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Rigidity of the canonical isometric imbedding of the Cayley projective plane $P^2(\boldsymbol{Cay})$

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Abstract. In [7], we have proved that $P^2(\boldsymbol{Cay})$ cannot be isometrically immersed into $\boldsymbol{R}^{25}$ even locally. In this paper, we investigate isometric immersions of $P^2(\boldsymbol{Cay})$ into $\boldsymbol{R}^{26}$ and prove that the canonical isometric imbedding $\boldsymbol{f}_0$ of $P^2(\boldsymbol{Cay})$ into $\boldsymbol{R}^{26}$, which is defined in Kobayashi [17], is rigid in the following strongest sense: Any isometric immersion $\boldsymbol{f}_1$ of a connected open set $U(\subset P^2(\boldsymbol{Cay}))$ into $\boldsymbol{R}^{26}$ coincides with $\boldsymbol{f}_0$ up to a euclidean transformation of $\boldsymbol{R}^{26}$, i.e., there is a euclidean transformation a of $\boldsymbol{R}^{26}$ satisfying $\boldsymbol{f}_1 = a\boldsymbol{f}_0$ on U.

Key words: curvature invariant, isometric immersion, Cayley projective plane, rigidity.

1. Introduction

In the previous paper [7], we investigated the problem of (local) isometric immersions of the quaternion projective plane $P^2(\boldsymbol{H})$ and the Cayley projective plane $P^2(\boldsymbol{Cay})$. In particular, we proved the following non-existence theorem of (local) isometric immersions:

Theorem 1 *Any open set of the Cayley projective plane $P^2(\boldsymbol{Cay})$ cannot be isometrically immersed into $\boldsymbol{R}^{25}$.*

As is well-known, there is an isometric immersion $\boldsymbol{f}_0$ of $P^2(\boldsymbol{Cay})$ into the euclidean space $\boldsymbol{R}^{26}$, which is called the canonical isometric imbedding of $P^2(\boldsymbol{Cay})$ (Kobayashi [17]). This fact, together with Theorem 1, implies that $\boldsymbol{R}^{26}$ is the least dimensional euclidean space into which $P^2(\boldsymbol{Cay})$ can be (locally) isometrically immersed.

In this paper, we consider (local) isometric immersions of $P^2(\boldsymbol{Cay})$ into $\boldsymbol{R}^{26}$ and discuss the rigidity of the canonical isometric imbedding $\boldsymbol{f}_0$. Concerning the rigidity of $\boldsymbol{f}_0$ Kaneda [15] has shown that the canonical isometric imbedding $\boldsymbol{f}_0$ is of finite type, i.e., the space of local infinitesimal isometric deformations of $\boldsymbol{f}_0$ is of finite dimension. However, it seems to the authors that any further result concerning the rigidity of $\boldsymbol{f}_0$ has not been

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obtained.

In the present paper, we will show the rigidity of the canonical isometric imbedding $\boldsymbol{f}_0$ in the following strongest form:

Theorem 2 *Let $\boldsymbol{f}_0$ be the canonical isometric imbedding of $P^2(\boldsymbol{Cay})$ into the euclidean space $\boldsymbol{R}^{26}$. Then, for any isometric immersion $\boldsymbol{f}_1$ defined on a connected open set U of $P^2(\boldsymbol{Cay})$ into $\boldsymbol{R}^{26}$, there exists a euclidean transformation a of $\boldsymbol{R}^{26}$ satisfying $\boldsymbol{f}_1 = a\boldsymbol{f}_0$ on U.*

To prove Theorem 2, we first establish a rigidity theorem for an isometric immersion of a Riemannian manifold. Let M be an n-dimensional Riemannian manifold and let $\boldsymbol{f}_0$ be an isometric immersion of M into the m-dimensional euclidean space $\boldsymbol{R}^m$. We will prove that if the Gauss equation in codimension r $(= m - n)$ admits essentially one solution everywhere on M, then $\boldsymbol{f}_0$ is rigid, i.e., for any isometric immersion $\boldsymbol{f}_1$ of M into $\boldsymbol{R}^m$ there exists a euclidean transformation a of $\boldsymbol{R}^m$ such that $\boldsymbol{f}_1 = a\boldsymbol{f}_0$ (see Theorem 5). This theorem may be established by various methods; for example, by combining the results of Nomizu [19] and Szczarba [21], [22] (cf. Agaoka [1]) or by solving a differential system of Pfaff (cf. Bishop–Crittenden [10], Ch. X). In this paper, we will give a simple proof based on a congruence theorem of differentiable mappings, which is easy to understand and gives a clear view on the geometric meaning (see Theorem 6).

Next, we will show that for the Cayley projective plane $P^2(\boldsymbol{Cay})$ the Gauss equation in codimension 10 $(= 26 - \dim P^2(\boldsymbol{Cay}))$ admits essentially one solution (see Theorem 10). To show this, we utilize the results obtained in [6] and [7]. Among all, the result concerning pseudo-abelian subspaces (Proposition 8) plays an important role in our proof.

Then, Theorem 2 is a direct consequence of Theorem 5 and Theorem 10.

Throughout this paper we assume the differentiability of class C^∞. Notations for Lie algebras are the same as those used in [6] and [7].

2. The Gauss equation

Let M be a Riemannian manifold and $T(M)$ the tangent bundle of M. We denote by g the Riemannian metric of M and by R the Riemannian curvature tensor of type (1, 3) with respect to g.

Let $\boldsymbol{N}$ be a euclidean vector space, i.e., $\boldsymbol{N}$ is a vector space over $\boldsymbol{R}$ endowed with an inner product $\langle\ ,\ \rangle$. Let $p \in M$ and let $S^2T_p^*(M) \otimes \boldsymbol{N}$ be the space of $\boldsymbol{N}$-valued symmetric bilinear forms on $T_p(M)$. We call the following equation on $\boldsymbol{\Psi} \in S^2T_p^*(M) \otimes \boldsymbol{N}$ the *Gauss equation* at $p \in M$:

$$-g_p(R_p(x, y)z, w) = \langle \boldsymbol{\Psi}(x, z), \boldsymbol{\Psi}(y, w) \rangle - \langle \boldsymbol{\Psi}(x, w), \boldsymbol{\Psi}(y, z) \rangle, \quad (2.1)$$

where $x,\, y,\, z,\, w \in T_p(M)$. We denote by $\mathcal{G}_p(\boldsymbol{N})$ the set of all solutions of (2.1), which is called the *Gaussian variety* associated with $\boldsymbol{N}$ at $p \in M$. As is well-known, $\mathcal{G}_p(\boldsymbol{N}) = \emptyset$ happens in case the dimensionality r $(= \dim \boldsymbol{N})$ is so small, however, $\mathcal{G}_p(\boldsymbol{N}) \neq \emptyset$ if r is sufficiently large (see Cartan [11] or Kaneda–Tanaka [16]).

Let $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ be two euclidean vector spaces and let φ be a linear mapping of $\boldsymbol{N}_1$ to $\boldsymbol{N}_2$. Define a linear map $\widehat{\varphi}$ of $S^2T_p^*(M) \otimes \boldsymbol{N}_1$ to $S^2T_p^*(M) \otimes \boldsymbol{N}_2$ by

$$(\widehat{\varphi}\boldsymbol{\Psi})(x, y) = \varphi(\boldsymbol{\Psi}(x, y)), \ \boldsymbol{\Psi} \in S^2T_p^*(M) \otimes \boldsymbol{N}_1, \ x, y \in T_p(M). \quad (2.2)$$

Then, we can easily verify

Lemma 3 *Let φ be a linear mapping of a euclidean vector space $\boldsymbol{N}_1$ to a euclidean vector space $\boldsymbol{N}_2$. Assume that φ is isometric, i.e., $\langle \varphi(x),\, \varphi(y) \rangle_2 = \langle x,\, y \rangle_1$ $(x, y \in \boldsymbol{N}_1)$, where $\langle\ ,\ \rangle_i$ $(i = 1, 2)$ denotes the inner product of $\boldsymbol{N}_i$. Then $\widehat{\varphi}\,\mathcal{G}_p(\boldsymbol{N}_1) \subset \mathcal{G}_p(\boldsymbol{N}_2)$. In particular, if* $\dim \boldsymbol{N}_1 = \dim \boldsymbol{N}_2$, *then $\widehat{\varphi}\,\mathcal{G}_p(\boldsymbol{N}_1) = \mathcal{G}_p(\boldsymbol{N}_2)$.*

In view of Lemma 3, the solvability of the Gauss equation (2.1) substantially depends on the dimensionality of $\boldsymbol{N}$. To emphasize $\dim \boldsymbol{N}$ we call (2.1) the Gauss equation in codimension r $(= \dim \boldsymbol{N})$.

Let $\boldsymbol{N}$ be a euclidean vector space and let $O(\boldsymbol{N})$ be the orthogonal transformation group of $\boldsymbol{N}$. We define an action of $O(\boldsymbol{N})$ on $S^2T_p^*(M) \otimes \boldsymbol{N}$ by

$$(h\boldsymbol{\Psi})(x,\, y) = h(\boldsymbol{\Psi}(x,\, y)),$$

where $\boldsymbol{\Psi} \in S^2T_p^*(M) \otimes \boldsymbol{N}$, $h \in O(\boldsymbol{N})$, $x,\, y \in T_p(M)$. We say that two elements $\boldsymbol{\Psi}$ and $\boldsymbol{\Psi}' \in S^2T_p^*(M) \otimes \boldsymbol{N}$ are *equivalent* if there is an element $h \in O(\boldsymbol{N})$ such that $\boldsymbol{\Psi}' = h\boldsymbol{\Psi}$. It is easily seen that if $\boldsymbol{\Psi}$ and $\boldsymbol{\Psi}' \in S^2T_p^*(M) \otimes \boldsymbol{N}$ are equivalent and $\boldsymbol{\Psi} \in \mathcal{G}_p(\boldsymbol{N})$, then $\boldsymbol{\Psi}' \in \mathcal{G}_p(\boldsymbol{N})$. We say that the Gaussian variety $\mathcal{G}_p(\boldsymbol{N})$ is *EOS* if $\mathcal{G}_p(\boldsymbol{N}) \neq \emptyset$ and if it consists of essentially one solution, i.e., any solutions of the Gauss equation (2.1) are equivalent to each other under the action of $O(\boldsymbol{N})$.

Proposition 4 *Let M be a Riemannian manifold and let $p \in M$. Let $\boldsymbol{N}$ be an r-dimensional euclidean vector space such that $\mathcal{G}_p(\boldsymbol{N})$ is EOS. Then*:
(1) *Let $\boldsymbol{\Psi}$ be an arbitrary element of $\mathcal{G}_p(\boldsymbol{N})$. Then, the vectors $\boldsymbol{\Psi}(x, y)$ $(x, y \in T_p(M))$ span the whole space $\boldsymbol{N}$.*
(2) *Let $\boldsymbol{N}_1$ be a euclidean vector space. Then*:
(2*a*) $\mathcal{G}_p(\boldsymbol{N}_1) = \emptyset$ *if* $\dim \boldsymbol{N}_1 < r$;
(2*b*) $\mathcal{G}_p(\boldsymbol{N}_1)$ *is EOS if* $\dim \boldsymbol{N}_1 = r$;
(2*c*) $\mathcal{G}_p(\boldsymbol{N}_1)$ *is not EOS if* $\dim \boldsymbol{N}_1 > r$.

Proof. Note that if $\boldsymbol{\Psi'} \in S^2T_p^*(M) \otimes \boldsymbol{N}$ is equivalent to $\boldsymbol{\Psi}$, then we have $|\boldsymbol{\Psi'}(x, y)| = |\boldsymbol{\Psi}(x, y)|$ for any $x, y \in T_p(M)$, where $|\boldsymbol{n}|$ denotes the norm of $\boldsymbol{n} \in \boldsymbol{N}$ with respect to $\langle\ ,\ \rangle$.

Now, suppose that the vectors $\boldsymbol{\Psi}(x, y)$ $(x, y \in T_p(M))$ do not span the whole space $\boldsymbol{N}$. Then, there is a non-zero vector $\boldsymbol{n} \in \boldsymbol{N}$ satisfying $\langle \boldsymbol{n}, \boldsymbol{\Psi}(x, y) \rangle = 0$ for any $x, y \in T_p(M)$. Define an element $\boldsymbol{\Psi'} \in S^2T_p^*(M) \otimes \boldsymbol{N}$ by

$$\boldsymbol{\Psi'} = \boldsymbol{\Psi} + (\xi^*)^2 \otimes \boldsymbol{n},$$

where ξ^* is a non-zero element of $T_p^*(M)$. Then, it is easy to see that $\boldsymbol{\Psi'} \in \mathcal{G}_p(\boldsymbol{N})$. However, by a simple calculation, we have $|\boldsymbol{\Psi'}(x, x)|^2 = |\boldsymbol{\Psi}(x, x)|^2 + |\boldsymbol{n}|^2 \xi^*(x)^2$. Therefore, if we take $x \in T_p(M)$ such that $\xi^*(x) \neq 0$, then we have $|\boldsymbol{\Psi'}(x, x)| \neq |\boldsymbol{\Psi}(x, x)|$. This proves that $\boldsymbol{\Psi'}$ is not equivalent to $\boldsymbol{\Psi}$ and hence $\mathcal{G}_p(\boldsymbol{N})$ is not EOS. Thus, we obtain (1).

Next we prove (2). First assume $\dim \boldsymbol{N}_1 = r$. Let φ be an isometric linear isomorphism of $\boldsymbol{N}$ onto $\boldsymbol{N}_1$. Then we have $O(\boldsymbol{N}_1) = \varphi \cdot O(\boldsymbol{N}) \cdot \varphi^{-1}$. Moreover, by Lemma 3 we have $\widehat{\varphi}\, \mathcal{G}_p(\boldsymbol{N}) = \mathcal{G}_p(\boldsymbol{N}_1)$. Since $\mathcal{G}_p(\boldsymbol{N})$ is EOS, $O(\boldsymbol{N})$ acts transitively on $\mathcal{G}_p(\boldsymbol{N})$. Therefore, it is easily seen that $O(\boldsymbol{N}_1)$ acts transitively on $\mathcal{G}_p(\boldsymbol{N}_1)$. This proves that $\mathcal{G}_p(\boldsymbol{N}_1)$ is EOS.

We next consider the case $\dim \boldsymbol{N}_1 < r$. Suppose that $\mathcal{G}_p(\boldsymbol{N}_1) \neq \emptyset$ and $\boldsymbol{\Psi}_1 \in \mathcal{G}_p(\boldsymbol{N}_1)$. Let φ be an isometric linear mapping of $\boldsymbol{N}_1$ to $\boldsymbol{N}$. Then, we know that $\widehat{\varphi}\, \boldsymbol{\Psi}_1 \in \mathcal{G}_p(\boldsymbol{N})$ and the vectors $(\widehat{\varphi}\, \boldsymbol{\Psi}_1)(x, y)$ $(x, y \in T_p(M))$ are contained in the proper subspace $\varphi(\boldsymbol{N}_1)$ $(\subsetneq \boldsymbol{N})$. This contradicts (1). The case $\dim \boldsymbol{N}_1 > r$ is similarly dealt with. □

We say that a Riemannian manifold M is *formally rigid* in codimension r if there is a euclidean vector space $\boldsymbol{N}$ with $\dim \boldsymbol{N} = r$ such that the Gaussian variety $\mathcal{G}_p(\boldsymbol{N})$ is EOS at each $p \in M$. By virtue of Proposition 4 (2), we know that if M is formally rigid in codimension r, then it is not formally rigid in any other codimension r' ($\neq r$).

Remark 1 It should be noted that there is a Riemannian manifold M that is not formally rigid in any codimension r. For example, assume that M is the space of negative constant curvature of dimension n. Let $\boldsymbol{N}$ be a euclidean vector space of dimension r. Then, by Ôtsuki's lemma we have $\mathcal{G}_p(\boldsymbol{N}) = \emptyset$ if $r < n-1$ (see Ôtsuki [20]). On the other hand, Kaneda [13] proved that if $r = n-1$, then $\mathcal{G}_p(\boldsymbol{N}) \neq \emptyset$ and around a suitable $\boldsymbol{\Psi}_0 \in \mathcal{G}_p(\boldsymbol{N})$, $\mathcal{G}_p(\boldsymbol{N})$ forms a submanifold of $S^2T_p^*(M) \otimes \boldsymbol{N}$ of dimension $n(n-1)$ (see Theorem 3.1 of [13]). Since $n(n-1) > \dim O(\boldsymbol{N})$, $\mathcal{G}_p(\boldsymbol{N})$ cannot be EOS. If $r \geq n$, then by Proposition 4 ($2a$) we know that $\mathcal{G}_p(\boldsymbol{N})$ is not EOS. Accordingly, the space of negative constant curvature M is not formally rigid in any codimension r.

Remark 2 For each Riemannian submanifold $M \subset \boldsymbol{R}^m$ listed below, $\mathcal{G}_p(\boldsymbol{N})$ is known to be EOS at each $p \in M$, where $\boldsymbol{N}$ is the normal vector space of M at p in $\boldsymbol{R}^m$:

(1) The sphere $S^n \subset \boldsymbol{R}^{n+1}$ ($n \geq 3$);
(2) The symplectic group $Sp(2) \subset \boldsymbol{R}^{16}$ (see Agaoka [1]);
(3) A submanifold $M \subset \boldsymbol{R}^m$ with type number ≥ 3 (see Allendoerfer [9], Kobayashi–Nomizu [18]).

Consequently, these submanifolds are formally rigid in our sense and it has been proved that they are actually rigid in $\boldsymbol{R}^m$ (see [1], [9]).

However, we note that the formal rigidness of M in codimension r does not imply the existence of an isometric immersion of M into $\boldsymbol{R}^{n+r}$ ($n = \dim M$). Indeed, Kaneda [14] gave an example of three dimensional Riemannian manifold M that is formally rigid in codimension 1 but cannot be locally isometrically immersed into $\boldsymbol{R}^4$.

We will prove in the next section that if a connected Riemannian manifold M is formally rigid in codimension r and if there is an isometric immersion $\boldsymbol{f}$ of M into $\boldsymbol{R}^{n+r}$ ($n = \dim M$), then M (precisely, $\boldsymbol{f}(M)$) is actually rigid in $\boldsymbol{R}^{n+r}$ (see Theorem 5).

3. Rigidity theorem

In this section, we will prove the following rigidity theorem:

Theorem 5 *Let M be an n-dimensional Riemannian manifold and let $\boldsymbol{f}_0$ be an isometric immersion of M into the euclidean space $\boldsymbol{R}^m$. Assume*:

(1) *M is connected;*
(2) *M is formally rigid in codimension $r = m - n$.*
Then, any isometric immersion $\boldsymbol{f}_1$ of M into the euclidean space $\boldsymbol{R}^m$ coincides with $\boldsymbol{f}_0$ up to a euclidean transformation of $\boldsymbol{R}^m$, i.e., there exists a euclidean transformation a of $\boldsymbol{R}^m$ such that $\boldsymbol{f}_1 = a\boldsymbol{f}_0$.

Before proceeding to the proof of Theorem 5, we make some preparations. Let $M(m, m')$ be the space of real matrices of degree $m \times m'$, where m and m' are non-negative integers. In what follows we identify $M(m, 1)$ with the m-dimensional euclidean space $\boldsymbol{R}^m$ in a natural way. Then, we note that the canonical inner product $\langle\ ,\ \rangle$ of $\boldsymbol{R}^m$ is given by $\langle \boldsymbol{v}, \boldsymbol{w}\rangle = {}^t\boldsymbol{v} \cdot \boldsymbol{w}$ for $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{R}^m$.

Let us define an operation of $M(m, m)$ on $\boldsymbol{R}^m$ by

$$M(m, m) \times \boldsymbol{R}^m \ni \big(H, \boldsymbol{v}\big) \longmapsto H \cdot \boldsymbol{v} \in \boldsymbol{R}^m,$$

where $\cdot$ means the usual matrix multiplication.

Let ∇ be the Riemannian connection associated with M. Let $\boldsymbol{f} = {}^t(f^1, \ldots, f^m)$ be a differentiable map of M into the euclidean space $\boldsymbol{R}^m$. By $\overbrace{\nabla \cdots \nabla}^{k} \boldsymbol{f}$ we denote the k-th order covariant derivative of $\boldsymbol{f}$, which is defined as follows:

$$\overbrace{\nabla_{x_1} \cdots \nabla_{x_k}}^{k} \boldsymbol{f} = {}^t(\ldots, \overbrace{\nabla_{x_1} \cdots \nabla_{x_k}}^{k} f^i, \ldots) \in \boldsymbol{R}^m,$$

where $p \in M$; $x_1, \ldots, x_k \in T_p(M)$. (Precisely, see Tanaka [23], Kaneda–Tanaka [16] or Kaneda [14].) It is known that $\nabla\nabla\boldsymbol{f}$ and $\nabla\nabla\nabla\boldsymbol{f}$ satisfy the following integrability conditions:

$$\nabla_x \nabla_y \boldsymbol{f} = \nabla_y \nabla_x \boldsymbol{f}, \tag{3.1}$$

$$\nabla_z \nabla_x \nabla_y \boldsymbol{f} = \nabla_x \nabla_z \nabla_y \boldsymbol{f} - \nabla_{R(z,x)y} \boldsymbol{f}. \tag{3.2}$$

We say that a differentiable map $\boldsymbol{f}$ of M into $\boldsymbol{R}^m$ is 2-*generic* if at each $p \in M$, the whole space $\boldsymbol{R}^m$ is spanned by the vectors of the form $\nabla_x \boldsymbol{f}$ ($x \in T_p(M)$), $\nabla_y \nabla_z \boldsymbol{f}$ ($y, z \in T_p(M)$). It is clear that if $\boldsymbol{f}$ is 2-generic, then we have the inequality $m \leq (1/2)n(n+3)$. Note that a 2-generic map $\boldsymbol{f}$ is not necessarily an immersion.

We first show the following congruence theorem:

Theorem 6 *Let M be an n-dimensional Riemannian manifold and let $\boldsymbol{f}_i$ $(i=0,\,1)$ be two differentiable maps of M into the euclidean space $\boldsymbol{R}^m$. Assume*:

(1) *M is connected*;

(2) *$\boldsymbol{f}_0$ is 2-generic*;

(3) *At each $p \in M$ there is an element $H(p) \in O(m)$ satisfying*

$$\nabla_x \boldsymbol{f}_1 = H(p)\cdot(\nabla_x \boldsymbol{f}_0), \qquad \forall x \in T_p(M), \tag{3.3}$$

$$\nabla_y \nabla_z \boldsymbol{f}_1 = H(p)\cdot(\nabla_y \nabla_z \boldsymbol{f}_0), \qquad \forall y,\, z \in T_p(M). \tag{3.4}$$

Then, $\boldsymbol{f}_1$ coincides with $\boldsymbol{f}_0$ up to a euclidean transformation of $\boldsymbol{R}^m$. More precisely, $H(p)$ is identically equal to a constant value $H_0 \in O(m)$ everywhere on M and $\boldsymbol{f}_1$ can be written as $\boldsymbol{f}_1 = H_0\boldsymbol{f}_0 + \boldsymbol{c}_0$, where $\boldsymbol{c}_0$ is a constant vector of $\boldsymbol{R}^m$.

Proof. We first note that, since $\boldsymbol{f}_0$ is 2-generic, $H(p)$ satisfying (3.3) and (3.4) is uniquely determined at each $p \in M$ and the map $H\colon M \ni p \longmapsto H(p) \in O(m)$ is differentiable. Via the canonical inclusion $O(m) \subset M(m,\,m)$, we can regard H as an $M(m,\,m)$-valued function on M satisfying

$${}^tHH = I_m, \tag{3.5}$$

where I_m denotes the identity matrix of degree m. Differentiate (3.5) covariantly. Then by Leibnitz' law we get

$$\nabla_x({}^tH)H(p) + {}^tH(p)(\nabla_x H) = 0, \quad \forall x \in T_p(M). \tag{3.6}$$

In this equality, the covariant derivative $\nabla_x H$ means the element of $M(m,m)$ given by $\nabla_x H = \left(\nabla_x h_i^j\right)$, where h_i^j denotes the $(i,\,j)$-component of H. By the very definition of $\nabla_x H$ we have $\nabla_x({}^tH) = {}^t(\nabla_x H)$.

Let us define an $M(m,\,m)$-valued 1-form L by

$$L(x) = {}^tH(p)(\nabla_x H), \quad x \in T_p(M). \tag{3.7}$$

Then, by (3.6) we have

$${}^tL(x) + L(x) = 0, \quad \forall x \in T_p(M), \tag{3.8}$$

implying that the matrix $L(x) \in M(m,\,m)$ is skew-symmetric.

We now show that the equality $L(x) = 0$ holds for any $x \in T_p(M)$. Since $\boldsymbol{f}_0$ is 2-generic, it suffices to prove

$$L(y)\cdot(\nabla_x \boldsymbol{f}_0) = 0, \quad \forall x,\, y \in T_p(M), \tag{3.9}$$

$$L(z) \cdot (\nabla_y \nabla_x \boldsymbol{f}_0) = 0, \quad \forall x,\, y,\, z \in T_p(M). \tag{3.10}$$

Differentiating (3.3) and (3.4) covariantly, we have

$$\nabla_y \nabla_x \boldsymbol{f}_1 = \nabla_y H \cdot (\nabla_x \boldsymbol{f}_0) + H(p) \cdot (\nabla_y \nabla_x \boldsymbol{f}_0), \quad \forall x,\, y \in T_p(M), \tag{3.11}$$

$$\nabla_z \nabla_y \nabla_x \boldsymbol{f}_1 = \nabla_z H \cdot (\nabla_y \nabla_x \boldsymbol{f}_0) + H(p) \cdot (\nabla_z \nabla_y \nabla_x \boldsymbol{f}_0), \quad \forall x,\, y,\, z \in T_p(M). \tag{3.12}$$

Then by (3.4) and (3.11) we have $\nabla_y H \cdot (\nabla_x \boldsymbol{f}_0) = 0$ for each $x,\, y \in T_p(M)$. Consequently, multiplying ${}^t H(p)$ from the left, we have (3.9).

We now prove (3.10). Exchanging z and y in (3.12), we have

$$\nabla_y \nabla_z \nabla_x \boldsymbol{f}_1 = \nabla_y H \cdot (\nabla_z \nabla_x \boldsymbol{f}_0) + H(p) \cdot (\nabla_y \nabla_z \nabla_x \boldsymbol{f}_0), \quad \forall x,\, y,\, z \in T_p(M). \tag{3.13}$$

Subtract (3.13) from (3.12). Then, using the integrability condition (3.2) and the equality (3.3), we have

$$\nabla_z H(\nabla_y \nabla_x \boldsymbol{f}_0) = \nabla_y H(\nabla_z \nabla_x \boldsymbol{f}_0), \quad \forall x,\, y,\, z \in T_p(M). \tag{3.14}$$

Consequently, multiplying ${}^t H(p)$ from the left, we get

$$L(z) \cdot (\nabla_y \nabla_x \boldsymbol{f}_0) = L(y) \cdot (\nabla_z \nabla_x \boldsymbol{f}_0), \quad \forall x,\, y,\, z \in T_p(M). \tag{3.15}$$

Since $L(z)$ is a skew-symmetric matrix, we have

$$\big\langle L(z) \cdot (\nabla_y \nabla_x \boldsymbol{f}_0),\, \nabla_u \boldsymbol{f}_0 \big\rangle = -\big\langle \nabla_y \nabla_x \boldsymbol{f}_0,\, L(z) \cdot (\nabla_u \boldsymbol{f}_0) \big\rangle = 0.$$

Therefore, to prove (3.10), we have to show

$$\big\langle L(z) \cdot (\nabla_y \nabla_x \boldsymbol{f}_0),\, \nabla_v \nabla_w \boldsymbol{f}_0 \big\rangle = 0, \quad \forall x,\, y,\, z,\, v,\, w \in T_p(M). \tag{3.16}$$

Define an element $X \in \otimes^5 T_p^*(M)$ by

$$X(z,\, y,\, x,\, v,\, w) = \big\langle L(z) \cdot (\nabla_y \nabla_x \boldsymbol{f}_0),\, \nabla_v \nabla_w \boldsymbol{f}_0 \big\rangle, \quad x,\, y,\, z,\, v,\, w \in T_p(M). \tag{3.17}$$

In the following, we will show $X(z,\, y,\, x,\, v,\, w) = 0$ for $x,\, y,\, z,\, v,\, w \in T_p(M)$. By the integrability condition (3.1) and by (3.15), we easily know that $X(z,\, y,\, x,\, v,\, w)$ is symmetric with respect to the pairs $\{x,\, y\}$, $\{v,\, w\}$ and $\{z,\, y\}$. Further, since $L(z)$ is a skew-symmetric endomorphism of $\boldsymbol{R}^m$

(see (3.8)), it follows that

$$X(z, y, x, v, w) = -X(z, v, w, y, x). \tag{3.18}$$

Therefore, $X(z, y, x, v, w)$ is anti-symmetric with respect to the pair $\{x, w\}$, because

$$\begin{aligned} X(z, y, x, v, w) &= -X(z, v, w, y, x) = -X(v, z, w, y, x) \\ &= \quad X(v, y, x, z, w) = \quad X(y, v, x, z, w) \\ &= -X(y, z, w, v, x) = -X(z, y, w, v, x). \end{aligned}$$

Consequently, we get

$$\begin{aligned} X(z, y, x, v, w) &= -X(z, y, w, v, x) = -X(z, w, y, x, v) \\ &= \quad X(z, w, v, x, y) = \quad X(z, v, w, y, x). \end{aligned}$$

This, together with (3.18), proves $X(z, y, x, v, w) = 0$. Thus we get (3.10).

By the above argument, we know that $L(x) = {}^tH(p)(\nabla_x H) = 0$ for any $x \in T_p(M)$. This implies that H is a locally constant function and hence H is identically equal to an element $H_0 \in O(m)$ on M, because M is connected. Consequently, the difference $\boldsymbol{c} = \boldsymbol{f}_1 - H_0 \cdot \boldsymbol{f}_0$ satisfies

$$\nabla_x \boldsymbol{c} = \nabla_x(\boldsymbol{f}_1 - H_0 \cdot \boldsymbol{f}_0) = \nabla_x \boldsymbol{f}_1 - H_0 \cdot (\nabla_x \boldsymbol{f}_0) = 0, \quad \forall x \in T_p(M).$$

Therefore, $\boldsymbol{c}$ is also identically equal to a constant vector $\boldsymbol{c}_0 \in \boldsymbol{R}^m$, completing the proof of the theorem. □

Remark 3 The argument in the proof of the equality $X = 0$ is essentially the same that is developed in the proof of the uniqueness of the metric connection of the normal bundle associated with an isometric imbedding (see the proof of Theorem 1 of [19]); It is almost the same that is used to calculate the third prolongation of the symbol of the operator L (see Proposition 2.2 of [16]). Here we remark that $X = 0$ can be proved without assuming the existence of (isometric) immersions.

We are now in a position to prove Theorem 5.

Proof of Theorem 5. We show that the map $\boldsymbol{f}_i$ $(i = 0, 1)$ is 2-generic and for each $p \in M$ there is an element $H(p) \in O(m)$ satisfying the equalities (3.3) and (3.4).

Let $i = 0$ or 1. Let $\boldsymbol{f}_{i*}T_p(M)$ (resp. $\boldsymbol{N}_i$) be the tangent vector space (resp. normal vector space) of $\boldsymbol{f}_i(M)$ at $\boldsymbol{f}_i(p) \in \boldsymbol{R}^m$. Then, we have

$\dim \boldsymbol{f}_{i*}T_p(M) = n$ and $\dim \boldsymbol{N}_i = m - n$. We regard $\boldsymbol{f}_{i*}T_p(M)$ and $\boldsymbol{N}_i$ as euclidean vector spaces endowed with the inner products induced from the inner product $\langle\ ,\ \rangle$ of $\boldsymbol{R}^m$. By a natural parallel displacement from $\boldsymbol{f}_i(p)$ to the origin $o \in \boldsymbol{R}^m$, we regard $\boldsymbol{f}_{i*}T_p(M)$ and $\boldsymbol{N}_i$ as linear subspaces of $\boldsymbol{R}^m$. Since $\boldsymbol{f}_i$ is an isometric immersion, $\boldsymbol{f}_{i*}T_p(M)$ is spanned by the vectors $\nabla_x \boldsymbol{f}_i$ $(x \in T_p(M))$ and

$$\big\langle \nabla_x \boldsymbol{f}_i,\, \nabla_y \boldsymbol{f}_i \big\rangle = g_p(x,\, y), \quad \forall x,\, y \in T_p(M). \tag{3.19}$$

The second order derivative $\nabla\nabla\boldsymbol{f}_i$, which is so called the *second fundamental form* of $\boldsymbol{f}_i$, satisfies $\nabla\nabla\boldsymbol{f}_i \in S^2T_p^*(M) \otimes \boldsymbol{N}_i$ and $\nabla\nabla\boldsymbol{f}_i \in \mathcal{G}_p(\boldsymbol{N}_i)$ (see [23], [16]). Since $\mathcal{G}_p(\boldsymbol{N}_i)$ is EOS, the vectors $\nabla_x\nabla_y\boldsymbol{f}_i$ $(x,\, y \in T_p(M))$ span $\boldsymbol{N}_i$, implying that $\boldsymbol{f}_i$ is 2-generic (see Proposition 4 (1)). Take an isometric linear isomorphism φ_2 of $\boldsymbol{N}_0$ onto $\boldsymbol{N}_1$. Since $\widehat{\varphi_2}\,\nabla\nabla\boldsymbol{f}_0 \in \mathcal{G}_p(\boldsymbol{N}_1)$ and since $\mathcal{G}_p(\boldsymbol{N}_1)$ is EOS (see Proposition 4 (2*b*)), there is an element $h_1 \in O(\boldsymbol{N}_1)$ such that $h_1(\widehat{\varphi_2}\nabla\nabla\boldsymbol{f}_0) = \nabla\nabla\boldsymbol{f}_1$. On the other hand, in view of (3.19) we also know that there is an isometric linear isomorphism φ_1 of $\boldsymbol{f}_{0*}T_p(M)$ onto $\boldsymbol{f}_{1*}T_p(M)$ satisfying $\varphi_1(\nabla_x\boldsymbol{f}_0) = \nabla_x\boldsymbol{f}_1$ $(x \in T_p(M))$. Define a linear endomorphism $H(p)$ of $\boldsymbol{R}^m$ satisfying $H(p)|_{\boldsymbol{f}_{0*}T_p(M)} = \varphi_1$ and $H(p)|_{\boldsymbol{N}_0} = h_1 \cdot \varphi_2$. Then, it is easily seen that $H(p) \in O(m)$ and the equalities (3.3) and (3.4) are satisfied.

Therefore, by Theorem 6 we know that $\boldsymbol{f}_1$ can be written as $\boldsymbol{f}_1 = a\boldsymbol{f}_0$, where a denotes the euclidean transformation of $\boldsymbol{R}^m$ defined by $\boldsymbol{R}^m \ni \boldsymbol{x} \longmapsto H_0 \cdot \boldsymbol{x} + \boldsymbol{c}_0 \in \boldsymbol{R}^m$. Thus, we obtain the theorem. □

4. The Cayley projective plane $P^2(Cay)$

Let $M = G/K$ be a compact Riemannian symmetric space. Let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of G (resp. K). We denote by $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ the canonical decomposition of $\mathfrak{g}$ associated with the symmetric pair $(G,\, K)$. We denote by $(\ ,\ )$ the inner product of $\mathfrak{g}$ given by the (-1)-multiple of the Killing form of $\mathfrak{g}$. As usual, we can identify $\mathfrak{m}$ with the tangent space $T_o(G/K)$ at the origin $o = \{K\}$. We assume that the G-invariant Riemannian metric g of G/K satisfies

$$g_o(X,\, Y) = \big(X,\, Y\big), \quad \forall X,\, Y \in \mathfrak{m}.$$

Then, it is well-known that at the origin o the Riemannian curvature tensor R of type (1, 3) is given by

$$R_o(X, Y)Z = -\big[[X, Y], Z\big], \quad X, Y, Z \in \mathfrak{m}.$$

Hereafter, we consider the case of the Cayley projective plane $P^2(\boldsymbol{Cay})$. As is well-known, $P^2(\boldsymbol{Cay})$ can be represented by $P^2(\boldsymbol{Cay}) = G/K$, where $G = F_4$ and $K = Spin(9)$. Take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{m}$ and fix it in the following discussions. We note that since $\mathrm{rank}(P^2(\boldsymbol{Cay})) = 1$, we have $\dim \mathfrak{a} = 1$.

For each element $\lambda \in \mathfrak{a}$ we define two subspaces $\mathfrak{k}(\lambda) \subset \mathfrak{k}$ and $\mathfrak{m}(\lambda) \subset \mathfrak{m}$ by

$$\begin{aligned}\mathfrak{k}(\lambda) &= \big\{X \in \mathfrak{k} \mid \big[H, [H, X]\big] = -(\lambda, H)^2 X, \quad \forall H \in \mathfrak{a}\big\},\\ \mathfrak{m}(\lambda) &= \big\{Y \in \mathfrak{m} \mid \big[H, [H, Y]\big] = -(\lambda, H)^2 Y, \quad \forall H \in \mathfrak{a}\big\}.\end{aligned}$$

We call λ a *restricted root* if $\mathfrak{m}(\lambda) \neq 0$. Let Σ be the set of all non-zero restricted roots. In the case of $P^2(\boldsymbol{Cay})$, there is a restricted root μ such that $\Sigma = \{\pm\mu, \pm 2\mu\}$. We take and fix such a restricted root μ. Then we have $\mathfrak{m}(0) = \mathfrak{a} = \boldsymbol{R}\mu$ and

$$\begin{aligned}\mathfrak{k} &= \mathfrak{k}(0) + \mathfrak{k}(\mu) + \mathfrak{k}(2\mu) \quad \text{(orthogonal direct sum)},\\ \mathfrak{m} &= \mathfrak{m}(0) + \mathfrak{m}(\mu) + \mathfrak{m}(2\mu) \ \text{(orthogonal direct sum)}.\end{aligned}$$

(For details, see [6], [7].) For simplicity, for each integer i we set $\mathfrak{k}_i = \mathfrak{k}(|i|\mu)$, $\mathfrak{m}_i = \mathfrak{m}(|i|\mu)$ $(|i| \leq 2)$, $\mathfrak{k}_i = \mathfrak{m}_i = 0$ $(|i| > 2)$. Then we have

Proposition 7 ([7]) (1) *Let $i, j = 0, 1, 2$. Then*:

$$\begin{aligned}&\big[\mathfrak{k}_i, \mathfrak{k}_j\big] \subset \mathfrak{k}_{i+j} + \mathfrak{k}_{i-j},\\ &\big[\mathfrak{m}_i, \mathfrak{m}_j\big] \subset \mathfrak{k}_{i+j} + \mathfrak{k}_{i-j},\\ &\big[\mathfrak{k}_i, \mathfrak{m}_j\big] \subset \mathfrak{m}_{i+j} + \mathfrak{m}_{i-j}.\end{aligned} \tag{4.1}$$

(2) $\dim \mathfrak{m} = 16$, $\dim \mathfrak{k}_1 = \dim \mathfrak{m}_1 = 8$, $\dim \mathfrak{k}_2 = \dim \mathfrak{m}_2 = 7$.

In what follows, we recall the results obtained in [7], which will be needed in the proof of Theorem 2. Let V be a subspace of $\mathfrak{m}$. V is called *pseudo-abelian* if it satisfies $\big[V, V\big] \subset \mathfrak{k}_0$ (or equivalently $\big[[V, V], \mathfrak{a}\big] = 0$). (Precisely, see [6].) As is easily seen, $\mathfrak{m}_2$ is a pseudo-abelian subspace of $\mathfrak{m}$, because $\big[\mathfrak{m}_2, \mathfrak{m}_2\big] \subset \mathfrak{k}_0$ (see (4.1)).

On the contrary, we have

Proposition 8 *Let $G/K = P^2(\boldsymbol{Cay})$. Then, any pseudo-abelian subspace V of $\mathfrak{m}$ with $\dim V > 2$ must be contained in $\mathfrak{m}_2$.*

For the proof, see Lemma 6 of [7]. The following proposition summarizes the results of [7] (see Proposition 7, Proposition 10 and Lemma 17 of [7]).

Proposition 9 (1) *Let* $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ *and* $Y_1 \in \mathfrak{m}_1$. *Assume that* $Y_0 \neq 0$, $Y_1 \neq 0$. *Then, there are elements* $k_0, k_1 \in K$ *satisfying*

$$\mathrm{Ad}(k_0)\mu \in \boldsymbol{R}Y_0, \quad \mathrm{Ad}(k_0)\mathfrak{m}_2 = \bigl\{Y_0' \in \mathfrak{a} + \mathfrak{m}_2 \bigm| (Y_0', Y_0) = 0\bigr\}, \tag{4.2}$$

$$\mathrm{Ad}(k_1)\mu \in \boldsymbol{R}Y_1, \quad \mathrm{Ad}(k_1)\mathfrak{m}_2 = \bigl\{Y_1' \in \mathfrak{m}_1 \bigm| (Y_1', Y_1) = 0\bigr\}. \tag{4.3}$$

(2) *Let* $Y_0, Y_0' \in \mathfrak{a} + \mathfrak{m}_2$, $Y_1, Y_1' \in \mathfrak{m}_1$ *and* $X_1 \in \mathfrak{k}_1$. *Then*:

$$\bigl[Y_0, [Y_0, Y_0']\bigr] = \begin{cases} -4\bigl(\mu, \mu\bigr)\bigl(Y_0, Y_0\bigr)Y_0', & \text{if } \bigl(Y_0, Y_0'\bigr) = 0, \\ 0, & \text{if } Y_0' \in \boldsymbol{R}Y_0, \end{cases} \tag{4.4}$$

$$\bigl[Y_0, [Y_0, Y_1]\bigr] = -\bigl(\mu, \mu\bigr)\bigl(Y_0, Y_0\bigr)Y_1, \tag{4.5}$$

$$\bigl[Y_1, [Y_1, Y_0]\bigr] = -\bigl(\mu, \mu\bigr)\bigl(Y_1, Y_1\bigr)Y_0, \tag{4.6}$$

$$\bigl[Y_1, [Y_1, Y_1']\bigr] = \begin{cases} -4\bigl(\mu, \mu\bigr)\bigl(Y_1, Y_1\bigr)\, Y_1', & \text{if } \bigl(Y_1, Y_1'\bigr) = 0, \\ 0, & \text{if } Y_1' \in \boldsymbol{R}Y_1, \end{cases} \tag{4.7}$$

$$\bigl[X_1, [X_1, Y_0]\bigr] = -\bigl(\mu, \mu\bigr)\bigl(X_1, X_1\bigr)Y_0. \tag{4.8}$$

5. Solutions of the Gauss equation

In this and the next sections, we prove

Theorem 10 *The projective plane* $P^2(\boldsymbol{Cay})$ *is formally rigid in codimension* 10 $(= 26 - \dim P^2(\boldsymbol{Cay}))$.

If this theorem is established, then Theorem 2 immediately follows from Theorem 5.

On account of homogeneity of $P^2(\boldsymbol{Cay})$, in order to show Theorem 10 we have only to prove that the Gaussian variety $\mathcal{G}_o(\boldsymbol{N})$ is EOS at the origin o for any euclidean vector space $\boldsymbol{N}$ with $\dim \boldsymbol{N} = 10$.

In what follows we assume that $M = P^2(\boldsymbol{Cay})$ and that $\boldsymbol{N}$ is a euclidean vector space with $\dim \boldsymbol{N} = 10$. We will prove the following theorem:

Theorem 11 *Let* $\boldsymbol{\Psi} \in \mathcal{G}_o(\boldsymbol{N})$. *Then*:

(1) *There are linearly independent vectors* $\mathbf{A}$ *and* $\mathbf{B} \in \boldsymbol{N}$ *satisfying*

(1*a*) $\langle \mathbf{A}, \mathbf{A}\rangle = \langle \mathbf{B}, \mathbf{B}\rangle = 4(\mu, \mu)$ *and* $\langle \mathbf{A}, \mathbf{B}\rangle = 2(\mu, \mu)$;

(1*b*) $\boldsymbol{\Psi}(Y_0, Y_0') = (Y_0, Y_0')\mathbf{A}, \quad \forall Y_0, Y_0' \in \mathfrak{a} + \mathfrak{m}_2$;

(1*c*) $\mathbf{\Psi}(Y_1, Y_1') = (Y_1, Y_1')\mathbf{B}, \quad \forall Y_1, Y_1' \in \mathfrak{m}_1;$
(1*d*) $\langle \mathbf{A}, \mathbf{\Psi}(\mu, \mathfrak{m}_1)\rangle = \langle \mathbf{B}, \mathbf{\Psi}(\mu, \mathfrak{m}_1)\rangle = 0.$
(2) $\mathbf{\Psi}(Y_1, Y_2) + \big(1/(\mu, \mu)^2\big)\mathbf{\Psi}\big(\mu, \big[[\mu, Y_1], Y_2\big]\big) = 0, \quad \forall Y_1 \in \mathfrak{m}_1, \ \forall Y_2 \in \mathfrak{m}_2.$
(3) $\langle \mathbf{\Psi}(\mu, Y_1), \mathbf{\Psi}(\mu, Y_1')\rangle = (\mu, \mu)^2 (Y_1, Y_1'), \quad \forall Y_1, Y_1' \in \mathfrak{m}_1.$

Before proceeding to the proof of Theorem 11 we make a somewhat lengthy preparation. Let $\boldsymbol{N}$ be a euclidean vector space and let $S^2\mathfrak{m}^* \otimes \boldsymbol{N}$ be the space of $\boldsymbol{N}$-valued symmetric bilinear forms on $\mathfrak{m}$. Let $\mathbf{\Psi} \in S^2\mathfrak{m}^* \otimes \boldsymbol{N}$ and $Y \in \mathfrak{m}$. We define a linear map $\mathbf{\Psi}_Y$ of $\mathfrak{m}$ to $\boldsymbol{N}$ by

$$\mathbf{\Psi}_Y \colon \mathfrak{m} \ni Y' \longmapsto \mathbf{\Psi}(Y, Y') \in \boldsymbol{N}$$

and denote by $\mathbf{Ker}(\mathbf{\Psi}_Y)$ the kernel of $\mathbf{\Psi}_Y$. We say that an element $Y \in \mathfrak{m}$ is *singular* (resp. *non-singular*) with respect to $\mathbf{\Psi}$ if $\mathbf{\Psi}_Y(\mathfrak{m}) \neq \boldsymbol{N}$ (resp. $\mathbf{\Psi}_Y(\mathfrak{m}) = \boldsymbol{N}$). Apparently, 0 ($\in \mathfrak{m}$) is a singular element for any $\mathbf{\Psi} \in S^2\mathfrak{m}^* \otimes \boldsymbol{N}$.

Proposition 12 *Let $\mathbf{\Psi} \in \mathcal{G}_o(\boldsymbol{N})$. Let $Y \in \mathfrak{m}$ ($Y \neq 0$) and let k be an element of K satisfying* $\mathrm{Ad}(k)\mu \in \boldsymbol{R}Y$. *Then*:
(1) $\mathbf{Ker}(\mathbf{\Psi}_Y) \subset \mathrm{Ad}(k)\mathfrak{m}_2$. *Consequently,* $\dim \mathbf{Ker}(\mathbf{\Psi}_Y) \leq 7$.
(2) *Assume that Y is non-singular with respect to $\mathbf{\Psi}$. Then, it holds that* $\dim \mathbf{Ker}(\mathbf{\Psi}_Y) = 6$ *and* $\mathbf{Ker}(\mathbf{\Psi}_Y) \subsetneq \mathrm{Ad}(k)\mathfrak{m}_2$.
(3) *Assume that Y is singular with respect to $\mathbf{\Psi}$. Then, it holds that* $\mathbf{Ker}(\mathbf{\Psi}_Y) = \mathrm{Ad}(k)\mathfrak{m}_2$, $\dim \mathbf{Ker}(\mathbf{\Psi}_Y) = 7$ *and* $\dim \mathbf{\Psi}_Y(\mathfrak{m}) = 9$.

Proof. First, note that $\dim \mathbf{Ker}(\mathbf{\Psi}_Y) \geq \dim \mathfrak{m} - \dim \boldsymbol{N} = 6$. Consequently, it is easy to see that Y is singular (resp. non-singular) with respect to $\mathbf{\Psi}$ if and only if $\dim \mathbf{Ker}(\mathbf{\Psi}_Y) > 6$ (resp. $\dim \mathbf{Ker}(\mathbf{\Psi}_Y) = 6$).

Multiplying Y by a non-zero scalar if necessary, we may assume that $Y = \mathrm{Ad}(k)\mu$. From the Gauss equation (2.1) it follows that

$$R_o\big(\mathbf{Ker}(\mathbf{\Psi}_Y), \mathbf{Ker}(\mathbf{\Psi}_Y)\big)Y = 0.$$

In our terminology we have

$$\big[[\mathbf{Ker}(\mathbf{\Psi}_Y), \mathbf{Ker}(\mathbf{\Psi}_Y)], Y\big] = 0.$$

Applying $\mathrm{Ad}(k^{-1})$ to the both sides of the above equality, we have

$$\big[[\mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\mathbf{\Psi}_Y), \mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\mathbf{\Psi}_Y)], \mu\big] = 0.$$

Since $\mathfrak{a} = \boldsymbol{R}\mu$, it follows that $\mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_Y)$ is a pseudo-abelian subspace of $\mathfrak{m}$. By Proposition 8 and by the fact $\dim \mathbf{Ker}(\boldsymbol{\Psi}_Y) \geq 6$, we have $\mathrm{Ad}(k^{-1})\,\mathbf{Ker}(\boldsymbol{\Psi}_Y) \subset \mathfrak{m}_2$ and hence $\mathbf{Ker}(\boldsymbol{\Psi}_Y) \subset \mathrm{Ad}(k)\mathfrak{m}_2$, proving (1).

Assume that Y is non-singular with respect to $\boldsymbol{\Psi}$. Then, as we have stated above, we have $\dim \mathbf{Ker}(\boldsymbol{\Psi}_Y) = 6$. Since $\dim \mathfrak{m}_2 = 7$ (see Proposition 7 (2)), it follows that $\mathbf{Ker}(\boldsymbol{\Psi}_Y) \subsetneq \mathrm{Ad}(k)\mathfrak{m}_2$, proving (2).

Finally, we assume Y is singular with respect to $\boldsymbol{\Psi}$. Then, we have $\dim \mathbf{Ker}(\boldsymbol{\Psi}_Y) > 6$. Since $\mathbf{Ker}(\boldsymbol{\Psi}_Y) \subset \mathrm{Ad}(k)\mathfrak{m}_2$ and since $\dim \mathfrak{m}_2 = 7$, we have $\dim \mathbf{Ker}(\boldsymbol{\Psi}_Y) = 7$ and $\mathbf{Ker}(\boldsymbol{\Psi}_Y) = \mathrm{Ad}(k)\mathfrak{m}_2$. This proves (3). □

Corollary 13 *Let* $\boldsymbol{\Psi} \in \mathcal{G}_o(\boldsymbol{N})$. *Let* $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ $(Y_0 \neq 0)$ *and* $Y_1 \in \mathfrak{m}_1$ $(Y_1 \neq 0)$. *Then*:

(1) $\mathbf{Ker}(\boldsymbol{\Psi}_{Y_0}) \subset \{Y_0' \in \mathfrak{a} + \mathfrak{m}_2 \mid (Y_0', Y_0) = 0\}$. *In particular, if* Y_0 *is singular with respect to* $\boldsymbol{\Psi}$, *then* $\mathbf{Ker}(\boldsymbol{\Psi}_{Y_0}) = \{Y_0' \in \mathfrak{a} + \mathfrak{m}_2 \mid (Y_0', Y_0) = 0\}$.

(2) $\mathbf{Ker}(\boldsymbol{\Psi}_{Y_1}) \subset \{Y_1' \in \mathfrak{m}_1 \mid (Y_1', Y_1) = 0\}$. *In particular, if* Y_1 *is singular with respect to* $\boldsymbol{\Psi}$, *then* $\mathbf{Ker}(\boldsymbol{\Psi}_{Y_1}) = \{Y_1' \in \mathfrak{m}_1 \mid (Y_1', Y_1) = 0\}$.

Proof. Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ $(Y_0 \neq 0)$. By Proposition 9 (1), we know that there is an element $k_0 \in K$ satisfying (4.2). Applying Proposition 12 to Y_0, we easily get $\mathbf{Ker}(\boldsymbol{\Psi}_{Y_0}) \subset \{Y_0' \in \mathfrak{a} + \mathfrak{m}_2 \mid (Y_0', Y_0) = 0\}$. Assume that Y_0 is singular with respect to $\boldsymbol{\Psi}$. Then, by the equality $\mathbf{Ker}(\boldsymbol{\Psi}_{Y_0}) = \mathrm{Ad}(k_0)\mathfrak{m}_2$, we get (1).

The assertion (2) is similarly dealt with. □

Let $\boldsymbol{\Psi} \in S^2\mathfrak{m}^* \otimes \boldsymbol{N}$. A subspace U of $\mathfrak{m}$ is called *singular* with respect to $\boldsymbol{\Psi}$ if each element of U is singular with respect to $\boldsymbol{\Psi}$.

Proposition 14 *Let* $\boldsymbol{\Psi} \in \mathcal{G}_o(\boldsymbol{N})$. *Let* $Y \in \mathfrak{m}$ $(Y \neq 0)$ *and let* $k \in K$ *satisfy* $\mathrm{Ad}(k)\mu \in \boldsymbol{R}Y$. *Assume that* Y *is non-singular with respect to* $\boldsymbol{\Psi}$. *Then*:

(1) $\mathbf{Ker}(\boldsymbol{\Psi}_Y)$ *is a singular subspace with respect to* $\boldsymbol{\Psi}$.

(2) *There is an element* $Y' \in \mathrm{Ad}(k)\mathfrak{m}_2$ *satisfying* $\boldsymbol{\Psi}(Y, Y') \neq 0$ *and*

$$\boldsymbol{N} = \boldsymbol{R}\boldsymbol{\Psi}(Y, Y') + \boldsymbol{\Psi}_{Y''}(\mathfrak{m}) \quad (\textit{orthogonal direct sum}), \tag{5.1}$$

where Y'' *is an arbitrary non-zero element of* $\mathbf{Ker}(\boldsymbol{\Psi}_Y)$.

Proof. Since Y is non-singular with respect to $\boldsymbol{\Psi}$, we have $\mathbf{Ker}(\boldsymbol{\Psi}_Y) \subsetneq \mathrm{Ad}(k)\mathfrak{m}_2$ (see Proposition 12). Take a non-zero element $Y' \in \mathrm{Ad}(k)\mathfrak{m}_2$ such that $\big(Y', \mathbf{Ker}(\boldsymbol{\Psi}_Y)\big) = 0$. Then, since $Y' \notin \mathbf{Ker}(\boldsymbol{\Psi}_Y)$, we have $\boldsymbol{\Psi}(Y, Y') \neq 0$.

Let $Y'' \in \mathbf{Ker}(\mathbf{\Psi}_Y)$ $(Y'' \neq 0)$. Then, by the Gauss equation (2.1) we have

$$\begin{aligned}&\big([[Y', Y''], Y], W\big)\\ &= \langle \mathbf{\Psi}(Y', Y), \mathbf{\Psi}(Y'', W)\rangle - \langle \mathbf{\Psi}(Y', W), \mathbf{\Psi}(Y'', Y)\rangle,\end{aligned} \tag{5.2}$$

where W is an arbitrary element of $\mathfrak{m}$. Note that the left hand side of (5.2) vanishes, because

$$\begin{aligned}\big[[Y', Y''], Y\big] &\in \big[[\mathrm{Ad}(k)\mathfrak{m}_2, \mathrm{Ad}(k)\mathfrak{m}_2], \mathrm{Ad}(k)\mu\big]\\ &= \mathrm{Ad}(k)\big[[\mathfrak{m}_2, \mathfrak{m}_2], \mu\big] = 0.\end{aligned}$$

We also note that $\mathbf{\Psi}(Y'', Y) = 0$, because $Y'' \in \mathbf{Ker}(\mathbf{\Psi}_Y)$. Consequently, we have $\langle \mathbf{\Psi}(Y', Y), \mathbf{\Psi}(Y'', W)\rangle = 0$. This implies that each element of $\mathbf{\Psi}_{Y''}(\mathfrak{m})$ is orthogonal to $\mathbf{\Psi}(Y', Y)$. Therefore, $\mathbf{\Psi}_{Y''}(\mathfrak{m}) \neq \boldsymbol{N}$, implying that Y'' is singular with respect to $\mathbf{\Psi}$. Hence, by Proposition 12 (3) we have $\dim \mathbf{\Psi}_{Y''}(\mathfrak{m}) = 9$, which proves (5.1). □

The following lemma assures that for each $\mathbf{\Psi} \in \mathcal{G}_o(\boldsymbol{N})$ there are many high dimensional singular subspaces with respect to $\mathbf{\Psi}$.

Lemma 15 *Let $\mathbf{\Psi} \in \mathcal{G}_o(\boldsymbol{N})$. Then, there are singular subspaces U and V with respect to $\mathbf{\Psi}$ satisfying $U \subset \mathfrak{a}+\mathfrak{m}_2$, $V \subset \mathfrak{m}_1$, $\dim U \geq 6$ and $\dim V \geq 6$.*

Proof. If $\mathfrak{a} + \mathfrak{m}_2$ contains no non-singular element with respect to $\mathbf{\Psi}$, then we can take $U = \mathfrak{a} + \mathfrak{m}_2$. (Note that $\dim(\mathfrak{a} + \mathfrak{m}_2) = 8$.) On the contrary, if $\mathfrak{a} + \mathfrak{m}_2$ contains a non-singular element Y_0, then we set $U = \mathbf{Ker}(\mathbf{\Psi}_{Y_0})$. Then, we know that $U \subset \mathfrak{a} + \mathfrak{m}_2$, $\dim U = 6$ (see Proposition 12 (2) and Corollary 13 (1)) and that U is a singular subspace with respect to $\mathbf{\Psi}$ (see Proposition 14 (1)). Similarly, we can select a singular subspace $V \subset \mathfrak{m}_1$ with $\dim V \geq 6$. □

Proposition 16 *Let $\mathbf{\Psi} \in \mathcal{G}_o(\boldsymbol{N})$. Let U and V be arbitrary singular subspaces with respect to $\mathbf{\Psi}$ satisfying $U \subset \mathfrak{a} + \mathfrak{m}_2$, $V \subset \mathfrak{m}_1$, $\dim U \geq 6$ and $\dim V \geq 6$. Then there are two vectors $\mathbf{A}$ and $\mathbf{B} \in \boldsymbol{N}$ satisfying*:

(1) $\langle \mathbf{A}, \mathbf{A}\rangle = \langle \mathbf{B}, \mathbf{B}\rangle = 4(\mu, \mu)$;
(2) $\mathbf{\Psi}(\xi, Y_0) = (\xi, Y_0)\mathbf{A}, \quad \forall \xi \in U, \ \forall Y_0 \in \mathfrak{a} + \mathfrak{m}_2$;
(3) $\mathbf{\Psi}(\eta, Y_1) = (\eta, Y_1)\mathbf{B}, \quad \forall \eta \in V, \ \forall Y_1 \in \mathfrak{m}_1$;
(4) $\langle \mathbf{A}, \mathbf{\Psi}_{Y_0}(\mathfrak{m}_1)\rangle = \langle \mathbf{B}, \mathbf{\Psi}_{Y_0}(\mathfrak{m}_1)\rangle = 0, \quad \forall Y_0 \in \mathfrak{a} + \mathfrak{m}_2$.

Proof. Let $\xi \in U$ ($\xi \neq 0$). Since ξ is singular with respect to $\boldsymbol{\Psi}$, $\mathbf{Ker}(\boldsymbol{\Psi}_\xi)$ coincides with the orthogonal complement of $\boldsymbol{R}\xi$ in $\mathfrak{a}+\mathfrak{m}_2$ (see Corollary 13 (1)). Hence, the equality $\boldsymbol{\Psi}(\xi, Y_0) = 0$ holds for each $Y_0 \in \mathfrak{a}+\mathfrak{m}_2$ satisfying $(\xi, Y_0) = 0$. In particular, we have

$$\boldsymbol{\Psi}(\xi, \xi') = 0, \quad \forall \xi, \xi' \in U \text{ with } (\xi, \xi') = 0.$$

Then, applying the same argument as in the proof of Proposition 9 of [7], we can prove that there is a vector $\mathbf{A} \in \boldsymbol{N}$ satisfying

$$\boldsymbol{\Psi}(\xi, \xi') = (\xi, \xi')\mathbf{A}, \quad \forall \xi, \xi' \in U. \tag{5.3}$$

Let $Y_0 \in \mathfrak{a}+\mathfrak{m}_2$ satisfy $(Y_0, U) = 0$. Then, since $(\xi, Y_0) = 0$, we have $\boldsymbol{\Psi}(\xi, Y_0) = 0$ and $(\xi, Y_0)\mathbf{A} = 0$. This, together with (5.3), proves (2). The assertion (3) can be proved in the same way.

We now prove (1). Let $\xi, \xi' \in U$ satisfy $(\xi, \xi') = 0$ and $(\xi, \xi) = (\xi', \xi') = 1$. Put $X = Z = \xi$ and $Y = W = \xi'$ into the Gauss equation (2.1). Then, we have

$$\big([[\xi, \xi'], \xi], \xi'\big) = \langle \boldsymbol{\Psi}(\xi, \xi), \boldsymbol{\Psi}(\xi', \xi')\rangle - \langle \boldsymbol{\Psi}(\xi, \xi'), \boldsymbol{\Psi}(\xi', \xi)\rangle.$$

Since $\big[[\xi, \xi'], \xi\big] = 4(\mu, \mu)\xi'$ (see (4.4)), $\boldsymbol{\Psi}(\xi, \xi) = \boldsymbol{\Psi}(\xi', \xi') = \mathbf{A}$ and $\boldsymbol{\Psi}(\xi, \xi') = 0$, we have $\langle \mathbf{A}, \mathbf{A}\rangle = 4(\mu, \mu)$. Similarly, by (4.7) we can prove $\langle \mathbf{B}, \mathbf{B}\rangle = 4(\mu, \mu)$, proving (1).

Finally, we prove (4). Let $Y_1 \in \mathfrak{m}_1$ and $Y_0 \in \mathfrak{a}+\mathfrak{m}_2$. Take an element $\xi \in U$ satisfying $(\xi, Y_0) = 0$ and $(\xi, \xi) = 1$. Such ξ can exist, because $\dim U \geq 6$. Put $X = Z = \xi$, $Y = Y_0$ and $W = Y_1$ into the Gauss equation (2.1). Then we have

$$\big([[\xi, Y_0], \xi], Y_1\big) = \langle \boldsymbol{\Psi}(\xi, \xi), \boldsymbol{\Psi}(Y_0, Y_1)\rangle - \langle \boldsymbol{\Psi}(\xi, Y_1), \boldsymbol{\Psi}(Y_0, \xi)\rangle.$$

Since $(\xi, Y_0) = 0$, we have $\boldsymbol{\Psi}(\xi, Y_0) = 0$ and $\big[[\xi, Y_0], \xi\big] = 4(\mu, \mu)Y_0$ (see (4.4)). Moreover, since $\boldsymbol{\Psi}(\xi, \xi) = \mathbf{A}$ and $(Y_0, Y_1) = 0$, we have

$$\begin{aligned} \langle \mathbf{A}, \boldsymbol{\Psi}_{Y_0}(Y_1)\rangle &= \langle \boldsymbol{\Psi}(\xi, \xi), \boldsymbol{\Psi}(Y_0, Y_1)\rangle \\ &= \langle \boldsymbol{\Psi}(\xi, Y_1), \boldsymbol{\Psi}(Y_0, \xi)\rangle + 4(\mu, \mu)(Y_0, Y_1) \\ &= 0. \end{aligned}$$

Since Y_1 is an arbitrary element of $\mathfrak{m}_1$, we have $\langle \mathbf{A}, \boldsymbol{\Psi}_{Y_0}(\mathfrak{m}_1)\rangle = 0$. In a similar way, the equality $\langle \mathbf{B}, \boldsymbol{\Psi}_{Y_0}(\mathfrak{m}_1)\rangle = 0$ can be proved. □

Remark 4 As seen in the proof of Lemma 15, singular subspaces U and V may not be uniquely determined. However, it is noted that the vectors $\mathbf{A}$ and $\mathbf{B}$ in Proposition 16 do not depend on the choice of U and V. In fact, let U' and V' be different singular subspaces with respect to $\mathbf{\Psi}$ satisfying $U' \subset \mathfrak{a} + \mathfrak{m}_2$ and $V' \subset \mathfrak{m}_1$ with $\dim U' \geq 6$, $\dim V' \geq 6$. Let $\mathbf{A}'$ and $\mathbf{B}'$ be vectors of $\boldsymbol{N}$ satisfying (1) $\sim$ (4) of Proposition 16. Then, since $\dim(\mathfrak{a} + \mathfrak{m}_2) = \dim \mathfrak{m}_1 = 8$, we have $U \cap U' \neq 0$, $V \cap V' \neq 0$. Take $\xi \in U \cap U'$ and $\eta \in V \cap V'$ such that $(\xi, \xi) = (\eta, \eta) = 1$. Then we have $\mathbf{A} = \mathbf{\Psi}(\xi, \xi) = \mathbf{A}'$ and $\mathbf{B} = \mathbf{\Psi}(\eta, \eta) = \mathbf{B}'$, showing our assertion.

In the following discussions, we fix an element $\mathbf{\Psi} \in \mathcal{G}_o(\boldsymbol{N})$, singular subspaces U, V and vectors $\mathbf{A}$, $\mathbf{B}$ stated in Proposition 16 and prove several lemmas which are indispensable to the proof of Theorem 11.

Lemma 17 *Let $\xi \in U$, $\eta \in V$, $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$ and $Y_1 \in \mathfrak{m}_1$. Set $C = \langle \mathbf{A}, \mathbf{B} \rangle - (\mu, \mu)$. Then $C > 0$ and:*
(1) $\langle \mathbf{\Psi}_{Y_0}(\eta), \mathbf{\Psi}_{Y_0}(Y_1) \rangle = \{\langle \mathbf{\Psi}(Y_0, Y_0), \mathbf{B} \rangle - (\mu, \mu)(Y_0, Y_0)\}(\eta, Y_1)$;
(2) $\langle \mathbf{\Psi}_\xi(\eta), \mathbf{\Psi}_\xi(Y_1) \rangle = C(\xi, \xi)(\eta, Y_1)$.

Proof. Putting $X = Z = Y_0$, $Y = Y_1$ and $W = \eta$ into (2.1), we have

$$\big([[Y_0, Y_1], Y_0], \eta\big) = \langle \mathbf{\Psi}(Y_0, Y_0), \mathbf{\Psi}(Y_1, \eta) \rangle - \langle \mathbf{\Psi}(Y_0, \eta), \mathbf{\Psi}(Y_1, Y_0) \rangle.$$

Since $\big[[Y_0, Y_1], Y_0\big] = (\mu, \mu)(Y_0, Y_0)Y_1$ (see (4.5)) and $\mathbf{\Psi}(Y_1, \eta) = (Y_1, \eta)\mathbf{B}$, we easily get (1). Putting $Y_0 = \xi \in U$ into (1), we easily have (2). If we set $Y_1 = \eta \in V$ in (2), we have $\langle \mathbf{\Psi}_\xi(\eta), \mathbf{\Psi}_\xi(\eta) \rangle = C(\xi, \xi)(\eta, \eta)$. Since $\mathbf{Ker}(\mathbf{\Psi}_\xi) \cap \mathfrak{m}_1 = 0$ (see Corollary 13 (1)), we have $\mathbf{\Psi}_\xi(\eta) \neq 0$ if $\eta \neq 0$. Consequently, we have $C > 0$. □

Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$. Let ξ^0 be a non-zero element of U satisfying $(\xi^0, Y_0) = 0$. (Such ξ^0 exists, because $\dim U \geq 6$.) We define a linear mapping $\mathbf{\Theta}_{Y_0, \xi^0} : V \longrightarrow \boldsymbol{N}$ by

$$\mathbf{\Theta}_{Y_0, \xi^0}(\eta) = \mathbf{\Psi}_{Y_0}(\eta) + \frac{1}{C(\xi^0, \xi^0)} \mathbf{\Psi}_{\xi^0}\big([[\xi^0, \eta], Y_0]\big), \quad \eta \in V.$$

Then we have

Lemma 18 $\langle \mathbf{A}, \mathbf{\Theta}_{Y_0, \xi^0}(V) \rangle = \langle \mathbf{\Psi}_{\xi^0}(V), \mathbf{\Theta}_{Y_0, \xi^0}(V) \rangle = 0.$

Proof. We first note that $\big[[\xi^0, \eta], Y_0\big] \in \mathfrak{m}_1$ for $\eta \in V$ and note that $\mathbf{\Theta}_{Y_0, \xi^0}(V) \subset \mathbf{\Psi}_{Y_0}(\mathfrak{m}_1) + \mathbf{\Psi}_{\xi^0}(\mathfrak{m}_1)$. By Proposition 16 (4), we have

$\langle \mathbf{A}, \mathbf{\Psi}_{Y_0}(\mathfrak{m}_1)\rangle = \langle \mathbf{A}, \mathbf{\Psi}_{\xi^0}(\mathfrak{m}_1)\rangle = 0$ and hence $\langle \mathbf{A}, \mathbf{\Theta}_{Y_0, \xi^0}(V)\rangle = 0$.

Let $\eta, \eta' \in V$. Then by putting $X = Y_0$, $Y = \eta'$, $Z = \eta$ and $W = \xi^0$ into the Gauss equation (2.1), we have

$$\begin{aligned}\big(\big[[Y_0, \eta'], \eta\big], \xi^0\big) &= \langle \mathbf{\Psi}(Y_0, \eta), \mathbf{\Psi}(\eta', \xi^0)\rangle - \langle \mathbf{\Psi}(Y_0, \xi^0), \mathbf{\Psi}(\eta', \eta)\rangle \\ &= \langle \mathbf{\Psi}_{Y_0}(\eta), \mathbf{\Psi}_{\xi^0}(\eta')\rangle - \langle \mathbf{A}, \mathbf{B}\rangle(Y_0, \xi^0)(\eta', \eta).\end{aligned}$$

Since $(Y_0, \xi^0) = 0$, we have

$$\langle \mathbf{\Psi}_{Y_0}(\eta), \mathbf{\Psi}_{\xi^0}(\eta')\rangle = \big(\big[[Y_0, \eta'], \eta\big], \xi^0\big). \tag{5.4}$$

On the other hand, we have

$$\big\langle \mathbf{\Psi}_{\xi^0}\big(\big[[\xi^0, \eta], Y_0\big]\big), \mathbf{\Psi}_{\xi^0}(\eta')\big\rangle = C(\xi^0, \xi^0)\big(\big[[\xi^0, \eta], Y_0\big], \eta'\big)$$

(see Lemma 17 (2)). Therefore,

$$\begin{aligned}&\langle \mathbf{\Theta}_{Y_0, \xi^0}(\eta), \mathbf{\Psi}_{\xi^0}(\eta')\rangle \\ &\quad = \left\langle \mathbf{\Psi}_{Y_0}(\eta) + \frac{1}{C(\xi^0, \xi^0)} \mathbf{\Psi}_{\xi^0}\big(\big[[\xi^0, \eta], Y_0\big]\big), \mathbf{\Psi}_{\xi^0}(\eta') \right\rangle \\ &\quad = \big(\big[[Y_0, \eta'], \eta\big], \xi^0\big) + \big(\big[[\xi^0, \eta], Y_0\big], \eta'\big) \\ &\quad = -([Y_0, \eta'], [\xi^0, \eta]) + ([\xi^0, \eta], [Y_0, \eta']) \\ &\quad = 0.\end{aligned}$$

This completes the proof. □

We can further show

Lemma 19 *Let* $\eta \in V$. *Assume that* $\big[[\xi^0, \eta], Y_0\big] \in V$. *Then*:

$$\begin{aligned}&|\mathbf{\Theta}_{Y_0,\xi^0}(\eta)|^2 \\ &\quad = \left[\langle \mathbf{\Psi}(Y_0, Y_0), \mathbf{B}\rangle - (\mu, \mu)(Y_0, Y_0)\left\{1 + \frac{(\mu, \mu)}{C}\right\}\right](\eta, \eta).\end{aligned} \tag{5.5}$$

Proof. Set $\eta' = \big[[\xi^0, \eta], Y_0\big]$. By Lemma 18, Lemma 17 and the equality (5.4) we have

$$\begin{aligned}&\langle \mathbf{\Theta}_{Y_0,\xi^0}(\eta), \mathbf{\Theta}_{Y_0,\xi^0}(\eta)\rangle \\ &\quad = \left\langle \mathbf{\Psi}_{Y_0}(\eta) + \frac{1}{C(\xi^0, \xi^0)} \mathbf{\Psi}_{\xi^0}(\eta'), \mathbf{\Theta}_{Y_0,\xi^0}(\eta) \right\rangle \\ &\quad = \langle \mathbf{\Psi}_{Y_0}(\eta), \mathbf{\Theta}_{Y_0,\xi^0}(\eta)\rangle\end{aligned}$$

$$
\begin{aligned}
&= \langle \mathbf{\Psi}_{Y_0}(\eta),\ \mathbf{\Psi}_{Y_0}(\eta)\rangle + \frac{1}{C(\xi^0,\ \xi^0)}\langle \mathbf{\Psi}_{Y_0}(\eta),\ \mathbf{\Psi}_{\xi^0}(\eta')\rangle \\
&= \{\langle \mathbf{\Psi}(Y_0,\ Y_0),\ \mathbf{B}\rangle - (\mu,\ \mu)(Y_0,\ Y_0)\}(\eta,\ \eta) \\
&\qquad\qquad + \frac{1}{C(\xi^0,\ \xi^0)}\big([[Y_0,\ \eta'],\ \eta],\ \xi^0\big).
\end{aligned}
$$

Since $[\xi^0,\ \eta] \in \mathfrak{k}_1$, by (4.8) and (4.5) we have

$$
\begin{aligned}
\big([[Y_0,\ \eta'],\ \eta],\ \xi^0\big) &= -([Y_0,\ \eta'],\ [\xi^0,\ \eta]) \\
&= \big(Y_0,\ [[\xi^0,\ \eta],\ \eta']\big) \\
&= \big(Y_0,\ [[\xi^0,\ \eta],\ [[\xi^0,\ \eta],\ Y_0]]\big) \\
&= -(\mu,\ \mu)([\xi^0,\ \eta],\ [\xi^0,\ \eta])(Y_0,\ Y_0) \\
&= (\mu,\ \mu)\big([\xi^0,\ [\xi^0,\ \eta]],\ \eta\big)(Y_0,\ Y_0) \\
&= -(\mu,\ \mu)^2(\xi^0,\ \xi^0)(\eta,\ \eta)(Y_0,\ Y_0).
\end{aligned}
$$

Therefore, we obtain (5.5). □

Lemma 20 *Let $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$. Then*:
(1) $\langle \mathbf{\Psi}(Y_0,\ Y_0),\ \mathbf{B}\rangle = (\mu,\ \mu)(Y_0,\ Y_0)\{1 + (\mu,\ \mu)/C\}$.
(2) *Let ξ^0 be a non-zero element of U satisfying* $(Y_0,\ \xi^0) = 0$. *Then,* $\mathbf{\Theta}_{Y_0,\xi^0}(\eta) = 0$*, i.e., the equality*

$$
\mathbf{\Psi}(Y_0,\ \eta) + \frac{1}{C(\xi^0,\ \xi^0)}\mathbf{\Psi}\big(\xi^0,\ [[\xi^0,\ \eta],\ Y_0]\big) = 0 \tag{5.6}
$$

holds for each $\eta \in V$ satisfying $\big[[\xi^0,\ \eta],\ Y_0\big] \in V$.

Proof. We first show that there is a non-zero element $\eta^0 \in V$ satisfying $\mathbf{\Theta}_{Y_0,\,\xi^0}(\eta^0) = 0$ and $\big[[\xi^0,\ \eta^0],\ Y_0\big] \in V$. Let $\boldsymbol{D}$ be the orthogonal complement of $\boldsymbol{R}\mathbf{A} + \mathbf{\Psi}_{\xi^0}(V)$ in $\boldsymbol{N}$ and let V' be the orthogonal complement of V in $\mathfrak{m}_1$. By Lemma 18, we easily have $\mathbf{\Theta}_{Y_0,\xi^0}(V) \subset \boldsymbol{D}$. Therefore, to obtain η^0 satisfying the above condition, it suffices to find a non-zero solution $\eta = \eta^0 \in V$ of the system of linear homogeneous equations

$$
\langle \mathbf{\Theta}_{Y_0,\xi^0}(\eta),\ \boldsymbol{D}\rangle = \big(\big[[\xi^0,\ \eta],\ Y_0\big],\ V'\big) = 0. \tag{5.7}
$$

Since $\mathbf{Ker}(\mathbf{\Psi}_{\xi^0}) \cap \mathfrak{m}_1 = 0$ (see Corollary13 (1)) and $\langle \mathbf{A},\ \mathbf{\Psi}_{\xi^0}(\mathfrak{m}_1)\rangle = 0$ (see Proposition 16 (4)), we have $\dim(\boldsymbol{R}\mathbf{A} + \mathbf{\Psi}_{\xi^0}(V)) = 1 + \dim V \geq 7$. (Recall that we are assuming $V \subset \mathfrak{m}_1$ and $\dim V \geq 6$.) Hence, we have $\dim \boldsymbol{D} \leq \dim \boldsymbol{N} - 7 = 3$. Moreover, we have $\dim V' = 8 - \dim V \leq 2$. Consequently, the rank of the system (5.7) is less than or equal to 5. Therefore, we can find a non-zero solution $\eta^0 \in V$ of (5.7). Putting $\eta = \eta^0$ into (5.5), we obtain the equality (1). Further, putting (1) into (5.5), we have $\mathbf{\Theta}_{Y_0,\xi^0}(\eta) = 0$ for any $\eta \in V$ satisfying $\big[[\xi^0,\, \eta],\, Y_0\big] \in V$. □

Lemma 21 *The vectors* $\mathbf{A}$ *and* $\mathbf{B}$ *are linearly independent and* $\langle \mathbf{A},\, \mathbf{B}\rangle = 2(\mu,\, \mu)$, $C = (\mu,\, \mu)$.

Proof. Let $\xi \in U$ with $(\xi,\, \xi) = 1$. Since $\mathbf{\Psi}(\xi,\, \xi) = \mathbf{A}$ (see (5.3)), by putting $Y_0 = \xi$ into the equality in Lemma 20 (1), we easily have $\langle \mathbf{A},\, \mathbf{B}\rangle = (\mu,\, \mu)\{1 + (\mu,\, \mu)/C\}$. Since $C = \langle \mathbf{A},\, \mathbf{B}\rangle - (\mu,\, \mu)$, it immediately follows that $C^2 = (\mu,\, \mu)^2$. Since $C > 0$, we get $C = (\mu,\, \mu)$ and hence $\langle \mathbf{A},\, \mathbf{B}\rangle = 2(\mu,\, \mu)$. This, together with Proposition 16 (1), proves that $\mathbf{A}$ and $\mathbf{B}$ are linearly independent. □

These being prepared, we show Theorem 11.

Proof of Theorem 11. First we show that μ is singular with respect to any element $\mathbf{\Psi} \in \mathcal{G}_o(\boldsymbol{N})$. Suppose that there is an element $\mathbf{\Psi}_0 \in \mathcal{G}_o(\boldsymbol{N})$ such that μ is non-singular with respect to $\mathbf{\Psi}_0$. Then, $\mathbf{Ker}((\mathbf{\Psi}_0)_\mu)$ is a singular subspace with respect to $\mathbf{\Psi}_0$ and it satisfies $\dim \mathbf{Ker}((\mathbf{\Psi}_0)_\mu) = 6$ and $\mathbf{Ker}((\mathbf{\Psi}_0)_\mu) \subset \mathfrak{m}_2$ (see Proposition 12 and Proposition 14).

Now, set $\mathbf{\Psi} = \mathbf{\Psi}_0$ and $U = \mathbf{Ker}((\mathbf{\Psi}_0)_\mu)$ in Proposition 16. Let $\mathbf{A}$, $\mathbf{B}$ be the vectors of $\boldsymbol{N}$ satisfying (1)–(4) of Proposition 16. Let $\xi \in U = \mathbf{Ker}((\mathbf{\Psi}_0)_\mu)$ with $\xi \neq 0$. First, we show $\mathbf{B} \in (\mathbf{\Psi}_0)_\xi(\mathfrak{m})$. In fact, there is a non-zero element $Y_2^0 \in \mathfrak{m}_2$ satisfying $\mathbf{\Psi}_0(\mu,\, Y_2^0) \neq 0$ and $\boldsymbol{N} = \boldsymbol{R}\mathbf{\Psi}_0(\mu,\, Y_2^0) + (\mathbf{\Psi}_0)_\xi(\mathfrak{m})$ (orthogonal direct sum) (see Proposition 14). By Lemma 20 (1) and by the relation

$$\mathbf{\Psi}_0(\mu,\, Y_2^0) = \frac{1}{2}\Big(\mathbf{\Psi}_0(\mu + Y_2^0,\, \mu + Y_2^0) - \mathbf{\Psi}_0(\mu,\, \mu) - \mathbf{\Psi}_0(Y_2^0,\, Y_2^0)\Big),$$

we easily have $\langle \mathbf{\Psi}_0(\mu,\, Y_2^0),\, \mathbf{B}\rangle = 0$, which proves $\mathbf{B} \in (\mathbf{\Psi}_0)_\xi(\mathfrak{m})$. Since $(\mathbf{\Psi}_0)_\xi(\mathfrak{m}) = \boldsymbol{R}\mathbf{A} + (\mathbf{\Psi}_0)_\xi(\mathfrak{m}_1)$ (orthogonal direct sum) and $\langle \mathbf{B},\, (\mathbf{\Psi}_0)_\xi(\mathfrak{m}_1)\rangle = 0$ (see Proposition 16 (2), (4)), we have $\mathbf{B} \in \boldsymbol{R}\mathbf{A}$. This contradicts Lemma 21. Accordingly, we can conclude that μ is singular with respect to any element $\mathbf{\Psi} \in \mathcal{G}_o(\boldsymbol{N})$.

Now we show that any element of $\mathfrak{m}$ is singular with respect to any $\mathbf{\Psi} \in \mathcal{G}_o(\boldsymbol{N})$. Let Y be a non-zero element of $\mathfrak{m}$. Take an element $k \in K$ such that $\mathrm{Ad}(k)\mu \in \boldsymbol{R}Y$ and define $\mathbf{\Psi}' \in S^2\mathfrak{m}^* \otimes \boldsymbol{N}$ by

$$\boldsymbol{\Psi}'(Y', Y'') = \boldsymbol{\Psi}(\mathrm{Ad}(k)Y', \mathrm{Ad}(k)Y''), \quad Y', Y'' \in \mathfrak{m}.$$

Then, it is easily seen that $\boldsymbol{\Psi}' \in \mathcal{G}_o(\boldsymbol{N})$. Applying the arguments developed above, we know that μ is also singular with respect to $\boldsymbol{\Psi}'$. Note that $\boldsymbol{\Psi}'_\mu(\mathfrak{m}) = \boldsymbol{\Psi}_{\mathrm{Ad}(k)\mu}(\mathrm{Ad}(k)\mathfrak{m}) = \boldsymbol{\Psi}_Y(\mathfrak{m})$. Then, since $\boldsymbol{\Psi}'_\mu(\mathfrak{m}) \neq \boldsymbol{N}$, we have $\boldsymbol{\Psi}_Y(\mathfrak{m}) \neq \boldsymbol{N}$, implying that Y is singular with respect to $\boldsymbol{\Psi}$.

Accordingly, in Proposition 16 and in the discussion after it, we may allow to put $U = \mathfrak{a} + \mathfrak{m}_2$ and $V = \mathfrak{m}_1$. Therefore, by Proposition 16 and Lemma 21, we get (1) of Theorem 11. Further, putting $Y_0 = Y_2 \in \mathfrak{m}_2$, $\xi^0 = \mu$ and $\eta = Y_1$ into (5.6), we get (2) of Theorem 11. The assertion (3) of Theorem 11 follows from Lemma 17 (2) and Lemma 21. This completes the proof of the theorem. □

6. Proof of Theorem 10

Let $\{E_i\ (1 \leq i \leq 8)\}$ be an orthonormal basis of $\mathfrak{m}_1$. (Note that $\dim \mathfrak{m}_1 = 8$.) Let $\boldsymbol{\Psi} \in \mathcal{G}_o(\boldsymbol{N})$ and let $\mathbf{A}$, $\mathbf{B}$ be the vectors of $\boldsymbol{N}$ stated in Theorem 11. We define vectors $\{\mathbf{F}_i\ (1 \leq i \leq 10)\}$ of $\boldsymbol{N}$ by setting $\mathbf{F}_i = \boldsymbol{\Psi}(\mu, E_i)/(\mu, \mu)$ $(1 \leq i \leq 8)$, $\mathbf{F}_9 = (\mathbf{A}+\mathbf{B})/2\sqrt{3}|\mu|$ and $\mathbf{F}_{10} = (\mathbf{A}-\mathbf{B})/2|\mu|$. We now show that $\{\mathbf{F}_i\ (1 \leq i \leq 10)\}$ forms an orthonormal basis of $\boldsymbol{N}$. By Theorem 11 (3) we have $\langle \mathbf{F}_i, \mathbf{F}_j \rangle = \delta_{ij}$ $(1 \leq i, j \leq 8)$, where δ_{ij} denotes Kronecker's delta. Moreover, since $\langle \mathbf{A}, \mathbf{F}_i \rangle = \langle \mathbf{B}, \mathbf{F}_i \rangle = 0$ $(1 \leq i \leq 8)$ (see Theorem 11 (1*d*)), we have $\langle \mathbf{F}_9, \mathbf{F}_i \rangle = \langle \mathbf{F}_{10}, \mathbf{F}_i \rangle = 0$ $(1 \leq i \leq 8)$. The equalities $\langle \mathbf{F}_9, \mathbf{F}_9 \rangle = \langle \mathbf{F}_{10}, \mathbf{F}_{10} \rangle = 1$ and $\langle \mathbf{F}_9, \mathbf{F}_{10} \rangle = 0$ immediately follow from Theorem 11 (1*a*).

Now let $\boldsymbol{\Psi}'$ be another element of $\mathcal{G}_o(\boldsymbol{N})$. Let $\mathbf{A}'$ and $\mathbf{B}'$ be the vectors stated in Theorem 11 for $\boldsymbol{\Psi}'$. As in the case of $\boldsymbol{\Psi}$ we can also define an orthonormal basis $\{\mathbf{F}'_i\ (1 \leq i \leq 10)\}$ of $\boldsymbol{N}$. Then, there is an element $h \in O(10)$ satisfying $\mathbf{F}'_i = h\mathbf{F}_i$ $(1 \leq i \leq 10)$. Here we note that $\mathbf{A}' = h\mathbf{A}$, $\mathbf{B}' = h\mathbf{B}$ and $\boldsymbol{\Psi}'(\mu, E_i) = h\boldsymbol{\Psi}(\mu, E_i)$ $(1 \leq i \leq 8)$. Set $\boldsymbol{\Phi} = \boldsymbol{\Psi}' - h\boldsymbol{\Psi} \in S^2\mathfrak{m}^* \otimes \boldsymbol{N}$. Then, by Theorem 11 (1) we have

$$\boldsymbol{\Phi}(\mathfrak{a} + \mathfrak{m}_2, \mathfrak{a} + \mathfrak{m}_2) = \boldsymbol{\Phi}(\mathfrak{m}_1, \mathfrak{m}_1) = \boldsymbol{\Phi}(\mathfrak{a}, \mathfrak{m}_1) = 0.$$

By the fact $\big[[\mu, \mathfrak{m}_1], \mathfrak{m}_2\big] \subset \mathfrak{m}_1$ and Theorem 11 (2), we have

$$\boldsymbol{\Phi}(\mathfrak{m}_2, \mathfrak{m}_1) \subset \boldsymbol{\Phi}\big(\mu, \big[[\mu, \mathfrak{m}_1], \mathfrak{m}_2\big]\big) \subset \boldsymbol{\Phi}(\mathfrak{a}, \mathfrak{m}_1) = 0,$$

which proves that $\boldsymbol{\Phi}(\mathfrak{m}_2, \mathfrak{m}_1) = 0$. Therefore, we have $\boldsymbol{\Phi} = 0$, i.e., $\boldsymbol{\Psi}' = h\boldsymbol{\Psi}$. This implies that the Gaussian variety $\mathcal{G}_o(\boldsymbol{N})$ is EOS. This completes the proof of Theorem 10. □

References

[1] Agaoka Y., *Isometric immersions of SO(5)*. J. Math. Kyoto Univ. **24** (1984), 713–724.

[2] Agaoka Y. and Kaneda E., *On local isometric immersions of Riemannian symmetric spaces*. Tôhoku Math. J. **36** (1984), 107–140.

[3] Agaoka Y. and Kaneda E., *An estimate on the codimension of local isometric imbeddings of compact Lie groups*. Hiroshima Math. J. **24** (1994), 77–110.

[4] Agaoka Y. and Kaneda E., *Local isometric imbeddings of symplectic groups*. Geometriae Dedicata **71** (1998), 75–82.

[5] Agaoka Y. and Kaneda E., *Strongly orthogonal subsets in root systems*. Hokkaido Math. J. **31** (2002), 107–136.

[6] Agaoka Y. and Kaneda E., *A lower bound for the curvature invariant $p(G/K)$ associated with a Riemannian symmetric space G/K*. Hokkaido Math. J. **33** (2004), 153–184.

[7] Agaoka Y. and Kaneda E., *Local isometric imbeddings of $P^2(\boldsymbol{H})$ and $P^2(\boldsymbol{Cay})$*. Hokkaido Math. J. **33** (2004), 399–412.

[8] Agaoka Y. and Kaneda E., *Local isometric imbeddings of Grassmann manifolds*. in preparation.

[9] Allendoerfer C.B., *Rigidity for spaces of class greater than one*. Amer. J. Math. **61** (1939), 633–644.

[10] Bishop R.L. and Crittenden R.J., *Geometry of Manifolds*. Academic Press, New York, 1964.

[11] Cartan E., *Sur la possibilité de plonger un espace riemannien donné dans un espace euclidien*. Ann. Soc. Pol. Math. **6** (1927), 1–7.

[12] Helgason S., *Differential Geometry, Lie Groups, and Symmetric Spaces*. Academic Press, New York, 1978.

[13] Kaneda E., *On local isometric immersions of the spaces of negative constant curvature into the euclidean spaces*. J. Math. Kyoto Univ. **19** (1979), 269–284.

[14] Kaneda E., *On the Gauss-Codazzi equations*. Hokkaido Math. J. **19** (1990), 189–213.

[15] Kaneda E., *Types of the canonical isometric imbeddings of symmetric R-spaces*. Hokkaido Math. J. **22** (1993), 35–61.

[16] Kaneda E. and Tanaka N., *Rigidity for isometric imbeddings*. J. Math. Kyoto Univ. **18** (1978), 1–70.

[17] Kobayashi S., *Isometric imbeddings of compact symmetric spaces*. Tôhoku Math. J. **20** (1968), 21–25.

[18] Kobayashi S. and Nomizu K., *Foundations of Differential Geometry* II. Interscience Publishers, New York, 1969.

[19] Nomizu K., *Uniqueness of the normal connections and congruence of isometric immersions*. Tôhoku Math. J. **28** (1976), 613–617.

[20] Ôtsuki T., *Isometric imbedding of Riemann manifolds in a Riemann manifold*. J. Math. Soc. Japan **6** (1954), 221–234.
[21] Szczarba R.H., *On existence and rigidity of isometric immersions*. Bull. Amer. Math. Soc. **75** (1969), 783–787.
[22] Szczarba R.H., *On isometric immersions of Riemannian manifolds in euclidean space*. Bol. Soc. Brasil. Mat. **1** (2) (1970), 31–45.
[23] Tanaka N., *Rigidity for elliptic isometric imbeddings*. Nagoya Math. J. **51** (1973), 137–160.

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