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Bounded circular distortion curves and quasidisks*

Yuming Chu and Zhenjiang Zhao

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Abstract. Let D be a Jordan domain in $\overline{R}^2$ and $\Gamma = \partial D$ be the boundary of D. Then D is a quasidisk if and only if Γ is a bounded circular distortion curve.

Key words: bounded circular distortion curve, quasidisk, quasiconformal, mapping.

1. Introduction

Let D be a Jordan domain in $\overline{R}^2$ and $f\colon \overline{R}^2 \to \overline{R}^2$ be a k-quasiconformal mapping, where $1 \le k < +\infty$. D is called a quasidisk if D is the image of the unit disk B^2 under f.

It is well-known that quasidisks play a very important role in quasiconformal mapping theory, complex dynamics, Fuchsian groups, Teichmüller space theory and low dimensional topology, see [2, 3, 7, 9, 11] etc.

In 1963, L.V. Ahlfors obtained the three-point property of quasidisks ([1]). Later, F.W. Gehring [5], B.G. Osgood [10], J.G. Krzyz [8], Y. Chu and J. Cheng [4] studied the quasidisks extensively. Several characterizations of quasidisks were obtained. In this paper, we shall prove a new characterization of quasidisks.

Definition 1 Let E be a set in $\overline{R}^2$ and $c \ge 1$ be a constant. E is called a c-linearly locally connected set if for any $x \in R^2$ and $0 < r < +\infty$, the following are satisfied:
(1) any two points in $E \cap \overline{B}^2(x,r)$ can be joined by a curve in $E \cap \overline{B}^2(x,cr)$;
(2) any two points in $E \backslash B^2(x,r)$ can be joined by a curve in $E \backslash B^2(x,r/c)$.

E is called a linearly locally connected set if E is a c-linearly locally connected set for some $c \ge 1$.

F.W. Gehring and O. Martio obtained the following result ([6]):

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Theorem A *If D is a Jordan domain in $\overline{R}^2$, then D is a quasidisk if and only if D is a linearly locally connected domain.*

Definition 2 A curve $\Gamma \subset \overline{R}^2$ is said to be of c-bounded circular distortion, where $0 < c \leq 1$, if for all $x \in \Gamma \cap R^2$ and $r > 0$, the disk $B^2(x, cr)$ meets only the x-component of $\Gamma \cap \overline{B}^2(x, r)$. Γ is called a bounded circular distortion curve if Γ is a c-bounded circular distortion curve for some c, where $0 < c \leq 1$.

Remark 1 It follows from definitions 1 and 2 that a Jordan curve with c-bounded circular distortion is b-linearly locally connected, where $b = 1/c$.

The main aim of this paper is to prove the following result:

Theorem 1.1 *Let D be a Jordan domain in $\overline{R}^2$ and $\Gamma = \partial D$ be the boundary of D. Then D is a quasidisk if and only if Γ is a bounded circular distortion curve.*

2. The proof of Theorem 1.1

Lemma 2.1 *Let Γ be a Jordan curve. If Γ is a bounded circular distortion curve, then Γ is a quasicircle.*

Proof. Since Γ is a bounded circular distorsion curve, by definition 2, there exists a constant c $(0 < c \leq 1)$ such that Γ is a c-bounded circular distortion curve.

Suppose first that $\infty \in \Gamma$. Let x_1, x_2, x_3 be three points on Γ in this order. If $c|x_1 - x_2| > |x_1 - x_3|$, then obviously Γ is not a c-bounded circular distortion curve. Consequently,

$$\frac{|x_1 - x_2|}{|x_1 - x_3|} \leq \frac{1}{c}. \tag{2.1}$$

It follows from [1, Theorem1] that Γ is a quasicircle.

Then suppose that $\Gamma \in R^2$. Without loss of generality, we may assume $0 < c \leq 1/2$. By [1, P_{295}], Γ is a quasicircle if

$$\frac{|x_1 - x_2|\,|x_3 - x_4|}{|x_1 - x_3|\,|x_2 - x_4|} \leq b, \tag{2.2}$$

where $x_i \in \Gamma$ $(i = 1, 2, 3, 4)$, x_2 and x_4 lie in different components of $\Gamma \setminus \{x_1, x_3, \}$. In the following we shall show that (2.2) holds for $b = c^{-4}$.

Let x_i $(i = 1, 2, 3, 4)$ be the above stated four points on Γ, and let $u = |x_1 - x_2|/|x_1 - x_3|$. Suppose $u > 1/c^2$. Now $|x_1 - x_4| \leq |x_1 - x_3|/c$ since otherwise Γ is not a c-bounded circular distortion curve. It follows that

$$|x_1 - x_4| \leq \frac{|x_1 - x_3|}{c} = \frac{|x_1 - x_2|}{cu} < c|x_1 - x_2|. \tag{2.3}$$

On the other hand, if we set $a = |x_1 - x_2|/|x_2 - x_4|$, then (2.3) implies that

$$a \leq \frac{|x_1 - x_4| + |x_2 - x_4|}{|x_2 - x_4|} \leq \frac{c|x_1 - x_2| + |x_2 - x_4|}{|x_2 - x_4|} = ac + 1. \tag{2.4}$$

Obviously $a \leq 1/(1 - c) \leq 2$. Combining the following inequalities:

$$|x_3 - x_4| \leq |x_3 - x_1| + |x_1 - x_4| \leq \Big(1 + \frac{1}{c}\Big)|x_1 - x_3| \leq \frac{2}{c}|x_1 - x_3|, \tag{2.5}$$

we conclude that (2.2) holds with $b = 4/c \leq 1/c^4$.

The cases where $v = |x_3 - x_4|/|x_2 - x_4| > 1/c^2$ and $u, v \leq 1/c^2$ can be proved in analogous way. These complete the proof.

Remark 2 The result that a Jordan domain $D \subset \overline{R}^2$ is a quasidisk if and only if ∂D is linearly locally connected had been proved by M.F. Walker in [12, Corollary 4.4], but the method in the proof of Lemma 2.1 is different from that in [12].

Lemma 2.2 *Let D be a Jordan domain and $\Gamma = \partial D$ be the boundary of D. If D is a quasidisk, then Γ is a bounded circular distortion curve.*

Proof. Since D is a quasidisk, by Theorem A, D is a linearly locally connected domain. Then there exists a constant $c \geq 1$ such that D is a c-linearly locally connected domain. In the following we shall prove that $\Gamma = \partial D$ is a $1/c$-bounded circular distortion curve.

Suppose that Γ is not a $1/c$-bounded circular distortion curve. Then there exist $x \in \Gamma \cap R^2$ and r $(0 < r < +\infty)$ such that $B^2(x, r/c)$ meets a component E_1 of $\Gamma \cap \overline{B}^2(x, r)$, which isn't the x-component E_2 of $\Gamma \cap \overline{B}^2(x, r)$. Let G_i be the component of $B^2(x, r) \cap D$ which contains E_i as a part of a boundary $(i = 1, 2)$. There are two possibilities:

(1) $G_1 = G_2$. It is easy to see that there exist points $x_1, x_2 \in D \backslash B^2(x, r)$ which can be joined by a curve in D only through $B^2(x, r/c)$. Hence x_1, x_2 cannot be joined by a curve in $D \setminus B^2(x, r/c)$.

(2) $G_1 \neq G_2$. Choose points $x_i \in B^2(x, r/c) \cap G_i$ $(i = 1, 2)$. If x_1 and x_2 can be joined by a curve α in D, then α will meet $\overline{R}^2 \setminus \overline{B}^2(x, r)$. Hence x_1, x_2 cannot be joined by a curve in $D \cap \overline{B}^2(x, r)$.

The above shows that D isn't a c-linearly locally connected domain. This is a contradiction. Hence Γ is a bounded circular distortion curve.

Proof of Theorem 1.1. If D is a quasidisk, then $\Gamma = \partial D$ is a bounded circular distortion curve by Lemma 2.2. On the other hand, if $\Gamma = \partial D$ is a bounded circular distortion curve, then Γ is a quasicircle by Lemma 2.1, hence D is a quasidisk.

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Y. Chu
Department of Mathematics
Huzhou Normal college
Huzhou, Zhejiang 313000
P. R. China
E-mail: chuyuming@hutc.zj.cn

Z. Zhao
Department of Mathematics
Huzhou Normal college
Huzhou, Zhejiang 313000
P. R. China
E-mail: zhaozjcn@163.com