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Well-posedness of the Cauchy problem for the semilinear Schrödinger equation with quadratic nonlinearity in Besov spaces

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Abstract. Well-posedness of the Cauchy problem for the semilinear Schrödinger equation with quadratic nonlinear terms is studied. By making use of Besov spaces we can improve the regularity assumption on the initial data. When the nonlinear term is $c_1u^2 + c_2\bar{u}^2$, our results are as follows: When $d = 1$ or 2, for any initial data $u_0 \in H^{-3/4}(\mathbb{R}^d)$ there exists a unique local-in-time solution $u \in B^{(-3/4,1/2)}_{2,(2,1),-|\xi|^2}(\mathbb{R}^d \times I_T)$. When $d \geq 3$, for any small data $u_0 \in H^{\rho}(\mathbb{R}^d)$, where $\rho(z) = z^{d/2-2}\log(2+z)$, there exists a unique local-in-time solution $u \in B^{(\rho,1/2)}_{2,(2,1),-|\xi|^2}(\mathbb{R}^d \times I_T)$, and for any $u_0 \in H^s(\mathbb{R}^d)$, $s > d/2-2$, there exists a unique local-in-time solution $u \in B^{(s,1/2)}_{2,(2,1),-|\xi|^2}(\mathbb{R}^d \times I_T)$. Here $I_T = (-T,T)$. We also have results for the equation with the nonlinear term $c_3u\bar{u}$.

Key words: semilinear Schrödinger equation, Besov type norm, initial value problem.

1. Introduction

This paper is a continuation of our study on the Cauchy problem for the semilinear Schrödinger equation

$$\begin{cases} \partial_t u = i\Delta u + N(u,\bar{u}), & x \in \mathbb{R}^d,\ t \in \mathbb{R}, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $N(u,\bar{u}) = c_1u^2 + c_2\bar{u}^2$ or $N(u,\bar{u}) = c_3u\bar{u}$, c_1, c_2, c_3 are constants. In [5] we proved that the Cauchy problem (1.1) with $N(u,\bar{u}) = c_1u^2 + c_2\bar{u}^2$ in one space dimension is locally well-posed in $B^{-3/4}_{2,1}(\mathbb{R})$. In this paper we improve this result, that is, this Cauchy problem is locally well-posed in $H^{-3/4}(\mathbb{R})$. Further, by the same method we show that when $N(u,\bar{u}) = c_1u^2+c_2\bar{u}^2$ the Cauchy problem (1.1) in two space dimensions is locally well-posed in $H^{-3/4}(\mathbb{R}^2)$ and when $N(u,\bar{u}) = c_3u\bar{u}$ it is well-posed in $H^{\rho}(\mathbb{R}^2)$ with $\rho(z) = z^{-1/4}\log(2+z)$. For 3-D case we prove that the Cauchy problem (1.1) with $N(u,\bar{u}) = c_1u^2+c_2\bar{u}^2$ has a unique local-in-time solution

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for small data in $H^\rho(\mathbb{R}^3)$ with $\rho(z) = z^{-1/2}\log(2+z)$ and it has a unique local-in-time solution for any data in $H^s(\mathbb{R}^3)$ if $s > -1/2$. As far as we know, there is no paper in which the well-posedness of (1.1) in $H^s(\mathbb{R}^3)$ with $s > -1/2$ is proved. We also give the results for higher-dimensional case.

By means of the Fourier restriction norm method (due to Bourgain [1]) Kenig, Ponce and Vega ([3]) have studied the Cauchy problem for the 1-D semilinear Schrödinger equation. They used the following bilinear estimates

$$\|fg\|_{X_{s,b-1}} \le c\|f\|_{X_{s,b}}\|g\|_{X_{s,b}}, \tag{1.2}$$

$$\|\bar f\bar g\|_{X_{s,b-1}} \le c\|f\|_{X_{s,b}}\|g\|_{X_{s,b}}. \tag{1.3}$$

They showed that the 1-D Cauchy problem with $N(u,\bar u) = c_1u^2 + c_2\bar u^2$ (or $N(u,\bar u) = c_3u\bar u$) has a unique local-in-time solution in H^s if $s > -3/4$ (or $s > -1/4$). Our results in [5] are improvement of their results.

Staffilani ([6]) generalized the results of Kenig, Ponce and Vega to the 2-dimensional case, who showed that a unique local-in-time solution of the 2-D Cauchy problem exists in H^s if $s > -1/2$ when $N(u,\bar u) = c_3\bar u^2$. To prove this, she showed that the estimate (1.3) holds for any $s > -1/2$ in $\mathbb{R}^2 \times \mathbb{R}$. The study for the 2-dimensional case was improved further, that is, Colliander, Delort, Kenig and Staffilani ([2]) proved local well-posedness of the 2-D Cauchy problem with nonlinearity $N(u,\bar u) = c_1u^2 + c_2\bar u^2$ (or $N(u,\bar u) = c_3u\bar u$) in H^s if $s > -3/4$ (or $s > -1/4$). In this paper we improve their results.

When the space dimension $d \ge 2$ in the case $N(u,\bar u) = u^2$ there arises a difficulty that

$-\xi^2 + (\xi-\eta)^2 + \eta^2 = 2\eta(\xi-\eta)$ goes near to 0 while $|\eta|$ and $|\xi-\eta|$ are large

(ξ, η are the Fourier variables). We resolve this difficulty by expressing a function $\hat f(\xi,\tau)$ as an integral sum of functions $\hat f_\nu^{[\omega]}(\xi,\tau)$ supported in the set $\mathbb{R}_+D(\nu,\omega)\times\mathbb{R}$, where $D(\nu,\omega) := \{\xi \in S^{d-1}; \arccos(\xi\cdot\omega) \le 2^{-\nu}\}$ (see proof of Lemma 4.6 for detail). By this method we can get sharp estimates.

This paper is organized as follows. In §2 we give the definition of the spaces we use, and state our main results and our bilinear estimates. In §4 we give the lemmas which give the estimates of the norm of bilinear operators which are needed to calculate the norm of products. The bilinear estimates for $c_1u^2 + c_2\bar u^2$ (Theorem 1) is proved in §5 and §6, and that for the case $c_3u\bar u$ (Theorem 2) is proved in §7. Finally, in §8 we explain the method to get our main results.

Notations We use the following notations:
L^p_x denotes L^p-space of functions of x,
$\|f(x,y,z)\|_{L^p_x\times L^q_y\times L^r_z} := \|[\|\{\|f(x,y,z)\|_{L^r_z}\}\|_{L^q_y}]\|_{L^p_x}$,
$\ell^{(q_1,q_2)}$ denotes the sequence space defined by the norm

$$\|\{a_{jk}\}\|_{\ell^{(q_1,q_2)}} = \left\{\sum_k\left(\sum_j |a_{jk}|^{q_1}\right)^{q_2/q_1}\right\}^{1/q_2}. \tag{1.4}$$

For a Banach space X and an open set Ω in the Euclidean space $B^\sigma_{p,q}(\Omega;X)$ denotes the X-valued Besov space.
χ_M denotes the defining function of a set M.
$j\wedge k=\min(j,k)$ and $j\vee k=\max(j,k)$.

2. Definition and Main results

The Besov type norm which corresponds to the Fourier restriction norm is defined as follows:

Definition 1 *For a weight ρ on $\mathbb{R}_+$, $b\in\mathbb{R}$, $1\le p\le\infty$, $1\le q_1\le\infty$, $1\le q_2\le\infty$ and a real-valued C^∞-function $P(\xi)$ the space $B^{(\rho,b)}_{p,(q_1,q_2),P}(\mathbb{R}^{d+1})$ is the space of tempered distributions f such that the norm*

$$\|f\|_{B^{(\rho,b)}_{p,(q_1,q_2),P}} := \left\|\{\rho(2^j)\,2^{bk}\|f_{jk,P}(x,t)\|_{L^p(R^{d+1})}\}\right\|_{\ell^{(q_1,q_2)}(\bar{\mathbb{N}}\times\bar{\mathbb{N}})} \tag{2.1}$$

is finite. Here, $\bar{\mathbb{N}} := \mathbb{N}\cup\{0\}$,

$$\hat{f}_{jk,P}(\xi,\tau)=\varphi_j(|\xi|)\varphi_k(\tau-P(\xi))\hat{f}(\xi,\tau), \tag{2.2}$$

and $\varphi_j(z)$, $j=0,1,\dots,$ are C^∞-functions of $z\in\mathbb{R}$ with the following relations:

$$\varphi_j(z)=\varphi_j(-z),\ \operatorname{supp}\varphi_0\subset\{z;|z|<2\},\ \operatorname{supp}\varphi_1\subset\{z;1<|z|<4\},$$

$$\varphi_k(z)=\varphi_1(2^{-k+1}z)\quad(\text{for } k\ge 1),\qquad \sum_{j=0}^\infty\varphi_j(z)=1.$$

We write the space by $B^{(s,b)}_{p,(q_1,q_2),P}(\mathbb{R}^{d+1})$ when $\rho(z)=z^s$ and $B^{(\rho,b)}_{p,(q,q),P} = B^{(\rho,b)}_{p,q,P}$, and omit the subscript P when $P=0$.

For a function space $X(\mathbb{R}^{d+1})$ and an open set Ω in $\mathbb{R}^{d+1}$ the space $X(\Omega)$ is the set of all distributions f which have an extension $\tilde{f}\in X(\mathbb{R}^{d+1})$, and *its norm is defined by*

$$\|f\|_{X(\Omega)} := \inf\{\|\tilde{f}\|_{X(\mathbb{R}^{d+1})}; f = \tilde{f}|_{\Omega}\}. \tag{2.3}$$

In order to solve the Cauchy problem in $H^\rho(\mathbb{R}^d)$ (instead of $B^\rho_{2,1}$) we are forced to use the spaces with indices (q_1, q_2).

Note that

$$B^\rho_{2,2}(\mathbb{R}^d) = H^\rho(\mathbb{R}^d), \tag{2.4}$$

$$B^{(\rho,b)}_{2,(2,1)}(\mathbb{R}^d \times I) = B^b_{2,1}(I; H^\rho(\mathbb{R}^d)), \tag{2.5}$$

and it is easy to see that $f \in H^\rho(\mathbb{R}^d)$ if and only if $\rho(\sqrt{1+|\xi|^2}\,)\hat{f}(\xi) \in L^2$.

Assume that $|\partial^\alpha_\xi P(\xi)| \le C(\alpha)(1+|\xi|)^{\nu-|\alpha|}$ holds for any α, where ν is a constant independent of α and $C(\alpha)$ is a constant depend on α. Then $B^{(\rho,b)}_{p,(q_1,q_2),P}(\mathbb{R}^{d+1})$ is a Banach space and $\mathcal{S}(\mathbb{R}^{d+1})$ is dense in the space if $p < \infty$, $q_1 < \infty$, $q_2 < \infty$ (see Theorem 2.1 in [5]). Also we see that $B^{(\rho,1/2)}_{2,(q,1),P}(\mathbb{R}^{d+1})$ s continuously imbedded into the space of bounded continuous $B^\rho_{2,q}(\mathbb{R}^d)$-valued functions of $t \in \mathbb{R}$, which guarantees the initial condition makes sense in this space (see Theorem 2.2 in [5]).

Using these spaces, our **MAIN RESULTS** are stated as follows:

We write $I_T := (-T, T)$ here.

Part (I). *The case $d = 1$ or 2.*

(a) *If $N(u,\bar{u}) = c_1u^2 + c_2\bar{u}^2$, then for any $u_0 \in H^{-3/4}(\mathbb{R}^d)$ there exists $T = T(\|u_0\|_{H^{-3/4}(\mathbb{R}^d)}) > 0$ and a unique solution $u(x,t) \in B^{(-3/4,1/2)}_{2,(2,1),-|\xi|^2}(\mathbb{R}^d \times I_T)$ to* (1.1) *satisfying $u(x,t) - W(t)u_0(x) \in B^{(\rho,1/2)}_{2,(2,1),-|\xi|^2}(\mathbb{R}^d \times I_T)$. Here $\rho(z) = z^{-3/4}\log(2+z)$* and $\{W(t)f\}(x,t) := \mathcal{F}_x^{-1}e^{itP(\xi)}\mathcal{F}_x f(x,t)$.

(b) *If $N(u,\bar{u}) = c_3u\bar{u}$, then for any $u_0 \in H^\rho(\mathbb{R}^d)$ there exists $T = T(\|u_0\|_{H^\rho(\mathbb{R}^d)}) > 0$ and a unique solution $u(x,t) \in B^{(\rho,1/2)}_{2,(2,1),-|\xi|^2}(\mathbb{R}^d \times I_T)$ to* (1.1). *Here $\rho(z) = z^{-1/4}\log(2+z)$.*

Part (II). *The case $d \ge 3$.*

(a) *If $N(u,\bar{u}) = c_1u^2 + c_2\bar{u}^2$ and $s = d/2 - 2$, then for any $T > 0$ there exists $\delta(T) > 0$ such that* (1.1) *has a unique solution $u(x,t) \in B^{(\rho,1/2)}_{2,(2,1),-|\xi|^2}(\mathbb{R}^d \times I_T)$ for any $u_0 \in H^\rho(\mathbb{R}^d)$ with $\|u_0\|_{H^\rho} \le \delta(T)$. Here $\rho(z) = z^s\log(2+z)$.*

Also, if $N(u,\bar{u}) = c_1u^2 + c_2\bar{u}^2$ and $s > d/2 - 2$, then for any $u_0 \in H^s(\mathbb{R}^d)$ there exists $T = T(\|u_0\|_{H^s(\mathbb{R}^d)}) > 0$ and a unique solution

$u(x,t) \in B^{(s,1/2)}_{2,(2,1),-|\xi|^2}(\mathbb{R}^d \times I_T)$ *to* (1.1).
(b) *The case* $d = 3$. *If* $N(u,\bar{u}) = c_3 u\bar{u}$, *then the same result as* Part (I) (b) *holds.*
The case $d \geq 4$. *If* $N(u,\bar{u}) = c_3 u\bar{u}$, *then the same result as* Part (II) (a) *holds.*

As in [2], [3], [5], the keys to prove these results are the following bilinear estimates:

Theorem 1 *Let* $P(\xi) = \pm|\xi|^2$, *and let* Q *be* P *or* $-P$, *and define* $\rho(z) = \log(2+z)z^s$.
(a) *Let* $d = 1$ *or* 2, $s \geq -3/4$ *and let* $b > 1/2$. *Then we have the inequalities*

$$\|fg\|_{B^{(\rho,-1/2)}_{2,1,P}} \leq c\|f\|_{B^{(\rho,1/2)}_{2,(2,1),Q}} \|g\|_{B^{(s,1/2)}_{2,(2,1),Q}}, \tag{2.6}$$

$$\|fg\|_{B^{(\rho,-1/2)}_{2,1,P}} \leq c\big\{\|f\|_{B^{(s,b)}_{2,(2,1),Q}} \|g\|_{B^{(s,1/2)}_{2,(2,1),Q}} + \|f\|_{B^{(s,1/2)}_{2,(2,1),Q}} \|g\|_{B^{(s,b)}_{2,(2,1),Q}}\big\}. \tag{2.7}$$

(b) *Let* $d \geq 3$, $s = d/2 - 2$. *Then, we have the inequality*

$$\|fg\|_{B^{(\rho,-1/2)}_{2,1,P}} \leq c\|f\|_{B^{(\rho,1/2)}_{2,(2,1),Q}} \|g\|_{B^{(\rho,1/2)}_{2,(2,1),Q}}. \tag{2.8}$$

Also, if $d \geq 3$, $s > d/2 - 2$, *then we have the inequality*

$$\|fg\|_{B^{(\rho,-1/2)}_{2,1,P}} \leq c\|f\|_{B^{(s,1/2)}_{2,(2,1),Q}} \|g\|_{B^{(s,1/2)}_{2,(2,1),Q}}. \tag{2.9}$$

Theorem 2 *Let* $P(\xi) = \pm|\xi|^2$, $\xi \in \mathbb{R}^d$, *and define* $\rho(z) = \log(2+z)z^s$.
(a) *Let* $d = 1,\ 2$ *or* 3, $-1/4 \leq s$. *Then, we have the inequality*

$$\|fg\|_{B^{(\rho,-1/2)}_{2,1,P}} \leq c\min\big\{\|f\|_{B^{(\rho,1/2)}_{2,(2,1),P}} \|g\|_{B^{(s,1/2)}_{2,(2,1),-P}}, \|f\|_{B^{(s,1/2)}_{2,(2,1),P}} \|g\|_{B^{(\rho,1/2)}_{2,(2,1),-P}}\big\}. \tag{2.10}$$

(b) *Let* $d \geq 4$, $s \geq (d-4)/2$. *Then, we have the inequality*

$$\|fg\|_{B^{(\rho,-1/2)}_{2,1,P}} \leq c\|f\|_{B^{(\rho,1/2)}_{2,(2,1),P}} \|g\|_{B^{(\rho,1/2)}_{2,(2,1),-P}}. \tag{2.11}$$

Also, if $s > (d-4)/2$, *then we have*

$$\|fg\|_{B^{(\rho,-1/2)}_{2,1,P}} \leq c\|f\|_{B^{(s,1/2)}_{2,(2,1),P}} \|g\|_{B^{(s,1/2)}_{2,(2,1),-P}}. \tag{2.12}$$

3. Norm of bilinear operators

Definition 2 For a kernel $K(x,y)$ defined on $\mathbb{R}^d \times \mathbb{R}^d$ the bilinear operators $B(K;f,g)$ is defined by

$$B(K;f,g)(x) := \int K(x,y)f(y)g(x-y)dy, \tag{3.1}$$

and its norm is denoted by $N_{bl}(K)$.

First note the basic properties of bilinear operators which have been proved in [5].

Lemma 3.1 *Let $K(x,y)$ be a measurable function on $\mathbb{R}^{2d}$.*

(a) *Put $K_1(x,y) = K(y,x)$, $K_2(x,y) = K(x,x-y)$, $K_3(x,y) = K(y,y-x)$, $K_4(x,y) = K(x-y,-y)$, $K_5(x,y) = K(x-y,x)$. Then we have $N_{bl}(K) = N_{bl}(K_1) = N_{bl}(K_2) = N_{bl}(K_3) = N_{bl}(K_4) = N_{bl}(K_5)$.*

(b) *If $M(x,y)$ is a non-negative measurable function such that $|K(x,y)| \le M(x,y)$ for almost everywhere, then $N_{bl}(K) \le N_{bl}(M)$.*

Next, we prove the following lemma which gives sharp estimates of the norm of bilinear operators:

Lemma 3.2 *Let $d = d' + d''$, and write $x = (x', x'')$, $y = (y', y'')$, x', $y' \in \mathbb{R}^{d'}$, x'', $y'' \in \mathbb{R}^{d''}$. Then for a kernel $K(x,y) = K(x', x'', y', y'')$ we have $N_{bl}(K) \le \min\{C_1, C_2\}$, where*

$$C_1 = \|K(x', x'', y', y'')\|_{L^2_{x'} \times L^\infty_{(x'',y')} \times L^2_{y''}}, \tag{3.2}$$

$$C_2 = \|K(x', x'', y', y'')\|_{L^2_{y'} \times L^\infty_{(x',y'')} \times L^2_{x''}}. \tag{3.3}$$

In particular, taking $d'' = 0$, $d' = d$, we have

$$N_{bl}(K) \le \min\{\operatorname*{ess.sup}_x \|K(x,y)\|_{L^2_y}, \operatorname*{ess.sup}_y \|K(x,y)\|_{L^2_x}\}. \tag{3.4}$$

Proof. Put

$$K_1(x') := \|K(x', x'', y', y'')\|_{L^\infty_{(x'',y')} \times L^2_{y''}},$$

$$F(x', x'') := \iint K(x', x'', y', y'')f(y', y'')g(x'-y', x''-y'')dy'\,dy''.$$

Then, by Schwarz's inequality we have

$$\begin{aligned}
&|F(x',x'')| \\
&\le \int dy'\, \|K(x',x'',y',y'')\|_{L^2_{y''}} \Big(\int |f(y',y'')g(x'-y',x''-y'')|^2 dy''\Big)^{1/2} \\
&\le \int dy'\, K_1(x') \Big(\int |f(y',y'')g(x'-y',x''-y'')|^2 dy''\Big)^{1/2}.
\end{aligned}$$

This combined with the identity

$$\begin{aligned}
&\Big\| \Big(\int |f(y',y'')g(x'-y',x''-y'')|^2 dy''\Big)^{1/2} \Big\|_{L^2_{x''}} \\
&\quad = \Big(\int \Big| \int |f(y',y'')g(x'-y',x''-y'')|^2 dy'' \Big| dx''\Big)^{1/2} \\
&\quad = \Big(\int |f(y',y'')|^2 dy''\Big)^{1/2} \Big(\int |g(x'-y',x'')|^2 dx''\Big)^{1/2},
\end{aligned}$$

yields

$$\begin{aligned}
&\|F(x',x'')\|_{L^2_{x''}} \\
&\quad \le K_1(x') \int dy' \Big(\int |f(y',y'')|^2 dy''\Big)^{1/2} \Big(\int |g(x'-y',x'')|^2 dx''\Big)^{1/2} \\
&\quad \le K_1(x') \Big(\iint |f(y',y'')|^2 dy'\, dy''\Big)^{1/2} \Big(\iint |g(y',x'')|^2 dy'\, dx''\Big)^{1/2} \\
&\quad = K_1(x') \|f\|_{L^2} \|g\|_{L^2}.
\end{aligned}$$

Take $L^2_{x'}$-norm. Then we get $\|F\|_{L^2_x} \le C_1 \|f\|_{L^2} \|g\|_{L^2}$.

Moreover, by Lemma 3.1 we have $N_{bl}(K(x,y)) = N_{bl}(K(y,x))$, and hence

$$\begin{aligned}
N_{bl}(K(x,y)) &\le \|K(y',y'',x',x'')\|_{L^2_{x'} \times L^\infty_{(x'',y')} \times L^2_{y''}} \\
&= \|K(x',x'',y',y'')\|_{L^2_{y'} \times L^\infty_{(y'',x')} \times L^2_{x''}} = C_2. \qquad \square
\end{aligned}$$

Since $L^p(\mathbb{R}^d)$-norm is invariant with respect to orthogonal transformations, that is, $\|f(x)\|_{L^p} = \|f(Tx)\|_{L^p}$ for any orthogonal transformation T, we have the following lemma:

Lemma 3.3 *Let $K(x,y)$ be a kernel, T a orthogonal transformation on $\mathbb{R}^d$, and define $K^{[T]}(x,y) = K(Tx,Ty)$. Then $N_{bl}(K) = N_{bl}(K^{[T]})$.*

4. A special class of bilinear operators

In this section we write $\xi=(\xi_1,\dots,\xi_d)$, $\eta=(\eta_1,\dots,\eta_d)\in\mathbb{R}^d$, τ, $\sigma\in\mathbb{R}$, and define $\gamma_j:=\chi_{\{z\in\mathbb{R};\,2^{j-1}<|z|<2^{j+1}\}}$ for $j>0$, $\gamma_0:=\chi_{\{z\in\mathbb{R};\,|z|<2\}}$, $\tilde{\gamma}_j(z)=\gamma_0(2^{-j}z)$. Further, for any real-valued function P, Q, R we define $\gamma_{jk}^{[P]}(\xi,\tau):=\gamma_j(|\xi|)\gamma_k(\tau-P(\xi))$,

$$H_{jk\ell m}^{[P,Q]}(\xi,\tau,\eta,\sigma)=\gamma_{jk}^{[P]}(\eta,\sigma)\gamma_{\ell m}^{[Q]}(\xi-\eta,\tau-\sigma), \tag{4.1}$$

$$H_{hjk\ell m}^{[P,Q]}(\xi,\tau,\eta,\sigma)=\gamma_h(|\xi|)H_{jk\ell m}^{[P,Q]}(\xi,\tau,\eta,\sigma), \tag{4.2}$$

$$H_{hnjk\ell m}^{[P,Q,R]}(\xi,\tau,\eta,\sigma)=\gamma_{hn}^{[P]}(\xi,\tau)H_{jk\ell m}^{[Q,R]}(\xi,\tau,\eta,\sigma). \tag{4.3}$$

The following lemma has been proved in [5].

Lemma 4.1 *Assume that $\gamma_h(|\xi|)\gamma_j(|\eta|)\gamma_\ell(|\xi-\eta|)\neq 0$ for some ξ, η. Then $h\leq j\vee\ell+2$.*

Moreover, $h\geq j\vee\ell-2$ when $|j-\ell|\geq 3$ and $|j-\ell|\leq 2$ when $h\leq j\vee\ell-3$.

The following seven lemmas concern with the norm of bilinear operators of special kind in $(d+1)$-dimensional space. In these lemmas, c denotes a constant depending only on d.

First, Lemma 3.1 and Lemma 3.2 together with inequalities

$$\begin{aligned}
&\|H_{jk\ell m}^{[P,Q]}(\xi,\tau,\eta,\sigma)\|_{L^2_{(\eta,\sigma)}}^2\\
&\quad=\int\gamma_j(|\xi|)\gamma_\ell(|\xi-\eta|)d\eta\int\gamma_k(\sigma-P(\eta))\gamma_m(\tau-\sigma-Q(\xi-\eta))d\sigma\\
&\quad\leq 2^{d(j\wedge\ell)+k\wedge m+2(d+1)},\\
&\|H_{hjk\ell m}^{[P,Q]}(\xi,\tau,\eta,\sigma)\|_{L^2_{(\xi,\tau)}}^2\leq\int\gamma_h(|\xi|)d\xi\int\gamma_m(\tau-\sigma-Q(\xi-\eta))d\tau\\
&\qquad\qquad\leq 2^{dh+m+2(d+1)},\\
&\|H_{hnjk\ell m}^{[P,Q,R]}(\xi,\tau,\eta,\sigma)\|_{L^2_{(\xi,\tau)}}^2\leq\int\gamma_h(|\xi|)\gamma_\ell(|\xi-\eta|)d\xi\int\gamma_n(\tau-P(\xi))d\tau\\
&\qquad\qquad\leq 2^{d(h\wedge\ell)+n+2(d+1)},
\end{aligned}$$

and identities

$$\begin{aligned}
&H_{hjk\ell m}^{[P,Q]}(\xi,\tau,\xi-\eta,\tau-\sigma)=H_{h\ell mjk}^{[P,Q]}(\xi,\tau,\eta,\sigma),\\
&H_{hnjk\ell m}^{[P,Q,R]}(\xi,\tau,\xi-\eta,\tau-\sigma)=H_{hn\ell mjk}^{[P,Q,R]}(\xi,\tau,\eta,\sigma),
\end{aligned}$$

yield the following

Lemma 4.2 *Let P, Q and R be real-valued functions. Then*

$$N_{bl}(H^{[P,Q]}_{jk\ell m}) \le c\, 2^{\{k\wedge m+d(j\wedge\ell)\}/2}, \tag{4.4}$$

$$N_{bl}(H^{[P,Q]}_{hjk\ell m}) \le c\, 2^{(dh+k\wedge m)/2}, \tag{4.5}$$

$$N_{bl}(H^{[P,Q,R]}_{hnjk\ell m}) \le c\, 2^{\{d\min(h,j,\ell)+n\}/2}. \tag{4.6}$$

Next,

Lemma 4.3 *Let $P(\xi) = \pm|\xi|^2$, $d \ge 2$. Then,*

$$N_{bl}(H^{[P,P]}_{jk\ell m}) \le c\, 2^{\{k+m+(d-1)(j\wedge\ell)-j\vee\ell\}/2}, \tag{4.7}$$

$$N_{bl}(H^{[P,P]}_{hjk\ell m}) \le c\, 2^{\{k+m+(d-1)\min(h,j,\ell)-j\vee\ell\}/2}. \tag{4.8}$$

Proof. By changing variables $\sigma' = \sigma - P(\eta)$, $\eta_1 - \xi_1/2 = r\cos\theta$, $\eta_2 - \xi_2/2 = r\sin\theta$, $s = \tau - \sigma' \mp 2r^2 \mp |\xi|^2/2 \mp 2|\eta'' - \xi''/2|^2$, where $\xi = (\xi_1, \xi_2, \xi'')$, $\eta = (\eta_1, \eta_2, \eta'')$, ξ_1, ξ_2, η_1, $\eta_2 \in \mathbb{R}$, ξ'', $\eta'' \in \mathbb{R}^{d-2}$, we have

$$\begin{aligned}
&\|H^{[P,P]}_{jk\ell m}(\xi,\tau,\eta,\sigma)\|^2_{L^2_{(\eta,\sigma)}} \\
&\quad\le \int \gamma_j(|\eta''|)\gamma_\ell(|\xi''-\eta''|)d\eta'' \int \gamma_k(\sigma')d\sigma' \int_0^{2\pi} d\theta \\
&\qquad\times \int \gamma_m(\tau - \sigma' \mp 2r^2 \mp |\xi|^2/2 \mp 2|\eta''-\xi''/2|^2) r\, dr \\
&\quad\le 2\pi\cdot 2^k \int \gamma_j(|\eta''|)\gamma_\ell(|\xi''-\eta''|)d\eta'' \int \gamma_m(s)ds \\
&\quad= c\, 2^{k+m+(d-2)(j\wedge\ell)} \qquad (c = 2^{2d-1}\pi).
\end{aligned}$$

Thus we have (4.7) for the case $|j-\ell| < 3 + (\log_2 d)/2$.

Now, we consider the case where $j \ge \ell + 3 + (\log_2 d)/2$. For $\alpha \in \{1, 2, \ldots, d\}$ we define

$$\Gamma_\alpha := \{\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d; |\xi_\alpha| \ge |\xi_\beta| \text{ for every } \beta \in \{1, 2, \ldots, d\}\} \tag{4.9}$$

(the cone in $\mathbb{R}^d$ where α-th coordinate is the largest in absolute value).

Then we see that

$$\sum_{\alpha=1}^{d} \chi_{\Gamma_\alpha}(\xi) = 1 \tag{4.10}$$

holds for almost everywhere. We show

$$\|H^{[P,P]}_{jk\ell m}(\xi,\tau,\eta,\sigma)\chi_{\Gamma_\alpha}(\eta)\|_{L^2_{(\eta,\tau)}} \le c\, 2^{\{k+m-j+(d-1)\ell\}/2},$$

where c is a constant independent of j, k, ℓ, m, α. We may assume that $\alpha = 1$ since $\|H^{[P,P]}_{jk\ell m}(\xi,\tau,\eta,\sigma)\chi_{\Gamma_\alpha}(\eta)\|_{L^2_{(\eta,\tau)}} = \|H^{[P,P]}_{jk\ell m}(\xi,\tau,\eta,\sigma)\chi_{\Gamma_1}(\eta)\|_{L^2_{(\eta,\tau)}}$ follows from Lemma 3.3.

We write $\xi = (\xi_1, \xi')$, $\eta = (\eta_1, \eta')$, ξ_1, $\eta_1 \in \mathbb{R}$, ξ', $\eta' \in \mathbb{R}^{d-1}$. Since $|\eta_1| \ge |\eta|/\sqrt{d}$ when $\gamma_j(|\eta|)\chi_{\Gamma_1}(\eta) \ne 0$, by setting $\sigma' = \sigma - P(\eta)$, we have

$$\begin{aligned}&\iint |H^{[P,P]}_{jk\ell m}(\xi,\tau,\eta,\sigma)\chi_{\Gamma_1}(\eta)|^2 d\eta_1\, d\sigma \\ &\quad\le \int \gamma_k(\sigma')d\sigma' \int_{|\eta_1|\ge 2^{j-1}/\sqrt{d}} \gamma_m(\tau-\sigma'-P(\xi-\eta)-P(\eta))d\eta_1.\end{aligned}$$

Set $\zeta_1 = \tau - \sigma' - P(\xi-\eta) - P(\eta)$. Then we have

$$\Big|\frac{d\zeta_1}{d\eta_1}\Big| = |4\eta_1 - 2\xi_1| \ge \frac{2^{j-1}}{\sqrt{d}} \quad \text{if } |\xi-\eta| < 2^{\ell+1},$$

and consequently

$$\begin{aligned}&\iint |H^{[P,P]}_{jk\ell m}(\xi,\tau,\eta,\sigma)\chi_{\Gamma_1}(\eta)|^2 d\eta\, d\sigma \\ &\quad\le \sqrt{d}\, 2^{-j+1} \int \gamma_\ell(|\xi'-\eta'|)d\eta' \int \gamma_k(\sigma')d\sigma' \int \gamma_m(\zeta_1)d\zeta_1 \\ &= c\, 2^{k+m-j+(d-1)\ell} \qquad (c = \sqrt{d}\, 2^{2d+3}).\end{aligned}$$

Therefore

$$\|H^{[P,P]}_{jk\ell m}(\xi,\tau,\eta,\sigma)\chi_{\Gamma_1}(\eta)\|_{L^2_{(\eta,\sigma)}} \le c\, 2^{\{k+m-j+(d-1)\ell\}/2}.$$

Also, by the same consideration we have

$$\int d\xi' \sup_{\xi_1,\eta',\tau} \iint |H^{[P,P]}_{hjk\ell m}(\xi,\tau,\eta,\sigma)\chi_{\Gamma_1}(\eta)|^2 d\eta_1\, d\sigma \le c\, 2^{k+m-j+(d-1)h},$$

which implies

$$\|H^{[P,P]}_{hjk\ell m}(\xi,\tau,\eta,\sigma)\chi_{\Gamma_1}(\eta)\|_{L^2_{\xi'}\times L^\infty_{(\tau,\xi_1,\eta')}\times L^2_{(\eta_1,\sigma)}} \le c\,2^{\{k+m-j+(d-1)h\}/2}.$$

With the aid of Lemma 3.2 and (4.10), this gives the conclusion of the lemma. □

Lemma 4.4 *Let* $P(\xi)=\pm|\xi|^2$*. Then*

$$N_{bl}(H^{[P,-P]}_{jk\ell m}) \le c\,2^{\{k+m-(j\vee\ell)+(d-1)(j\wedge\ell)\}/2} \tag{4.11}$$

holds when $|j-\ell|\ge 4+(\log_2 d)/2$*, and*

$$N_{bl}(H^{[P,-P]}_{hjk\ell m}) \le c\,2^{\{k+m-h+(d-1)\min(h,j,\ell)\}/2}. \tag{4.12}$$

Proof. Assume that $|j-\ell|\ge 4+(\log_2 d)/2$ and consider (4.11). We may assume that $j\ge \ell+4+(\log_2 d)/2$, since $N_{bl}(H^{[P,-P]}_{hjk\ell m})=N_{bl}(H^{[-P,P]}_{h\ell mjk})$. By changing variables $\zeta_1=\tau-\sigma'+P(\xi-\eta)-P(\eta)$, $\sigma'=\sigma-P(\eta)$, and noting that $\left|d\zeta_1/d\eta_1\right|=|2\xi_1|\ge 2^{j-1}/\sqrt{d}$, (In fact, $|\xi_1|\ge|\eta_1|-|\xi_1-\eta_1|\ge 2^{j-1}/\sqrt{d}-2^{\ell+1}>2^{j-2}/\sqrt{d}$ when $\gamma_j(\eta)\gamma_\ell(\xi-\eta)\chi_{\Gamma_1}(\eta)\ne 0$) we obtain

$$\begin{aligned}&\|H^{[P,-P]}_{jk\ell m}(\xi,\tau,\eta,\sigma)\chi_{\Gamma_1}(\eta)\|^2_{L^2_{(\eta,\sigma)}}\\ &\le c\,2^{-j}\int\tilde\gamma_\ell(|\xi'-\eta'|)d\eta'\int\gamma_k(\sigma')d\sigma'\int\gamma_m(\zeta_1)d\zeta_1\\ &= c\,2^{k+m-j+(d-1)\ell}.\end{aligned}$$

Combining this with (4.10) and $N_{bl}(H^{[P,-P]}_{hjk\ell m}(\xi,\tau,\eta,\sigma)\Gamma_\alpha(\eta)) = N_{bl}(H^{[P,-P]}_{hjk\ell m}(\xi,\tau,\eta,\sigma)\Gamma_1(\eta))$, we obtain (4.11).

Next we consider (4.12). We may assume $h>0$. When $\xi\in\Gamma_1$ we have $\left|d\zeta_1/d\eta_1\right|=|2\xi_1|\ge 2^h/\sqrt{d}$. Therefore,

$$\iint |H^{[P,-P]}_{hjk\ell m}(\xi,\tau,\eta,\sigma)|^2 d\eta_1\,d\sigma \le c\tilde\gamma_j(\eta')\tilde\gamma_\ell(|\xi'-\eta'|)\,2^{k+m-h},$$

which implies

$$\iint |H^{[P,-P]}_{hjk\ell m}(\xi,\tau,\eta,\sigma)|^2 d\eta\,d\sigma \le c\,2^{k+m-h+(d-1)(j\wedge\ell)}. \tag{4.13}$$

In the same way, we have the estimate (4.13) when $\xi \in \Gamma_\alpha$. We also have

$$\int d\xi' \sup_{\tau,\eta'} \iint |H^{[P,-P]}_{hjk\ell m}(\xi,\tau,\eta,\sigma)\chi_{\Gamma_\alpha}(\xi)|^2 d\eta_\alpha\, d\sigma \le c\, 2^{k+m+(d-2)h}, \tag{4.14}$$

where $\xi' = (\xi_1,\dots,\xi_{\alpha-1},\xi_{\alpha+1},\dots,\xi_d)$, $\eta' = (\eta_1,\dots,\eta_{\alpha-1},\eta_{\alpha+1},\dots,\eta_d)$, which implies, with the help of Lemma 3.2,

$$\begin{aligned} N_{bl}(H^{[P,-P]}_{hjk\ell m}) &\le \sum_{\alpha=1}^{d} N_{bl}(H^{[P,-P]}_{hjk\ell m}(\xi,\tau,\eta,\sigma)\chi_{\Gamma_\alpha}(\xi)) \\ &\le c\, 2^{\{k+m+(d-2)h\}/2}. \end{aligned}$$

□

Lemma 4.5 *Let $P(\xi) = \pm|\xi|^2$, Q a real-valued function. Then we have*

$$N_{bl}(H^{[P,Q,P]}_{hnjk\ell m}) \le c\, 2^{\{n+m+(d-1)\min(h,j,\ell)-j\}/2}, \tag{4.15}$$

$$N_{bl}(H^{[P,Q,-P]}_{hnjk\ell m}) \le c\, 2^{\{n+m+(d-1)\min(h,j,\ell)-\max(h,j,\ell)\}/2}, \tag{4.16}$$

$$N_{bl}(H^{[P,-P,-P]}_{hnjk\ell m}) \le c\, 2^{\{n+k\wedge m+(d-1)\min(h,j,\ell)-\max(h,j,\ell)\}/2}, \tag{4.17}$$

$$N_{bl}(H^{[P,P,P]}_{hnjk\ell m}) \le c\, 2^{\{n+k\wedge m-(j\vee\ell)+(d-1)h\}/2}. \tag{4.18}$$

Proof. Since Lemma 3.1 implies that $N_{bl}(H^{[P,Q,P]}_{hnjk\ell m}) = N_{bl}(H^{[Q,P,-P]}_{jkhn\ell m})$, (4.15) follows from (4.12).

By $N_{bl}(H^{[P,Q,-P]}_{hnjk\ell m}) = N_{bl}(H^{[Q,P,P]}_{jkhn\ell m})$ and (4.8) we have

$$N_{bl}(H^{[P,Q,-P]}_{hnjk\ell m}) \le c\, 2^{\{n+m+(d-1)\min(h,j,\ell)-h\vee\ell\}/2}.$$

If $h < j\vee\ell-2$, then $|j-\ell| \le 2$ by Lemma 4.1. Hence $\max(h,j,\ell) \le h\vee\ell+2$. If $h \ge j\vee\ell-2$, then it is clear that $\max(h,j,\ell) \le h\vee\ell+2$. Thus we have (4.16).

Since $N_{bl}(H^{[P,-P,-P]}_{hnjk\ell m}) = N_{bl}(H^{[P,-P,-P]}_{hn\ell mjk})$, (4.17) follows from (4.16).

Finally, we consider (4.18). By (4.15) and $N_{bl}(H^{[P,P,P]}_{hnjk\ell m}) = N_{bl}(H^{[P,P,P]}_{hn\ell mjk})$ we have

$$N_{bl}(H^{[P,P,P]}_{hnjk\ell m}) \le c\, 2^{\{n+(k-\ell)\wedge(m-j)+(d-1)\min\{h,j,\ell\}\}/2}. \tag{4.19}$$

If $h < j\vee\ell-2$, then $|j-\ell| \le 2$, and hence (4.19) implies (4.18).

Also, by (4.19) we have $N_{bl}(H^{[P,P,P]}_{hnjk\ell m}) \le c\, 2^{\{n+k\wedge m+(d-2)(j\wedge\ell)\}/2}$. Therefore (4.18) holds when $h \ge j\vee\ell-2$. □

In this section we use the following symbols: $S^{d-1} = \{\xi \in \mathbb{R}^d; |\xi| = 1\}$. For any $\omega \in S^{d-1}$,

$$D(\nu,\omega) := \{\xi \in S^{d-1}; \arccos(\xi\cdot\omega) \le 2^{-\nu}\},$$
$$E(\nu,\omega) := \{\xi \in S^{d-1}; |\pi/2 - \arccos(\xi\cdot\omega)| < 2^{-\nu+4}\}.$$

Here we take $\arccos 1 = 0$, $0 \le \arccos z \le \pi$.

The following estimate (4.21) is crucial to prove Theorem 1 for the case $Q = P$.

Lemma 4.6 *Let $j \wedge \ell > 0$, and let $P(\xi) = \pm|\xi|^2$. Define*

$$H^{[P,P]}_{jk\ell m\nu}(\xi,\tau,\eta,\sigma) := \varphi_{j+\ell-\nu}((\xi-\eta)\cdot\eta)\gamma^{[P]}_{jk}(\eta,\sigma)\gamma^{[P]}_{\ell m}(\xi-\eta,\tau-\sigma). \tag{4.20}$$

Then we have

$$N_{bl}(H^{[P,P]}_{jk\ell m\nu}) \le c\, 2^{\{k+m+(d-2)(j\wedge\ell)-\nu\}/2}. \tag{4.21}$$

To show this lemma we need the following:

Lemma 4.7 $\gamma_j(|\eta|)\gamma_\ell(|\xi-\eta|)\varphi_{j+\ell-\nu}((\xi-\eta)\cdot\eta)\chi_{D(\nu,\omega)}(|\eta|^{-1}\eta) \ne 0$ *implies* $|\xi-\eta|^{-1}(\xi-\eta) \in E(\nu,\omega)$.

Proof. If $\gamma_j(|\eta|)\gamma_\ell(|\xi-\eta|)\varphi_{j+\ell-\nu}((\xi-\eta)\cdot\eta)\chi_{D(\nu,(1,0,\dots,0))}(|\eta|^{-1}\eta) \ne 0$, then

$$\eta = (|\eta|\cos\theta, \omega|\eta|\sin\theta),\ \omega \in S^{d-2},$$
$$\xi-\eta = (|\xi-\eta|\cos\theta', \omega'|\xi-\eta|\sin\theta'),\ \omega' \in S^{d-2},$$
$$|(\xi-\eta)\cdot\eta| \le 2^{j+\ell-\nu+1},\ 0 \le \theta \le 2^{-\nu}.$$

Hence

$$|\eta|\,|\xi-\eta|\,|\cos(\theta-\theta')| \le 2^{j+\ell-\nu+1} + 2|\xi-\eta|\,|\eta|2^{-\nu},$$

which gives

$$|\cos(\theta-\theta')| \le 2^{-\nu+3} + 2^{-\nu+1} = 5\cdot 2^{-\nu+1}.$$

Therefore, $|\pi/2+\theta-\theta'| < 12\times 2^{-\nu}$, that is, $|\xi-\eta|^{-1}(\xi-\eta) \in E(\nu,(1,0,\dots,0))$.

Assume now that $\gamma_j(|\eta|)\gamma_\ell(|\xi-\eta|)\varphi_{j+\ell-\nu}((\xi-\eta)\cdot\eta)\chi_{D(\nu,\omega)}(|\eta|^{-1}\eta) \ne 0$. Then it follows that $\gamma_j(|T_\omega\eta|)\gamma_\ell(|T_\omega\xi - T_\omega\eta|)\varphi_{j+\ell-\nu}((T_\omega\xi - T_\omega\eta)\cdot T_\omega\eta)$ $\chi_{D(\nu,(1,0,\dots,0))}(|\eta|^{-1}T_\omega\eta) \ne 0$, where T_ω is the rotation with $T_\omega\omega =$

$(1, 0, \ldots, 0)$. Hence, $|\xi - \eta|^{-1} T_\omega(\xi - \eta) \in E(\nu, (1, 0, \ldots, 0))$, so that $|\xi - \eta|^{-1}(\xi - \eta) \in T_\omega^{-1} E(\nu, (1, 0, \ldots, 0)) = E(\nu, \omega)$. □

Proof of Lemma 4.6. As $N_{bl}(H^{[P,P]}_{jk\ell m\nu}) = N_{bl}(H^{[P,P]}_{\ell m jk\nu})$, we may assume that $\ell \geq j > 0$.

Since $N_{bl}(H^{[P,P]}_{jk\ell m\nu}) \leq N_{bl}(H^{[P,P]}_{jk\ell m}) \leq c\, 2^{(k+m+(d-2)(j\wedge\ell))/2}$ follows from (4.7), we also may assume that $\nu \geq 6 + (\log_2 d)/2$.

Define

$$\hat{f}^{[\omega]}_\nu(\xi, \tau) := \chi_{D(\nu,\omega)}(\xi|\xi|^{-1}) \hat{f}(\xi, \tau), \tag{4.22}$$

$$H^{[P,P]}_{jk\ell m\nu,\omega}(\xi, \tau, \eta, \sigma) := \chi_{D(\nu,\omega)}(\eta|\eta|^{-1}) H^{[P,P]}_{jk\ell m\nu}(\xi, \tau, \eta, \sigma). \tag{4.23}$$

Then we have

$$\hat{f}(\xi, \tau) = A_\nu \int_{S^{d-1}} \hat{f}^{[\omega]}_\nu(\xi, \tau) d\omega, \qquad A_\nu = c_d\, 2^{(d-1)\nu}. \tag{4.24}$$

We show first

$$N_{bl}(H^{[P,P]}_{jk\ell m\nu,\omega}) \leq c\, 2^{(k+m+(d-2)j-(d-1)\nu)/2}. \tag{4.25}$$

Since, taking T_ω to be the rotation in ξ-space which goes ω to $(1, 0, \ldots, 0)$, we have $H^{[P,P]}_{jk\ell m\nu,\omega}(T_\omega\xi, \tau, T_\omega\eta, \sigma) = H^{[P,P]}_{jk\ell m\nu,(1,0,\ldots,0)}(\xi, \tau, \eta, \sigma)$, we see by Lemma 3.3 that it is sufficient to consider only the case where $\omega = (1, 0, \ldots, 0)$. The estimate of $N_{bl}(H^{[P,P]}_{jk\ell m\nu,(1,0,\ldots,0)})$ follows from Lemma 3.2 combined with the following inequality

$$\sup_{\xi,\tau} \iint |H^{[P,P]}_{jk\ell m\nu,(1,0,\ldots,0)}(\xi, \tau, \eta, \sigma)|^2 d\eta\, d\sigma \leq c\, 2^{k+m+(d-2)j-(d-1)\nu}, \tag{4.26}$$

which is proved as follows:

Assume that $H^{[P,P]}_{jk\ell m\nu,(1,0,\ldots,0)}(\xi, \tau, \eta, \sigma) \neq 0$. Then, we have $2^{j-1} < |\eta| < 2^{j+1}$, $\eta = (|\eta|\cos\theta, \omega|\eta|\sin\theta)$, $\omega \in S^{d-2}$, $0 \leq \theta < 2^{-\nu}$, hence $\eta_1 \geq |\eta|(1 - 2^{-2\nu}/2) > 2^{j-1}(1 - 2^{-9})$, $|\eta'| \leq |\eta|\, 2^{-\nu}$. Since $|\eta \cdot (\xi - \eta)| < 2^{j+\ell-\nu+1}$, we have $|(\xi_1 - \eta_1)\eta_1| < 2^{j+\ell-\nu+1} + |\xi' - \eta'|\,|\eta'| \leq 2^{j+\ell-\nu+1} + |\xi - \eta|\,|\eta|\, 2^{-\nu}$, which implies that $|\xi_1 - \eta_1| < 2^{\ell-\nu+1}(3 \cdot (1 - 2^{-9})^{-1}) < 2^{\ell-\nu+3}$.

We consider the case where $H^{[P,P]}_{jk\ell m\nu,\omega}(\xi, \tau, \eta, \sigma)\chi_{\Gamma_2}(0, \xi' - \eta') \neq 0$. We use change of variables $\sigma' = \sigma - P(\xi)$, $\zeta_1 = \tau - \sigma' - P(\xi - \eta) - P(\eta)$. Since $|\xi_2 - \eta_2| \geq 2^{\ell-1}/\sqrt{d-1} > 2^{\ell-1-(\log_2 d)/2}$, $|d\zeta_1/d\eta_2| = |2\xi_2 - 4\eta_2| \geq 2|\xi_2 - \eta_2| - 2|\eta_2| \geq 2^{\ell-(\log_2 d)/2} - 2^{j-\nu+2} > 2^{\ell-1}/\sqrt{d}$ when $\nu \geq 5 + (\log_2 d)/2$.

Therefore we have

$$\begin{aligned}
&\sup_{\xi,\tau}\iint |H^{[P,P]}_{jk\ell m\nu,(1,0,\dots,0)}(\xi,\tau,\eta,\sigma)\chi_{\Gamma_2}(0,\xi'-\eta')|^2 d\eta\, d\sigma \\
&\quad\le c\,2^{k+m-\ell}\int_{|\eta''|<2^{j-\nu+1}} d\eta'' \int_{|\xi_1-\eta_1|<2^{\ell-\nu+3}} d\eta_1 \\
&\quad\le c\,2^{k+m-\ell+(d-2)(j-\nu)+\ell-\nu} \\
&\quad= c\,2^{k+m+(d-2)j-(d-1)\nu},
\end{aligned}$$

which leads to (4.26), because

$$H^{[P,P]}_{jk\ell m\nu,\omega}(\xi,\tau,\eta,\sigma)=\sum_{\beta=2}^{d} H^{[P,P]}_{jk\ell m\nu,\omega}(\xi,\tau,\eta,\sigma)\chi_{\Gamma_\beta}(0,\xi'-\eta') \quad \text{a.e.}$$

Next, it follows from Lemma 4.7 that

$$H_{jk\ell m\nu,\omega}(\xi,\tau,\eta,\sigma)=H_{jk\ell m\nu,\omega}(\xi,\tau,\eta,\sigma)\chi_{E(\nu,\omega)}(|\xi-\eta|^{-1}(\xi-\eta)),$$

which implies

$$B(H^{[P,P]}_{jk\ell m\nu};\hat f,\hat g)(\xi,\tau)=A_\nu\int_{S^{d-1}} B(H^{[P,P]}_{jk\ell m\nu,\omega};\hat f^{[\omega]}_\nu,\hat g^{[\omega]}_{\nu *})(\xi,\tau)d\omega, \tag{4.27}$$

where

$$\hat g^{[\omega]}_{\nu *}(\xi,\tau)=\chi_{E(\nu,\omega)}(\xi|\xi|^{-1})\hat g(\xi,\tau). \tag{4.28}$$

Schwarz's inequality gives that

$$\begin{aligned}
&\|B(H^{[P,P]}_{jk\ell m\nu};\hat f,\hat g)\|_{L^2} \\
&\quad\le A_\nu\int_{S^{d-1}}\|B(H^{[P,P]}_{jk\ell m\nu,\omega};\hat f^{[\omega]}_\nu,\hat g^{[\omega]}_{\nu *})\|_{L^2}\,d\omega \\
&\quad\le c2^{(k+m+(d-2)j)/2}\left\{\int_{S^{d-1}}A_\nu\|\hat f^{[\omega]}_\nu\|^2_{L^2}\,d\omega\right\}^{1/2}\left\{\int_{S^{d-1}}\|\hat g^{[\omega]}_{\nu *}\|^2_{L^2}\,d\omega\right\}^{1/2} \\
&\quad\le c'\,2^{(k+m+(d-2)j-\nu)/2}\|f\|_{L^2}\|g\|_{L^2}.
\end{aligned}$$

Here we used the following facts:

$$\int_{S^{d-1}}A_\nu\|\hat f^{[\omega]}_\nu\|^2_{L^2}\,d\omega=\iint|\hat f(\xi,\tau)|^2\,d\xi\,d\tau\int_{D(\nu,|\xi|^{-1}\xi)}A_\nu\,d\omega=\|\hat f\|^2_{L^2},$$

$$\int_{S^{d-1}} \|\hat{g}_{\nu *}^{[\omega]}\|_{L^2}^2 \, d\omega = \iint |\hat{g}(\xi,\tau)|^2 \, d\xi \, d\tau \int_{E(\nu,|\xi|^{-1}\xi)} d\omega \le c\, 2^{-\nu} \|\hat{g}\|_{L^2}^2 .$$

□

Similarly, a crucial estimate to prove Theorem 2 is the following:

Lemma 4.8 *Let $j \wedge \ell > 0$, $h > 0$, and let $P(\xi) = \pm|\xi|^2$. Define*

$$H_{hjk\ell m\nu}^{[P,-P]*}(\xi,\tau,\eta,\sigma) := \varphi_{h+\ell-\nu}((\xi-\eta)\cdot\xi) H_{hjk\ell m}^{[P,-P]}(\xi,\tau,\eta,\sigma). \quad (4.29)$$

Then we have

$$N_{bl}(H_{hjk\ell m\nu}^{[P,-P]*}) \le c\, 2^{\{k+m+(d-2)h\wedge\ell-\nu\}/2}. \quad (4.30)$$

Proof. First, consider the case $h \le \ell$. In view of (4.12) we may assume that $\nu \ge 6$. Writing

$$H_{hjk\ell m\nu,\omega}^{[P,-P]*}(\xi,\tau,\eta,\sigma) = H_{hjk\ell m\nu}^{[P,-P]*}(\xi,\tau,\eta,\sigma)\chi_{D(\nu,\omega)}(|\xi|^{-1}\xi),$$

we have by Lemma 4.7

$$\begin{aligned} &\iint \hat{\psi}(\xi,\tau) B(H_{hjk\ell m\nu}^{[P,-P]*}; \hat{f}, \hat{g})(\xi,\tau) d\xi \, d\tau \quad (4.31)\\ &= A_\nu \iiint_{S^{d-1}} \hat{\psi}_\nu^{[\omega]}(\xi,\tau) B(H_{hjk\ell m\nu,\omega}^{[P,-P]*}; \hat{f}, \hat{g}_{\nu *}^{[\omega]})(\xi,\tau) d\xi \, d\tau \, d\omega \end{aligned}$$

for $\psi \in \mathcal{S}$, where $\psi_\nu^{[\omega]}$ is defined by (4.22) with f replaced by ψ.

Thus the conclusion follows from the fact that

$$N_{bl}(H_{hjk\ell m\nu,\omega}^{[P,-P]*}) \le c\, 2^{(k+m+(d-2)h-(d-1)\nu)/2}. \quad (4.32)$$

In fact, Schwarz's inequality gives that

$$\begin{aligned} &\left| \iint \hat{\psi}(\xi,\tau) B(H_{hjk\ell m\nu}^{[P,-P]*}; \hat{f}, \hat{g}) d\xi \, d\tau \right| \\ &\le c\sqrt{A_\nu}\, 2^{(k+m+(d-2)h)/2} \|\hat{f}\|_{L^2} \int \|\hat{\psi}_\nu^{[\omega]}\|_{L^2} \|\hat{g}_{\nu *}^{[\omega]}\|_{L^2} \, d\omega \\ &\le c'\, 2^{(k+m+(d-2)h-\nu)/2} \|f\|_{L^2} \|\psi\|_{L^2} \|g\|_{L^2}. \end{aligned}$$

Now we proceed to prove (4.32). The estimate (4.32) is verified by Lemma 3.3 if we give the proof for the case $\omega = (1,0,\ldots,0)$, and (4.32) for this case follows from Lemma 3.2 and the inequality

$$\int d\xi' \Big\{ \int d\xi_1 \sup_{\eta',\tau} \iint |H^{[P,-P]*}_{hjk\ell m\nu,(1,0,\dots,0)}(\xi,\tau,\eta,\sigma)|^2 d\eta_1\, d\sigma \Big\} \\ \le c\, 2^{k+m+(d-2)h-(d-1)\nu}, \tag{4.33}$$

which is proved as follows:

Assume that $H^{[P,-P]*}_{hjk\ell m\nu,(1,0,\dots,0)}(\xi,\tau,\eta,\sigma) \neq 0$. Then, we have $|\xi_1| \ge 2^{h-2}$, $|\xi'| \le 2^{h-\nu+2}$. Therefore, by changing variables $\sigma' = \sigma - P(\eta)$, $\zeta = \tau - \sigma' + P(\xi-\eta) - P(\eta)$, in view of the fact that $|d\zeta/d\eta_1| = |2\xi_1| > 2^{h-1}$, we obtain (4.33).

Next, consider the case $\ell \le h$. Using the identity

$$\iint \hat{\psi}(\xi,\tau) B(H^{[P,-P]*}_{hjk\ell m\nu}; \hat{f}, \hat{g})(\xi,\tau) d\xi\, d\tau \\ = A_\nu \iiint_{S^{d-1}} \hat{\psi}^{[\omega]}_{\nu *}(\xi,\tau) B(H^{[P,-P]**}_{hjk\ell m\nu,\omega}; \hat{f}, \hat{g}^{[\omega]}_\nu)(\xi,\tau) d\xi\, d\tau\, d\omega, \tag{4.34}$$

where

$$H^{[P,-P]**}_{hjk\ell m\nu,\omega}(\xi,\tau,\eta,\sigma) = H^{[P,-P]*}_{hjk\ell m\nu}(\xi,\tau,\eta,\sigma)\chi_{D(\nu,\omega)}(|\xi-\eta|^{-1}(\xi-\eta)),$$

(4.30) is a consequence of the inequality

$$N_{bl}(H^{[P,-P]**}_{hjk\ell m\nu,\omega}) \le c\, 2^{(k+m+(d-2)\ell-(d-1)\nu)/2}, \tag{4.35}$$

which can be proved as follows:

Assume that $H^{[P,-P]**}_{hjk\ell m\nu,(1,0,\dots,0)}(\xi,\tau,\eta,\sigma) \neq 0$. Then, we have $|\xi_1 - \eta_1| \ge 2^{\ell-1}(1-2^{-9})$, $|\xi' - \eta'| \le 2^{\ell-\nu+1}$, $|\xi_1| \le 2^{h-\nu+3}$, $|\xi'| \ge 2^{h-2}$. We make use of the identity $H^{[P,-P]**}_{hjk\ell m\nu,(1,0,\dots,0)}(\xi,\tau,\eta,\sigma) = \sum_{\beta=2}^d H^{[P,-P]**}_{hjk\ell m\nu,(1,0,\dots,0)}(\xi,\tau,\eta,\sigma)$ $\chi_{\Gamma_\beta}(0,\xi')$. When $\beta = 2$, we put $\sigma' = \sigma - P(\eta)$, $\zeta = \tau - \sigma' + P(\xi-\eta) - P(\eta)$. Since $|d\zeta/d\eta_2| = |2\xi_2| > 2^{h-2-(\log_2 d)/2}$, we obtain that

$$\int_{|\xi_1| \le 2^{h-\nu+3}} d\xi_1 \sup_{\xi',\eta_1,\tau} \int \tilde{\gamma}_{\ell-\nu}(\xi''-\eta'') d\eta'' \\ \times \iint |H^{[P,-P]**}_{hjk\ell m\nu,(1,0,\dots,0)}(\xi,\tau,\eta,\sigma)\chi_{\Gamma_2}(0,\xi')|^2 d\eta_2\, d\sigma \\ \le c\, 2^{k+m+(d-2)\ell-(d-1)\nu},$$

which means (4.35). □

Lemma 4.9 *Let $j \wedge \ell > 0$, $h > 0$ and let $P(\xi) = \pm|\xi|^2$. Then we have*

$$N_{bl}\big(H^{[P,P,P]}_{hnjk\ell m}(\xi,\tau,\eta,\sigma)\varphi_{j+\ell-\nu}(\eta\cdot(\xi-\eta))\big) \le c\,2^{\{n+k\wedge m+(d-2)j\wedge\ell-\nu\}/2}, \tag{4.36}$$

$$N_{bl}\big(H^{[P,P,-P]}_{hnjk\ell m}(\xi,\tau,\eta,\sigma)\varphi_{\ell+h-\nu}(\xi\cdot(\xi-\eta))\big) \le c\,2^{(n+k\wedge m+(d-2)\min(h,j,\ell)-\nu)/2}. \tag{4.37}$$

Proof. By (4.30), we have

$$\begin{aligned} &N_{bl}\big(H^{[P,P,P]}_{hnjk\ell m}(\xi,\tau,\eta,\sigma)\varphi_{j+\ell-\nu}(\eta\cdot(\xi-\eta))\big)\\ &\quad = N_{bl}\big(H^{[P,P,-P]}_{jkhn\ell m}(\xi,\tau,\eta,\sigma)\varphi_{j+\ell-\nu}(\xi\cdot(\xi-\eta))\big)\\ &\quad \le c\,2^{\{n+m+(d-2)j\wedge\ell-\nu\}/2}. \end{aligned}$$

Since $N_{bl}\big(H^{[P,P,P]}_{hnjk\ell m}\varphi_{j+\ell-\nu}(\eta\cdot(\xi-\eta))\big) = N_{bl}\big(H^{[P,P,P]}_{hn\ell mjk}\varphi_{j+\ell-\nu}(\eta\cdot(\xi-\eta))\big)$, we also have $N_{bl}\big(H^{[P,P,P]}_{hnjk\ell m}\varphi_{j+\ell-\nu}(\eta\cdot(\xi-\eta))\big) \le c\,2^{\{n+k+(d-2)j\wedge\ell-\nu\}/2}$.

Next we consider (4.37). By (4.30), we see that

$$\begin{aligned} &N_{bl}\big(H^{[P,P,-P]}_{hnjk\ell m}(\xi,\tau,\eta,\sigma)\varphi_{\ell+h-\nu}(\xi\cdot(\xi-\eta))\big)\\ &\quad = N_{bl}\big(H^{[P,-P,P]}_{hn\ell mjk}(\xi,\tau,\eta,\sigma)\varphi_{\ell+h-\nu}(\xi\cdot\eta)\big)\\ &\quad = N_{bl}\big(H^{[-P,P,-P]}_{\ell mhnjk}(\xi,\tau,\eta,\sigma)\varphi_{\ell+h-\nu}(\xi\cdot\eta)\big)\\ &\quad = N_{bl}\big(H^{[-P,-P,P]}_{\ell mjkhn}(\xi,\tau,\eta,\sigma)\varphi_{\ell+h-\nu}(\xi\cdot(\xi-\eta))\big)\\ &\quad \le N_{bl}\big(H^{[-P,P]}_{\ell jkhn}(\xi,\tau,\eta,\sigma)\varphi_{\ell+h-\nu}(\xi\cdot(\xi-\eta))\big)\\ &\quad \le c\,2^{(n+k+(d-2)h\wedge\ell-\nu)/2}. \end{aligned}$$

Using (4.21), we obtain that

$$\begin{aligned} &N_{bl}\big(H^{[P,P,-P]}_{hnjk\ell m}(\xi,\tau,\eta,\sigma)\varphi_{\ell+h-\nu}(\xi\cdot(\xi-\eta))\big)\\ &\quad = N_{bl}\big(H^{[P,P,P]}_{jkhn\ell m}(\xi,\tau,\eta,\sigma)\varphi_{\ell+h-\nu}(\eta\cdot(\xi-\eta))\big)\\ &\quad \le N_{bl}\big(H^{[P,P]}_{hn\ell m}(\xi,\tau,\eta,\sigma)\varphi_{\ell+h-\nu}(\eta\cdot(\xi-\eta))\big)\\ &\quad \le c\,2^{\{n+m+(d-2)(h\wedge\ell)-\nu\}/2}. \end{aligned}$$

Hence, we get (4.37) if $h\wedge\ell$ can be replaced by $\min(h,j,\ell)$. To do so we may assume that $\nu \ge 6$ because of (4.16), and show that $h\wedge\ell \le \min(h,j,\ell)+4$, which can be done as follows: Obviously, it is sufficient to consider the case

$j = \min(h, j, \ell)$. If $h < \ell - 2$, then Lemma 4.1 says that $|j - \ell| \leq 2$, and hence $j \geq \ell - 2$. If $h \geq \ell - 2$, it follows that $2^{j+h+2} \geq |\xi \cdot \eta| \geq |\xi|^2 - |\xi \cdot (\xi - \eta)| > 2^{2h-2} - 2^{h+\ell-\nu+1} > 2^{2h-3}$, so that $j \geq h - 4$. $\square$

5. Proof of Theorem 1. Part I.

In this section we put $\varphi_{hn}^{[P]}(\xi,\tau) := \varphi_h(|\xi|)\varphi_n(\tau - P(\xi))$, and write $\hat{f}_{jk,Q}(\xi,\tau) := \varphi_{jk}^{[Q]}(\xi,\tau)\hat{f}(\xi,\tau)$. Then we have $f = \sum_{j,k} f_{jk,Q}$, $g = \sum_{\ell,m} g_{\ell m,Q}$, which give

$$fg = \sum_{j,k,\ell,m} f_{jk,Q}\, g_{\ell m,Q}. \tag{5.1}$$

We need the following lemmas:

Lemma 5.1 (Lemma 5.2 in [5]) *Let* $P(\xi) = \pm|\xi|^2$. *Assume that* $H_{hnjk\ell m}^{[P,-P,-P]}(\xi,\tau,\eta,\sigma) \not\equiv 0$. *Then, we have* $\max(n,k,m) \geq 2(j \vee \ell) - 3$.

Lemma 5.2 (A special case of Lemma 3.1 in [5]) *Let* $c_1 = \sup_j \sum_\ell |c_{j\ell}|$, $c_2 = \sup_\ell \sum_j |c_{j\ell}|$. *Then we have*

$$\left| \sum_j \sum_\ell c_{j\ell} a_j b_\ell \right| \leq \sqrt{c_1 c_2}\, \|\{a_j\}\|_{\ell^2} \|\{b_\ell\}\|_{\ell^2}. \tag{5.2}$$

Let f, $g \in \mathcal{S}(\mathbb{R}^{d+1})$. To prove Theorem 1 we divide the norm of the terms of the formula (5.1) into five parts in consideration of Lemma 4.1. That is, we set

$$\begin{cases} I(1) := \{(h,n,j,k,\ell,m);\ k \vee m \leq n \leq 4(j+\ell), \\ \qquad\qquad\qquad\qquad\qquad j,\ \ell \geq 5,\ 0 \leq h \leq j \vee \ell - 5\}, \\ I(2) := \{(h,n,j,k,\ell,m);\ k \vee m \leq n \leq 4(j+\ell), \\ \qquad\qquad\qquad\qquad\qquad j,\ \ell \geq 5,\ j \vee \ell - 4 \leq h \leq j \vee \ell + 2\}, \\ I(3) := \{(h,n,j,k,\ell,m);\ k \vee m > n,\ n \leq 4(j+\ell),\ j,\ \ell \geq 5, \\ \qquad\qquad\qquad\qquad\qquad 0 \leq h \leq j \vee \ell - 5\}, \\ I(4) := \{(h,n,j,k,\ell,m);\ k \vee m > n,\ n \leq 4(j+\ell),\ j,\ \ell \geq 5, \\ \qquad\qquad\qquad\qquad\qquad j \vee \ell - 4 \leq h \leq j \vee \ell + 2\}, \\ I(5) := \{(h,n,j,k,\ell,m);\ n > 4(j+\ell),\ j,\ \ell \geq 5\} \\ \qquad\quad \cup \{(h,n,j,k,\ell,m);\ j \leq 5 \text{ or } \ell \leq 5\}, \end{cases} \tag{5.3}$$

and put $F_\nu := \sum_{I(\nu)} \rho(2^h)\, 2^{-n/2} \|\varphi_{hn}^{[P]} \hat{f}_{jk,Q} * \hat{g}_{\ell m,Q}\|_{L^2}$, $\nu = 1, \dots, 5$. Then, by (5.1) we have

$$\|fg\|_{B^{(\rho,-1/2)}_{2,1,P}} \le F_1 + F_2 + F_3 + F_4 + F_5. \tag{5.4}$$

Now we proceed to **prove Theorem 1 for the case $Q = -P$.**

It follows from (4.8), Lemma 4.1 and Lemma 5.1 that

$$\begin{aligned} F_1 &\le c \sum_{|j-\ell|\le 2} \sum_{k,m} \sum_{n \ge 2(j\vee\ell)-3} \sum_{h=0}^{j\vee\ell-5} \rho(2^h)\, 2^{(-n+k+m-(j\vee\ell)+(d-1)h)/2} \\ &\quad \times \|f_{jk,-P}\|_{L^2} \|g_{\ell m,-P}\|_{L^2} \\ &\le c' \sum_{|j-\ell|\le 2} \sum_{k,m} \sum_{h=0}^{j\vee\ell-5} \rho(2^h)\, 2^{(k+m-3(j\vee\ell)+(d-1)h)/2} \\ &\quad \times \|f_{jk,-P}\|_{L^2} \|g_{\ell m,-P}\|_{L^2}. \end{aligned}$$

When $d = 2$ this gives that $F_1 \le c\|f\|_{B^{(s,1/2)}_{2,(2,1),-P}} \|g\|_{B^{(s,1/2)}_{2,(2,1),-P}}$ for any $s \ge -3/4$, since

$$\sum_h (h+1)\, 2^{(s+1/2)h} \le \begin{cases} c < +\infty & \text{if } s < -1/2, \\ c(j \vee \ell + 1)^2\, 2^{(s+1/2)(j\vee\ell)} & \text{if } s \ge -1/2, \end{cases} \tag{5.5}$$

and

$$\sum_{|j-\ell|\le 2} a_j b_\ell \le c\|\{a_j\}\|_{\ell^2} \|\{b_\ell\}\|_{\ell^2}. \tag{5.6}$$

When $d \ge 3$ F_1 can be bounded by $c\min\{\|f\|_{B^{(s,1/2)}_{2,(2,1),-P}} \|g\|_{B^{(\rho,1/2)}_{2,(2,1),-P}}, \|f\|_{B^{(\rho,1/2)}_{2,(2,1),-P}} \|g\|_{B^{(s,1/2)}_{2,(2,1),-P}}\}$ for $s = d/2 - 2$, since $|j - \ell| \le 2$ and (5.6). For $s > d/2 - 2$ F_1 is estimated by $\|f\|_{B^{(s,1/2)}_{2,(2,1),-P}} \|g\|_{B^{(s,1/2)}_{2,(2,1),-P}}$ since $(j+1)\, 2^{-\delta j} \le C(\delta) < \infty$ if $\delta > 0$.

Lemma 5.1 and (4.7) imply that

$$F_2 \le c \sum_{j,\ell} \sum_{k,m} \sum_{n \ge 2(j\vee\ell)-3} \sum_{h=j\vee\ell-4}^{j\vee\ell+2} \rho(2^h)\, 2^{\{-n+k+m-(j\vee\ell)+(d-1)(j\wedge\ell)\}/2}$$

$$
\begin{aligned}
&\times \|f_{jk,-P}\|_{L^2}\|g_{\ell m,-P}\|_{L^2}\\
&\le c' \sum_{j,\ell}\sum_{k,m}(j\vee\ell+1)\,2^{(k+m)/2+s(j\vee\ell)+(d/2-1/2)(j\wedge\ell)-3(j\vee\ell)/2}\\
&\times \|f_{jk,-P}\|_{L^2}\|g_{\ell m,-P}\|_{L^2}\\
&\le c'' \sum_{j,\ell}\sum_{k,m}(j\wedge\ell+1)\,2^{(k+m)/2+s(j+\ell)+(d/2-2-s)(j\wedge\ell)-|j-\ell|}\\
&\times \|f_{jk,-P}\|_{L^2}\|g_{\ell m,-P}\|_{L^2},
\end{aligned}
$$

since

$$
\frac{j\vee\ell+1}{j\wedge\ell+1}\,2^{-\delta|j-\ell|}\le C(\delta)\quad\text{if } \delta>0, \tag{5.7}
$$

where $C(\delta)$ is a constant depend only on δ.

When $d\ge 3$, $s=d/2-2$, we have by Lemma 5.2 $F_2\le c\|f\|_{B^{(\rho,1/2)}_{2,(2,1),-P}}\|g\|_{B^{(s,1/2)}_{2,(2,1),-P}}$, since $\sum_j 2^{-|j-\ell|}<3$, $\sum_\ell 2^{-|j-\ell|}<3$.

When $d=2$, $s\ge -3/4$ or when $d\ge 3$, $s>d/2-2$, we have $F_2\le c\|f\|_{B^{(s,1/2)}_{2,(2,1),-P}}\|g\|_{B^{(s,1/2)}_{2,(2,1),-P}}$, since $(j+1)\,2^{-\delta j}$ is bounded if $\delta>0$.

By (4.17), Lemma 4.1 and Lemma 5.1, we see that

$$
\begin{aligned}
F_3 &\le c\sum_{|j-\ell|\le 2}\ \sum_{k\vee m\ge 2(j\vee\ell)-3}\ \sum_{n\le 4(j+\ell)}\ \sum_{h=0}^{j\vee\ell-5}\rho(2^h)\\
&\quad\times 2^{(-n+n+k\wedge m-(j\vee\ell)+(d-1)h)/2}\|f_{jk,-P}\|_{L^2}\|g_{\ell m,-P}\|_{L^2}\\
&\le c'\sum_{|j-\ell|\le 2}\ \sum_{k,m}\ \sum_{h=0}^{j\vee\ell-5}(4(j+\ell)+1)\rho(2^h)\,2^{(k+m-3(j\vee\ell)+(d-1)h)/2}\\
&\quad\times \|f_{jk,-P}\|_{L^2}\|g_{\ell m,-P}\|_{L^2}.
\end{aligned}
$$

When $d=2$ we have by this estimate and (5.5) that $F_3\le c\min\{\|f\|_{B^{(s,1/2)}_{2,(2,1),-P}}\|g\|_{B^{(\rho,1/2)}_{2,(2,1),-P}}, \|f\|_{B^{(\rho,1/2)}_{2,(2,1),-P}}\|g\|_{B^{(s,1/2)}_{2,(2,1),-P}}\}$ if $s=-3/4$, and that $F_3\le c\|f\|_{B^{(s,1/2)}_{2,(2,1),-P}}\|g\|_{B^{(s,1/2)}_{2,(2,1),-P}}$ if $s>-3/4$.

When $d\ge 3$ we have

$$
\begin{aligned}
F_3 &\le c\sum_{|j-\ell|\le 2}\ \sum_{k,m}(j\vee\ell+1)^2\,2^{s(j\vee\ell)+(k+m)/2+(d/2-2)(j\vee\ell)}\\
&\quad\times \|f_{jk,-P}\|_{L^2}\|g_{\ell m,-P}\|_{L^2}.
\end{aligned}
$$

Thus when $s=d/2-2$ we have $F_3 \le c\|f\|_{B^{(\rho,1/2)}_{2,(2,1),-P}}\|g\|_{B^{(\rho,1/2)}_{2,(2,1),-P}}$, and when $s>d/2-2$ we have $F_3 \le c\|f\|_{B^{(s,1/2)}_{2,(2,1),-P}}\|g\|_{B^{(s,1/2)}_{2,(2,1),-P}}$, since $(j+1)^2\,2^{-\delta j}$ is bounded if $\delta>0$.

It follows from (4.17) and Lemma 5.1 that

$$\begin{aligned} F_4 &\le c\sum_{j,\,\ell}\sum_{k\vee m\ge 2(j\vee\ell)-3}\sum_{n\le 4(j+\ell)}\sum_{h=j\vee\ell-4}^{j\vee\ell+2}\rho(2^h)\\ &\quad\times 2^{\{k\wedge m+(d-1)(j\wedge\ell)-j\vee\ell\}/2}\|f_{jk,-P}\|_{L^2}\|g_{\ell m,-P}\|_{L^2}\\ &\le c'\sum_{j,\,\ell}\sum_{k,\,m}(4(j+\ell)+1)\rho(2^{j\vee\ell})\,2^{\{k+m+(d-1)(j\wedge\ell)-3(j\vee\ell)\}/2}\\ &\quad\times\|f_{jk,-P}\|_{L^2}\|g_{\ell m,-P}\|_{L^2}. \end{aligned}$$

Hence, by (5.7) we have

$$\begin{aligned} F_4 &\le c\sum_{j,\,\ell}\sum_{k,\,m}(j\vee\ell+1)^2\,2^{(k+m)/2+s(j+\ell)+(d/2-2-s)(j\wedge\ell)-3|j-\ell|/2}\\ &\quad\times\|f_{jk,-P}\|_{L^2}\|g_{\ell m,-P}\|_{L^2}\\ &\le c'\sum_{j,\,\ell}\sum_{k,\,m}(j\wedge\ell+1)^2\,2^{(k+m)/2+s(j+\ell)+(d/2-2-s)(j\wedge\ell)-|j-\ell|}\\ &\quad\times\|f_{jk,-P}\|_{L^2}\|g_{\ell m,-P}\|_{L^2}. \end{aligned}$$

Therefore, when $d\ge 3$, F_4 is estimated in the same way as F_3.

Estimate of F_5 follows from the lemma below.

Lemma 5.3 *Let P, Q, R be real-valued C^∞-functions, $\rho(z)=z^s\log(2+z)$, $s\ge d/2-2$. Then*

$$\begin{aligned} &\sum_{j,\,\ell}\sum_{k,\,m}\sum_{n\ge 4(j+\ell)-40}\sum_{h}\rho(2^h)\,2^{-n/2}\|\varphi_{hn,P}(\hat f_{jk,Q}*\hat g_{\ell m,R})\|_{L^2}\\ &\quad\le c\|f\|_{B^{(s,1/2)}_{2,(2,1),Q}}\|g\|_{B^{(s,1/2)}_{2,(2,1),R}}. \end{aligned} \tag{5.8}$$

Proof. It follows from (4.4) that

$$\sum_{j,\,\ell}\sum_{k,\,m}\sum_{n\ge 4(j+\ell)-40}\sum_{h}\rho(2^h)\,2^{-n/2}\|\varphi_{hn,P}(\hat f_{jk,Q}*\hat g_{\ell m,R})\|_{L^2}$$

$$\leq c \sum_{j,\ell} \sum_{k,m} \sum_{h} \rho(2^h)\, 2^{-2(j+\ell)+\{k\wedge m+d(j\wedge\ell)\}/2} \|f_{jk,Q}\|_{L^2} \|g_{\ell m,R}\|_{L^2}.$$

For the case $s < 0$ (5.8) is proved by the observations that $\sum_h \rho(2^h)$ is bounded and that $-2(j+\ell)+(d/2)(j\wedge\ell) = (d/2-2)(j+\ell)-(d/2)(j\vee\ell)$.

For the case $s \geq 0$ (5.8) follows from the observations that $\sum_{h=0}^{j\vee\ell+2} \rho(2^h) \leq c(j\vee\ell+1)^2\, 2^{s(j\vee\ell)}$, and that $-2(j+\ell)+(d/2)(j\wedge\ell) = (d/2-2)(j\wedge\ell)-2(j\vee\ell)$.

The above estimates also give (2.7). (See proof of Theorem 5.1 in [5]).

□

6. Proof of Theorem 1. Part II

Here we give the proof of Theorem 1 for the case $Q = P$. We start with

Lemma 6.1 *Let* $P(\xi) = \pm|\xi|^2$. *Assume that* $H^{[P,P,P]}_{hnjk\ell m\nu}(\xi,\tau,\eta,\sigma) \not\equiv 0$. *Then, we have* $\max(n,k,m) \geq j+\ell-\nu-2$.

Moreover, if $h < j\vee\ell-5$ *and* $H^{[P,P,P]}_{hnjk\ell m}(\xi,\tau,\eta,\sigma) \not\equiv 0$, *then we have* $\max(n,k,m) \geq j+\ell-4$.

Proof. Assume that $H^{[P,P,P]}_{hnjk\ell m\nu}(\xi,\tau,\eta,\sigma) \neq 0$. Then we see that

$$\begin{aligned} 2^{j+\ell-\nu} &< 2|\eta\cdot(\xi-\eta)| < |\tau'| + |\tau'-\sigma' \pm 2(\xi-\eta)\cdot\eta| + |\sigma'| \\ &< 2^{n+1} + 2^{m+1} + 2^{k+1} \leq 3\cdot 2^{\max(n,k,m)+1}, \end{aligned}$$

where $\tau' = \tau - P(\xi)$, $\sigma' = \sigma - P(\eta)$, so that we have $\max(n,k,m) \geq j+\ell-\nu-2$.

Assume that $h < j\vee\ell-5$. When $j \geq \ell$, $|(\xi-\eta)\cdot\eta| \geq |\eta|^2 - |\xi|\,|\eta| \geq 2^{2j-2} - 2^{h+j+2} \geq 2^{j+\ell-3}$. When $\ell \geq j$, $|(\xi-\eta)\cdot\eta| \geq |\xi-\eta|^2 - |\xi|\,|\xi-\eta| \geq 2^{2\ell-2} - 2^{h+\ell+2} \geq 2^{j+\ell-3}$. So we see $|(\xi-\eta)\cdot\eta| \geq 2^{j+\ell-3}$. This implies that

$$\begin{aligned} 2^{j+\ell-2} &\leq 2|\eta\cdot(\xi-\eta)| \leq |\tau'| + |\tau'-\sigma' \pm 2(\xi-\eta)\cdot\eta| + |\sigma'| \\ &< 2^{n+1} + 2^{m+1} + 2^{k+1} \leq 3\cdot 2^{\max(n,k,m)+1}. \end{aligned}$$

Therefore we have $\max(n,k,m) \geq j+\ell-4$. This completes the proof of Lemma 6.1.

□

Let $f,\, g \in \mathcal{S}(\mathbb{R}^{d+1})$, and use the inequality (5.4) with $Q = P$.

It follows from (4.8), Lemma 4.1 and Lemma 6.1 that

$$F_1 \le c \sum_{|j-\ell|\le 2} \sum_{k,m} \sum_{n\ge j+\ell-4} \sum_{h=0}^{j\vee\ell-5} \rho(2^h)\, 2^{(-n+k+m-(j\vee\ell)+(d-1)h)/2} \times \|f_{jk,P}\|_{L^2}\|g_{\ell m,P}\|_{L^2}.$$

This estimate is the same one as for the case $Q=-P$, since $|j-\ell|\le 2$. Therefore we have the same estimate of F_1 as for the case $Q=-P$.

F_2 is estimated in the same way as F_4 in the case $Q=-P$, since (4.21) together with Lemma 6.1 gives

$$\begin{aligned} F_2 &\le c \sum_{j,\ell} \sum_{k,m} \sum_{\nu=-2}^{j+\ell} \sum_{n\ge j+\ell-\nu-2} \sum_{h=j\vee\ell-4}^{j\vee\ell+2} \rho(2^h) \\ &\quad\times 2^{\{-n+k+m+(d-2)(j\wedge\ell)-\nu\}/2}\|f_{jk,P}\|_{L^2}\|g_{\ell m,P}\|_{L^2} \\ &\le c' \sum_{j,\ell} \sum_{k,m} (j\vee\ell+1)\rho(2^{j\vee\ell})\, 2^{\{k+m+(d-2)(j\wedge\ell)-j-\ell\}/2} \\ &\quad\times \|f_{jk,P}\|_{L^2}\|g_{\ell m,P}\|_{L^2} \\ &\le c'' \sum_{j,\ell} \sum_{k,m} (j\vee\ell+1)^2\, 2^{(k+m)/2+s(j+\ell)+(d/2-2-s)(j\wedge\ell)-|j-\ell|/2} \\ &\quad\times \|f_{jk,P}\|_{L^2}\|g_{\ell m,P}\|_{L^2}. \end{aligned}$$

We have the same estimate for F_3 as for F_1 in the case $Q=-P$, since (4.18) combined with Lemma 4.1 and Lemma 6.1 implies that

$$F_3 \le c \sum_{|j-\ell|\le 2} \sum_{k,m} \sum_{h=0}^{j\vee\ell-5} (j\vee\ell+1)\rho(2^h)\, 2^{(k+m-3(j\vee\ell)+(d-1)h)/2} \times \|f_{jk,P}\|_{L^2}\|g_{\ell m,P}\|_{L^2}.$$

It follows from (4.36) and Lemma 6.1 that

$$\begin{aligned} F_4 &\le c \sum_{j,\ell} \sum_{k\vee m>n} \sum_{n\le 4(j+\ell)} \sum_{\nu} \sum_{k\vee m\ge j+\ell-\nu-2} \sum_{h=j\vee\ell-4}^{j\vee\ell+2} \rho(2^h) \\ &\quad\times 2^{\{k\wedge m+(d-2)(j\wedge\ell)-\nu\}/2}\|f_{jk,P}\|_{L^2}\|g_{\ell m,P}\|_{L^2} \\ &\le c' \sum_{j,\ell} \sum_{k\vee m>n} \sum_{n\le 4(j+\ell)} \sum_{\nu\ge j+\ell-k\vee m-2} \rho(2^{j\vee\ell}) \end{aligned}$$

$$\begin{aligned}&\times 2^{\{k\wedge m+(d-2)(j\wedge\ell)-\nu\}/2}\|f_{jk,P}\|_{L^2}\|g_{\ell m,P}\|_{L^2}\\ \le\ & c''\sum_{j,\ell}\sum_{k,m}(j\vee\ell+1)\rho(2^{j\vee\ell})\,2^{\{k+m+(d-2)(j\wedge\ell)-j-\ell\}/2}\\ &\times\|f_{jk,P}\|_{L^2}\|g_{\ell m,P}\|_{L^2}.\end{aligned}$$

Therefore F_4 can be estimated in the same way as F_2.

Estimate of F_5 follows from Lemma 5.3. □

7. Proof of Theorem 2.

To prove Theorem 2 we need the following lemma:

Lemma 7.1 *Let* $P(\xi)=\pm|\xi|^2$.
(a) $H^{[P,P,-P]}_{hnjk\ell m}(\xi,\tau,\eta,\sigma)\not\equiv 0$ *implies that* $n\le\max(h+\ell+2,k,m)+3$. *Moreover,* $\max\{k,m,n\}>2\ell-4$ *if* $j\le\ell-5$.
(b) $H^{[P,P,-P]}_{hnjk\ell m}(\xi,\tau,\eta,\sigma)\varphi_{h+\ell-\nu}(\xi\cdot(\xi-\eta))\not\equiv 0$ *implies that* $\max(n,k,m)\ge h+\ell-\nu-2$.

Proof. Let $H^{[P,P,-P]}_{hnjk\ell m}(\xi,\tau,\eta,\sigma)\neq 0$. Since

$$\begin{aligned}2^{n-1}&<|\tau'|\le|\tau'-\sigma'\pm 2(\xi-\eta)\cdot\xi|+|\sigma'|+2|\xi\cdot(\xi-\eta)|\\ &<3\cdot 2^{\max(m,k,h+\ell+2)+1},\end{aligned}$$

we have $n\le\max(m,k,h+\ell+2)+3$, where $\tau'=\tau-P(\xi)$, $\sigma'=\sigma-P(\eta)$.

Assume that $j\le\ell-5$. Then we have
$2|\xi\cdot(\xi-\eta)|\ge 2|\xi-\eta|^2-2|\eta|\cdot|\xi-\eta|>2^{2\ell-1}-2^{j+\ell+3}\ge 2^{2\ell-2}$,
which gives $3\cdot 2^{\max\{k,m,n\}+1}\ge|\tau'|+|\tau'-\sigma'\pm 2(\xi-\eta)\cdot\xi|+|\sigma'|\ge 2|\xi\cdot(\xi-\eta)|>2^{2\ell-2}$.
Hence we have $\max\{k,m,n\}>2\ell-4$ if $j\le\ell-5$.

Assume that $H^{[P,P,-P]}_{hnjk\ell m}(\xi,\tau,\eta,\sigma)\varphi_{h+\ell-\nu}(\xi\cdot(\xi-\eta))\neq 0$. Then we see that $2^{h+\ell-\nu}<2|\xi\cdot(\xi-\eta)|<|\tau'|+|\tau'-\sigma'\pm 2(\xi-\eta)\cdot\xi|+|\sigma'|<3\cdot 2^{\max(n,k,m)+1}$, so that we have $\max(n,k,m)>h+\ell-\nu-3$. □

Now we proceed to **prove Theorem 2.** Let f, $g\in\mathcal{S}(\mathbb{R}^{d+1})$, and define

$$G_{0i}:=c_d\sum_{(h,n,j,k,\ell,m)\in I(0i)}\rho(2^h)\,2^{-n/2}\|\varphi^{[P]}_{hn}\hat f_{jk,P}*\hat g_{\ell m,-P}\|_{L^2},\qquad\text{for } i=0,1,2,3,\tag{7.1}$$

$$G_i := c_d \sum_{(h,n,j,k,\ell,m)\in I(i)} \rho(2^h)\, 2^{-n/2} \|\varphi_{hn}^{[P]} \hat{f}_{jk,P} * \hat{g}_{\ell m,-P}\|_{L^2}, \quad \text{for } i = 1,2,3, \tag{7.2}$$

where

$$\begin{cases} I(00) := \{(h,n,j,k,\ell,m);\ j \le 2,\ \ell \le 2\}, \\ I(01) := \{(h,n,j,k,\ell,m);\ j \le 2,\ \ell \ge 3\}, \\ I(02) := \{(h,n,j,k,\ell,m);\ j \ge 3,\ \ell \le 2\}, \\ I(03) := \{(h,n,j,k,\ell,m);\ h = 0,\ j \ge 3,\ \ell \ge 3\}, \\ I(1) := \{(h,n,j,k,\ell,m);\ k \vee m \le n \le 8(h+\ell), \\ \qquad\qquad j > \ell - 5 - \log_2 d/2,\ j \ge 3,\ \ell \ge 3,\ h \ge 1\}, \\ I(2) := \{(h,n,j,k,\ell,m);\ k \vee m > n,\ n \le 8(h+\ell), \\ \qquad\qquad j \ge 3,\ \ell \ge 3,\ h \ge 1\}, \\ I(3) := \{(h,n,j,k,\ell,m);\ k \vee m \le n \le 8(h+\ell), \\ \qquad\qquad 3 \le j \le \ell - 5 - \log_2 d/2\}, \\ I(4) := \{(h,n,j,k,\ell,m);\ n > 8(h+\ell),\ j \ge 3,\ \ell \ge 3,\ h \ge 1\}, \end{cases} \tag{7.3}$$

and $c_d = (2\pi)^{-d/2}$. Then $\|fg\|_{B^{(\rho,-1/2)}_{2,1,P}} \le \sum_{i=0}^{3} G_{0i} + \sum_{i=1}^{4} G_i$.

Estimate of G_{00}. By (4.4), it is easy to see that

$$\begin{aligned} G_{00} &\le c \sum_{I(00)} \rho(2^h)\, 2^{(-n+k\wedge m)/2} \|f_{jk,P}\|_{L^2} \|g_{\ell m,-P}\|_{L^2} \\ &\le c' \|f\|_{B^{(s,1/2)}_{2,(2,1),P}} \|g\|_{B^{(s,1/2)}_{2,(2,1),-P}}. \end{aligned}$$

Estimate of G_{01} and G_{02}. By (4.12) and Lemma 4.1 we have

$$\begin{aligned} G_{01} &\le c \sum_{j=0}^{2} \sum_{\ell \ge 3} \sum_{k,m} \sum_{h=\ell-2}^{\ell+2} \sum_{n} \rho(2^h)\, 2^{(-n+k+m-h)/2} \\ &\quad \times \|f_{jk,P}\|_{L^2} \|g_{\ell m,-P}\|_{L^2} \\ &\le c' \sum_{j=0}^{2} \sum_{\ell \ge 3} \sum_{k,m} \rho(2^\ell)\, 2^{(k+m-\ell)/2} \|f_{jk,P}\|_{L^2} \|g_{\ell m,-P}\|_{L^2} \\ &\le c'' \|f\|_{B^{(s,1/2)}_{2,(2,1),P}} \|g\|_{B^{(s,1/2)}_{2,(2,1),-P}}. \end{aligned}$$

In the same way we have $G_{02} \le c\|f\|_{B^{(s,1/2)}_{2,(2,1),P}}\|g\|_{B^{(s,1/2)}_{2,(2,1),-P}}$.

Estimate of G_{03}. Since $N_{bl}(H^{[P,P,-P]}_{hnjk\ell m}) = N_{bl}(H^{[P,-P,P]}_{hn\ell mjk})$, it follows from Lemma 4.1, (4.15) and (4.16) that

$$\begin{aligned} G_{03} &\le c\sum_{j\ge 3}\sum_{\ell\ge 3}\sum_{k,m}\sum_{n} 2^{-n/2}\|\varphi^{[P]}_{0n}\hat f_{jk,P} * \hat g_{\ell m,-P}\|_{L^2} \\ &\le c'\sum_{|j-\ell|\le 2}\sum_{k,m}\sum_{n} 2^{(k\wedge m - j\vee\ell)/2}\|f_{jk,P}\|_{L^2}\|g_{\ell m,-P}\|_{L^2}. \end{aligned}$$

Hence, by Lemma 7.1 we have

$$\begin{aligned} G_{03} &\le c\sum_{|j-\ell|\le 2}\sum_{k\vee m\le \ell+2}\sum_{n=0}^{\ell+5} 2^{(k\wedge m - j\vee\ell)/2}\|f_{jk,P}\|_{L^2}\|g_{\ell m,-P}\|_{L^2} \\ &\quad + c\sum_{|j-\ell|\le 2}\sum_{k\vee m>\ell+2}\sum_{n=0}^{k\vee m+3} 2^{(k\wedge m - j\vee\ell)/2}\|f_{jk,P}\|_{L^2}\|g_{\ell m,-P}\|_{L^2} \\ &\le c'\sum_{k,m}\sum_{|j-\ell|\le 2}\big\{(\ell+6)\,2^{(k\wedge m - j\vee\ell)/2} \\ &\quad + (k\vee m+4)\,2^{(k+m-j\vee\ell-k\vee m)/2}\big\}\|f_{jk,P}\|_{L^2}\|g_{\ell m,-P}\|_{L^2}. \end{aligned}$$

Therefore we have $G_{03} \le c\min\{\|f\|_{B^{(s,1/2)}_{2,(2,1),P}}\|g\|_{B^{(\rho,1/2)}_{2,(2,1),-P}}, \|f\|_{B^{(\rho,1/2)}_{2,(2,1),P}}\|g\|_{B^{(s,1/2)}_{2,(2,1),-P}}\}$ when $s = -1/4$ and $G_{03} \le c\|f\|_{B^{(s,1/2)}_{2,(2,1),P}}\|g\|_{B^{(s,1/2)}_{2,(2,1),-P}}$ when $s > -1/4$.

Estimate of G_1 and G_2. Put $H^{[P,P,-P]}_{hnjk\ell m\nu}(\xi,\tau,\eta,\sigma) = H^{[P,P,-P]}_{hnjk\ell m}(\xi,\tau,\eta,\sigma)\,\varphi_{h+\ell-\nu}(\xi\cdot(\xi-\eta))$. Then we have

$$|\varphi^{[P]}_{hn}(\xi,\tau)\hat f_{jk,P} * \hat g_{\ell m,-P}(\xi,\tau)| \le \sum_{\nu=-2}^{h+\ell} B(H^{[P,P,-P]}_{hnjk\ell m\nu}; |\hat f_{jk,P}|, |\hat g_{\ell m,-P}|), \tag{7.4}$$

which implies, with the aid of Lemma 7.1 and (4.30) that

$$G_1 \le c\Big\{\sum_{|j-\ell|\le 2}\sum_{h=1}^{j\vee\ell-3} + \sum_{j,\ell\ge 3}\sum_{h=j\vee\ell-2}^{j\vee\ell+2}\Big\}\rho(2^h)\,2^{(d-2)\min(h,j,\ell)/2}J_{hj\ell,1},$$

$$J_{hj\ell,1} := \sum_{k,\,m} \sum_{\nu=-2}^{h+\ell} \sum_{n \geq h+\ell-\nu-2} 2^{\{-n+k+m-\nu\}/2} \|f_{jk,P}\|_{L^2} \|g_{\ell m,-P}\|_{L^2}$$
$$\leq c' \sum_{k,\,m} (h+\ell+3)\, 2^{(k+m-\ell-h)/2} \|f_{jk,P}\|_{L^2} \|g_{\ell m,-P}\|_{L^2}.$$

(Note that $h \wedge \ell \leq \min\{h, j, \ell\} + 6 + \log_2 d/2$)

Also, it follows from Lemma 7.1 and (4.37) that

$$G_2 \leq c\Big\{ \sum_{|j-\ell|\leq 2} \sum_{h=1}^{j\vee\ell-3} + \sum_{j,\,\ell\geq 3} \sum_{h=j\vee\ell-2}^{j\vee\ell+2} \Big\} \rho(2^h)\, 2^{(d-2)\min(h,j,\ell)/2} J_{hj\ell,2},$$

$$J_{hj\ell,2} := \sum_{n\leq 8(h+\ell)} \sum_{\nu=-2}^{h+\ell} \sum_{k\vee m \geq h+\ell-\nu-2} 2^{(-n+n+k\wedge m-\nu)/2}$$
$$\times \|f_{jk,P}\|_{L^2} \|g_{\ell m,-P}\|_{L^2}$$
$$\leq c \sum_{k,\,m} \sum_{\nu \geq h+\ell-k\vee m-2} (8(h+\ell)+1)\, 2^{(-\nu+k\wedge m)/2}$$
$$\times \|f_{jk,P}\|_{L^2} \|g_{\ell m,-P}\|_{L^2}$$
$$\leq c' \sum_{k,\,m} (8(h+\ell)+1)\, 2^{(k+m-h-\ell)/2} \|f_{jk,P}\|_{L^2} \|g_{\ell m,-P}\|_{L^2}.$$

Thus we have $G_1 + G_2 \leq c(H_1 + H_2)$, where

$$H_1 := \sum_{|j-\ell|\leq 2} \sum_{h=1}^{j\vee\ell-3} \sum_{k,m} (\ell+h+3)\rho(2^h)\, 2^{\{k+m+(d-2)\min(h,j,\ell)-h-\ell\}/2}$$
$$\times \|f_{jk,P}\|_{L^2} \|g_{\ell m,-P}\|_{L^2}$$

$$H_2 := \sum_{j,\,\ell\geq 3} \sum_{h=j\vee\ell-2}^{j\vee\ell+2} \sum_{k,m} (\ell+h+3)\rho(2^h)\, 2^{\{k+m+(d-2)\min(h,j,\ell)-h-\ell\}/2}$$
$$\times \|f_{jk,P}\|_{L^2} \|g_{\ell m,-P}\|_{L^2}.$$

The case $d = 2$. This gives $G_1 + G_2 \leq c \min\{\|f\|_{B^{(s,1/2)}_{2,(2,1),P}} \|g\|_{B^{(\rho,1/2)}_{2,(2,1),-P}}, \|f\|_{B^{(\rho,1/2)}_{2,(2,1),P}} \|g\|_{B^{(s,1/2)}_{2,(2,1),-P}}\}$ when $s = -1/4$ and $G_1 + G_2 \leq c\|f\|_{B^{(s,1/2)}_{2,(2,1),P}} \|g\|_{B^{(s,1/2)}_{2,(2,1),-P}}$ when $s > -1/4$. In fact, if $s < 1/2$ $\sum_{h=1}^{j\vee\ell-3}(\ell+h+3)\rho(2^h)\, 2^{(-h-\ell)/2} \leq c(2(j\vee\ell)+1)\, 2^{-\ell/2}$, and if $s \geq 1/2$ $\sum_{h=1}^{j\vee\ell-3}(\ell+h+3)\, \rho(2^h)\, 2^{(-h-\ell)/2} \leq c(j\vee\ell+1)^3\, 2^{s\ell-\ell}$.

Also, we have $\sum_{h=j\vee\ell-2}^{j\vee\ell+2}(\ell + h + 3)\rho(2^h)\, 2^{(-h-\ell)/2} \le c(j \vee \ell + 2)^2\, 2^{(s-1/2)(j\vee\ell)-\ell/2} \le c'\, 2^{s(j+\ell)-|j-\ell|/4}$.

For the case $d = 3$ this also gives the same estimate of $G_1 + G_2$. In fact, we have

$$\begin{aligned}
&\sum_{h=1}^{j\vee\ell-3} (\ell + h + 3)\rho(2^h)\, 2^{\{-h-\ell+\min(h,j,\ell)\}/2} \\
&\quad \le c \sum_{h=1}^{j\vee\ell-3} (\ell + h + 3)\rho(2^h)\, 2^{-\ell/2} \\
&\quad \le c' \begin{cases} (2(j \vee \ell) + 4)\, 2^{-\ell/2} & \text{if } s < 0, \\ (j \vee \ell + 1)^3\, 2^{-(j\vee\ell)/4+s(j+\ell)} & \text{if } s \ge 0, \end{cases} \qquad \text{and} \\
&\sum_{h=j\vee\ell-2}^{j\vee\ell+2} (\ell + h + 3)\rho(2^h)\, 2^{\{-h-\ell+j\wedge\ell)\}/2} \\
&\quad \le c(j \vee \ell + 2)\rho(2^{j\vee\ell})\, 2^{-(j\vee\ell)/2} \\
&\quad \le c'\, 2^{s(j+\ell)-|j-\ell|/4}.
\end{aligned}$$

The case $d \ge 4$. Since

$$\begin{aligned}
&\sum_{h=1}^{j\vee\ell-3} (\ell + h + 3)\rho(2^h)\, 2^{\{-h-\ell+(d-2)\min(h,j,\ell)\}/2} \\
&\quad \le c(j \vee \ell + 1)\rho(2^{j\vee\ell})\, 2^{\{-\ell+(d-3)(j\vee\ell)\}/2},
\end{aligned}$$

we have

$$H_1 \le c \sum_{|j-\ell|\le 2} \sum_{k,\,m} (j + 1)^2\, 2^{(s+d/2-2)j+(k+m)/2} \|f_{jk,P}\|_{L^2} \|g_{\ell m,-P}\|_{L^2},$$

which gives

$$H_1 \le c \begin{cases} \|f\|_{B^{(\rho,1/2)}_{2,(2,1),P}} \|g\|_{B^{(\rho,1/2)}_{2,(2,1),-P}} & \text{when } s = d/2 - 2, \\ \|f\|_{B^{(s,1/2)}_{2,(2,1),P}} \|g\|_{B^{(s,1/2)}_{2,(2,1),-P}} & \text{when } s > d/2 - 2. \end{cases}$$

Also from

$$\sum_{h=j\vee\ell-2}^{j\vee\ell+2} (\ell + h + 3)\rho(2^h)\, 2^{\{-h-\ell+(d-2)\min(h,j,\ell)\}/2}$$

$$\leq c(j \vee \ell + 1)\rho(2^{j\vee\ell})\, 2^{(d/2-1)(j\wedge\ell)-(j\vee\ell)/2-\ell/2},$$

it follows that

$$H_2 \leq c \sum_{j,\,\ell} \sum_{k,\,m} (j \vee \ell + 1)^2\, 2^{s(j+\ell)+(d/2-2-s)(j\wedge\ell)-|j-\ell|/2+(k+m)/2} \times \|f_{jk,P}\|_{L^2} \|g_{\ell m,-P}\|_{L^2}.$$

Therefore H_2 can be estimated in the same way as F_2 in §5.

Estimate of G_3. Note first that $|\ell - h| \leq 2$ if $j \leq \ell - 5 - \log_2 d/2$. Lemma 7.1 and (4.11) give

$$\begin{aligned} G_3 &\leq c \sum_{j,\,\ell} \sum_{k,\,m} \sum_{h=\ell-2}^{\ell+2} \sum_{n\geq 2h-4} \rho(2^h)\, 2^{\{-n+k+m-\ell+(d-1)j\}/2} \\ &\quad \times \|f_{jk,P}\|_{L^2} \|g_{\ell m,-P}\|_{L^2} \\ &\leq c' \sum_{j,\,\ell} \sum_{k,\,m} (\ell+1)\, 2^{s\ell+(d/2-2)j-3|\ell-j|/2+(k+m)/2} \\ &\quad \times \|f_{jk,P}\|_{L^2} \|g_{\ell m,-P}\|_{L^2} \\ &\leq c'' \sum_{j,\,\ell} \sum_{k,\,m} (j+1)\, 2^{s\ell+(d/2-2)j-|\ell-j|+(k+m)/2} \\ &\quad \times \|f_{jk,P}\|_{L^2} \|g_{\ell m,-P}\|_{L^2}, \end{aligned}$$

which gives

$$G_3 \leq c \begin{cases} \min\{\|f\|_{B^{(\rho,1/2)}_{2,(2,1),P}} \|g\|_{B^{(s,1/2)}_{2,(2,1),-P}}, \|f\|_{B^{(s,1/2)}_{2,(2,1),P}} \|g\|_{B^{(\rho,1/2)}_{2,(2,1),-P}}\} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{when } s = d/2-2, \\ \|f\|_{B^{(s,1/2)}_{2,(2,1),P}} \|g\|_{B^{(s,1/2)}_{2,(2,1),-P}} \qquad\qquad \text{when } s > d/2-2. \end{cases}$$

Finally, by Lemma 5.3 we have $G_4 \leq c\|f\|_{B^{(s,1/2)}_{2,(2,1),P}} \|g\|_{B^{(s,1/2)}_{2,(2,1),-P}}$. In fact, if $h < j \vee \ell - 2$, then $|j-\ell| \leq 2$, hence $8(h+\ell) \geq 8\ell \geq 4(j+\ell) - 8$, and if $h \geq j \vee \ell - 2$, then $8(h+\ell) \geq 4(j+\ell) - 8$. □

8. Outline of proof of Main Results

Main results follow from Theorem 1 and Theorem 2 by making use of the method employed in [5]. However, for the sake of completeness we write here outline of their proof. Consider the case where $d = 2$, $N(u, \bar{u}) =$

$c_1u^2 + c_2\bar{u}^2$. Let

$$B(f,g) := \int_0^t W(t-t')\{c_1 f(x,t')g(x,t') + c_2\overline{f(x,t')g(x,t')}\}dt', \tag{8.1}$$

where $W(t)$ is defined by $\{W(t)f\}(x,t) := \mathcal{F}_x^{-1}e^{itP(\xi)}\mathcal{F}_x f(x,t)$. Then, by Theorem 1 we have

$$\|B(f,g)\|_{B^{(\rho,1/2)}_{2,1,P}(\mathbb{R}^d\times I)} \le C\|f\|_{B^{(\rho,1/2)}_{2,(2,1),P}(\mathbb{R}^d\times I)}\|g\|_{B^{(s,1/2)}_{2,(2,1),P}(\mathbb{R}^d\times I)} \tag{8.2}$$

holds for any $f \in B^{(\rho,1/2)}_{2,(2,1),P}(\mathbb{R}^d\times I)$, $g \in B^{(s,1/2)}_{2,(2,1),P}(\mathbb{R}^d\times I)$, and

$$\begin{aligned}\|B(f,g)\|&_{B^{(\rho,1/2)}_{2,1,P}(\mathbb{R}^d\times I)} \\ &\le C\big\{\|f\|_{B^{(s,b)}_{2,(2,1),P}(\mathbb{R}^d\times I)}\|g\|_{B^{(s,1/2)}_{2,(2,1),P}(\mathbb{R}^d\times I)} \\ &\qquad + \|f\|_{B^{(s,1/2)}_{2,(2,1),P}(\mathbb{R}^d\times I)}\|g\|_{B^{(s,b)}_{2,(2,1),P}(\mathbb{R}^d\times I)}\big\}\end{aligned} \tag{8.3}$$

holds for any f, $g \in B^{(s,b)}_{2,(2,1),P}(\mathbb{R}^d\times I)$ if $b > 1/2$. (see [5]) Here $I = (-a,a)$, $0 < a \le 1$, $P(\xi) = \pm|\xi|^2$, $s = -3/4$, $\rho(z) = z^s\log(2+z)$, and C is a constant independent of a, f and g. With the aid of the fixed point theorem, the existence of solutions to the integral equation

$$u(x) = W(t)u_0(x) + \int_0^t W(t-t')N(u,\bar{u})(x,t')dt', \tag{8.4}$$

follows from (8.2) for $u_0 \in H^\rho(\mathbb{R}^2)$. But, putting $u(x,t) = W(t)u_0(x) + v(x,t)$, we have

$$\begin{aligned} v = {}& B(W(t)u_0(x), W(t)u_0(x)) + B(W(t)u_0(x), v) \\ &+ B(v, W(t)u_0(x)) + B(v,v).\end{aligned}$$

Hence, by using (8.2) and (8.3), we can solve this equation for $u_0 \in H^{-3/4}(\mathbb{R}^2)$ with small norm. To remove the smallness assumption on the initial data, we employ the scaling method, that is, we put $v(x,t) = u(\delta x, \delta^2 t)$ and solve the equation for v.

Note that we could not remove the smallness assumption by the scaling method when $d = 3$, $s = -1/2$. But when $s > -1/2$ by Theorem 1 for any

$f,\ g \in B^{(s,1/2)}_{2,(2,1),P}(\mathbb{R}^d \times I)$ we have

$$\|B(f,g)\|_{B^{(s,1/2)}_{2,(2,1),P}(\mathbb{R}^d\times I)} \le C\|f\|_{B^{(s,1/2)}_{2,(2,1),P}(\mathbb{R}^d\times I)}\|g\|_{B^{(s,1/2)}_{2,(2,1),P}(\mathbb{R}^d\times I)}, \tag{8.5}$$

and we can use the scaling method to obtain local well-posedness of the Cauchy problem (1.1) in $H^s(\mathbb{R}^3)$.

Finally, we mention that our proof of the uniqueness needs the following facts (see [5]):

Let $b \ge 1/2$, and $I = (-a, a)$, $a > 0$. Then, for any $f \in B^{(\rho,b)}_{2,(2,1),P}(\mathbb{R}^d \times I)$ with $f(x, 0) = 0$, $\|f|_{\mathbb{R}^d\times(-\delta,\delta)}\|_{B^{(\rho,b)}_{2,(2,1),P}(\mathbb{R}^d\times(-\delta,\delta))} \to 0$ as $\delta \to +0$.

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