Hokkaido Mathematical Journal Vol. 34 (2005) p. 315–330

Polysuperharmonic functions on a harmonic space

M. Al-Qurashi and V. Anandam

(Received April 14, 2003; Revised January 19, 2004)

Abstract. In the context of the axiomatic potential theory, we introduce the notions of polyharmonic functions and polypotentials on a Brelot harmonic space Ω. For these functions, we prove some results analogous to the Riesz decomposition, balayage, domination principle, etc., which are usually associated with harmonic and superharmonic functions on Ω. We also consider the polyharmonic classifications of the harmonic spaces.

Key words: m-harmonic functions, m-potential domains.

1. Introduction

The potential theoretic study of polyharmonic functions u defined by $\Delta^m u = 0$ on $\mathbb{R}^n$ covers different aspects of polysuperharmonic functions v defined by $(-\Delta)^i v \geq 0$ for $1 \leq i \leq m$, the existence of polypotentials, the generalized Liouville-Picard theorem, the analogue of the Laurent development for polyharmonic functions defined on an annulus, etc. This analysis is facilitated by the fact that the functions satisfying $(-\Delta)^i v \geq 0$ are δ-subharmonic almost everywhere and that the continuous functions u satisfying the condition $(-\Delta)^m u = 0$ have an Almansi representation.

We initiate in this note a similar study in the framework of the axiomatic potential theory. After defining polyharmonic functions on a Brelot harmonic space Ω, we introduce the notions of polysuperharmonic functions and polypotentials on the domains in Ω. Then, for these functions, we obtain certain results analogous to the Laurent decomposition, the Liouville-Picard theorem, the Riesz decomposition, balayage, domination principle which are usually associated with harmonic and superharmonic functions on Ω. Also we remark on the classification of the harmonic spaces Ω based on the existence of polypotentials on Ω.

2000 Mathematics Subject Classification : 31D05, 31B30.

2. Preliminaries

Smyrnélis [12] has developed an axiomatic theory for biharmonic functions on a locally compact space X. A pair of continuous functions (h_1, h_2) defined on a domain ω in X is called biharmonic if h_1 and h_2 satisfy locally some mean value property related to the solution of the Riquier problem in $\mathbb{R}^n$. The space X along with this sheaf of biharmonic functions is called a biharmonic space if an axiom of regularity, an axiom of convergence and an axiom of separability are satisfied. Along with these hypotheses, it is assumed that there exists a special pair (p_1, p_2) of potentials on X. Consequently, such a biharmonic space resembles $\mathbb{R}^n$, $n \geq 5$, and the functions studied in this frame work on X are generlizations of the smooth functions u in $\mathbb{R}^n$ satisfying the condition $(-\Delta)^j u \geqslant 0$ for $j = 0$, 1 and 2 rather than the larger class of functions v satisfying the only condition $(-\Delta)^2 v \geqslant 0$. In such a space he obtained many results related to Riezs decomposition, balayage, and domination principle in the axiomatic case of a harmonic space with potentials > 0. However, this axiomatic set-up, specially devised to extend the study of biharmonic functions in $\mathbb{R}^n$, $n \geq 5$, to a locally compact space X, does not easily yield to the investigation of polyharmonic functions of order $m > 2$ and the associated polyharmonic classification theory in X.

For this purpose, we work here on a locally compact space Ω which is a harmonic space where the converse to the local Riesz representation of positive superharmonic functions is valid. In Ω, a polyharmonic function is a δ-superharmonic function by definition and hence may not necessarily be continuous. This allows a certain generality to the study of polyharmonic functions on Ω. (It is not rather easy to verify whether a Riemann surface R is a biharmonic space in the sense of Smyrnélis since the Laplacian Δ is not an invariant operator under a parametric change on R; see the remark on Sario et al. [11, p. 6]. However adding some assumptions occasionally, we can see that a polyharmonic space Ω of order 2 is also a biharmonic space in the sense of [12]).

Let Ω be a locally compact space with a countable base provided with a sheaf H of harmonic functions satisfying the axioms 1, 2 and 3 of Brelot [6, pp. 13–14]. Fix a Radon measure λ on Ω such that each superharmonic function on a domain ω in Ω is locally λ-integrable. Such measures can be constructed by using the harmonic measures on Ω (see [3]). Let us assume also that the axiom of local proportionality (see [6, p. 40]) and the axiom

A^* of quasi-analyticity (see De La Paradelle [8, p. 391]) are verified on Ω, and the constants are harmonic on Ω. With these restrictions we call $\Omega = (\Omega, H, \lambda)$ a *harmonic space.*

Examples of harmonic spaces

1) A Riemannian manifold R, where the harmonic functions are defined by means of the Laplace-Beltrami operator Δ, is a harmonic space; here we take $d\lambda$ as the volume measure.
2) A Riemann surface with the usual definition of harmonic functions.
3) The Euclidean space $\mathbb{R}^n$, $n \geq 1$, with $d\lambda$ as the Lebesgue measure.
4) A domain Ω in $\mathbb{R}^n$, $n \geq 2$, with harmonic functions defined by means of C^2-solutions of a second order elliptic differential operator

$$Lu = \sum a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum b_i \frac{\partial u}{\partial x_i}$$

with locally Lipschitz coefficients, as given in Mme.R.M.Herv é [7, pp. 560–563] and $d\lambda$ as the Lebesgue measure.

We start with the following lemma (originally proved in the classical case $\mathbb{R}^n$ by Brelot [5]; see also Arsov [4]), proved by using an approximation lemma given in De La Pradelle [8, Théorème 10].

Lemma 2.1 ([1, Theorem 4.2]) *Let μ be a positive Radon measure on an open set ω in a harmonic space $\Omega = (\Omega, H, \lambda)$. Then there exists a superharmonic function s on ω such that μ is the measure associated with s in a local Riesz representation.* (*We represent this correspondence by the equation $(-L)s = \mu$ on ω.*)

As a consequence, if f is a locally $d\lambda$-integrable function on an open set ω in Ω, there exists a δ-superharmonic function u on ω with the associated signed measure $f d\lambda$. We represent this as $(-L)u = f$ on ω. Since we are assuming that each superharmonic function is locally $d\lambda$-integrable, if u is a δ-superharmonic function on ω, then there exists a δ-superharmonic function v on ω such that $(-L)v = u$ on ω.

3. Polysuperharmonic functions

In this section, we define polyharmonic and polysuperharmonic functions on a domain ω in a harmonic space $\Omega = (\Omega, H, \lambda)$; the Laurent decomposition theorem and the Liouville-Picard theorem are proved for polyharmonic functions; and the notion of the greatest polyharmonic minorant of a polysuperharmonic function is made precise.

Definitions 3.1 1) Let $(u_i)_{m\geq i\geq 1}$ be m functions defined on an open set ω in a harmonic space $\Omega = (\Omega, H, \lambda)$ such that $(-L)u_{j+1} = u_j$, $1 \leqslant j \leqslant m-1$. We say that $u = (u_i)_{m\geq i\geq 1}$ *is a polysuperharmonic function of order* m or shortly m*-superharmonic* (resp. m*-subharmonic*, resp. m*-harmonic*) if u_1 is superharmonic (resp. subharmonic, resp. harmonic).
2) Given a superharmonic function s on ω, by using Lemma 2.1, we can construct an m-superharmonic function $u = (u_i)_{m\geq i\geq 1}$ on ω such that $u_1 = s$. We say that u is *generated by* s.
3) If $u = (u_i)_{m\geq i\geq 1}$ is m-superharmonic on ω, the harmonic support of u_1 is called the m*-harmonic support* of u. Let $v = (v_i)_{m\geq i\geq 1}$ be another such function. We say that $u \geq v$ if and only if $u_i \geq v_i$ for every i. In particular, $u \geq 0$ if and only if $u_i \geqslant 0$ for every i.

Theorem 3.2 *In a harmonic space* Ω*, let* ω *be an open set and* K *a compact set* $\subset \omega$*. Let* $h = (h_i)_{m\geq i\geq 1}$ *be an* m*-harmonic function on* $\omega\backslash K$*. Then there exists an* m*-harmonic function* s *on* $\Omega\backslash K$ *and an* m*-harmonic function* t *on* ω *such that* $h = s - t$ *on* $\omega\backslash K$*.*

Proof. Since h_1 is harmonic on $\omega\backslash K$, by [2, Lemma 5] there exist a harmonic function s_1 on $\Omega\backslash K$ and a harmonic function t_1 on ω such that $h_1 = s_1 - t_1$ on $\omega\backslash K$. Let $(-L)f = s_1$ on $\Omega\backslash K$ and $(-L)g = t_1$ on ω, so that $(-L)h_2 = h_1 = (-L)f - (-L)g$ on $\omega\backslash K$. Hence $h_2 = f - g +$ (a harmonic function H_2) on $\omega\backslash K$. Then as above, we write $H_2 = u_2 - v_2$ on $\omega\backslash K$ where u_2 is harmonic on $\Omega\backslash K$ and v_2 is harmonic on ω. Write $s_2 = f + u_2$ and $t_2 = g + v_2$ so that s_2 is defined on $\Omega\backslash K$ such that $(-L)s_2 = s_1$ and t_2 is defined on ω such that $(-L)t_2 = t_1$ on ω. Note $h_2 = s_2 - t_2$ on $\omega\backslash K$.

Proceeding in the same way, we construct $s = (s_i)_{m\geq i\geq 1}$ on $\Omega\backslash K$ and $t = (t_i)_{m\geq i\geq 1}$ on ω such that $h = s - t$ on $\omega\backslash K$. Since s_1 and t_1 are harmonic, s is m-harmonic on $\Omega\backslash K$ and t is m-harmonic on ω. □

Corollary 3.3 *Let* $u = (u_i)_{m\geq i\geq 1}$ *be an* m*-superharmonic function defined outside a compact set in* Ω*. Then there exist an* m*-superharmonic function* $s = (s_i)_{m\geq i\geq 1}$ *on* Ω *and an* m*-harmonic function* $h = (h_i)_{m\geq i\geq 1}$ *outside a compact set, such that* $u = s + h$ *outside a compact set in* Ω*.*

Proof. Given the superharmonic function $u_1$outside a compact set in Ω, by using the Dirichlet solution, we can assume that $u_1$is harmonic on $\omega\backslash K$, where K is a compact set in a relatively compact open set ω. Then, by using the above Laurent decoposition we can see that there exist a harmonic function h_1 on $\Omega\backslash K$ and a harmonic function t on ω such that $u_1 = h_1 - t$ on $\omega\backslash K$. Define $s_1 = u_1 - h_1$ on $\Omega\backslash K$ and $= -t$ on ω. Then s_1 is superharmonic on Ω and $u_1 = s_1 + h_1$ on $\Omega\backslash K$.

Let $(-L)s'_2 = s_1$ and $(-L)h'_2 = h_1$. Then $(-L)u_2 = u_1 = (-L)s'_2 + (-L)h'_2$ on $\Omega\backslash K$, so that $u_2 = s'_2 + h'_2 + v$ on $\Omega\backslash K$ where v is harmonic. Write $v = f + g$ outside a compact set, where f is harmonic on Ω and g is harmonic outside a compact set. Write $s_2 = s'_2 + f$ and $h_2 = h'_2 + g$. Then $(-L)s_2 = s_1$ on Ω and $(-L)h_2 = h_1$ outside a compact set; moreover, $u_2 = s_2 + h_2$ on $\Omega\backslash K$.

Proceeding similarly, construct s_3 and h_3 such that $(-L)s_3 = s_2$ on Ω and $(-L)h_3 = h_2$ outside a compact set; moreover, $u_3 = s_3 + h_3$ on $\Omega\backslash K$. This method leads to the construction of an m-superharmonic function $s = (s_i)_{m\geq i\geq 1}$ on Ω and an $m-$harmonic function $h = (h_i)_{m\geq i\geq 1}$ outside a compact set such that $u = s + h$ outside a compact set in Ω. □

Remark If we place some restrictions on the harmonic space Ω, the decompositions in the above theorem and corollary can be expressed in a unique fashion (by using Theorem 4.14).

The classical Liouville-Picard theorem states that every positive harmonic function on $\mathbb{R}^n$, $n \geq 2$, is a constant. As a consequence, there does not exist any positive locally integrable function u on $\mathbb{R}^n$ such that $(-\Delta)u = 1$ in the sense of distributions. For, since $\Delta(|x|^2) = 2n$, if $(-\Delta)u = 1$, then we should have $u(x) = -(|x|^2)/(2n) + h(x)$ a.e., where $h(x)$ is harmonic on $\mathbb{R}^n$; if $u \geq 0$ also, then $h(x) \geq (|x|^2)/(2n)$ and hence h is a constant, not possible.

Theorem 3.4 *The following are equivalent in* Ω:

1) *For any $m \geq 1$, a positive m-harmonic function $u = (u_i)_{m\geq i\geq 1}$ is a constant $u = (\alpha, 0, \ldots, 0)$.*
2) *Every positive harmonic function on Ω is a constant and there is no function $v \geq 0$ such that $(-L)v = 1$ on Ω.*

Proof. 1) $\Longrightarrow$ 2) Let $h \geq 0$ be harmonic on Ω. Since h is 1-harmonic, by (1), h is a constant. Now, suppose that there is a function $v > 0$ on Ω such that $(-L)v = 1$. Then $(v, 1)$ is a 2-harmonic function > 0. Hence by (1), $(v, 1)$ must be a constant of the form $(\alpha, 0)$, a contradiction.

2) $\Longrightarrow$ 1) Let $u = (u_i)_{m \geq i \geq 1}$ be a positive m-harmonic function on Ω. Then, u_1 is harmonic ≥ 0, so that u_1 is a constant $c \geq 0$. Suppose $c > 0$; then $(-L)u_2 = u_1 = c$ so that $(-L)v = 1$ if $v = (1/c)u_2$, a contradiction. Hence $c = 0$, that is $u_1 = 0$ so that u_2 is harmonic. Since $u_2 \geq 0$, it should be a constant. Proceeding as above, we should have $u_2 \equiv 0$, then $u_3 \equiv 0, \ldots,$ and $u_{m-1} \equiv 0$. Consequently, since $(-L)u_m = 0$, u_m is harmonic; also since $u_m \geq 0$, it is a constant $\alpha \geq 0$. Thus, $u = (\alpha, 0, \ldots, 0)$. □

Corollary 3.5 (Liouville-Picard theorem for polyharmonic functions on $\mathbb{R}^n$) *In $\mathbb{R}^n$, $n \geq 2$, if u is a locally integrable function such that $(-\Delta)^i u \geq 0$ for $0 \leq i \leq m-1$ and $(-\Delta)^m u = 0$, then u is a constant in the sense of distributions.*

Remark The above corollary in $\mathbb{R}^n$ can be deduced also from the results of Nicolescu [10, pp. 16–17].

Theorem 3.6 *Let s be an m-superharmonic function on a domain ω in Ω, and let t be an m-subharmonic function on ω such that $t \leq s$ on ω. Then there exists an m-harmonic function h on ω such that $t \leq h \leq s$ on ω.*

Proof. Let $s = (s_i)_{m \geq i \geq 1}$ be an m-superharmonic function on ω and $t = (t_i)_{m \geq i \geq 1}$ be an m-subharmonic function on ω such that $t \leq s$ on ω. Let h_1 be the greatest harmonic minorant of s_1 on ω so that $s_1 \geq h_1 \geq t_1$. Let $(-L)H_2 = h_1$; choose f_2 and g_2 such that $(-L)f_2 = s_1 - h_1$ and $(-L)g_2 = t_1 - h_1$. Then f_2 is superharmonic and g_2 is subharmonic such that $(-L)s_2 = s_1 = (-L)f_2 + (-L)H_2$ and $(-L)t_2 = t_1 = (-L)g_2 + (-L)H_2$. Consequently, $s_2 = f_2 + H_2 +$ (a harmonic function); write $s_2 = f_2' + H_2$, where f_2' is a superharmonic function on Ω. Similarly, write $t_2 = g_2' + H_2$, where g_2' is a subharmonic function on Ω. Since $s_2 \geq t_2$ by hypothesis, $f_2' \geq g_2'$. Let u be the greatest harmonic minorant of f_2' so that $f_2' \geq u \geq g_2'$. Define $h_2 = H_2 + u$. Then $(-L)h_2 = h_1$ and $s_2 \geq h_2 \geq t_2$.

Remark that if h_2' is such that $(-L)h_2' = h_1' \leq h_1$ and $s_2 \geq h_2' \geq t_2$, then $h_2 \geq h_2'$. For, in this case $(-L)h_2' = h_1' \leq (-L)H_2$ so that $h_2' = H_2 +$ (a subharmonic function v). This implies $f_2' + H_2 = s_2 \geq h_2' = H_2 + v$ so that v is a subharmonic minorant of f_2'. Since u is the greatest harmonic minorant of f'_2, $u \geq v$. Consequently, $h'_2 = H_2 + v \leq H_2 + u = h_2$.

Proceeding in the same way, we construct $h = (h_i)_{m \geq i \geq 1}$ which is an m-harmonic function such that $t \leq h \leq s$ on ω. This function h has the additional property that if h' is any m-harmonic function on ω such that $t \leq h' \leq s$, then $h' \leq h$. □

Remark The m-harmonic function h on ω constructed as above such that $t \leq h \leq s$ is called *the greatest m-harmonic minorant of s on ω.*

4. Polypotentials

In this section, we define polypotentials on domains ω in a harmonic space; they are needed while considering the Riesz decomposition, the balayage and the domination principle associated with positive m-harmonic functions on ω. We use the expression "near infinity in ω" to mean "outside a compact set in ω".

Definition 4.1 An m-superharmonic function defined on a domain ω in Ω is said to be a *polypotential of order m or simply an m-potential* if its greatest m-harmonic minorant on ω is 0. If there exists an m-potential > 0 on ω, we say that ω *is an m-potential domain.*

Theorem 4.2 *An m-superharmonic function $u = (u_i)_{m \geq i \geq 1}$ on a domain ω is an m-potential if and only if each u_i is a potential on ω.*

Proof. First note that since $u \geq 0$, each u_i is a positive superharmonic function.

1) Let the m-superharmonic function u be an m-potential, that is the greatest m-harmonic minorant of u is 0. Then, u_i is a potential for all i, $1 \leq i \leq m$. For otherwise, let i be the smallest index such that u_i is not a potential. If $i = m$, let $h_m > 0$ be the greatest harmonic minorant of u_m on ω. Then $h = (h_m, 0, \ldots, 0)$ is m-harmonic on ω and $0 \leq h \leq u$, a contradiction.

Suppose $i < m$. Let $h_i > 0$ be the greatest harmonic minorant of u_i on ω. Let $(-L)s_{i+1} = h_i$ and $(-L)t_{i+1} = u_i - h_i$ on ω. Since $(-L)u_{i+1} = u_i$, $u_{i+1} = s_{i+1} + t_{i+1} +$ (a harmonic function H) on ω. Note that s_{i+1} and t_{i+1} are superharmonic functions on ω and $u_{i+1} \geq 0$, so that s_{i+1} and t_{i+1} have subharmonic minorants. Let h_{i+1} and t'_{i+1} be the potential parts of s_{i+1} and t_{i+1} in the Riesz decomposition. We can then write $u_{i+1} = h_{i+1} + t'_{i+1} +$ (a harmonic function H') on ω. Since $-H' \leq h_{i+1} + t'_{i+1}$, $-H' \leq 0$, so that $h_{i+1} \leq u_{i+1}$ on ω and $(-L)h_{i+1} = (-L)s_{i+1} = h_i$.

Proceeding in the same way, we obtain $h = (h_m, \ldots, h_i, 0, \ldots, 0)$ on ω which is m-harmonic, nonnegative and $h \leq u$, a contradiction. Consequently, for each i, u_i is a potential on ω.

2) Conversely, suppose that each term in the m-superharmonic function $u = (u_i)_{m \geq i \geq 1}$ is a potential on ω. Clearly 0 is an m-harmonic minorant of u on ω. Suppose $h = (h_i)_{m \geq i \geq 1}$ is its greatest m-harmonic minorant. Then, since $0 \leq h_1 \leq u_1$ and u_1 is a potential, $h_1 \equiv 0$. This implies that h_2 is harmonic on ω, since $(-L)h_2 = h_1$. Again $0 \leq h_2 \leq u_2$, so that $h_2 \equiv 0$. Proceeding similarly, we show that each $h_i \equiv 0$, so that 0 is the greatest m-harmonic minorant of u; that is, u is an m-potential on ω. □

Corollary 4.3 *If $n \geq 2m + 1$, $\mathbb{R}^n$ is an m-potential domain.*

Proof. Let $p = (p_i)_{m \geq i \geq 1}$ where $(-\Delta)p_{j+1} = p_j$ for $1 \leq j \leq m - 1$ and $p_m(x) = |x|^{2m-n}$. Then p is an m-superharmonic function and each p_i is a potential. Hence p is an m-potential on $\mathbb{R}^n$, $n \geq 2m + 1$. □

Remarks 1) If $n \leq 2m$, $\mathbb{R}^n$ is not an m-potential domain (see Corollary 4.15).

2) A 2-potential (called a bipotential) corresponds to an $\mathcal{H}$-potential defined by Smyrnélis [12, Definition 5.9, p. 77].

Proposition 4.4 *Let $p = (p_i)_{m \geq i \geq 1}$ be an m-potential on a domain ω and $v = (v_i)_{m \geq i \geq 1}$ be an m-subharmonic function such that $v \leq p$ on ω, then $v \leq 0$.*

Proof. By the above Theorem 4.2, each p_i is a potential on ω. Since v_1 is subharmonic and $v_1 \leq p_1$, we should have $v_1 \leq 0$. Since $(-L)v_2 = v_1 \leq 0$, v_2 is a subharmonic function and $v_2 \leq p_2$; hence, we have $v_2 \leq 0$. Proceeding similarly, we show that each $v_i \leq 0$. □

Theorem 4.5 *An m-superharmonic function $s \geq 0$ on a domain ω in Ω is the unique sum of an m-potential p and an m-harmonic function h on ω.*

Proof. Since $s \geq 0$, we can find its greatest m-harmonic minorant h on ω (Theorem 3.6). Write $s - h = p = (p_i)_{m \geq i \geq 1}$. Since $p_1 = s_1 - h_1$ is a potential, and $(-\Delta)p_{j+1} = p_j$ for $1 \leq j \leq m-1$, p is an m-superharmonic function whose greatest m-harmonic minorant is 0. Hence p is an m-potential on ω and $s = p + h$. The uniqueness of the decomposition is a consequence of the above Proposition 4.4. □

Lemma 4.6 *Let u be a subharmonic function defined outside a compact set in a harmonic space Ω having a potential > 0. Then there exist a finite continuous potential p on Ω with compact harmonic support and a subharmonic function v on Ω, such that $u = v + p$ outside a compact set.*

Proof. Let u be defined outside a compact set A in Ω. Let ω be a relatively compact domain such that $A \subset \omega \subset \Omega$. By taking the Dirichlet solution on $\omega \backslash A$ with boundary value u, we can assume that u is harmonic on $\omega \backslash A$. By [2, Lemma 5], $u = s - t$ on $\omega \backslash A$, where t is harmonic on ω and s is harmonic on $\Omega \backslash A$ such that $s = Bs$ on $\Omega \backslash K$ for a suitable compact set $K \supset \mathring{K} \supset A$; here Bs stands for the Dirichlet solution on $\Omega \backslash K$ with boundary value s on ∂K and 0 at infinity.

If $u_1 = u - s$ on $\Omega \backslash A$ and $= -t$ on ω, then u_1 is subharmonic on Ω such that $u = u_1 + s$ on $\Omega \backslash K$. Since $s = Bs$ is harmonic on $\Omega \backslash K$, by [2, Lemma 6], $s = p_1 - p_2$ near infinity, where p_1 and p_2 are bounded continuous potentials on Ω with compact harmonic support. Consequently, $u = u_1 + p_1 - p_2$ near infinity. Now write $v = u_1 - p_2$ and $p = p_1$ to obtain the decomposition stated in the Lemma. □

Theorem 4.7 *Let $Q = (Q_i)_{m \geq i \geq 1}$ be an m-potential on a domain ω in Ω. Let p_1 be a potential on ω such that $p_1 \leq \alpha_1 Q_1$ outside a compact set in ω. Then p_1 generates a unique m-potential $p = (p_i)_{m \geq i \geq 1}$ on ω; moreover, $p_i \leq \beta Q_i$ outside a compact set in ω, for some constant β and all i.*

Proof. Let $(-L)u = p_1$ and $(-L)v = \alpha_1 Q_1 - p_1$ on ω. Then u is a superharmonic function on ω and v is a superharmonic function outside a compact set in ω. Hence, $u + v = \alpha_1 Q_2 +$ (a harmonic function) outside a compact set, since $(-L)Q_2 = Q_1$ on ω. This implies that u has a subharmonic minorant outside a compact set. Write $u = p_2 + h$ where p_2 is a potential on ω and h is (not necessarily positive) harmonic on ω.

Thus, $(-L)p_2 = p_1$ and $p_2 = \alpha_1 Q_2 +$ (a subharmonic function) outside a compact set in ω. Then by the above Lemma 4.6, $p_2 = \alpha_1 Q_2 + s + q$ outside a compact set in ω, where s is subharmonic on ω and q is a finite continuous potential with compact harmonic support. Since $s \leq p_2$ outside a compact set, $s \leq 0$ on ω; and since q is a finite continuous potential with compact harmonic support, $q \leq c_1 Q_2$ for some $c_1 > 0$. Thus, if $\alpha_2 = \alpha_1 + c_1$, then $p_2 \leq \alpha_2 Q_2$ outside a compact set in ω.

We repeat the above procedure to obtain a potential p_3 on ω such that

$(-L)p_3 = p_2$ and $p_3 \le \alpha_3 Q_3$ outside a compact set in ω. Continuing in the same way, we arrive at $p = (p_i)_{m \ge i \ge 1}$ which is an m-potential on ω such that $p_i \le \beta Q_i$ outside a compact set for every i, if $\beta = \max_{1 \le i \le m} \alpha_i$.

To show the uniqueness, assume that $q = (q_i)_{m \ge i \ge 1}$ is another m-potential on ω such that $p_1 = q_1$. Then, $(-L)p_2 = (-L)q_2$ on ω, so that $p_2 = q_2 +$ (a harmonic function h) on ω. But as p_2 and q_2 are potentials on ω, $h \equiv 0$. Proceeding similarly, we see that $p_i = q_i$ for every i. □

Corollary 4.8 *Let p_1 be a potential with compact harmonic support on an m-potential domain ω in Ω. Then, p_1 generates an m-potential $p = (p_i)_{m \ge i \ge 1}$ on ω.*

Proof. Let $Q = (Q_i)_{m \ge i \ge 1}$ be an m-potential on ω. Since p_1 has compact harmonic support, $p_1 \le \alpha_1 Q_1$ outside a compact set in ω. Hence we can apply the above Theorem 4.7. □

Corollary 4.9 *If ω is an m-potential domain, then for any z in ω, there exists an m-potential $p = (p_i)_{m \ge i \ge 1}$ on ω with point m-harmonic support at z.*

Proof. Choose a potential p_1 on ω with point m-harmonic support at z. Let $p = (p_i)_{m \ge i \ge 1}$ be the m-potential generated by p_1 on ω. Then the m-harmonic support of p is $\{z\}$. □

Suitably modifying the proof of Theorem 4.7, we prove the following theorem:

Theorem 4.10 *Let $s = (s_i)_{m \ge i \ge 1}$ be a positive m-superharmonic function on a domain ω in Ω. Let v_1 be a positive superharmonic function (resp. a potential) on ω such that $v_1 \le s_1$. Then, v_1 generates a positive m-superharmonic function (resp. an m-potential) $v = (v_i)_{m \ge i \ge 1}$ on ω such that $v \le s$.*

Proof. Let $(-L)u_2 = s_1 - v_1$ and $(-L)v_2 = v_1$. Then $(-L)s_2 = s_1 = (-L)u_2 + (-L)v_2$, so that $s_2 = u_2 + v_2 +$ (a harmonic function h_2) on ω. Since $s_2 \ge 0$, v_2 has a subharmonic minorant on ω and hence is a potential up to an additive harmonic function; u_2 also has a similar property, so that without loss of generality we can assume that u_2 and v_2 are potentials in the equation $s_2 = u_2 + v_2 + h_2$. Note $-h_2 \le u_2 + v_2$, which implies that $-h_2 \le 0$, so that $v_2 \le s_2$ and $(-L)v_2 = v_1$.

Proceeding similarly, we can construct potentials v_j, $2 \le j \le m$, such that $v_j \le s_j$ and $(-L)v_j = v_{j-1}$. Consequently, $v = (v_i)_{m\ge i\ge 1}$ is an m-superharmonic function ≥ 0 on ω *such that* $v \le s$. Note that if v_1 is a potential on ω, v is an m-potential. $\square$

Corollary 4.11 (Balayage) *Let* $s = (s_i)_{m\ge i\ge 1}$ *be a positive* m*-superharmonic function on a domain* ω *in* Ω*. Let* e *be a set in* ω*. Then there exists a positive* m*-superharmonic function* v *on* ω *such that* $v \le s$ *on* ω, $v = s+$ (*an* $(m-1)$*-harmonic function*) *on* $\mathring{e}$ *and* v *is* m*-harmonic on* $\omega\backslash\bar{e}$*. Moreover,* v *is an* m*-potential if* s *is an* m*-potential or if* e *is relatively compact in* ω*.*

Proof. Take $v_1 = \hat{R}^e_{s_1}$ on ω and construct $v = (v_i)_{m\ge i\ge 1}$ as in the above theorem. Since $v_1 = s_1$ on $\mathring{e}$, u_2 in the above construction is harmonic on $\mathring{e}$ so that v_2 equals s_2 up to an additive harmonic function on $\mathring{e}$. This means that $v = (v_i)_{m\ge i\ge 1}$ equals s up to an additive $(m-1)$-harmonic function on $\mathring{e}$. Also since v_1 is harmonic on $\omega\backslash\bar{e}$, v is m-harmonic on $\omega\backslash\bar{e}$. (Here we are identifying any $(m-1)$-harmonic function $(h_{m-1}, \ldots, h_1)$ as an m-harmonic function $(h_{m-1}, \ldots, h_1, 0)$.)

Moreover, if s is an m-potential (or more generally if only s_1 is a potential) or if e is relatively compact in ω, then v_1 is a potential on ω, and hence $v = (v_i)_{m\ge i\ge 1}$ is an m-potential on ω. $\square$

Note For Smyrnélis' definition of $\mathcal{H}$-balayage of a positive $\mathcal{H}$-superharmonic function (s_1, s_2), see [12, p. 73].

Theorem 4.12 *Let* $s = (s_i)_{m\ge i\ge 1}$ *be a positive* m*-superharmonic function and* $p = (p_i)_{m\ge i\ge 1}$ *be an* m*-potential on a domain* ω *in* Ω*. If* $s_1 \ge p_1$*, then* $s \ge p$*.*

Proof. Let $(-L)u_2 = s_1 - p_1$. Then $(-L)s_2 = s_1 = (-L)u_2 + (-L)p_2$ which implies that $s_2 = u_2 + p_2 +$ (a harmonic function h_2) on ω. Since $s_2 \ge 0$, u_2 has a subharmonic minorant on ω and hence is the sum of a potential and a harmonic function. Without loss of generality, we assume that u_2 is a potential in the expression $s_2 = u_2 + p_2 + h_2$. Then, $-h_2 \le u_2 + p_2$ implies that $-h_2 \le 0$ so that $s_2 \ge p_2$. Proceeding similarly, we find that $s_i \ge p_i$ for all i, that is $s \ge p$. $\square$

Corollary 4.13 (Domination Principle) *Suppose that the axiom D* (*see Brelot* [6, p. 65]) *is satisfied in the harmonic space* Ω*. Let* $s = (s_i)_{m\ge i\ge 1}$ *be a positive* m*-superharmonic function on* ω *and* $p = (p_i)_{m\ge i\ge 1}$ *be an* m*-potential on* ω*. Suppose* p_1 *is locally bounded and* $s_1 \ge p_1$ *on the harmonic* *support of* p_1. *Then,* $s \geq p$.

Theorem 4.14 *Let* Ω *be an* m*-potential domain. Then, given any* m*-harmonic function* $h = (h_i)_{m \geq i \geq 1}$ *outside a compact set, there exist a unique* m*-harmonic function* $H = (H_i)_{m \geq i \geq 1}$ *on* Ω *and a potential* $q > 0$ *on* Ω *such that* $|h_i - H_i| \leq q$ *outside a compact set for every* i.

Proof. Since h_1 is harmonic outside a compact set, there exist a harmonic function H_1 on Ω and a potential p_1 with compact harmonic support such that $|h_1 - H_1| \leq p_1$ near infinity. Let $(-L)H_2' = H_1$ on Ω. Let $p = (p_i)_{m \geq i \geq 1}$ be the m-potential on Ω generated by p_1.

Since $-(-L)p_2 \leq (-L)h_2 - (-L)H_2' \leq (-L)p_2$ near infinity, there exist a superharmonic function s_2 and a subharmonic function t_2 outside a compact set such that $h_2 - H_2' = p_2 + t_2$ and $h_2 - H_2' = -p_2 + s_2$ near infinity. Consequently, $t_2 \leq s_2$, and hence there exists a harmonic function u_2 near infinity such that $t_2 \leq u_2 \leq s_2$; also, there exists a harmonic function v_2 on Ω such that $|u_2 - v_2| \leq p_2'$ near infinity, where p_2' is a potential with compact harmonic support on Ω. Consequently, $-p_2 - p_2' \leq h_2 - H_2' - v_2 \leq p_2 + p_2'$ near infinity.

Write $H_2 = H_2' + v_2$, so that $(-L)H_2 = H_1$ on Ω. Let $(-L)H_3' = H_2$. Since p_2' is a potential with compact harmonic support, there exists a potential p_3' (as a consequence of Corollary 4.8) such that $(-L)p_3' = p_2'$ on Ω. Since $(-L)p_3 = p_2$, $(-L)(p_3 + p_3') = p_2 + p_2'$. Consequently, if we write $q_2 = p_2 + p_2'$, we have $|h_2 - H_2| \leq q_2$ near infinity, where q_2 is a potential such that q_2 generates a potential (since $q_2 = (-L)(p_3 + p_3')$).

Then, proceeding as above, we find H_3 on Ω such that $(-L)H_3 = H_2$ and $|h_3 - H_3| \leq q_3$ near infinity where q_3 generates a potential on Ω. Continuing thus, we obtain an m-harmonic function $H = (H_i)_{m \geq i \geq 1}$ on Ω such that for every i, $|h_i - H_i| \leq q$ near infinity, where $q = p_1 + q_2 + \cdots + q_m$ is a potential on Ω.

To prove the uniqueness of H, suppose $H' = (H_i')_{m \geq i \geq 1}$ is an m-harmonic function on Ω such that $|h_i - H_i'| \leq q'$ near infinity for some potential q' on Ω. Then, since H_1 and H_1' are harmonic on Ω and $|H_1 - H_1'| \leq q + q'$ near infinity, we have $H_1 - H_1' \equiv 0$. Consequently, $H_2 - H_2'$ is harmonic on Ω and since $|H_2 - H_2'| \leq q + q'$ near infinity, $H_2 - H_2' \equiv 0$ and so on. Thus $H = H'$ on Ω. □

Corollary 4.15 *For* $n \leq 2m$, $\mathbb{R}^n$ *is not an* m*-potential domain.*

Proof. Let $h(x) = |x|^{2m-n}$ if n is odd and $= |x|^{2m-n} \log |x|$ if n is even. Then $(h, (-\Delta)h, \ldots, (-\Delta)^{m-1}h)$ is an m-harmonic function on $\mathbb{R}^n \backslash \{0\}$.

Suppose $\mathbb{R}^n$ is an m-potential domain. Then by the above Theorem 4.14, there exist an m-harmonic function $(H, (-\Delta)H, \ldots, (-\Delta)^{m-1}H)$ on $\mathbb{R}^n$ and a potential q on $\mathbb{R}^n$ such that $|H - h| \leq q$ near infinity.

Fix a in $\mathbb{R}^n$ and let $d\rho_a^r(x)$ be the harmonic measure on $|x| = r > |a|$. Then $\int q(x) d\rho_a^r(x) \longrightarrow 0$ as $r \longrightarrow \infty$. But $\int (H - h) d\rho_a^r(x)$ does not tend to zero when $r \longrightarrow \infty$. For, $H(x)$ is of the form $H(x) = \sum_{i=0}^{m-1} |x|^{2i} h_i(x)$, where h_i is harmonic on $\mathbb{R}^n$ so that $\int (H - h) d\rho_a^r(x) = \sum_{i=0}^{m-1} r^{2i} h_i(a) + r^{2m-n}$ if n is odd, and $= \sum_{i=0}^{m-1} r^{2i} h_i(a) + r^{2m-n} \log r$ if n is even. Thus in any case $\int (H - h) d\rho_a^r(x)$ cannot tend to 0 when $r \longrightarrow \infty$, a contradiction. □

Corollary 4.16 *Let u be a locally integrable function on $\mathbb{R}^n$, $n \leq 2m$, such that $(-\Delta)^i u \geq 0$ for $0 \leq i \leq m$. Then u is a constant (in the sense of distributions).*

Proof. Let $u_i = (-\Delta)^{m-i} u$. Then $s = (u_i)_{m \geq i \geq 1}$ is a positive m-superharmonic function on $\mathbb{R}^n$. Since $\mathbb{R}^n$, $n \leq 2m$, is not an m-potential domain by the above corollary, s should be an m-harmonic function (Theorem 4.5). This implies that $(-\Delta)^m u = 0$. Then by the Liouville-Picard theorem for polyharmonic functions on $\mathbb{R}^n$ (Corollary 3.5), u is a constant. (This corollary generalizes the important result that a positive superharmonic function on $\mathbb{R}^2$ is a constant.) □

Remark The referee remarks that Corollary 4.16 can also be obtained from the integral representation of polysuperharmonic functions in $\mathbb{R}^n$ as given in Mizuta [9].

Corollary 4.17 *Let $n \geq 2m + 1$. Then given any continuous function u outside a compact set in $\mathbb{R}^n$ such that $\Delta^m u = 0$, there exists a unique m-harmonic function v on $\mathbb{R}^n$ (that is $\Delta^m v = 0$) such that $u - v$ tends to 0 at infinity.*

Proof. When $n \geq 2m+1$, there is a special m-potential $Q = (Q_i)_{m \geq i \geq 1}$ on $\mathbb{R}^n$ where $Q_m = |x|^{2m-n}$ so that each Q_i tends to 0 at infinity. Since p_1 in the proof of the above Theorem 4.14 is a potential with compact support, $p_1 \longrightarrow 0$ at infinity. Consequently, using Theorem 4.7, we can see that in the above proof $q_i \longrightarrow 0$ at infinity for each i, $2 \leq i \leq m$. Consequently, taking $u = h_m$ and $v = H_m$ in the above Theorem 4.14, we conclude that $u - v$ tends to 0. □

Theorem 4.18 *In the harmonic space Ω, let ω be a domain on which there exists a positive potential. Then ω is an m-potential domain if and only if there exist an m-superharmonic function $s = (s_i)_{m \geq i \geq 1}$, $s_1 \neq 0$, and a potential p on ω such that $|s_i| \leq p$ near infinity in ω for every i.*

Proof. 1) If ω is an m-potential domain, there exists an m-potential $s = (p_i)_{m \geq i \geq 1}$, $p_1 \neq 0$. Take $p = \sum_{i=1}^{m} p_i$.

2) Conversely, let $s = (s_i)$, $s_1 \neq 0$, be an m-superharmonic function on ω such that $|s_i| \leq p$ near infinity in ω for every i and for a potential p on ω. Since $s_1 \geqslant -p$ outside a compact set in ω, s_1 has a subharmonic minorant on ω. Hence $s_1 = p_1 + h_1$, where p_1 is a potential on ω and h_1 is harmonic, so that $|h_1| \leq p + p_1$ near infinity in ω. This implies that $h_1 = 0$ and hence s_1 is a potential on ω.

Since $(-L)s_2 = s_1 > 0$, s_2 is superharmonic on ω. Since $|s_2| \leq p$ near infinity in ω, we conclude as above that s_2 is a potential. Proceeding similarly, we see that each s_i is a potential on ω, for every i. Hence $s = (s_i)_{m \geq i \geq 1}$ is actually an m-potential on ω. □

We conclude with some characterizations of an m-potential domain ω in Ω.

Theorem 4.19 *In the harmonic space Ω, let ω be a domain on which there exists a positive potential. Then, ω is an m-potential domain if and only if given any m-harmonic function $h = (h_i)_{m \geq i \geq 1}$ outside a compact set, there exist an m-harmonic function $H = (H_i)_{m \geq i \geq 1}$ and a potential q on ω such that $|h_i - H_i| \leq q$ near infinity in ω for every i.*

Proof. In view of Theorem 4.14, only one direction remains to be proved, namely: If the stated approximation property holds, then ω is an m-potential domain.

Let p be a finite continuous potential > 0 with compact harmonic support in ω. Let $u = (u_i)_{m \geq i \geq 1}$ be an m-superharmonic function on ω generated by $p = u_1$. Then by hypothesis, there exists an m-harmonic function $H = (H_i)_{m \geq i \geq 1}$ such that $|u_i - H_i| \leq q$ near infinity in ω, for some potential q on ω. Let $s_i = u_i - H_i$, so that $s = (s_i)_{m \geq i \geq 1}$ is an m-superharmonic function on ω such that $|s_i| \leq q$ near infinity in ω. Hence, s is an m-potential on ω (Theorem 4.18). □

Definition 4.20 An m-potential domain ω is said to be *tapered* if there exists an m-potential $Q = (Q_i)_{m\geq i\geq 1}$, $Q_1 > 0$, on ω such that each Q_i is bounded outside a compact set in ω.

Remark 1) $\mathbb{R}^n$, $n \geqslant 2m+1$, is a tapered m-potential domain.

2) Let ω be a tapered m-potential domain. Then every potential p_1 with compact harmonic support generates an m-potential $p = (p_i)_{m\geq i\geq 1}$ such that, for all i, $p_i \leq \alpha$ outside a compact set in ω. (To prove this, use Theorem 4.7).

The following is a characterization of a tapered m-potential domain in Ω.

Theorem 4.21 *In the harmonic space Ω, ω is a tapered m-potential domain if and only if given any m-harmonic function $h = (h_i)_{m\geq i\geq 1}$ outside a compact set in ω, there exist an m-harmonic function $H = (H_i)_{m\geq i\geq 1}$ on ω and a positive potential q bounded near infinity such that for each i, $|h_i - H_i| \leq q$ near infinity in ω.*

Proof. 1) Let ω be a tapered m-potential domain. Suppose h is m-harmonic near infinity in ω. Then, we follow the proof of Theorem 4.14 for the construction of the potential q and the m-harmonic function $H = (H_i)_{m\geq i\geq 1}$ on ω such that, $|h_i - H_i| \leq q$ outside a compact set. We notice that q has been defined there as $q = p_1 + q_2 + \cdots + q_m$. Now ω being tapered, we can see that each one of the terms in this sum is bounded near infinity. Hence q is a potential bounded near infinity in ω.

2) Conversely, suppose that ω has the approximation property stated in the theorem. Then by Theorem 4.19, ω is an m-potential domain. Moreover, since q is bounded near infinity, ω is tapered. □

Corollary 4.22 *Given any m-harmonic function $h = (h_i)_{m\geq i\geq 1}$ outside a compact set in a tapered m-potential domain ω, there exists an m-harmonic function $H = (H_i)_{m\geq i\geq 1}$ on ω, such that $|h_i - H_i|$ is bounded near infinity in ω for each i.*

Acknowledgment We thank the referee for his many remarks, leading to a much improved version of the original draft of this paper.

References

[1] Anandam V., *Admissible superharmonic functions and associated measures*. J. London Math. Soc. **19** (1979), 65–78.

[2] Anandam V., *Superharmonic extensions and the flux*. Bull. Sci. Math. **106** (1982), 85–92.

[3] Anandam V., *Biharmonic classification of harmonic spaces*. Rev. Roumaine Math. Pures et Appl. **45** (2000), 383–395.

[4] Arsove M., *Functions representable as differences of subharmonic functions*. Trans. Amer. Math. Soc. **75** (1953), 327–365.

[5] Brelot M., *Fonctions sousharmoniques associées à une mesure*. Stud. Cerc. Sti. Mat. Iasi **2** (1951), 114–118.

[6] Brelot M., *Axiomatique des fonctions harmoniques*. Les Presses de l'Université de Montréal, 1966.

[7] Hervé R.M., *Recherches axiomatiques sur la théorie des fonctions surharmoniques et du potentiel*. Ann. Inst. Fourier **12** (1962), 415–571.

[8] De La Pradelle A., *Approximation et caractère de quasi-analycité dans la théorie axiomatique des fonctions harmoniques*. Ann. Inst. Fourier **17** (1967), 383–399.

[9] Mizuta Y., *An integral representation and fine limits at infinity for functions whose Laplacians iterated m times are measures*. Hiroshima Math. J. **27** (1997), 415–427.

[10] Nicolescu M., *Les fonctions polyharmoniques*. Paris, Hermann, 1936.

[11] Sario L., Nakai M., Wang C. and Chung L.O., *Classification theory of Riemannian manifolds*; *Harmonic, quasiharmonic and biharmonic functions*. Lecture Notes in Math., vol. 605, Springer-Verlag, 1977.

[12] Smyrnélis E.P., *Axiomatique des fonctions biharmoniques* I. Ann. Inst. Fourier **25** (1975), 35–97.

M. Al-Qurashi
Department of Mathematics
King Saud University
P.O.Box 2455
Riyadh 11451, Saudi Arabia
E-mail: maysaa_1971@hotmail.com

V. Anandam
Department of Mathematics
King Saud University
P.O.Box 2455
Riyadh 11451, Saudi Arabia
E-mail: vanandam@ksu.edu.sa