Hokkaido Mathematical Journal Vol. 34 (2005) p. 219–235

Examples of globally hypoelliptic operator on special dimensional spheres without the bracket condition*

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(Received June 13, 2003)

Abstract. This paper gives examples of globally hypoelliptic operators on S^3, S^7, and S^{15} which are sums of squares of real vector fields. These operators fail to satisfy the infinitesimal transitivity condition (the bracket condition) at any point and therefore they are not hypoelliptic in any subdomain.

Key words: global hypoellipticity, Omori-Kobayashi conjecture.

1. Introduction

Let M be a closed (compact connected without boundary) $\boldsymbol{C}^\infty$ manifold. For an open subset Ω of M, we denote by $\boldsymbol{C}^\infty(\Omega)$ the space of smooth functions in Ω. A differential operator L is said to be *hypoelliptic* in M if and only if, for any open subset Ω of M, $Lu \in \boldsymbol{C}^\infty(\Omega)$ for a distribution u on M implies $u \in \boldsymbol{C}^\infty(\Omega)$. On the other hand, L is said to be *globally hypoelliptic* on M if and only if $Lu \in \boldsymbol{C}^\infty(M)$ for a distribution u implies $u \in \boldsymbol{C}^\infty(M)$. By definition, hypoelliptic operators are globally hypoelliptic.

Let $Z_1, Z_2, \ldots, Z_m$ be smooth real tangent vector fields on M (m is an arbitrary positive integer). The differential operator L which we shall treat is of the form:

$$L = \sum_{j=1}^{m} {Z_j}^* Z_j, \tag{1.1}$$

where ${Z_j}^*$ is the formal adjoint operator of Z_j with respect to a fixed smooth Riemannian metric on M. In this paper, we study a sufficient condition on $Z_1, Z_2, \ldots, Z_m$ under which L is globally hypoelliptic on M.

2000 Mathematics Subject Classification : Primary 35H10; Secondary 58J99.

*This article is the detailed paper of [4] announced in Proceedings of the Japan Academy.

Let $V[Z_1, \dots, Z_m]$ be the linear space defined by

$$V[Z_1, \dots, Z_m] = \left\{ \sum_{j=1}^{m} f_j Z_j \,;\ f_j \in \boldsymbol{C}^\infty(M) \right\}.$$

For every $Y \in V[Z_1, \dots, Z_m]$, $\exp tY$ denotes the one-parameter diffeomorphism group generated through integral curves by Y, and let $\mathcal{H}[Z_1, \dots, Z_m]$ be the closed subgroup generated by $\{\exp Y; Y \in V\}$ in the group of $\boldsymbol{C}^\infty$ diffeomorphism of M onto itself.

Definition 1.1 We say that $\mathcal{H}[Z_1, \dots, Z_m]$ is *transitive* on M if for any x, $y \in M$, there exists a $g \in \mathcal{H}[Z_1, \dots, Z_m]$ such that $x = gy$.

Next, let $\mathcal{L}[Z_1, \dots, Z_m]$ be the Lie algebra generated by $V[Z_1, \dots, Z_m]$.

Definition 1.2 We say that $\mathcal{H}[Z_1, \dots, Z_m]$ is *infinitesimally transitive* at $p \in M$ if $\mathcal{L}[Z_1, \dots, Z_m]\big|_p = T_pM$. (If this is fulfilled, we also say that $\{Z_1, \dots, Z_m\}$ satisfies *the bracket condition* at p.)

It is not difficult to see that $\mathcal{H}[Z_1, \dots, Z_m]$ is transitive on M if $\mathcal{H}[Z_1, \dots, Z_m]$ is infinitesimally transitive at every $p \in M$. These geometric notions of transitivity and infinitesimal transitivity are closely related to global hypoellipticity and hypoellipticity. We mention a well-known result due to Hörmander and the conjecture given by Omori and Kobayashi.

Theorem (Hörmander [1]) *If $\mathcal{H}[Z_1, \dots, Z_m]$ is infinitesimally transitive at every $p \in M$, then L defined by* (1.1) *is hypoelliptic in M.*

Conjecture (Omori-Kobayashi [3]) *If $\mathcal{H}[Z_1, \dots, Z_m]$ is transitive on M, then L defined by* (1.1) *is globally hypoelliptic on M.*

Omori and Kobayashi give an affirmative answer to this conjecture under an additional condition (Condition **(D)** below).

Now we present an interesting question concerning the above conjecture: "Is it possible to construct a globally hypoelliptic operator L of the form (1.1) with transitive but nowhere infinitesimally transitive system of vector fields $\{Z_1, \dots, Z_m\}$?" The answer is affirmative in the case where $M = \boldsymbol{T}^3 = [0, 2\pi] \times [0, 2\pi] \times [0, 2\pi]$. In fact, as was studied in [3], the following vector fields satisfy the conditions in the question above:

$$Z_1 = \partial_x, \quad Z_2 = \zeta(x)\,\partial_y, \quad Z_3 = \eta(x, y)\,\partial_z,$$

where $\zeta(x)$ and $\eta(x, y)$ are non-negative smooth functions such that they do not vanish identically and their supports are mutually disjoint. This example suggests that there will probably exist such a system if M is decomposable to a direct product of three or more closed manifolds. So we are interested in the case where M is not decomposable. In this paper, we demonstrate the existence of such systems on special dimensional spheres S^3, S^7 and S^{15}, where S^m is the m-dimensional standard unit sphere.

Theorem 1.3 *For $n \in \{2, 4, 8\}$, there exist a positive integer $m = m(n)$ and a system of vector fields $\{Z_1, \ldots, Z_m\}$ on S^{2n-1} such that the following three conditions hold:*

(A) *$\mathcal{H}[Z_1, \ldots, Z_m]$ is transitive on S^{2n-1}.*

(B) *There is no point in S^{2n-1} at which $\mathcal{H}[Z_1, \ldots, Z_m]$ is infinitesimally transitive.*

(C) *The differential operator L defined by* (1.1) *is globally hypoelliptic on S^{2n-1}.*

Remark 1.4 Let $d(n)$ be the maximal dimension of $\mathcal{L}[Z_1, \ldots, Z_m]$ over S^{2n-1}. Then for the systems which we construct, the pair of integers $(m(n), d(n))$ is the following:

$$(m(n), d(n)) = \begin{cases} (3, 2) & (n = 2), \\ (6, 6) & (n = 4), \\ (10, 12) & (n = 8). \end{cases}$$

Notice that $d(n) < 2n - 1$. This means Condition **(B)**.

We prove this theorem by constructing $\{Z_1, \ldots, Z_m\}$ explicitly. The idea based on [3] is the following. The transitivity of $\mathcal{H}[Z_1, \ldots, Z_m]$ implies the a priori estimate

$$\|u\|_0 \leq C\|Lu\|_0 + D_N\|u\|_{-N} \quad \text{for all } u \in \boldsymbol{C}^\infty(M), \tag{1.2}$$

where $\|\cdot\|_s$ stands for the norm of the Sobolev space of order s (see Theorem 2.1 and Corollary 2.4 of [3]). It is not difficult to see that L is globally hypoelliptic on M if we can find a *regulator* Λ, that is, an elliptic pseudo-differential operator of order 1, which commutes with $Z_1, \ldots, Z_m$. Furthermore, global hypoellipticity still holds, if the commutativity condition for Λ is replaced by the following weaker condition introduced in Proposition 3.2 of [3]:

(D) There exists a regulator Λ such that, for every δ, $N > 0$ and for all $u \in \boldsymbol{C}^\infty(M)$, the following two estimates hold:

$$\|[\Lambda, L]u\|_{-1} \le \delta\|Lu\|_0 + C(\delta, N)\|u\|_{-N},$$
$$\|[\Lambda, [\Lambda, L]]u\|_{-2} \le \delta\|Lu\|_0 + C(\delta, N)\|u\|_{-N}.$$

These are trivial if Λ commutes with $Z_1, \dots, Z_m$. The point is that on S^{2n-1} ($n = 2, 4, 8$), we have a globally defined basis $\{W_{jk}^{(n)}\}$ which commutes with the Laplacian Δ on S^{2n-1} with respect to the induced metric from $\boldsymbol{R}^{2n}$. For the construction of a system satisfying the conditions in Theorem 1.3, we cut off the support of $W_{jk}^{(n)}$ to reduce the dimension of $\mathcal{L}[Z_1, \dots, Z_m]$, while preserving the transitivity of $\mathcal{H}[Z_1, \dots, Z_m]$.

The plan of this paper is as follows. In §2, we construct a global basis of non-vanishing smooth vector fields on S^{2n-1}. We will take the basis suitably for the study of the transitivity condition **(A)** by using the Hopf mapping. In §3, we present explicit forms of the systems by using these bases. Transitivity and nowhere infinitesimal transitivity conditions **(A)** and **(B)** are discussed in §4. In §5, we introduce and prove a slightly abstract theorem on global hypoellipticity which shows Condition **(C)** on the systems constructed above.

2. The basis of non-vanishing smooth vector fields

Let n be 2 or 4 or 8. Then there exists a global basis of non-vanishing vector fields on S^{2n-1}. We denote by $z = {}^t(\xi, \eta)$ a point of $\boldsymbol{R}^{2n}$, where ξ, $\eta \in \boldsymbol{R}^n$. Here z, ξ and η are column vectors. We construct this basis as restriction of vector fields on $\boldsymbol{R}_z^{2n}$ of the form ${}^tz\,{}^tV\nabla_z$ with an antisymmetric orthogonal matrix V.

We introduce the so-called *Hopf mapping* from $\boldsymbol{R}^{2n}$ to $\boldsymbol{R}^{n+1}$, which turns out to be also from S^{2n-1} to S^n. This enables us to reduce the study of the transitivity on S^{2n-1} to that on S^n and, if we choose the basis of vector fields as follows, to transform the one-parameter diffeomorphism groups on S^{2n-1} to rotations on S^n (see (2.7) and (2.8)). We identify $\boldsymbol{R}^n$ with the complex number field $\boldsymbol{C}$ ($n = 2$), the quaternion field $\boldsymbol{H}$ ($n = 4$) or Cayley's algebra Ca$[\boldsymbol{H}]$ ($n = 8$). The Hopf mapping $\pi^{(n)}$ is defined by

$$\boldsymbol{R}^{2n} \ni z = {}^t(\xi, \eta) \longmapsto \pi^{(n)}(z) = \left(|\xi|^2 - |\eta|^2, 2\xi\eta\right) \in \boldsymbol{R}^{n+1},$$

where $|\xi|$ stands for the Euclidian norm of ξ and $\xi\eta$ the product of ξ

and η in the sense of $\boldsymbol{C}$ or $\boldsymbol{H}$ or Ca[$\boldsymbol{H}$]. We denote the elements by $\pi^{(n)}(z) = (\pi_0^{(n)}(z), \pi_1^{(n)}(z), \ldots, \pi_n^{(n)}(z))$. $\pi^{(n)}$ can be regarded as the mapping from S^{2n-1} to S^n, because $|\pi^{(n)}(z)| = |z|^2$.

Each element $\pi_j^{(n)}(z)$ of the Hopf mapping is represented by a real symmetric $2n \times 2n$ matrix $H_j^{(n)}$ as the quadratic form ${}^t z H_j^{(n)} z$ because it is a homogeneous polynomial of degree 2 with respect to z. These matrices are orthogonal and satisfy the following:

$$H_j^{(n)} H_k^{(n)} = -H_k^{(n)} H_j^{(n)} \quad (j,\, k = 0, \ldots, n;\ j \neq k). \tag{2.1}$$

We define new matrices $V_{jk}^{(n)}$ to be

$$V_{jk}^{(n)} = H_j^{(n)} H_k^{(n)} \quad (j,\, k = 0, \ldots, n;\ j \neq k).$$

Then by means of (2.1), we have the following properties of $\{V_{jk}^{(n)}\}$:

$$V_{jk}^{(n)} = -V_{kj}^{(n)} \qquad \text{if } j \neq k. \tag{2.2}$$

$$V_{j\alpha}^{(n)} V_{\alpha k}^{(n)} = V_{jk}^{(n)} \qquad \text{if } j,\, k \text{ and } \alpha \text{ are mutually distinct.} \tag{2.3}$$

$$V_{jk}^{(n)} V_{\alpha\beta}^{(n)} = V_{\alpha\beta}^{(n)} V_{jk}^{(n)} \qquad \text{if } j,\, k,\, \alpha \text{ and } \beta \text{ are mutually distinct.} \tag{2.4}$$

The basis $W_{jk}^{(n)}$ on S^{2n-1} is defined as the restriction of the vector fields $W_{jk}^{(n)} = {}^t z\, {}^t V_{jk}^{(n)} \nabla$ on $\boldsymbol{R}^{2n}$, where $\nabla = {}^t(\partial_{z_1}, \ldots, \partial_{z_{2n}})$. These vector fields are well-defined on S^{2n-1} thanks to the antisymmetricity (2.2).

By (2.3) and (2.4), we see that $W_{jk}^{(n)}$ have the following relations which we need to observe the dimension of $\mathcal{L}[Z_1, \ldots, Z_m]$:

$$[W_{j\alpha}^{(n)}, W_{\alpha k}^{(n)}] = -2W_{jk} \qquad \text{if } j,\, k \text{ and } \alpha \text{ are mutually distinct.} \tag{2.5}$$

$$[W_{jk}^{(n)}, W_{\alpha\beta}^{(n)}] = 0 \qquad \text{if } j,\, k,\, \alpha \text{ and } \beta \text{ are mutually distinct.} \tag{2.6}$$

On the other hand, the one-parameter diffeomorphism group generated by $W_{jk}^{(n)}$ on S^{2n-1} is transformed by $\pi^{(n)}$ to a rotation on S^n:

$$\pi_k^{(n)}(\exp(tW_{jk}^{(n)})z) = (\cos 2t)\pi_k^{(n)}(z) - (\sin 2t)\pi_j^{(n)}(z) \quad \text{if } j \neq k. \tag{2.7}$$

$$\pi_\alpha^{(n)}(\exp(tW_{jk}^{(n)})z) = \pi_\alpha^{(n)}(z) \quad \text{if } j,\, k \text{ and } \alpha \text{ are mutually distinct.} \tag{2.8}$$

Furthermore, $W_{jk}^{(n)}$ commutes with the Laplacian on S^{2n-1} with respect to the induced metric from $\boldsymbol{R}^{2n}$, which plays a crucial role in proving the global hypoellipticity.

3. Explicit forms of vector fields

We represent here explicit forms of vector fields satisfying the conditions in Theorem 1.3. We prepare some cut-off functions on S^{2n-1}. Let $\varphi_1(t)$, $\varphi_2(t)$ and $\psi(t)$ be functions on $\boldsymbol{R}$ such that

$$\begin{cases} \varphi_1, \varphi_2, \psi \in \boldsymbol{C}^\infty(\boldsymbol{R}), & 0 \leq \varphi_1, \varphi_2, \psi \leq 1, \\ \varphi_1 = 1 \text{ on } \{t \geq 3/4\}, & \operatorname{supp} \varphi_1 \subset \{t \geq 1/2\}, \\ \varphi_2 = 1 \text{ on } \{t \leq 0\}, & \operatorname{supp} \varphi_2 \subset \{t \leq 1/4\}, \\ \psi = 1 \text{ on } \{t \geq 5/6\}, & \operatorname{supp} \psi \subset \{t > 2/3\}, \end{cases}$$

let $\Phi_1^{(n)}$, $\Phi_2^{(n)}$ $(n = 2, 4, 8)$ and $\Psi_1^{(n)}$, $\Psi_2^{(n)}$ $(n = 4, 8)$ cut-off functions on S^{2n-1} defined as follows:

$$\Phi_1^{(n)}(z) = \varphi_1\big(\pi_0^{(n)}(z)\big), \qquad \Phi_2^{(n)}(z) = \varphi_2\big(\pi_0^{(n)}(z)\big) \quad (n = 2, 4, 8),$$

$$\Psi_1^{(4)}(z) = \psi\left(\sum_{j=0}^{1} (\pi_j^{(4)}(z))^2\right), \quad \Psi_2^{(4)}(z) = \psi\left(\sum_{j=2}^{4} (\pi_j^{(4)}(z))^2\right),$$

$$\Psi_1^{(8)}(z) = \psi\left(\sum_{j=0}^{3} (\pi_j^{(8)}(z))^2\right), \quad \Psi_2^{(8)}(z) = \psi\left(\sum_{j=4}^{8} (\pi_j^{(8)}(z))^2\right).$$

$\Phi_1^{(n)}$ and $\Phi_2^{(n)}$ have their supports near the north pole and on the southern hemisphere respectively. $\Phi_2^{(n)}\Psi_1^{(n)}$ and $\Phi_2^{(n)}\Psi_2^{(n)}$ have their supports on the disjoint domains in the southern hemisphere.

We begin with the case $n = 4$, 8.

Proposition 3.1 *Let n be 4. The following system of six vector fields on S^7 satisfies Conditions* **(A)***,* **(B)** *and* **(C)** *in* Theorem 1.3:

$$\Big\{\boldsymbol{W}_{\mathbf{04}}^{(\mathbf{4})}, \boldsymbol{W}_{\mathbf{12}}^{(\mathbf{4})}, \boldsymbol{\Phi}_{\mathbf{1}}^{(\mathbf{4})}\boldsymbol{W}_{\mathbf{13}}^{(\mathbf{4})}, \boldsymbol{\Phi}_{\mathbf{1}}^{(\mathbf{4})}\boldsymbol{W}_{\mathbf{23}}^{(\mathbf{4})}, \boldsymbol{\Phi}_{\mathbf{2}}^{(\mathbf{4})}\boldsymbol{\Psi}_{\mathbf{1}}^{(\mathbf{4})}\boldsymbol{W}_{\mathbf{01}}^{(\mathbf{4})}, \boldsymbol{\Phi}_{\mathbf{2}}^{(\mathbf{4})}\boldsymbol{\Psi}_{\mathbf{2}}^{(\mathbf{4})}\boldsymbol{W}_{\mathbf{34}}^{(\mathbf{4})}\Big\}.$$

Proposition 3.2 *Let n be 8. The following system of ten vector fields on S^{15} satisfies Conditions* **(A)***,* **(B)** *and* **(C)** *in* Theorem 1.3:

$$\Big\{\boldsymbol{W^{(8)}_{08}},\ \boldsymbol{W^{(8)}_{14}},\ \boldsymbol{W^{(8)}_{25}},\ \boldsymbol{\Phi^{(8)}_1 W^{(8)}_{23}},\ \boldsymbol{\Phi^{(8)}_1 W^{(8)}_{34}},\ \boldsymbol{\Phi^{(8)}_2 W^{(8)}_{37}},$$
$$\boldsymbol{\Phi^{(8)}_2 \Psi^{(8)}_1 W^{(8)}_{01}},\ \boldsymbol{\Phi^{(8)}_2 \Psi^{(8)}_1 W^{(8)}_{23}},\ \boldsymbol{\Phi^{(8)}_2 \Psi^{(8)}_2 W^{(8)}_{67}},\ \boldsymbol{\Phi^{(8)}_2 \Psi^{(8)}_2 W^{(8)}_{78}}\Big\}.$$

In case $n = 2$, we need another vector field $W^{(2)}$ on $\boldsymbol{R}^4$ which can be regarded as a smooth vector field on S^3:

$$W^{(2)} = {}^t z \begin{pmatrix} O_2 & -I_2 \\ I_2 & O_2 \end{pmatrix} \nabla,$$

where I_2 and O_2 are the 2×2 identity matrix and the 2×2 zero matrix respectively.

Proposition 3.3 *Let n be 2. The following system of three vector fields on S^3 satisfies Conditions* **(A)**, **(B)** *and* **(C)** *in* Theorem 1.3:

$$\Big\{\boldsymbol{W^{(2)}},\ \boldsymbol{\Phi^{(2)}_1 W^{(2)}_{12}},\ \boldsymbol{\Phi^{(2)}_2 W^{(2)}_{01}}\Big\}.$$

We prove these propositions in the following sections. Let $\{W^{(n)}_1, \dots, W^{(n)}_{m(n)}\}$ be the same system as in Proposition 3.1 or 3.2 or 3.3. We write $\mathcal{H}[W^{(n)}_1, \dots, W^{(n)}_{m(n)}]$ and $\mathcal{L}[W^{(n)}_1, \dots, W^{(n)}_{m(n)}]$ as $\mathcal{H}^{(n)}$ and $\mathcal{L}^{(n)}$ respectively. The proof of the transitivity and the nowhere infinitesimal transitivity of $\mathcal{H}^{(n)}$ will be done in the next section. The global hypoellipticity of $\sum_{j=1}^{m(n)} W^{(n)*}_j W^{(n)}_j$ on S^{2n-1} will be studied in §5.

4. Transitivity and nowhere infinitesimal transitivity

4.1. Nowhere infinitesimal transitivity

We prove Condition **(B)** in Propositions 3.1, 3.2 and 3.3.

In case $\boldsymbol{n = 2}$ $W^{(2)}$ commutes with $W^{(2)}_{01}$ and $W^{(2)}_{12}$. In addition, $\Phi^{(2)}_1 W^{(2)}_{12}$ and $\Phi^{(2)}_2 W^{(2)}_{01}$ are commutative thanks to the disjointness of their supports. Therefore, the dimension of $\mathcal{L}^{(2)}$ at every point is less than two. And hence, Condition **(B)** in Proposition 3.3 applies.

Before going into the other cases, we study the dimension of the Lie algebra generated by $W^{(n)}_{jk}$'s at every $p \in S^{2n-1}$. To do this, we introduce the following abstract group. Let $G^{(n)}$ be a group with the unit element e generated by $\varepsilon, a_0, a_1, \dots, a_n$ which satisfy

$$\varepsilon^2 = e, \qquad a_p^2 = e \quad (p = 0, \dots, n),$$

$$\varepsilon a_p = a_p \varepsilon \quad (p = 0, \ldots, n),$$
$$a_p a_q = \varepsilon a_q a_p \quad (p,\, q = 0, \ldots, n;\; p \neq q).$$

This is a finite group consisting of 2^{n+1} elements, and is isomorphic to the subgroup generated by $H_0^{(n)}, \ldots, H_n^{(n)}$ in $SO(2n)$. For a subgroup G of $G^{(n)}$, we denote by $d^{(n)}(G)$ the number of elements of the form $a_p a_q$ $(p < q)$ in G. Note that $d^{(n)}(\sigma_1 \sigma_2) = d^{(n)}(\sigma_1) + d^{(n)}(\sigma_2)$ if two subgroups σ_1 and σ_2 of $G^{(n)}$ are commutative. Let $[(\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)]$ be the subgroup of $G^{(n)}$ generated by $a_{\alpha_1} a_{\beta_1}, \ldots, a_{\alpha_m} a_{\beta_m}$. Then (2.3) and (2.4) yield

the dimension of $\mathcal{L}[W_{j_1 k_1}^{(n)}, W_{j_2 k_2}^{(n)}, \ldots, W_{j_m k_m}^{(n)}]$ *at every* $p \in S^{2n-1}$ *is less than* $d^{(n)}([(j_1, k_1), \ldots, (j_m, k_m)])$.

This fact allows us to verify Condition **(B)** in the case $n = 4$, 8.

In case $n = 4$ We divide S^7 into four domains: the support of $\Phi_1^{(4)}$, that of $\Phi_2^{(4)} \Psi_1^{(4)}$, that of $\Phi_2^{(4)} \Psi_2^{(4)}$ and otherwise, and investigate the maximal dimension of $\mathcal{L}^{(4)}$ at a point belonging to each domain. First, suppose that $p \in \operatorname{supp} \Phi_1^{(4)}$. Then we have

$$\dim \mathcal{L}^{(4)}\big|_p \leq \dim \mathcal{L}[W_{04}^{(4)}, W_{12}^{(4)}, W_{13}^{(4)}, W_{23}^{(4)}]\big|_p.$$

The subgroup $[(0,4),(1,2),(1,3),(2,3)]$ of $G^{(4)}$ corresponding to the Lie algebra on the right hand side is decomposable to $[(0,4)][(1,2),(1,3),(2,3)]$. So by the fact mentioned above, $\dim \mathcal{L}^{(4)}\big|_p$ is less than four. In the same way, we obtain subgroups corresponding to the Lie algebras on the other domains and their values of $d^{(4)}$, which are illustrated with Table 1. This implies that the dimension of $\mathcal{L}^{(4)}$ at every point is less than six. And hence Condition **(B)** in Proposition 3.1 is verified.

Table 1. Dimension of Lie algebra in case $n = 4$

Domain	Corresponding subgroup of $G^{(4)}$	$d^{(4)}(\cdot)$
$\operatorname{supp} \Phi_1^{(4)}$	$[(0,4)][(1,2),(1,3),(2,3)]$	$1 + {}_3\mathrm{C}_2$
$\operatorname{supp} \Phi_2^{(4)} \Psi_1^{(4)}$	$[(0,4),(1,2),(0,1)]$	${}_4\mathrm{C}_2$
$\operatorname{supp} \Phi_2^{(4)} \Psi_2^{(4)}$	$[(1,2)][(0,4),(3,4)]$	$1 + {}_3\mathrm{C}_2$
otherwise	$[(0,4)][(1,2)]$	$1 + 1$

In case $n = 8$ As in the preceding case, we illustrate the subgroups of $G^{(8)}$

Table 2. Dimension of Lie algebra in case $n = 8$

Domain	Corresponding subgroup of $G^{(8)}$	$d^{(8)}(\cdot)$
$\operatorname{supp} \varPhi_1^{(8)}$	$[(0,8)][(1,4),(2,3),(2,5),(3,4)]$	$1 + {}_5\mathrm{C}_2$
$\operatorname{supp} \varPhi_2^{(8)} \varPsi_1^{(8)}$	$[(0,4),(0,1),(0,8),(1,4)]$ $\cdot [(2,3),(2,5),(3,7)]$	${}_4\mathrm{C}_2 + {}_4\mathrm{C}_2$
$\operatorname{supp} \varPhi_2^{(8)} \varPsi_2^{(8)}$	$[(0,8),(3,7),(6,7),(7,8)]$ $\cdot [(1,4)][(2,5)]$	$1 + 1 + {}_5\mathrm{C}_2$
$\operatorname{supp} \varPhi_2^{(8)} \setminus$ $(\operatorname{supp} \varPsi_1^{(8)} \cup \operatorname{supp} \varPsi_2^{(8)})$	$[(0,8)][(1,5)][(2,5)][(3,7)]$	$1 + 1 + 1 + 1$
otherwise	$[(0,8)][(1,4)][(2,5)]$	$1 + 1 + 1$

corresponding to $\mathcal{L}^{(8)}$ on each domain and their value of $d^{(8)}$ with Table 2. This reveals that the dimension of $\mathcal{L}^{(8)}$ at every point is less than twelve. Now the proof of Condition **(B)** is completed.

4.2. Transitivity

For the proof of Condition **(A)** in Propositions 3.1, 3.2 and 3.3, it suffices to show that there exists, for any $z \in S^{2n-1}$, a $g \in \mathcal{H}^{(n)}$ such that $gz = {}^t(1, 0, \ldots, 0)$. The verification of this consists of the following two steps:

Step 1: We construct a $g_1 \in \mathcal{H}^{(n)}$ such that $\pi^{(n)}(g_1 z) = {}^t(1, 0, \ldots, 0)$.
Step 2: We choose a $g_2 \in \mathcal{H}^{(n)}$ so that $g_2 g_1 z = {}^t(1, 0, \ldots, 0)$, where g_1 is as in Step 1.

All of vector fields multiplied by $\varPhi_1^{(n)}$ in the system are needed only for the proof of Step 2, and are not necessary in Step 1.

Let us begin with Step 1. Roughly speaking, this step is equivalent to showing the transitivity of $\mathcal{H}^{(n)}$ on S^n. Since the action of the one-parameter diffeomorphism group generated by $W_{jk}^{(n)}$ on S^{2n-1} is interpreted as a rotation on S^n due to (2.7) and (2.8), we can choose t so that $\pi_k^{(n)}(\exp(tW_{jk}^{(n)})z) = 0$ (or $\pi_j^{(n)}(\exp(tW_{jk}^{(n)})z) = 0$). From now on we shall construct g_1 as the form $\exp(t_r W_{j_r}^{(n)}) \exp(t_{r-1} W_{j_{r-1}}^{(n)}) \cdots \exp(t_1 W_{j_1}^{(n)})$, where a sequence of numbers $\{t_s\}_{s=1}^r$ are chosen successively.

Step 1 in case $n = 4$ We construct g_1 according to Table 3 as follows. In each row of Table 3, there are a vector field, a notation '$p \to q$' and

Table 3. Construction of g_1 in case $n = 4$

		$\pi_0^{(4)}$	$\pi_1^{(4)}$	$\pi_2^{(4)}$	$\pi_3^{(4)}$	$\pi_4^{(4)}$
$W_{12}^{(4)}$	$1 \to 2$	$*$	$\boxed{0}$	$\boxed{*}$	$*$	$*$
$W_{04}^{(4)}$	$0 \to 4$	$\boxed{0}$	0	$*$	$*$	$\boxed{*}$
$\Phi_2^{(4)}\Psi_2^{(4)}W_{34}^{(4)}$	$3 \to 4$	0	0	$*$	$\boxed{0}$	$\boxed{*}$
$W_{04}^{(4)}$	$4 \to 0$	$\boxed{*}$	0	$*$	0	$\boxed{0}$
$W_{12}^{(4)}$	$2 \to 1$	$*$	$\boxed{*}$	$\boxed{0}$	0	0
$\Phi_2^{(4)}\Psi_1^{(4)}W_{01}^{(4)}$	$1 \to 0$	$\boxed{-1}$	$\boxed{0}$	0	0	0
$W_{04}^{(4)}$	$0 \leftrightarrow 0$	$\boxed{1}$	0	0	0	$\boxed{0}$

five symbols '$*$' or '0' or '1' or '-1' in order. Let W_s be the vector field, '$\alpha_s \to \beta_s$' the notation in the $(s+1)$-st row in Table 3 ($s = 1, \dots, 7$). We choose a sequence $\{t_s\}_{s=1}^7$ inductively in the following way. Let $z_0 = z$. We take t_s so that $\pi_{\alpha_s}^{(4)}(\exp(t_s W_s) z_{s-1}) = 0$ and set $z_s = \exp(t_s W_s) z_{s-1}$. If $\{t_s\}_{s=1}^6$ is determined, we set $t_7 = \pi/2$. We repeat this procedure and obtain $\{t_s\}_{s=1}^7$. For this sequence, the explicit form of g_1 in question is equal to

$$\exp((\pi/2)W_{04}^{(4)}) \exp(t_6 \Phi_2^{(4)} \Psi_1^{(4)} W_{01}^{(4)}) \exp(t_5 W_{12}^{(4)}) \exp(t_4 W_{04}^{(4)}) \\ \exp(t_3 \Phi_2^{(4)} \Psi_2^{(4)} W_{34}^{(4)}) \exp(t_2 W_{04}^{(4)}) \exp(t_1 W_{12}^{(4)}).$$

Five symbols on the right hand side of the $(s+1)$-st row stand for the state of $\pi^{(4)}(z_s)$. If '$*$' is in the $\pi_\alpha^{(4)}$-column, $\pi_\alpha^{(4)}(z_s)$ is unknown. If '0' (resp. '± 1') is in the $\pi_\alpha^{(4)}$-column, $\pi_\alpha^{(4)}(z_s) = 0$ (resp. $= \pm 1$). Two boxes $\square$ in the $(s+1)$-st row mean the elements of $\pi^{(4)}$ given a change by $\exp(tW_s)$.

Step 1 in case $n = 8$ We construct g_1 by using Table 4 in the same way as in the case $n = 4$. The different point of the procedure in the case $n = 4$ is that, if '$-$' appears in the $\pi_0^{(8)}$-column of the $(s+1)$-st row, we take t_s so that $\pi_{\alpha_s}^{(8)}(\exp(t_s W_s) z_{s-1}) = 0$ and $\pi_0^{(8)}(\exp(t_s W_s) z_{s-1}) \leq 0$.

Step 1 in case $n = 2$ We proceed with a different consideration from that in cases $n = 4$, 8. We note that $\pi^{(2)}(\exp(tW^{(2)})z)$ draws a unit circle in S^2 when t runs over $\boldsymbol{R}$ for every $z \in S^3$. Given $z \in S^3$,

Table 4. Construction of g_1 in case $n = 8$

		$\pi_0^{(8)}$	$\pi_1^{(8)}$	$\pi_2^{(8)}$	$\pi_3^{(8)}$	$\pi_4^{(8)}$	$\pi_5^{(8)}$	$\pi_6^{(8)}$	$\pi_7^{(8)}$	$\pi_8^{(8)}$
$W_{14}^{(8)}$	$1 \to 4$	$*$	$\boxed{0}$	$*$	$*$	$\boxed{*}$	$*$	$*$	$*$	$*$
$W_{25}^{(8)}$	$2 \to 5$	$*$	0	$\boxed{0}$	$*$	$*$	$\boxed{*}$	$*$	$*$	$*$
$W_{08}^{(8)}$	$0 \to 8$	$\boxed{0}$	0	0	$*$	$*$	$*$	$*$	$*$	$\boxed{*}$
$\Phi_2^{(8)} W_{37}^{(8)}$	$3 \to 7$	0	0	0	$\boxed{0}$	$*$	$*$	$*$	$\boxed{*}$	$*$
$\Phi_2^{(8)} \Psi_2^{(8)} W_{67}^{(8)}$	$6 \to 7$	0	0	0	0	$*$	$*$	$\boxed{0}$	$\boxed{*}$	$*$
$\Phi_2^{(8)} \Psi_2^{(8)} W_{78}^{(8)}$	$7 \to 8$	0	0	0	0	$*$	$*$	0	$\boxed{0}$	$\boxed{*}$
$W_{14}^{(8)}$	$4 \to 1$	0	$\boxed{*}$	0	0	$\boxed{0}$	$*$	0	0	$*$
$W_{25}^{(8)}$	$5 \to 2$	0	$*$	$\boxed{*}$	0	0	$\boxed{0}$	0	0	$*$
$W_{08}^{(8)}$	$8 \to 0$	$\boxed{-}$	$*$	$*$	0	0	0	0	0	$\boxed{0}$
$\Phi_2^{(8)} \Psi_1^{(8)} W_{01}^{(8)}$	$1 \to 0$	$\boxed{-}$	$\boxed{0}$	$*$	0	0	0	0	0	0
$\Phi_2^{(8)} \Psi_1^{(8)} W_{23}^{(8)}$	$2 \to 3$	$-$	0	$\boxed{0}$	$\boxed{*}$	0	0	0	0	0
$W_{08}^{(8)}$	$0 \to 8$	$\boxed{0}$	0	0	$*$	0	0	0	0	$\boxed{*}$
$\Phi_2^{(8)} W_{37}^{(8)}$	$3 \to 7$	0	0	0	$\boxed{0}$	0	0	0	$\boxed{*}$	$*$
$\Phi_2^{(8)} \Psi_2^{(8)} W_{78}^{(8)}$	$7 \to 8$	0	0	0	0	0	0	0	$\boxed{0}$	$\boxed{1}$
$W_{08}^{(8)}$	$8 \to 0$	$\boxed{1}$	0	0	0	0	0	0	0	$\boxed{0}$

we choose a $t_1 \in \boldsymbol{R}$ such that $\pi^{(2)}(\exp(t_1 W^{(2)})z)$ lies in the half circle $\{(a, b, 0) \in S^2;\, a \leq 0\}$. This is possible, because every unit circle in S^2 intersects every half circle. Next, we set $z_1 = \exp(t_1 W^{(2)})z$ and take t_2 so that $\pi_0^{(2)}(\exp(t_2 \Phi_2^{(2)} W_{01}^{(2)})z_1) = -1$. Consequently, we obtain g_1 as the following form

$$g_1 = \exp((\pi/2)W^{(2)}) \exp(t_2 \Phi_2^{(2)} W_{01}^{(2)}) \exp(t_1 W^{(2)}).$$

Now we go to Step 2. Let $E^{(n)}$ be the inverse image ${\pi^{(n)}}^{-1}(1, 0, \ldots, 0)$. This is a closed submanifold of S^{2n-1} and can be identified with S^{n-1}. If $jk \neq 0$, we can regard $W_{jk}^{(n)}$ as a smooth vector field on $E^{(n)}$.

Step 2 in case $n = 2$ $E^{(2)} \simeq S^1$ and $W_{12}^{(2)}$ acts transitively on it. Since $\Phi_1^{(2)}$ is identically equal to 1 on $E^{(2)}$, g_2 is obtained by $g_2 = \exp(s_1 \Phi_1^{(2)} W_{12}^{(2)})$ for a suitable $s_1 \in \boldsymbol{R}$.

Step 2 in case $n = 4$ $E^{(4)} \simeq S^3$ and $\{W_{12}^{(4)}, W_{13}^{(4)}, W_{23}^{(4)}\}$ is a basis of $T_q E^{(4)}$ at every $q \in E^{(4)}$. So $\mathcal{H}[W_{12}^{(4)}, W_{13}^{(4)}, W_{23}^{(4)}]$ acts transitively on $E^{(4)}$, and hence g_2 is obtained by

$$g_2 = \exp(s_1 W_{12}^{(4)}) \exp(s_2 \Phi_1^{(4)} W_{13}^{(4)}) \exp(s_3 \Phi_1^{(4)} W_{23}^{(4)})$$

for suitable s_1, s_2, $s_3 \in \boldsymbol{R}$.

Step 2 in case $n = 8$ $E^{(8)} \simeq S^7$ and $\mathcal{L}[W_{14}^{(8)}, W_{23}^{(8)}, W_{25}^{(8)}, W_{34}^{(8)}]$ spans $T_q E^{(8)}$ at every $q \in E^{(8)}$. So $\mathcal{H}[W_{14}^{(8)}, W_{23}^{(8)}, W_{25}^{(8)}, W_{34}^{(8)}]$ acts transitively on $E^{(8)}$. Therefore g_2 is obtained as an element of $\mathcal{H}[W_{14}^{(8)}, \Phi_1^{(8)} W_{23}^{(8)}, W_{25}^{(8)}, \Phi_1^{(4)} W_{34}^{(8)}]$.

5. Global hypoellipticity

Here we state the following slightly abstract theorem.

Theorem 5.1 *Let M be a closed smooth manifold, $Z_1, \dots, Z_m$ smooth real tangent vector fields on M, and $\zeta_1, \dots, \zeta_m$ smooth non-negative functions on M. Assume that $\mathcal{H}[\zeta_1 Z_1, \dots, \zeta_m Z_m]$ acts transitively on M, and that Z_j commutes with the Laplacian Δ_M on M for every j. Then the operator $L = \sum_{j=1}^m (\zeta_j Z_j)^* \zeta_j Z_j$ is globally hypoelliptic on M.*

Condition **(C)** on the systems in Propositions 3.1, 3.2 and 3.3 follows from this theorem since they satisfy the assumptions in the theorem: the transitivity, the commutativity and the non-negativity (of cut-off functions). In what follows, we prove Theorem 5.1.

Proof of Theorem 5.1. The proof is done in the same way as in §4 of [3]. By the transitivity of $\mathcal{H}[\zeta_1 Z_1, \dots, \zeta_m Z_m]$, we have

$$\|u\|_0 \leq C_0 \|Lu\|_0 + D_N \|u\|_{-N}. \tag{5.1}$$

Thus, by means of Theorem 3.3 of [3], it is sufficient to check Condition **(D)** with some regulator Λ to hold. That is to show the next statement:

For any $\delta > 0$ and any $N > 0$, there exists a constant $C(\delta, N)$ such that the following two inequalities hold for all $u \in \boldsymbol{C}^\infty(M)$:

$$\|[\Lambda, L]u\|_{-1} \leq \delta\|Lu\|_0 + C(\delta, N)\|u\|_{-N}, \tag{5.2}$$
$$\|[\Lambda, [\Lambda, L]]u\|_{-2} \leq \delta\|Lu\|_0 + C(\delta, N)\|u\|_{-N}. \tag{5.3}$$

We shall show (5.2) and (5.3) provided that $\Lambda = (1 - \Delta_M)^{1/2}$. We remark that $\Lambda^s = (1 - \Delta_M)^{s/2}$ can be regarded as an element of $\psi\mathrm{DO}^s$ by the ellipticity of Δ_M, where we denote by $\psi\mathrm{DO}^r$ the space of pseudodifferential operators of order r (cf. [2]). Δ_M commutes with Z_j by the assumption in Theorem 5.1, so Λ^s does.

First we prove two inequalities needed later. We denote by $(f, g) = \int_M f\overline{g}\, d\mu$ the usual L^2-inner product on M, where $d\mu$ stands for the volume element on M. Integration by parts gives the following inequality:

$$\sum_{j=1}^{m} \Big\|\zeta_j Z_j u\Big\|_0^2 \leq \Big(Lu, u\Big). \tag{5.4}$$

This implies, together with (5.1), that for every $N > 0$

$$\sum_{j=1}^{m} \Big\|\zeta_j Z_j u\Big\|_0 \leq C_1\|Lu\|_0 + D_N\|u\|_{-N}. \tag{5.5}$$

Now let us begin with (5.2). By the commutativity, $\Lambda^{-1}[\Lambda, L]u$ can be rewritten as follows:

$$\Lambda^{-1}[\Lambda, L]u = 2\Lambda^{-1}\sum_{j=1}^{m} Z_j^*\Big[\Lambda, \zeta_j\Big]\zeta_j Z_j u + \Lambda^{-1}\sum_{j=1}^{m} Z_j^*\Big[\zeta_j, [\Lambda, \zeta_j]\Big] Z_j u.$$

Thus the right hand side of (5.2) is estimated

$$\begin{aligned}&\|[\Lambda, L]u\|_{-1} \\ &\quad \leq C_2\bigg(\sum_{j=1}^{m}\Big\|\Lambda^{-1}Z_j^*\Big[\Lambda, \zeta_j\Big]\zeta_j Z_j u\Big\|_0 + \sum_{j=1}^{m}\Big\|\Big[\zeta_j, [\Lambda, \zeta_j]\Big] Z_j u\Big\|_0\bigg).\end{aligned}$$

Set $\Sigma_j = \{p \in M;\, \zeta_j(p) = 0\}$. Let χ_j be a smooth function which is identically equal to 1 on Σ_j if Σ_j is not empty, and identically equal to 0 if Σ_j is empty ($j = 1, \ldots, m$). These functions will be chosen later. We divide $\zeta_j Z_j$ in the first term into $\chi_j\zeta_j Z_j$ and $(1 - \chi_j)\zeta_j Z_j$, Z_j in the second term into $Z_j\chi_j$ and $Z_j(1 - \chi_j)$. Then, an asymptotic expansion formula yields

$$\begin{aligned}
&\|[\Lambda, L]u\|_{-1} \\
&\quad \le C_3\Big(\sum_{j=1}^{m}\Big\||\nabla\zeta_j|\chi_j\zeta_jZ_ju\Big\|_0 + \Big\||\nabla\zeta_j|\chi_ju\Big\|_0\Big) \qquad (5.6)\\
&\qquad + C_4(\{\chi_j\})\Big(\sum_{j=1}^{m}\|(\zeta_jZ_j)^2u\|_{-1} + \sum_{j=1}^{m}\|\zeta_jZ_ju\|_{-1} + \|u\|_{-1}\Big),
\end{aligned}$$

where C_3 is independent of the choice of $\{\chi_j\}$ and $\nabla\zeta_j$ stands for the gradient of ζ_j. Here we identified $(1-\chi_j)$ with $(\zeta_j)^{-1}(1-\chi_j)\zeta_j \in \boldsymbol{C}^\infty(S^{2n-1})$. Given arbitrary positive numbers δ and N, we choose the support of χ_j so small that

$$\Big||\nabla\zeta_j|\chi_j\Big| \le \frac{\delta}{6C_3(C_0+C_1)}.$$

This is possible, because the inequality $|\nabla\zeta_j| \le C\sqrt{\zeta_j}$ follows from the non-negativity of ζ_j. Next we apply the interpolation inequality:

$$\|v\|_{-1} \le \varepsilon\|v\|_0 + C(\varepsilon, N)\|v\|_{-(N+1)}$$

to the fourth and fifth terms on the right hand side of (5.6) with $\varepsilon = \delta/(3C_4(C_0+C_1))$. Then we obtain by using (5.1) and (5.5)

$$\begin{aligned}
\|[\Lambda, L]u\|_{-1} \le \frac{2\delta}{3}\|Lu\|_0 &+ C_5(\{\chi_j\}, N)\|u\|_{-N}\\
&+ C_4(\{\chi_j\})\sum_{j=1}^{m}\|(\zeta_jZ_j)^2u\|_{-1}. \qquad (5.7)
\end{aligned}$$

To evaluate the third term on the right hand side of the above inequality, we need the following lemma.

Lemma 5.2 *For any positive integer N, there exists a constant $C(N)$ such that*

$$\sum_{j=1}^{m}\|(\zeta_jZ_j)^2u\|_{-1/2}^2 \le C(N)\Big(\|Lu\|_0^2 + \|u\|_{-N}^2\Big) \quad \text{for all } u \in \boldsymbol{C}^\infty(M). \qquad (5.8)$$

This lemma will be proved in the last of this section. If we admit this for the moment, the third term on the right hand side of (5.7) is evaluated as

$$C_4(\{\chi_j\}) \sum_{j=1}^{m} \|(\zeta_j Z_j)^2 u\|_{-1} \le \frac{\delta}{3}\|Lu\|_0 + C_6(\{\chi_j\}, N)\|u\|_{-N}. \quad (5.9)$$

Here we used the interpolation inequality:

$$\|v\|_{-1} \le \varepsilon \|v\|_{-1/2} + C(\varepsilon, N)\|v\|_{-N-1}.$$

Therefore we obtain (5.2) by combining (5.7) with (5.9).

Next we show (5.3). By simple calculation, we have the following equality:

$$\begin{aligned} \Lambda^{-2}\Big[\Lambda, [\Lambda, L]\Big]u = 2\sum_{j=1}^{m} \Lambda^{-2} Z_j^* [\Lambda, \zeta_j]^2 Z_j u \\ + 2\sum_{j=1}^{m} \Lambda^{-2} Z_j^* \Big[\Lambda, [\Lambda, \zeta_j]\Big]\zeta_j Z_j u + Ru, \end{aligned}$$

where $R \in \psi\mathrm{DO}^{-1}$. This implies

$$\begin{aligned} &\Big\|\Big[[\Lambda, L], \Lambda\Big]u\Big\|_{-2} \\ &\le C_7\Big(\sum_{j=1}^{m}\Big\|[\Lambda, \zeta_j]^2 Z_j u\Big\|_{-1} + \sum_{j=1}^{m}\|\zeta_j Z_j u\|_{-1} + \|u\|_{-1}\Big). \end{aligned}$$

So we can prove (5.3) in the same way as the proof of (5.2). Now (5.2) and (5.3) are verified. □

Proof of Lemma 5.2. Substituting u in (5.4) by $\Lambda^{-1/2}\zeta_j Z_j u$, we have

$$\sum_{j=1}^{m}\Big\|\zeta_j Z_j \Lambda^{-1/2}\zeta_j Z_j u\Big\|_0^2 \le \sum_{j=1}^{m}\Big(L\Lambda^{-1/2}\zeta_j Z_j u, \Lambda^{-1/2}\zeta_j Z_j u\Big). \quad (5.10)$$

The left hand side is evaluated from below as

$$\sum_{j=1}^{m}\Big\|\zeta_j Z_j \Lambda^{-1/2}\zeta_j Z_j u\Big\|_0^2 \ge \frac{1}{2}\sum_{j=1}^{m}\|(\zeta_j Z_j)^2 u\|_{-1/2}^2 - C\sum_{j=1}^{m}\|\zeta_j Z_j u\|_0^2. \quad (5.11)$$

We treat the right hand side of (5.10). By the commutativity, $L\Lambda^{-1/2}\zeta_j Z_j u$

can be represented as

$$L\Lambda^{-1/2}\zeta_j Z_j u = \Lambda^{-1/2}\zeta_j Z_j L + \sum_{k=1}^{m} M_{jk}\zeta_k Z_k + M_{j0},$$

where $M_{jk} \in \psi\mathrm{DO}^{1/2}$ $(k = 0, \ldots, m)$. Consequently we have

$$\begin{aligned}
&\sum_{j=1}^{m}\Big(L\Lambda^{-1/2}\zeta_j Z_j u, \Lambda^{-1/2}\zeta_j Z_j u\Big) \\
&\quad \leq \sum_{j=1}^{m}\Big|\Big(\Lambda^{-1}\zeta_j Z_j L u, \zeta_j Z_j u\Big)\Big| + \sum_{j,\,k=1}^{m}\Big|\Big(\Lambda^{-1/2} M_{jk}\zeta_k Z_k u, \zeta_j Z_j u\Big)\Big| \\
&\qquad + \sum_{j=1}^{m}\Big|\Big(\Lambda^{-1/2} M_{j0} u, \zeta_j Z_j u\Big)\Big|.
\end{aligned}$$

Since $\Lambda^{-1}\zeta_j Z_j$, $\Lambda^{-1/2}M_{jk} \in \psi\mathrm{DO}^0$, we have by Schwarz' inequality

$$\begin{aligned}
&\sum_{j=1}^{m}\Big(L\Lambda^{-1/2}\zeta_j Z_j u, \Lambda^{-1/2}\zeta_j Z_j u\Big) \\
&\quad \leq C\bigg(\|Lu\|_0^2 + \sum_{j=1}^{m}\|\zeta_j Z_j u\|_0^2 + \|u\|_0^2\bigg).
\end{aligned}$$

Combining this inequality with (5.10) and (5.11), we obtain

$$\sum_{j=1}^{m}\|(\zeta_j Z_j)^2 u\|_{-1/2}^2 \leq C\bigg(\|Lu\|_0^2 + \sum_{j=1}^{m}\|\zeta_j Z_j u\|_0^2 + \|u\|_0^2\bigg).$$

Applying (5.1) and (5.5) with the right hand side, we obtain (5.8). □

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