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Holonomic systems of general Clairaut type

Masatomo Takahashi

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Abstract. In this papar we consider an important class of first order partial differential equations (or, holonomic systems). The notion of general Clairaut type equations is one of the generalized notions of classical Clairaut equations. We give a generic classification of holonomic systems of general Clairaut type as an application of the theory of complete Legendrian unfoldings.

Key words: holonomic system, holonomic system of general Clairaut type, Legendrian singularity theory.

1. Introduction

In the classical theory of first order partial differential equations, the notion of complete solutions plays an important role. The Clairaut equation is one of the typical examples of first order differential equations with classical complete solutions. We say that a system of first order partial differential equations is general Clairaut type if it has a classical complete solution. In [5, 6, 7, 9], the system of general Clairaut type with a regular property (which are called systems of Clairaut type) has been investigated. In particular, a generic classification and characterization of holonomic systems of Clairaut type are given in [7]. Moreover, a generic classification of first order ordinary differential equations of general Clairaut type is given in [3].

In this paper we give a generic classification of holonomic systems of general Clairaut type in any dimension. Since our concern is the local classification of differential equations, we can formulate as follows (cf. [7]): Let $\pi\colon PT^*\mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ be the projective cotangent bundle over $\mathbb{R}^{n+1}$. We have a local coordinate $(x, y, p) = (x_1, \dots, x_n, y, p_1, \dots, p_n)$ of $PT^*\mathbb{R}^{n+1}$ such that $(x_1, \dots, x_n, y)$ gives the canonical coordinate of $\mathbb{R}^{n+1}$ and the hyperplane in $T_{(x,y)}\mathbb{R}^{n+1}$ given by $dy - \sum_{i=1}^{n} p_i\, dx_i = 0$. This coordinate is called *the canonical coordinate* of $PT^*\mathbb{R}^{n+1}$. The *canonical contact form* on

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canonical coordinate of $PT^*\mathbb{R}^{n+1}$ is defined by $\theta = dy - \sum_{i=1}^{n} p_i\, dx_i$. Using this approach, a first order differential equation is most naturally interpreted as being a closed subset of $PT^*\mathbb{R}^{n+1}$. We consider that a *holonomic system of first order differential equation germ* (or, briefly, a *holonomic system*) is defined to be a smooth map germ $f\colon (\mathbb{R}^{n+1}, 0) \longrightarrow PT^*\mathbb{R}^{n+1}$. The term "general" means that f is not necessarily a germ of immersion. We say that f is *completely integrable* if there exsists a submersion germ $\mu\colon (\mathbb{R}^{n+1}, 0) \longrightarrow (\mathbb{R}, 0)$ such that $d\mu \wedge f^*\theta = 0$. We call μ a *complete integral* of f and the pair $(\mu, f)\colon (\mathbb{R}^{n+1}, 0) \longrightarrow \mathbb{R} \times PT^*\mathbb{R}^{n+1}$ is called *a holonomic system with complete integral.* If $\pi \circ f|_{\mu^{-1}(s)}$ is a non-singular map for each $s \in (\mathbb{R}, 0)$, f is called a *holonomic system of general Clairaut type.* Furthermore, if f is an immersion germ, then we call f a *holonomic system of Clairaut type* (cf. [7, 9]). A first order ordinary differential equation of general Clairaut type $(\mu, f)\colon (\mathbb{R}^2, 0) \longrightarrow \mathbb{R} \times PT^*\mathbb{R}^2$ (i.e., $n = 1$) has appeared in [3] such that f has the Whitney umbrella singularity.

We now consider divergent diagrams. Let (μ, g) be a pair of germs of a map $g\colon (\mathbb{R}^{n+1}, 0) \longrightarrow (\mathbb{R}^{n+1}, 0)$ and a submersion $\mu\colon (\mathbb{R}^{n+1}, 0) \longrightarrow (\mathbb{R}, 0)$. Then the divergent diagram

$$(\mathbb{R}, 0) \xleftarrow{\ \mu\ } (\mathbb{R}^{n+1}, 0) \xrightarrow{\ g\ } (\mathbb{R}^{n+1}, 0)$$

or briefly (μ, g), is called a (*holonomic*) *integral diagram* if there exists a holonomic system $f\colon (\mathbb{R}^{n+1}, 0) \longrightarrow PT^*\mathbb{R}^{n+1}$ such that (μ, f) is a holonomic system with complete integral and $\pi \circ f = g$.

We introduce an equivalence relation among integral diagrams. Let (μ, g) and (μ', g') be integral diagrams. Then (μ, g) and (μ', g') are *equivalent as integral diagram* (respectively, *strictly equivalent*) if the diagram

$$\begin{array}{ccccc}
(\mathbb{R}, 0) & \xleftarrow{\ \mu\ } & (\mathbb{R}^{n+1}, 0) & \xrightarrow{\ g\ } & (\mathbb{R}^{n+1}, 0) \\
\kappa \big\downarrow & & \psi \big\downarrow & & \big\downarrow \phi \\
(\mathbb{R}, 0) & \xleftarrow[\ \mu'\ ]{} & (\mathbb{R}^{n+1}, 0) & \xrightarrow[\ g'\ ]{} & (\mathbb{R}^{n+1}, 0)
\end{array}$$

commutes for some germs of diffeomorphisms κ, ψ and ϕ (respectively, $\kappa = id_{\mathbb{R}}$). We also consider a natural equivalence relation among holonomic systems. Let f and f' be holonomic system of equations. Then f and f' are *equivalent as holonomic system* if the diagram

$$\begin{array}{ccccc} (\mathbb{R}^{n+1},0) & \xrightarrow{f} & (PT^*\mathbb{R}^{n+1},z) & \xrightarrow{\pi} & (\mathbb{R}^{n+1},0) \\ \psi\Big\downarrow & & \Big\downarrow\Phi & & \Big\downarrow\phi \\ (\mathbb{R}^{n+1},0) & \xrightarrow[f']{} & (PT^*\mathbb{R}^{n+1},z') & \xrightarrow[\pi]{} & (\mathbb{R}^{n+1},0) \end{array}$$

commutes for germs of diffeomorphisms ψ, ϕ and a contact diffeomorphism Φ (that is, Φ is a diffeomorphism and preserving contact structure). We shall show that two holonomic systems f and f' are equivalent as holonomic system if and only if induced integral diagrams $(\mu, \pi \circ f)$ and $(\mu', \pi \circ f')$ are equivalent for generic (μ, f) and (μ', f') (cf. Theorem 2.3). The main result in this paper is the following theorem which gives a generic classification of holonomic systems of general Clairaut type:

Theorem 1.1 *For a generic holonomic system of general Clairaut type*

$$(\mu, f)\colon (\mathbb{R}^{n+1},0) \longrightarrow \mathbb{R} \times PT^*\mathbb{R}^{n+1},$$

the integral diagram $(\mu, \pi \circ f)$ is strictly equivalent to one of germs in the following list:

DA_1: $\mu = u_{n+1}$, $g = (u_1, \dots, u_n, u_{n+1})$.

DA_2: $\mu = u_{n+1} - \frac{1}{2}u_1$, $g = (u_1, \dots, u_n, u_{n+1}^2)$.

DA_2^0: $\mu = u_{n+1} + \frac{1}{2}(u_1^2 + \cdots + u_n^2)$, $g = (u_1, \dots, u_n, u_{n+1}^2)$.

DA_2^k $(1 \le k \le n)$: $\mu = u_{n+1} - \frac{1}{2}(u_1^2 + \cdots + u_k^2 - u_{k+1}^2 - \cdots - u_n^2)$,
$g = (u_1, \dots, u_n, u_{n+1}^2)$.

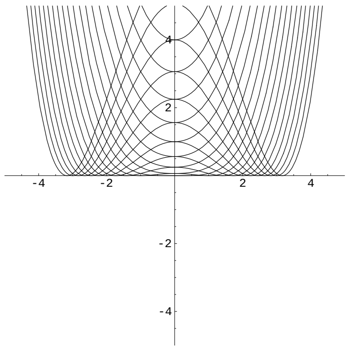
DA_ℓ $(3 \le \ell \le n+1)$: $\mu = u_{n+1}$,

$$g = \left(u_1, \dots, u_n, u_{n+1}^\ell + \sum_{i=1}^{\ell-1} u_i u_{n+1}^i\right).$$

$\widetilde{DA}_{n+2}$: $\mu = u_{n+1} + \alpha \circ g$ *for* $\alpha \in \mathfrak{M}_{(x,y)}$,

$$g = \left(u_1, \dots, u_n, u_{n+1}^{n+2} + \sum_{i=1}^{n} u_i u_{n+1}^i\right).$$

We call the germ of function $\alpha\colon (\mathbb{R}^{n+1},0) \longrightarrow (\mathbb{R},0)$ which appears in the normal forms $\widetilde{DA}_{n+2}$, a *functional moduli*. In this case, we have the characterization of functional moduli under the equivalence in [7]. The meaning of the genericity in the above theorem will be discribed in §2 (cf. Theorem 2.2). The normal forms DA_ℓ $(1 \leq \ell \leq n+1)$ and $\widetilde{DA}_{n+2}$ are holonomic systems of Clairaut type which have been already classified in [7]. The normal form DA_2^k $(0 \leq k \leq n)$ is the new one which contains many interesting new equations. If we take germs of diffeomorphism $\kappa = -id_{\mathbb{R}}$, $\psi(u_1, \dots, u_n, u_{n+1}) = (u_1, \dots, u_n, -u_{n+1})$ and $\phi = id_{\mathbb{R}}$, then DA_2^k and DA_2^{n-k} are equivalent. For $n = 1$, DA_2^0 and DA_2^1 are equivalent. In this case, f has the singularity of the Whitney umbrella, this equation is called the *Clairaut Whitney umbrella* in [3]. The phase portrait $\{\pi \circ f(\mu^{-1}(c))\}_{c \in \mathbb{R}}$ of the Clairaut Whitney umbrella is depicted as follows:



We now give typical examples of holonomic systems of general Clairaut type.

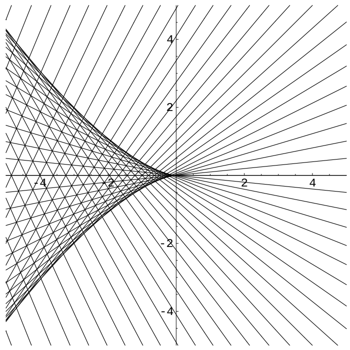
Example 1.2 (The holonomic Clairaut systems)

The holonomic Clairaut systems are given by $f\colon (\mathbb{R}^{n+1}, 0) \longrightarrow PT^*\mathbb{R}^{n+1}$;

$$f(u_1, \dots, u_{n+1}) = \left(u_1, \dots, u_n, \sum_{i=1}^{n} \gamma_i(u_{n+1}) u_i + g(\gamma(u_{n+1})), \gamma_1(u_{n+1}), \dots, \gamma_n(u_{n+1})\right)$$

and $\mu\colon (\mathbb{R}^{n+1}, 0) \longrightarrow (\mathbb{R}, 0)$; $\mu(u_1, \dots, u_{n+1}) = u_{n+1}$ where $\gamma\colon (\mathbb{R}, 0) \longrightarrow \mathbb{R}^n$ is an immersion germ and $g\colon (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$ is a smooth function germ. For more detailed properties, see [6, 7]. In this case, the complete solution $\pi \circ f(\mu^{-1}(c))$ $(c \in (\mathbb{R}, 0))$ is a family of affine hyperplanes.

If we take $\gamma(s) = (s, s^2, \dots, s^n)$ and $g(p_1, \dots, p_n) = p_1^{n+2}$, then it represents the normal form $\widetilde{DA}_{n+2}$ where $\alpha = 0$. The picture of the phase portrait $\{\pi \circ f(\mu^{-1}(c))\}_{c \in \mathbb{R}}$ for $n = 1$ is as follows:



Example 1.3 Here we consider the holonomic systems of integral diagram of DA_2^0 and DA_2^1 for $n = 2$. In the case of DA_2^0, the holonomic system f is the generalized cross cap singularity and the phase portrait is given by

$$\pi \circ f(\mu^{-1}(c)) = \Big(u_1, u_2, \Big(c - \frac{1}{2}(u_1^2 + u_2^2)\Big)^2\Big).$$

We can draw these pictures when $c = 1,\ 0,\ -1$, respectively in Fig. 1 and superimpose these pictures (that is, the phase portrait) $\{\pi \circ f(\mu^{-1}(c))\}_{c\in\mathbb{R}}$ of DA_2^0, see Fig. 2.

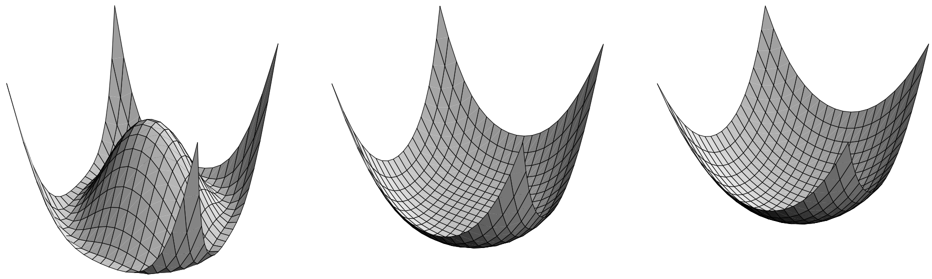
On the other hand, for DA_2^1, the phase portrait is given by

$$\pi \circ f(\mu^{-1}(c)) = \Big(u_1, u_2, \Big(c + \frac{1}{2}(u_1^2 - u_2^2)\Big)^2\Big).$$

We also draw these pictures when $c = 1,\ 0,\ -1$, respectively in Fig. 3 and superimpose these pictures (that is, the phase portrait) $\{\pi \circ f(\mu^{-1}(c))\}_{c\in\mathbb{R}}$ of DA_2^1, see Fig. 4.

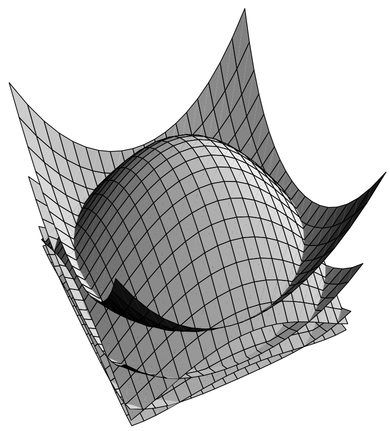
In §2, we prepare some basic tools and describe the meaning of the genericity. We also consider an equivalence relation among holonomic systems and the corresponding equivalence relation among integral diagrams. We consider generating families correspond to complete Legendrian unfolding associated to holonomic system of general Clairaut type (μ, f) in §3. For the proof of Theorem 1.1 in §4, we use the unfolding theory of function germs.

All germs of maps considered here are of class C^∞, unless stated otherwise.



$c = -1$ $c = 0$ $c = 1$

Fig. 1.



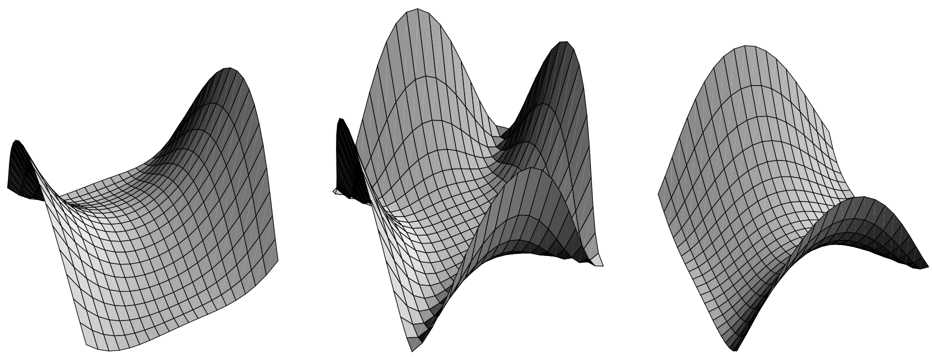
The phase portrait $\{\pi \circ f(\mu^{-1}(c))\}_{c\in\mathbb{R}}$ of DA_2^0 for $n = 2$.

Fig. 2.

2. Preparations

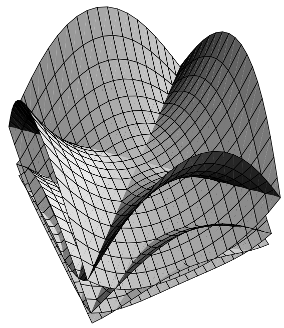
In this section we review some results on holonomic systems of general Clairaut type and establish the notion of the genericity.

We can construct a family of Legendrian immersions depending on (μ, f) as follows (see [7]): We consider the projective contangent bundle $\Pi\colon PT^*(\mathbb{R}\times\mathbb{R}^{n+1}) \longrightarrow \mathbb{R}\times\mathbb{R}^{n+1}$. Let $(s, x_1, \dots, x_n, y)$ be the canonical coordinate on $\mathbb{R}\times\mathbb{R}^{n+1}$ and $(s, x_1, \dots, x_n, y, q, p_1, \dots, p_n)$ be the corresponding local coordinate on $PT^*(\mathbb{R}\times\mathbb{R}^{n+1})$. Then the contact 1-form is given by $\Theta = dy - \sum_{i=1}^n p_i\, dx_i - q\, ds = \theta - q\, ds$. Let $(\mu, f)\colon (\mathbb{R}^{n+1}, 0) \longrightarrow \mathbb{R}\times PT^*\mathbb{R}^{n+1}$ be a holonomic system with complete integral, then there exsists a unique element $h \in \mathcal{E}_u$ such that $f^*\theta = h\, d\mu$, where $\mathcal{E}_u$ is the ring of function germs



$c=-1$ $c=0$ $c=1$

Fig. 3.



The phase portrait $\{\pi \circ f(\mu^{-1}(c))\}_{c\in\mathbb{R}}$ of DA_2^1 for $n=2$.

Fig. 4.

of $u=(u_1,\dots,u_{n+1})$-variables. Define a map germ

$$\ell_{(\mu,f)}\colon (\mathbb{R}^{n+1},0) \longrightarrow PT^*(\mathbb{R}\times\mathbb{R}^{n+1})$$

by

$$\ell_{(\mu,f)}(u) = \big(\mu(u), x_1\circ f(u),\dots,x_n\circ f(u)\, y\circ f(u), h(u), \\ p_1\circ f(u),\dots,p_n\circ f(u)\big).$$

If (μ,f) is a holonomic system of general Clairaut type, we can easily show that $\ell_{(\mu,f)}$ is a Legendrian immersion germ (that is, $\ell_{(\mu,f)}$ is an immersion germ with ${\ell_{(\mu,f)}}^*\Theta=0$). We call $\ell_{(\mu,f)}$ a *complete Legendrian unfolding associated to* (μ,f). By the aid of the notion of Legendrian unfoldings, holonomic systems of general Clairaut type are characterized as follows:

Proposition 2.1 *Let* $(\mu, f)\colon (\mathbb{R}^{n+1}, 0) \longrightarrow \mathbb{R} \times PT^*\mathbb{R}^{n+1}$ *be honolomic system with complete integral. Then* (μ, f) *is a holonomic system of general Clairaut type if and only if* $\Pi \circ \ell_{(\mu,f)}$ *is non-singular.*

The proof follows from a direct analogy of the proof for Proposition 4.1 in [5], so that we omit it.

We now establish the notion of *the genericity*. Let $U \subset \mathbb{R}^{n+1}$ be an open set. We denote by $\mathrm{Clair}(U, \mathbb{R} \times PT^*\mathbb{R}^{n+1})$ the set of holonomic system of general Clairaut type $(\mu, f)\colon U \longrightarrow \mathbb{R} \times PT^*\mathbb{R}^{n+1}$. We also define $L_R(U, PT^*(\mathbb{R} \times \mathbb{R}^{n+1}))$ to be the set of complete Legendrian unfolding $\ell_{(\mu,f)}\colon U \longrightarrow PT^*(\mathbb{R} \times \mathbb{R}^{n+1})$ such that $\Pi \circ \ell_{(\mu,f)}$ is non-singular.

These sets are topological spaces equipped with the Whitney C^∞-topology. A subset of either spaces is said to be *generic* if it is an open dense subset in the space.

The genericity of a property of germs are defined as follows: Let P be a property of holonomic system of general Clairaut type $(\mu, f)\colon U \longrightarrow \mathbb{R} \times PT^*\mathbb{R}^{n+1}$ (respectively, complete Legendrian unfolding $\ell_{(\mu,f)}\colon U \longrightarrow PT^*(\mathbb{R} \times \mathbb{R}^{n+1})$). For an open set $U \subset \mathbb{R}^{n+1}$, we define $\mathcal{P}(U)$ to be the set of $(\mu, f) \in \mathrm{Clair}(U, \mathbb{R} \times PT^*\mathbb{R}^{n+1})$ (respectively, $\ell_{(\mu,f)} \in L_R(U, PT^*(\mathbb{R} \times \mathbb{R}^{n+1}))$) such that the germ at u whose representative is given by (μ, f) (respectively, $\ell_{(\mu,f)}$) has property P for any $u \in U$.

The property P is said to be *generic* if for some neighbourhood U of 0 in $\mathbb{R}^{n+1}$, the set $\mathcal{P}(U)$ is a generic subset in $\mathrm{Clair}(U, \mathbb{R} \times PT^*\mathbb{R}^{n+1})$ (respectively, $L_R(U, PT^*(\mathbb{R} \times \mathbb{R}^{n+1}))$).

By the construction, we have a well-defined continuous mapping

$$(\Pi_1)_*\colon L_R(U, PT^*(\mathbb{R} \times \mathbb{R}^{n+1})) \longrightarrow \mathrm{Clair}(U, \mathbb{R} \times PT^*\mathbb{R}^{n+1})$$

defined by $(\Pi_1)_*(\ell_{(\mu,f)}) = \Pi_1 \circ \ell_{(\mu,f)} = (\mu, f)$, where $\Pi_1\colon PT^*(\mathbb{R} \times \mathbb{R}^{n+1}) \longrightarrow \mathbb{R} \times PT^*\mathbb{R}^{n+1}$ is the canonical projection. Then we have the following theorem.

Theorem 2.2 *The continuous map*

$$(\Pi_1)_*\colon L_R(U, PT^*(\mathbb{R} \times \mathbb{R}^{n+1})) \longrightarrow \mathrm{Clair}(U, \mathbb{R} \times PT^*\mathbb{R}^{n+1})$$

is a homeomorphism.

The proof follows from a direct analogy of the proof for Theorem 4.4 in [4], so that we omit it.

This theorem asserts that the genericity of a property of holonomic system of general Clairaut type can be interpreted by the genericity of the corresponding property of complete Legendrian unfolding.

We can assert the following theorem which reduces the equivalence problem for holonomic system with complete integral to that for the corresponding induced integral diagrams:

Theorem 2.3 ([5]) *Let* $(\mu, f)\colon (\mathbb{R}^{n+1}, 0) \longrightarrow (\mathbb{R} \times PT^*\mathbb{R}^{n+1}, (0, z))$ *and* $(\mu', f')\colon (\mathbb{R}^{n+1}, 0) \longrightarrow (\mathbb{R} \times PT^*\mathbb{R}^{n+1}, (0, z'))$ *be holonomic systems with complete integral such that the set of singular points of* $\pi \circ f$ *and* $\pi \circ f'$ *are closed sets without interior points. Then the following are equivalent*:

(1) f *and* f' *are equivalent as holonomic systems*

(2) $(\mu, \pi \circ f)$ *and* $(\mu', \pi \circ f')$ *are equivalent as integral diagrams.*

Remark: The condition that the set of singular points of $\pi \circ f$ is a closed set without interior points is satisfied for generic equations.

3. Equivalence of complete Ledendrian unfoldings and generating families

The main idea of the proof for Theorem 1.1 is to define an equivalence relation which can ignore functional modulus and to do everything in terms of generating families for complete Legendrian unfoldings of general Clairaut type.

Let (μ, f) be a holonomic system of general Clairaut type. Since $\ell_{(\mu,f)}$ is a germ of Legendrian immersion, there exists a generating family of $\ell_{(\mu,f)}$ by the theory of Legendrian singularities ([1, 10]). Let $F\colon ((\mathbb{R}\times\mathbb{R}^n)\times\mathbb{R}^k, 0) \longrightarrow (\mathbb{R}, 0)$ be a germ of a function such that $d_2F|_{0\times\mathbb{R}^n\times\mathbb{R}^k}$ is non-singular, where

$$d_2F(s, x, q) = \Big(\frac{\partial F}{\partial q_1}(s, x, q), \dots, \frac{\partial F}{\partial q_k}(s, x, q)\Big).$$

Then $C(F) = d_2F^{-1}(0)$ is a germ of smooth manifold of dimension $n+1$ and $\pi_F\colon (C(F), 0) \longrightarrow \mathbb{R}$ is a germ of a submersion, where $\pi_F(s, x, q) = s$. Define germs of maps

$$\widetilde{\mathcal{L}_F}\colon (C(F), 0) \longrightarrow J^1(\mathbb{R}^n, \mathbb{R})$$

by

$$\widetilde{\mathcal{L}_F}(s, x, q) = \Big(x, F(s, x, q), \frac{\partial F}{\partial x}(s, x, q)\Big),$$

and

$$\mathcal{L}_F\colon (C(F),0) \longrightarrow J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$$

by

$$\mathcal{L}_F(s,x,q) = \Big(s,x,F(s,x,q),\frac{\partial F}{\partial s}(s,x,q),\frac{\partial F}{\partial x}(s,x,q)\Big).$$

Since $\partial F/\partial q_i = 0$ $(i=1,\dots,k)$ on $C(F)$, we can easily show that

$$(\widetilde{\mathcal{L}_F}|_{\pi_F^{-1}(s)})^*\theta = 0.$$

By definition, $\mathcal{L}_F$ is a complete Legendrian unfolding associated to $(\pi_F,\widetilde{\mathcal{L}_F})$. By the same method of the theory of [1, 10], we can also show the following proposition.

Proposition 3.1 *All complete Legendrian unfolding germs are constructed by the above method.*

We say that F is *a generalized phase family* of the complete Legendrian unfolding $\mathcal{L}_F$.

Furthermore, by Proposition 2.1, $\ell_{(\mu,f)}$ is Legendrian non-singular. Then we can choose a family of germs of functions

$$F\colon (\mathbb{R}\times\mathbb{R}^n,0) \longrightarrow (\mathbb{R},0)$$

such that $\operatorname{Image} j^1F_s = f(\mu^{-1}(s))$ for any $s\in(\mathbb{R},0)$ where $F_s(x_1,\dots,x_n) = F(s,x_1,\dots,x_n)$. We remark that the map germ

$$j_1^1F\colon (\mathbb{R}\times\mathbb{R}^n,0) \longrightarrow J^1(\mathbb{R}^n,\mathbb{R})$$

is not necessarily an immersion germ, where $j_1^1F(s,x) = j^1F_s(x)$. In this case we have $(C(F),0) = (\mathbb{R}\times\mathbb{R}^n,0)$ and

$$\mathcal{L}_F = j^1F\colon (\mathbb{R}\times\mathbb{R}^n,0) \longrightarrow J^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}),$$

so that it is a complete Legendrian unfolding of general Clairaut type associated to (π_F, j_1^1F). Thus the generalized phase family of a complete Legendrian unfolding of general Clairaut type $\mathcal{L}_F$ is given by the above germ. We define $\widetilde{F}\colon (\mathbb{R}\times\mathbb{R}^n\times\mathbb{R},0) \longrightarrow (\mathbb{R},0)$ by $\widetilde{F}(s,x,y) = F(s,x) - y$ and call $\widetilde{F}$ *a generating family* of a complete Legengrian unfolding of general Clairaut type.

We now consider an equivalence relation among integral diagrams which ignore functional moduli. Let (μ, g) and (μ', g') be integral diagrams. Then (μ, g) and (μ', g') are $\mathcal{R}^+$*-equivalent* if there exist a germ of a diffeomorphism $\Psi\colon (\mathbb{R} \times \mathbb{R}^{n+1}, 0) \longrightarrow (\mathbb{R} \times \mathbb{R}^{n+1}, 0)$ of the form $\Psi(s, x) = (s + \alpha(x), \psi(x))$ and a germ of a diffeomorphism $\Phi\colon (\mathbb{R}^{n+1}, 0) \longrightarrow (\mathbb{R}^{n+1}, 0)$ such that $\Psi \circ (\mu, g) = (\mu', g') \circ \Phi$. We remark that if (μ, g) and (μ', g') are $\mathcal{R}^+$-equivalent by the above diffeomorphisms, then we have $\mu(u)+\alpha\circ g(u) = \mu' \circ \Phi(u)$ and $\psi \circ g(u) = g' \circ \Phi(u)$ for any $u \in (\mathbb{R}^{n+1}, 0)$. Thus the integral diagram $(\mu + \alpha \circ g, g)$ is strictly equivalent to (μ', g').

We now define the corresponding equivalence relation among Legendrian unfoldings. Let $\ell_{(\mu,f)}\colon (\mathbb{R}^{n+1}, 0) \longrightarrow (PT^*(\mathbb{R} \times \mathbb{R}^{n+1}), z)$ and $\ell_{(\mu',f')}\colon (\mathbb{R}^{n+1}, 0) \longrightarrow (PT^*(\mathbb{R} \times \mathbb{R}^{n+1}), z')$ be complete Legendrian unfoldings. We say that $\ell_{(\mu,f)}$ and $\ell_{(\mu',f')}$ are SP^+*-Legendrian equivalent* (respectively, SP*-Legendrian equivalent*) if there exist a germ of a contact diffeomorphism $K\colon (PT^*(\mathbb{R} \times \mathbb{R}^{n+1}), z) \longrightarrow (PT^*(\mathbb{R} \times \mathbb{R}^{n+1}), z')$, a germ of a diffeomorphism $\Phi\colon (\mathbb{R}^{n+1}, 0) \longrightarrow (\mathbb{R}^{n+1}, 0)$ and a germ of a diffeomorphism $\Psi\colon (\mathbb{R} \times \mathbb{R}^{n+1}, \Pi(z)) \longrightarrow (\mathbb{R} \times \mathbb{R}^{n+1}, \Pi(z'))$ of the form $\Psi(s, x) = (s + \alpha(x), \psi(x))$ (respectively, $\Psi(s, x) = (s, \psi(x))$) such that $\Pi \circ K = \Psi \circ \Pi$ and $K \circ \ell_{(\mu,f)} = \ell_{(\mu',f')} \circ \Phi$. where $\Pi\colon (PT^*(\mathbb{R} \times \mathbb{R}^{n+1}), z) \longrightarrow (\mathbb{R} \times \mathbb{R}^{n+1}, \Pi(z))$ is projection. It is clear that if $\ell_{(\mu,f)}$ and $\ell_{(\mu',f')}$ are SP^+-Legendrian equivalent (respectively, SP-Legendrian equivalent), then $(\mu, \pi\circ f)$ and $(\mu', \pi\circ f')$ are $\mathcal{R}^+$-equivalent (respectively, strictly equivalent). By [10, Theorem 1.1], the converse is also true for generic (μ, f) and (μ', f').

The notion of *stability* of complete Legendrian unfoldings with respect to SP^+-Legendrian equivalence (respectively, SP-Legendrian equivalence) is analogous to the usual notion of the stability of germs of Legendrian immersions with respect to the Legendrian equivalence. (cf. [1, Part *III*]).

On the other hand, we can interpret the above equivalence relation in terms of generating families. We now give a quick review of the theory of unfoldings of function germs in [2, 7].

Let $\widetilde{F}, \widetilde{F}'\colon (\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ be generating families of Legendrian unfolding of general Clairaut type, where $\widetilde{F}(s, x, y) = F(s, x) - y$, $\widetilde{F}'(s, x, y) = F'(s, x) - y$. We say that $\widetilde{F}$ and $\widetilde{F}'$ are P-$\mathcal{C}^+$*-equivalent* (respectively, P-$\mathcal{C}$*-equivalent*) if there exists a germ of a diffeomorphism $\Phi\colon (\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}, 0) \longrightarrow (\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}, 0)$ of the form

$$\Phi(s, x, y) = (s + \alpha(x, y), \phi_1(x, y), \phi_2(x, y))$$

(respectively,

$$\Phi(s,x,y) = (s, \phi_1(x,y), \phi_2(x,y)))$$

such that $\langle \widetilde{F} \circ \Phi \rangle_{\mathcal{E}_{(s,x,y)}} = \langle \widetilde{F}' \rangle_{\mathcal{E}_{(s,x,y)}}$, where $\langle \widetilde{F}' \rangle_{\mathcal{E}_{(s,x,y)}}$ is the ideal generated by $\widetilde{F}'$ in $\mathcal{E}_{(s,x,y)}$.

We say that $\widetilde{F}(s,x,y)$ is $\mathcal{C}^+$ (respectively, $\mathcal{C}$)-versal deformation of $f = F|_{\mathbb{R}\times 0}$ if

$$\mathcal{E}_s = \left\langle \frac{df}{ds} \right\rangle_{\mathbb{R}} + \langle f \rangle_{\mathcal{E}_s} + \left\langle \frac{\partial F}{\partial x_1}\Big|_{\mathbb{R}\times 0}, \dots, \frac{\partial F}{\partial x_n}\Big|_{\mathbb{R}\times 0}, 1 \right\rangle_{\mathbb{R}}$$

(respectively,

$$\mathcal{E}_s = \langle f \rangle_{\mathcal{E}_s} + \left\langle \frac{\partial F}{\partial x_1}\Big|_{\mathbb{R}\times 0}, \dots, \frac{\partial F}{\partial x_n}\Big|_{\mathbb{R}\times 0}, 1 \right\rangle_{\mathbb{R}}).$$

By the similar arguments like as those of [1, Theorems 20.8 and 21.4], we can show the following:

Theorem 3.2 *Let $\widetilde{F}$, $\widetilde{F}'\colon (\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ be generating families of complete Legendrian unfoldings of general Clairaut type $\mathcal{L}_F$, $\mathcal{L}_{F'}$ respectively. Then*

(1) $\mathcal{L}_F$ and $\mathcal{L}_{F'}$ are SP^+ (respectively, SP)-Legendrian equivalent if and only if $\widetilde{F}$ and $\widetilde{F}'$ are P-$\mathcal{C}^+$ (respectively, P-$\mathcal{C}$)-equivalent.

(2) $\mathcal{L}_F$ is SP^+ (respectively, SP)-Legendrian stable if and only if $\widetilde{F}$ is $\mathcal{C}^+$ (respectively, $\mathcal{C}$)-versal deformation of $f = F|_{\mathbb{R}\times 0}$.

Let f, $f'\colon (\mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ be germs of functions. We say that f and f' are *$\mathcal{C}$-equivalent* if and only if $\langle f \rangle_{\mathcal{E}_s} = \langle f' \rangle_{\mathcal{E}_s}$. Then the classification theory of germs of functions by the $\mathcal{C}$-equivalence is quite useful for our purpose. For each germ of a function $f\colon (\mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$, we set

$$\mathcal{C}\text{-cod}(f) = \dim_{\mathbb{R}} \mathcal{E}_s / \langle f \rangle_{\mathcal{E}_s},$$

$$\mathcal{C}^+\text{-cod}(f) = \dim_{\mathbb{R}} \mathcal{E}_s / \langle f \rangle_{\mathcal{E}_s} + \left\langle \frac{df}{ds} \right\rangle_{\mathbb{R}}.$$

Then we have the following well-known classification.

Lemma 3.3 *Let $f\colon (\mathbb{R}, 0) \longrightarrow (\mathbb{R}, 0)$ be a germ of a function with $\mathcal{K}$-cod$(f) < \infty$. Then f is $\mathcal{C}$-equivalent to the germ $s^{\ell+1}$ for some $\ell \in \mathbb{N}$.*

By a direct calculation, we have

$$\mathcal{C}\text{-cod}(s^{\ell+1}) = \ell + 1,$$
$$\mathcal{C}^{+}\text{-cod}(s^{\ell+1}) = \ell.$$

Thus we can easily determine $\mathcal{C}$ (respectively, $\mathcal{C}^+$)-versal deformations of the above germs by using the usual method:

- The $\mathcal{C}$-versal deformation: $s^{\ell+1} + \sum_{i=0}^{\ell} u_{i+1} s^i$.
- The $\mathcal{C}^+$-versal deformation: $s^{\ell+1} + \sum_{i=0}^{\ell-1} u_{i+1} s^i$.

The following theorem is useful and important for our purpose (cf. [2]).

Theorem 3.4 *Let $\widetilde{F}, \widetilde{F}' \colon (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0) \longrightarrow (\mathbb{R}, 0)$ be germs of functions such that $\widetilde{F}$, $\widetilde{F}'$ are $\mathcal{C}^+$ (respectively, $\mathcal{C}$)-versal deformations of $f = F|_{\mathbb{R}\times 0}$, $f' = F'|_{\mathbb{R}\times 0}$ respectively. Then $\widetilde{F}$ and $\widetilde{F}'$ are P-$\mathcal{C}^+$-equivalent (respectively, P-$\mathcal{C}$-equivalent) if and only if f and f' are $\mathcal{C}$-equivalent.*

Let $\widetilde{F}(s, x, y)$ be a $\mathcal{C}^+$-versal deformation of $f = F|_{\mathbb{R}\times 0}$. By Lemma 3.3 and Theorem 3.4, $\widetilde{F}(s, x, y)$ is P-$\mathcal{C}^+$-equivalent to one of the germs in the following list:

$$A_\ell \ (1 \leqq \ell \leqq n+1)\colon \ s^\ell + \sum_{i=1}^{\ell-1} x_i s^i + \sum_{j=\ell}^{n} x_j - y,$$

$$\widetilde{A}_{n+2}\colon \ s^{n+2} + \sum_{i=1}^{n} x_i s^i - y.$$

4. Proof of Theorem 1.1

The set of SP^+-Legendrian stable complete Legendrian unfoldings is an open and dense subset in $L_R(U, PT^*(\mathbb{R} \times \mathbb{R}^{n+1}))$. Therefore by Theorem 2.2, it gives a classification of SP^+-Legendrian stable complete Legendrian unfoldings of general Clairaut type under the SP^+-Legendrian equivalence (or SP-Legendrian equiavelence). Let (μ, f) be a holonomic system of general Clairaut type such that the corresponding complete Legendrian unfolding $\ell_{(\mu,f)}$ is SP^+-Legendrian stable. By the assumption, in Theorem 3.2, the generating family $\widetilde{F}$ of $\ell_{(\mu,f)}$ is $\mathcal{C}^+$-versal deformation of

$f = F|_{\mathbb{R}\times 0}$. Therefore $\widetilde{F}(s, x_1, \dots, x_n, y)$ is P-$\mathcal{C}^+$-equivalent to one of the germs A_ℓ $(1 \le \ell \le n+1)$ or $\widetilde{A}_{n+2}$ in the previous section.

We also consider a generic condition of generalized phase family F. Let $J^{n+2}(n+1, 1)$ be the set of $(n+2)$-jets of function $h\colon (\mathbb{R}^{n+1}, 0) \longrightarrow (\mathbb{R}, 0)$. We consider the following two algebraic subset of $J^{n+2}(n+1, 1)$:

$$\begin{aligned}\Sigma_1 = \Bigl\{ j^{n+1}h(0) \Bigm| \frac{\partial h}{\partial s}(0) = \frac{\partial^2 h}{\partial s^2}(0) = \cdots = \frac{\partial^{n+1} h}{\partial s^{n+1}}(0) \\ = \frac{\partial^{n+2} h}{\partial s^{n+2}}(0)\frac{\partial^2 h}{\partial s\,\partial x_1} = 0 \Bigr\},\end{aligned}$$

$$\begin{aligned}\Sigma_2 = \Bigl\{ j^{n+1}h(0) \Bigm| \frac{\partial h}{\partial s}(0) = \frac{\partial^2 h}{\partial s\,\partial x_1}(0) = \frac{\partial^3 h}{\partial s^3}(0) = \cdots = \frac{\partial^{n+1} h}{\partial s^{n+1}}(0) \\ = \frac{\partial^2 h}{\partial s^2}(0)\frac{\partial^3 h}{\partial s\,\partial x_1^2}(0)\cdots\frac{\partial^3 h}{\partial s\,\partial x_n^2}(0) = 0 \Bigr\}.\end{aligned}$$

We consider the union $W = \Sigma_1 \cup \Sigma_2$, then it is also an algebraic subset of $J^{n+2}(n+1, 1)$. We can stratify the algebraic set W by submanifolds whose codimensions are at least $n+2$. By Thom's jet transversality theorem, $j^{n+2}F(\mathbb{R}^{n+1}) \cap (\mathbb{R}^{n+1} \times \mathbb{R} \times W) = \emptyset$ for generic function $F(s, x_1, \dots, x_n)$. Therefore, we might assume that $\widetilde{F}$ is P-$\mathcal{C}^+$-equivalent to one of the germ A_ℓ $(1 \le \ell \le n+1)$ or $\widetilde{A}_{n+2}$ and F is satisfied one of the condition of (a_i) $(1 \le i \le n+2)$ or (b) for generic $\ell_{(\mu,f)}$:

(a_1): $\dfrac{\partial F}{\partial s}(0) \ne 0$,

(a_i) $(2 \le i \le n+1)$: $\dfrac{\partial F}{\partial s}(0) = \cdots = \dfrac{\partial^{i-1} F}{\partial s^{i-1}}(0) = 0$,

$$\frac{\partial^i F}{\partial s^i}(0) \ne 0 \ \Bigl(\text{and } \frac{\partial^2 F}{\partial s\,\partial x_1}(0) \ne 0\Bigr),$$

(a_{n+2}): $\dfrac{\partial F}{\partial s}(0) = \cdots = \dfrac{\partial^{n+1} F}{\partial s^{n+1}}(0) = 0$, $\dfrac{\partial^{n+2} F}{\partial s^{n+2}}(0)\dfrac{\partial^2 F}{\partial s\,\partial x_1}(0) \ne 0$,

(b): $\dfrac{\partial F}{\partial s}(0) = \dfrac{\partial^2 F}{\partial s\,\partial x_1} = 0$, $\dfrac{\partial^2 F}{\partial s^2}(0)\dfrac{\partial^3 F}{\partial s\,\partial x_1^2}(0)\dots\dfrac{\partial^3 F}{\partial s\,\partial x_n^2}(0) \ne 0$.

We now consider the condition (a_i) $(1 \le i \le n+1)$. Since A_i $(1 \le i \le n+1)$ is $\mathcal{C}$-versal deformation, $\widetilde{F}$ is P-$\mathcal{C}$-equivalent to

$$A_i\colon\ s^i+\sum_{j=1}^{i-1}x_js^j+\sum_{k=\ell}^{n}x_k-y.$$

We also consider the condition (a_{n+2}), then $\widetilde{F}$ is P-$\mathcal{C}^+$-equivalent to

$$\widetilde{A}_{n+2}\colon\ s^{n+2}+\sum_{i=1}^{n}x_is^i-y.$$

Next we consider the condition (b). It follows that $\widetilde{F}$ is P-$\mathcal{C}$-equivalent to $s^2-y+\phi(x_1,\dots,x_n)s$, where ϕ is function germ. By the condition, we might assume that ϕ is non-degenerate singular point at 0 (i.e., $(\partial\phi/\partial x_i)(0)=0$ $(i=1,\dots,n)$ and Hessian of ϕ at 0 is regular). Therefore, ϕ is $\mathcal{R}$-equivalent to $x_1^2+\cdots+x_k^2-x_{k+1}^2-\cdots-x_n^2$ for some integer k $(0\le k\le n)$ by Morse's lemma (cf. [8]). We remark that the concept of $\mathcal{R}$-equivalence is in the usual sence (cf. [2]) i.e., f and $g\colon(\mathbb{R}^n,0)\longrightarrow(\mathbb{R},0)$ are $\mathcal{R}$-equivalent if there exsist a diffeomorphim germ $\varphi\colon(\mathbb{R}^n,0)\longrightarrow(\mathbb{R}^n,0)$ such that $f\circ\varphi=g$. Hence $\widetilde{F}$ is P-$\mathcal{C}$-equivalent to

$$A_2^k\colon\ s^2-y+(x_1^2+\cdots+x_k^2-x_{k+1}^2-\cdots-x_n^2)s.$$

We detect the corresponding normal forms of integral diagrams as follows:

For the case A_ℓ $(1\le\ell\le n+1)$, we can choose

$$F(s,x_1,\dots,x_n)=s^\ell+\sum_{i=1}^{\ell-1}x_is^i+\sum_{j=\ell}^{n}x_j$$

as a generalized phase family, so that

$$\mathcal{L}_F=\Big(s,x_1,\dots,x_n,s^\ell+\sum_{i=1}^{\ell-1}x_is^i+\sum_{j=\ell}^{n}x_j,\\ \ell s^{\ell-1}+\sum_{i=1}^{\ell-1}ix_is^{i-1},s,\dots,s^{\ell-1},1,\dots,1\Big).$$

Then we can easily caluculate that the corresponding integral diagram is strictly equivalent to

$$DA_\ell\ (1\le\ell\le n+1)\colon\ \mu=u_{n+1},\\ g=\Big(u_1,\dots,u_n,u_{n+1}^\ell+\sum_{i=1}^{\ell-1}u_iu_{n+1}^i\Big).$$

This is the normal form in the case of DA_ℓ $(1 \leq \ell \leq n+1)$. If $\ell = 2$, we have

$$\mu = u_{n+1}, \quad g = (u_1, \dots, u_n, u_{n+1}^2 + u_1 u_{n+1}).$$

We define a local coordinate transformulation by $U_i = u_i$ $(i = 1, \dots, n)$, $U_{n+1} = u_{n+1} + (1/2)u_1$, then (μ, g) is strictly equivalent to

$$\mu = u_{n+1} - \frac{1}{2}u_1, \quad g = \Big(u_1, \dots, u_n, u_{n+1}^2 - \frac{1}{4}u_1^2\Big).$$

We also apply a local coordinate transformation which is defined by

$$X_i = x_i \ (i = 1, \dots, n), \quad Y = y + \frac{1}{4}x_1^2,$$

then we have the normal form DA_2.

For the case $\widetilde{A}_{n+2}$, we can choose

$$F(s, x_1, \dots, x_n) = s^{n+2} + \sum_{i=1}^{n} x_i s^i$$

as a generalized phase family. By the same calculations as those of the case $1 \leq \ell \leq n+1$, we can show that the corresponding integral diagram is $\mathcal{R}^+$-equivalent to

$$\mu = u_{n+1}, \quad g = \left(u_1, \dots, u_n, u_{n+1}^{n+2} + \sum_{i=1}^{n} u_i u_{n+1}^i\right).$$

Since the generating family for $\widetilde{A}_{n+2}$ is $\mathcal{C}^+$-versal and not $\mathcal{C}$-versal, the integral diagram is strictly equivalent to the normal form $\widetilde{DA}_{n+2}$.

Finally, for A_2^k the generalized phase family is given by

$$F(s, x_1, \dots, x_n) = s^2 + (x_1^2 + \dots + x_k^2 - x_{k+1}^2 - \dots - x_n^2)s.$$

Then we can also caluculate that the corresponding integral diagram is strictly equivalent to

$$\mu = u_{n+1}, \quad g = \big(u_1, \dots, u_n, u_{n+1}^2 + (u_1^2 + \dots + u_k^2 - u_{k+1}^2 - u_n^2)u_{n+1}\big).$$

We define a local coordinate transformation by

$$U_i = u_i \quad (i = 1, \dots, n),$$
$$U_{n+1} = u_{n+1} + (1/2)(u_1^2 + \dots + u_k^2 - u_{k+1}^2 - \dots - u_n^2),$$

then (μ, g) is strictly equivalent to

$$\mu = u_{n+1} - \frac{1}{2}(u_1^2 + \cdots + u_k^2 - u_{k+1}^2 - \cdots - u_n^2),$$
$$g = \Big(u_1, \ldots, u_n, u_{n+1}^2 - \frac{1}{4}(u_1^4 + \cdots + u_k^4 - u_{k+1}^4 - \cdots - u_n^4)\Big).$$

We again apply a local coordinate transformation which defined by

$$X_i = x_i \ (i = 1, \ldots, n), \quad Y = y + \frac{1}{4}(x_1^4 + \cdots + x_k^4 - x_{k+1}^4 - \cdots - x_n^4),$$

then we have the normal form DA_2^k in Theorem 1.1.

This completes the proof of Theorem 1.1. □

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Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan
E-mail: takahashi@math.sci.hokudai.ac.jp