho^2 ct\frac{\partial\varphi}{\partial a}\\
&+ \frac{(1-\rho^2)^{3/2}}{2\rho c}\big\{\cos\big(\rho^2 ct + \varphi(a,\,b)\big) - \cos\varphi(a,\,b)\big\}(1 + \cos\rho^2 ct)\frac{\partial\varphi}{\partial b}\\
&+ \rho\sqrt{1-\rho^2}\cos\big(\rho^2 ct + \varphi(a,\,b)\big).
\end{aligned}
$$

Moreover, if $\Delta \neq 0$, the entries in the inverse matrix are given as follows:

$$\frac{\partial t}{\partial u_1} = \frac{1}{\Delta}\Big\{\Big(1+\frac{\sqrt{1-\rho^2}}{\rho c}\Big)(\sin * - \sin\varphi)\varphi_a - \frac{a\sqrt{1-\rho^2}}{2\rho}(\cos * - \cos\varphi)\varphi_b\Big\},$$

$$\frac{\partial t}{\partial u_2} = \frac{1}{\Delta}\Big\{-\frac{\sqrt{1-\rho^2}}{\rho c}(\cos * - \cos\varphi)\varphi_a\Big\},$$

$$\frac{\partial t}{\partial u_3} = \frac{1}{\Delta}\Big\{-\frac{\sqrt{1-\rho^2}}{\rho c}(\cos * - \cos\varphi)\varphi_b\Big\},$$

$$\frac{\partial a}{\partial u_1} = \frac{1}{\Delta}\Big\{-\frac{a(1-\rho^2)}{2}\sin\rho^2 ct\varphi_b - \rho\sqrt{1-\rho^2}\sin * - \frac{(1-\rho^2)^{3/2}}{2\rho c}(\sin * - \sin\varphi)(1+\cos\rho^2 ct)\varphi_b\Big\},$$

$$\frac{\partial a}{\partial u_2} = \frac{1}{\Delta}\Big\{\rho\sqrt{1-\rho^2}\cos * + \frac{(1-\rho^2)^{3/2}}{2\rho c}(\cos * - \cos\varphi)(1+\cos\rho^2 ct)\varphi_b\Big\},$$

$$\frac{\partial a}{\partial u_3} = \frac{1}{\Delta}\Big\{-\frac{1-\rho^2}{c}\sin\rho^2 ct\varphi_b\Big\},$$

$$\frac{\partial b}{\partial u_1} = \frac{1}{\Delta}\Big\{\frac{a(1-\rho^2)}{2}\sin\rho^2 ct\varphi_a + \frac{(1-\rho^2)^{3/2}}{2\rho c}(\sin * - \sin\varphi)(1+\cos\rho^2 ct)\varphi_a + \frac{1-\rho^2}{2}(1+\cos\rho^2 ct) + \frac{ac\rho}{2}\sqrt{1-\rho^2}\cos *\Big\},$$

$$\frac{\partial b}{\partial u_2} = \frac{1}{\Delta}\Big\{-\frac{(1-\rho^2)^{3/2}}{2\rho c}(\cos * - \cos\varphi)(1+\cos\rho^2 ct)\varphi_a\Big\},$$

$$\frac{\partial b}{\partial u_3} = \frac{1}{\Delta}\Big\{\rho\sqrt{1-\rho^2}\cos * + \frac{1-\rho^2}{c}\sin\rho^2 ct\varphi_a\Big\},$$

where for convenience we use $*$, φ, φ_a, and φ_b in place of $\rho^2 ct + \varphi(a, b)$, $\varphi(a, b)$, $\partial\varphi/\partial a$, and $\partial\varphi/\partial b$, respectively.

If $\Delta \neq 0$, using these calculations, we can represent the derivatives $E_1\theta$ $(= \partial\theta/\partial u_1 - (1/2)cu_2\partial\theta/\partial u_3)$ and $E_2\theta$ $(= \partial\theta/\partial u_2 + (1/2)cu_1\partial\theta/\partial u_3)$ by the variables t, a, b. Then, the constant mean curvature equation

$$\sin\theta(E_1\theta) - \cos\theta(E_2\theta) = k \tag{4.13}$$

with a nonnegative constant k is represented by the variables t, a, b as follows:

$$\begin{aligned}
&c\rho^2\sin\bigl(\rho^2 ct+\varphi(a,\,b)\bigr)-\rho\sqrt{1-\rho^2}\varphi_a\cos\rho^2 ct \\
&+\frac{1-\rho^2}{2}\varphi_b\bigl\{\sin\bigl(\rho^2 ct+\varphi(a,\,b)\bigr)+\sin\rho^2 ct\bigr\} \\
&=k\Bigl[\frac{1-\rho^2}{c}\varphi_a\sin\rho^2 ct+\Bigl(1-\frac{\rho^{3/2}}{2\rho c}\Bigr)\bigl(\cos(\rho^2 ct+\varphi(a,\,b)) \\
&-\cos\varphi(a,\,b)\bigr)(1+\cos\rho^2 ct)+\rho\sqrt{1-\rho^2}\cos\bigl(\rho^2 ct+\varphi(a,\,b)\bigr)\Bigr]
\end{aligned} \tag{4.14}$$

We here recall that if the initial function φ satisfies that $\cos\varphi(a,\,b)\neq 0$, $\Delta\neq 0$ when $t=0$ and therefore $\Delta\neq 0$ for all t near 0. Under this condition, we put $r=\sin\rho^2 ct$. Then r also takes all values near 0 and it holds that $\cos\rho^2 ct=\sqrt{1-r^2}$. Rewrite the equation (4.14) as follows:

$$Ar+B(1-r^2)+C=D\sqrt{1-r^2}+Er\sqrt{1-r^2}$$

where

$$\begin{aligned}
A&=c\rho^2\cos\varphi-\frac{k(1-\rho^2)}{c}\varphi_a-\frac{k(1-\rho^2)}{2\rho c}\sin\varphi\varphi_b+k\rho\sqrt{1-\rho^2}\sin\varphi,\\
B&=\frac{1-\rho^2}{2}\varphi_b\sin\varphi-\frac{k(1-\rho^2)^{3/2}}{2\rho c}\cos\varphi\varphi_b,\\
C&=\frac{k(1-\rho^2)^{3/2}}{2\rho c}\cos\varphi\varphi_b,\\
D&=-c\rho^2\sin\varphi+\rho\sqrt{1-\rho^2}\varphi_a-\frac{1-\rho^2}{2}\sin\varphi\varphi_b+k\rho\sqrt{1-\rho^2}\cos\varphi,\\
E&=-\frac{1-\rho^2}{2}\varphi_b\cos\varphi-\frac{k(1-\rho^2)^{3/2}}{2\rho c}\sin\varphi\varphi_b.
\end{aligned}$$

If we take the square of both sides in the above equality, we have a polynomial with one variable r. Then, comparing the coefficients of the polynomial with each other, we obtain that $A=B=C=D=E=0$, and it follows by the equation $E=0$ that $\varphi_b=0$ and moreover by the equations $A=D=0$ that φ_a is constant. Again from the equations $A=D=0$ it follows that φ is constant and thus $\varphi_a=0$. This induces that $c\rho^2=0$, which is a contradiction. Summing up these arguments, we have the following theorem.

Theorem 4.5 *For α such that $-3c^2/4 < \alpha < c^2/4$, there exist no $\mathcal{O}(\alpha)$-surfaces with constant mean curvature, and in particular there exist no minimal $\mathcal{O}(\alpha)$-surfaces.*

Proof. Let S be an $\mathcal{O}(\alpha)$-surface of the Heisenberg group H and take a point p in S. Moreover, deform S for a direction transversal to the tangent space T_pS locally and isometrically by a left translation, and construct an involutive distribution D around p whose leaves are $\mathcal{O}(\alpha)$-surfaces. We here assume that S has constant mean curvature k and thus D has the same property. Also, we may suppose that p is the origin of H and take a smooth function θ around the origin p such that $D = D^\theta$. Let φ be the restriction of θ into the u_2u_3-plane in the $u_1u_2u_3$-cordinate space of H, i.e., $\theta(0, a, b) = \varphi(a, b)$. Then, if there exists a point $(0, a, b)$ near the origin such that $\cos\varphi(a, b) \neq 0$, we have a contradiction by applying the above argument to the restriction of θ into a sufficientlly small neighbourhood around the point $(0, a, b)$. Hence we may assume that $\cos\varphi = 0$ around $(0, 0)$. This implies that θ is constant along the u_2u_3-plane. Selecting the direction of the deformation of S as a direction transversal to S, we may moreover assume that θ is constant on a neighbourhood around the origin, since the deformation of S is done by a left translation. Particularly, the vector fields X and Y defined in the first of this section are left invariant and it holds that $X = \pm\rho E_2 - \sqrt{1-\rho^2}E_3$ and $Y = \pm E_1$. Since $[X, Y] = -\rho c E_3$, the distribution D is not involutive. This is a contradiction. □

We last seek typical examples of the $\mathcal{O}(\alpha)$-geometries where $-3c^2/4 < \alpha < c^2/4$. We consider an involutive distribution D^θ for a local function θ which satisfies that $E_3\theta = 0$. Then, by using (4.9), it holds that

$$\nabla_X X = \frac{c\rho}{\sqrt{1-\rho^2}}Y, \quad \nabla_X Y = \frac{c}{2}N - \frac{c\rho}{\sqrt{1-\rho^2}}X, \quad \nabla_X N = -\frac{c}{2}Y,$$

namely,

$$\begin{pmatrix} \nabla_X X \\ \nabla_X Y \\ \nabla_X N \end{pmatrix} = \begin{pmatrix} 0 & c\rho/\sqrt{1-\rho^2} & 0 \\ -c\rho/\sqrt{1-\rho^2} & 0 & c/2 \\ 0 & -c/2 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ N \end{pmatrix}.$$

Hence the integral curves of X is a helix in the Heisenberg group H with curvature $c\rho/\sqrt{1-\rho^2}$ and torsion $c/2$. Moreover, since

$$\nabla^\theta_X X = \frac{c\rho}{\sqrt{1-\rho^2}} Y, \quad \nabla^\theta_X Y = -\frac{c\rho}{\sqrt{1-\rho^2}} X,$$

it is a circle in each leaf S of D^θ with curvature $c\rho/\sqrt{1-\rho^2}$.

Next assume that the initial function φ of θ at the initial u_2u_3-plane Σ satisfies that $\cos\varphi(a, b) \neq 0$ for variables a and b. Then we can see that $E_3\theta = 0$ if and only if $\varphi_b = 0$, namely, φ is independent on the variable b. In fact, by the assumption, we can locally exchange the (u_1, u_2, u_3)-cordinates of H to the (t, a, b)-cordinates under the correspondence (4.11). Then, it holds that

$$E_3\theta = \frac{\partial\theta}{\partial u_3} = \frac{\partial t}{\partial u_3}\frac{\partial\theta}{\partial t} + \frac{\partial a}{\partial u_3}\frac{\partial\theta}{\partial a} + \frac{\partial b}{\partial u_3}\frac{\partial\theta}{\partial b}.$$

We here note that $\theta = \rho^2 ct + \varphi(a, b)$ by (4.12). Assume first that $E_3\theta = 0$ and substitute 0 as t into the above equation. Then, from the explicit expression of the inverse of Jacobi matrix of the mapping (4.11), it follows that $\partial t/\partial u_3|_{t=0} = \partial a/\partial u_3|_{t=0} = 0$, and $\partial b/\partial u_3|_{t=0} = 1$, and thus it holds that $\varphi_b = 0$. Assume next that $\varphi_b = 0$. Then, again by the expression of the inverse of Jacobi matrix, it follows that $\partial t/\partial u_3 = \partial a/\partial u_3 = 0$, and moreover $\partial\theta/\partial b = \varphi_b = 0$. These imply that $E_3\theta = 0$. Summing up these arguments, we have the following theorem.

Theorem 4.6 *For any α such that $-3c^2/4 < \alpha < c^2/4$, there exist local $\mathcal{O}(\alpha)$-surfaces foliated by circles of curvature $c\rho/\sqrt{1-\rho^2}$ which are helices of H with the same curvature $c\rho/\sqrt{1-\rho^2}$ and the torsion $c/2$.*

Remark Surfaces in the Heisenberg group H which are invariant under 1-parameter subgroups of G are called *helicoidal surfaces*. In particular, surfaces which are invariant under rotational isometries are called *rotational surfaces*. Rotational surfaces with constant mean curvature are classified by Caddeo, Piu and Ratto [1] and Tomter [10]. Figueroa, Mercuri and Pedorosa [4] classified helicoidal surfaces with constant mean curvature in H. Rotational surfaces of constant curvature are clasified in [2].

For more informations and elementary examples of minimal surfaces in H, we refer to [5].

References

[1] Caddeo R., Piu P. and Ratto A., *SO(2)-invariant minimal and constant mean curvature surfaces in 3-dimensional homogeneous spaces*. Manuscripta Math. **87** (1995), 1–12.

[2] Caddeo R., Piu P. and Ratto A., *Rotational surfaces in H_3 with constant Gauss curvature*. Boll. Un. Mat. Ital. B **10** (7) (1996), 341–357.

[3] Eberlein P., *Geometry of 2-step nilpotent groups with left invariant metrics* II. Trans. Amer. Math. Soc. **343** (2) (1994), 805–828.

[4] Figueroa C.B., Mercuri F. and Pedrosa R.H.L., *Invariant minimal surfaces of the Heisenberg group*. Ann. Mat. Pura Appl. **180** (2001), 211–221.

[5] Inoguchi J., Kumamoto T., Ohsugi N. and Suyama Y., *Differential geometry of curves and surfaces in 3-dimensional homogeneous spaces* II. Fukuoka Univ. Sci. Rep. **30** (1) (2000), 17–47.

[6] Kobayashi S. and Nomizu K., *Foundations of Differential Geometry,* I, II. Interscience, New York, 1963 and 1969.

[7] Milnor J., *Curvatures of left invariant metrics on Lie groups*. Advances in Math. **21** (1976), 293–329.

[8] Naitoh H. and Sakane Y., *On conjugate points of a nilpotent Lie group*. Tsukuba J. Math. **5** (1) (1981), 143–152.

[9] Sanini A., *Gauss map of a surface of the Heisenberg group*. Boll. Un. Mat. Ital. B, **11-2** (7) (1997), suppl., 79–93.

[10] Tomter P., *Constant mean curvature surfaces in the Heisenberg group, Differential Geometry: Partial Differential Equations on Manifolds*. (Los Angeles, CA, 1990), Proc. Sympos. Pure Math. **54**, Part 1, Amer. Math. Soc., Providence, RI, 1993, pp. 485–495.

J. Inoguchi
Department of Mathemtics Education
Faculty of Education
Utsunomiya University
Utsunomiya 321-8505, Japan
E-mail: inoguchi@cc.utsunomiya-u.ac.jp

K. Kuwabara
Department of Mathematics
Yamaguchi University
Yamaguchi 753-8512, Japan

H. Naitoh
Department of Mathematics
Yamaguchi University
Yamaguchi 753-8512, Japan
E-mail: naitoh@po.cc.yamaguchi-u.ac.jp