

Dirac-Weyl operators with a winding gauge potential

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Abstract. Considered is a quantum system of N (≥ 2) charged particles moving in the plane \mathbb{R}^2 under the influence of a perpendicular magnetic field. Each particle feels the magnetic field concentrated on the positions of the other particles. The gauge potential which gives this magnetic field is called a winding gauge potential. Properties of the Dirac-Weyl operators with a winding gauge potential are investigated. Notions of local quantization and partial quantization are introduced to determine them. Especially, it is proven that existence of the zero-energy states of the Dirac-Weyl operators with a winding gauge potential is well determined by the local quantization and the partial quantization.

Key words: Dirac-Weyl operators with a winding gauge potential, strong anticommutativity, decomposable operator.

1. Introduction

In a quantum system of a charged particle moving in the plane \mathbb{R}^2 under the influence of a perpendicular magnetic field, where the gauge potential of this magnetic field is strongly singular at some fixed isolated points, it is well-known that the Aharonov-Bohm effect (the AB effect) occurs when the particle goes around the magnetic field.

In a previous paper [13], the author considered a quantum system of N (≥ 2) charged particles moving in the plane \mathbb{R}^2 under the influence of a perpendicular magnetic field. Each particle feels the magnetic field concentrated on the positions of the other particles. The gauge potential which gives this magnetic field is called a winding gauge potential. If two particles go around each other, then an AB effect occurs in an extended sense. Hence, to investigate the properties of this system is very interesting. We also note that this system is closely related to the fractional statistics gas [7, 8]. In Ref. [13], the author discussed this system from a view point of the representation of the canonical commutation relations (CCR).

A motivation of this work comes from a paper [2] by A. Arai who studied the Dirac-Weyl operator with a singular gauge potential and showed some interesting behaviors of its zero-energy states. Our main aim of this paper is to investigate behaviors of zero-energy states of Dirac-Weyl operators with

a winding gauge potential.

In analyzing the system with wound AB effect, we encounter some mathematical objects: strongly anticommuting self-adjoint operators, decomposable operators and compositions of them. Therefore, we must develop a theory of these objects at the same time. By this reason, this work is interesting from not only a physical view point but also a mathematical view point.

The outline of the present paper is as follows. In Section 2, we describe the quantum system we are going to study and present preliminary results. First, we define the magnetic momentum operators with a winding gauge potential and discuss some fundamental properties. Then, we introduce the notions of the local quantization, partial quantization, and characterize the strong commutativity of the momentum operators with a winding gauge potential. We show that the momentum operators with a winding gauge potential can be expressed by the fibre direct integral of Arai's momentum operators [1]. In Section 3, we define the Dirac-Weyl operators with a winding gauge potential and present some basic properties. Sections 4 and 5 are concerned with zero-energy states of the Dirac-Weyl operators. In Section 4, we consider the case where the magnetic flux is locally quantized. In this case, we show that the Dirac-Weyl operators with a winding gauge potential have no zero-energy states. We also show that strong anticommutativity plays an important role. In Section 5, we discuss the case where the magnetic flux is partially quantized. In this case, the existence of the zero-energy states depends on the magnetic flux at each particle. A theory of decomposable operators is useful in this case. In Appendix A, we give a fundamental definitions and properties of strongly anticommuting self-adjoint operators in a \mathbb{Z}_2 -graded Hilbert space and in Appendix B, we summarize a theory of decomposable operators.

2. Preliminaries

2.1. Momentum operators with a winding gauge potential

We consider a quantum system of N charged particles with charge $q \in \mathbb{R} \setminus \{0\}$ moving in the plane \mathbb{R}^2 under the influence of a perpendicular magnetic field. The j th particle feels the magnetic field B_j given by a real distribution of the form

$$B_j(\mathbf{r}_1, \dots, \mathbf{r}_N) = \sum_{i \neq j} \gamma_{ij} \delta(\mathbf{r}_i - \mathbf{r}_j), \quad (\mathbf{r}_1, \dots, \mathbf{r}_j \in \mathbb{R}^2), \quad (1)$$

where $\gamma_{ij} \in \mathbb{R}$ ($i, j = 1, \dots, N$, $i \neq j$) and $\delta(\mathbf{r})$ is the Dirac's delta distribution. For simplicity, we assume the following.

Assumption

For $i \neq j$ ($i, j = 1, \dots, N$),

$$\gamma_{ij} = \gamma_{ji}.$$

Gauge potential \mathbb{A}_j ($j = 1, \dots, N$) of the magnetic field B_j are defined to be \mathbb{R}^2 -valued functions $\mathbb{A}_j = (A_{j1}, A_{j2})$ on the domain

$$\mathcal{M}_N := \{(\mathbf{r}_1, \dots, \mathbf{r}_N) \in \mathbb{R}^{2N} \mid \mathbf{r}_i \neq \mathbf{r}_j\}$$

such that

$$B_j = D_{x_j} A_{j2} - D_{y_j} A_{j1}$$

in the sense of distribution on \mathbb{R}^{2N} , where D_{x_j} and D_{y_j} denote the distributional partial differential operators in x_j and y_j , respectively. We denote by Δ_j ($j = 1, \dots, N$) the 2-dimensional Laplacian

$$\Delta_j := D_{x_j}^2 + D_{y_j}^2.$$

Using the well-known formula

$$\Delta_j \log |\mathbf{r}_j - \mathbf{r}_k| = 2\pi \delta(\mathbf{r}_j - \mathbf{r}_k) \quad (k \neq j),$$

we see that the distribution

$$\phi_N(\mathbf{r}_1, \dots, \mathbf{r}_N) := \sum_{i < j} \frac{\gamma_{ij}}{2\pi} \log |\mathbf{r}_i - \mathbf{r}_j|$$

satisfies

$$\Delta_j \phi_N(\mathbf{r}_1, \dots, \mathbf{r}_N) = B_j(\mathbf{r}_1, \dots, \mathbf{r}_N).$$

From this fact, we can take as a gauge potential of the magnetic field

$$\mathbb{A}_j = (A_{j1}, A_{j2}) = (-D_{y_j} \phi_N, D_{x_j} \phi_N), \quad j = 1, \dots, N.$$

Explicitly, we have

$$A_{j1}(\mathbf{r}_1, \dots, \mathbf{r}_N) = - \sum_{i \neq j} \frac{\gamma_{ij}}{2\pi} \frac{y_j - y_i}{|\mathbf{r}_i - \mathbf{r}_j|^2}, \quad (2)$$

$$A_{j2}(\mathbf{r}_1, \dots, \mathbf{r}_N) = \sum_{i \neq j} \frac{\gamma_{ij}}{2\pi} \frac{x_j - x_i}{|\mathbf{r}_i - \mathbf{r}_j|^2}, \quad (3)$$

for $(\mathbf{r}_1, \dots, \mathbf{r}_N) \in \mathcal{M}_N$.

Definition 2.1 The gauge potential $\mathbb{A}_j = (A_{j1}, A_{j2})$ ($j = 1, \dots, N$) given by (2) and (3) is called the *winding gauge potential*.

We use a system of units where the light speed c and the Planck constant \hbar are equal to 1. Let

$$p_{j1} := -iD_{x_j}, \quad p_{j2} := -iD_{y_j} \quad (j = 1, \dots, N),$$

acting in $L^2(\mathbb{R}^{2N})$. We introduce the operators P_{j1} and P_{j2} defined by

$$P_{j\alpha} := p_{j\alpha} - qA_{j\alpha}, \quad (j = 1, \dots, N, \alpha = 1, 2)$$

acting in $L^2(\mathbb{R}^{2N})$ with domain $\text{dom}(P_{j\alpha}) = \text{dom}(p_{j\alpha}) \cap \text{dom}(A_{j\alpha})$.

Definition 2.2 The pair of operators $\mathbb{P}_j = (P_{j1}, P_{j2})$ ($j = 1, \dots, N$) is called the *momentum operator with the winding gauge potential*.

Let

$$\begin{aligned} \mathcal{S}_1^{(N)} &:= \{(\mathbf{r}_1, \dots, \mathbf{r}_N) \in \mathbb{R}^{2N} \mid \mathbf{r}_i = (x_i, y_i) \in \mathbb{R}^2, y_i \neq y_j \quad (i \neq j)\}, \\ \mathcal{S}_2^{(N)} &:= \{(\mathbf{r}_1, \dots, \mathbf{r}_N) \in \mathbb{R}^{2N} \mid \mathbf{r}_i = (x_i, y_i) \in \mathbb{R}^2, x_i \neq x_j \quad (i \neq j)\} \end{aligned}$$

and let

$$\begin{aligned} \psi_{j1}(\mathbf{r}_1, \dots, \mathbf{r}_N) &:= - \sum_{i \neq j} \frac{\gamma_{ij}}{2\pi} \text{Arctan} \left(\frac{x_j - x_i}{y_j - y_i} \right), \\ \psi_{j2}(\mathbf{r}_1, \dots, \mathbf{r}_N) &:= \sum_{i \neq j} \frac{\gamma_{ij}}{2\pi} \text{Arctan} \left(\frac{y_j - y_i}{x_j - x_i} \right). \end{aligned}$$

Then we can prove the following theorem.

Theorem 2.3 For each $j = 1, \dots, N$ and $\alpha = 1, 2$, $P_{j\alpha}$ is essentially self-adjoint on $C_0^\infty(\mathcal{S}_\alpha^{(N)})$. Moreover, the following hold:

$$\overline{P_{j\alpha}} = e^{iq\psi_{j\alpha}} p_{j\alpha} e^{-iq\psi_{j\alpha}}, \quad (4)$$

where we denote the closure of $P_{j\alpha}$ by $\overline{P_{j\alpha}}$.

Proof. In the similar way as in proof of [13, Theorem 2.3], we can show the assertion. \square

By (1), it is clear that

$$D_{x_j} A_{j2}(\mathbf{r}_1, \dots, \mathbf{r}_N) - D_{y_j} A_{j1}(\mathbf{r}_1, \dots, \mathbf{r}_N) = 0, \quad j = 1, \dots, N$$

for each $(\mathbf{r}_1, \dots, \mathbf{r}_N) \in \mathcal{M}_N$. Therefore, we have

$$[P_{j1}, P_{j2}] = 0 \quad \text{on } C_0^\infty(\mathcal{M}_N).$$

Similarly, we can check that

$$[P_{j\alpha}, P_{k\beta}] = 0 \quad \text{on } C_0^\infty(\mathcal{M}_N) \quad (5)$$

for $j \neq k$ and $\alpha, \beta = 1, 2$.

We say that self-adjoint operators $\{T_j\}_{j=1}^n$ strongly commute if their spectral projections commute each other. It is well-known that $\{T_j\}_{j=1}^n$ strongly commute if and only if

$$e^{iaT_j} e^{ibT_k} = e^{ibT_k} e^{iaT_j}$$

for all $a, b \in \mathbb{R}$ and $j, k = 1, \dots, n$.

The above facts suggest that $\{\overline{P}_{j\alpha} \mid j = 1, \dots, N, \alpha = 1, 2\}$ may strongly commute.

To investigate the strong commutativity of $\{\overline{P}_{j\alpha} \mid j = 1, \dots, N, \alpha = 1, 2\}$, we introduce some notations. Let $a, b \in \mathbb{R}$ and $C(x, y; a, b)$ be the rectangular closed curve: $(x, y) \rightarrow (x+a, y) \rightarrow (x+a, y+b) \rightarrow (x, y+b) \rightarrow (x, y)$ in \mathbb{R}^2 and $D(x, y; a, b)$ be its interior domain. Then, for each $(\mathbf{r}_1, \dots, \mathbf{r}_N) \in \mathbb{R}^{2N}$ with $\mathbf{r}_i = (x_i, y_i) \in \mathbb{R}^2$, we define

$$\begin{aligned} \Phi_{j,k}^{(s,t)}(\mathbf{r}_1, \dots, \mathbf{r}_N) \\ := \begin{cases} \epsilon(s)\epsilon(t) \sum_{i \neq k} \gamma_{ik} \# \{i \mid i \neq k, \mathbf{r}_i \in D(\mathbf{r}_k; s, t)\} & (j = k) \\ -\epsilon(s)\epsilon(t) \gamma_{jk} \# \{(j, k) \mid (x_k, y_j) \in D((x_j, y_k); s, t)\} & (j \neq k) \end{cases} \end{aligned}$$

where we use the symbol $\epsilon(t)$ ($t \in \mathbb{R}$) defined by

$$\epsilon(s) := \begin{cases} 1 & (s \geq 0) \\ -1 & (s < 0) \end{cases}$$

and $\#A$ means the cardinality of the set A . The function $\Phi_{j,k}^{(s,t)}$ defines a unique self-adjoint multiplication operator on $L^2(\mathbb{R}^{2N})$. We denote it by the same symbol $\Phi_{j,k}^{(s,t)}$.

Theorem 2.4 For each $s, t \in \mathbb{R}$ and $j, k = 1, \dots, N$, we have

- (i) $e^{is\bar{P}_{j1}}e^{it\bar{P}_{k2}} = \exp(-iq\Phi_{j,k}^{(s,t)})e^{it\bar{P}_{k2}}e^{is\bar{P}_{j1}}$,
- (ii) $e^{is\bar{P}_{j\alpha}}e^{it\bar{P}_{k\alpha}} = e^{it\bar{P}_{k\alpha}}e^{is\bar{P}_{j\alpha}} \quad (\alpha = 1, 2)$,

Proof. In the similar way as in the proof of [13, Theorem 2.4], we can prove this theorem. \square

Definition 2.5 Let

$$\Lambda^{(N)} := \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid i < j, i, j = 1, \dots, N\}.$$

- (i) The magnetic flux is *locally quantized* if $\Phi_{j,k}^{(s,t)}$ is a $2\pi\mathbb{Z}/q$ -valued function for all $s, t \in \mathbb{R}$ and $(j, k) \in \Lambda^{(N)}$, equivalently, $\gamma_{ij}/\theta_0 \in \mathbb{Z}$ for all $(j, k) \in \Lambda^{(N)}$, where $\theta_0 := 2\pi/q$ the flux quanta.
- (ii) For each subset Λ of $\Lambda^{(N)}$, we say that the magnetic flux is *partially quantized with respect to Λ* if $\Phi_{j,k}^{(s,t)}$ is a $2\pi\mathbb{Z}/q$ -valued function for all $s, t \in \mathbb{R}$ and $(j, k) \in \Lambda$, equivalently, $\gamma_{ij}/\theta_0 \in \mathbb{Z}$ for all $(j, k) \in \Lambda$.

Corollary 2.6 For each $j, k = 1, \dots, N$, we have the following facts:

- (i) $\bar{P}_{j\alpha}$ and $\bar{P}_{k\beta}$ ($j < k$) strongly commute if and only if the magnetic flux is partially quantized with respect to $\{(j, k)\} \subset \Lambda^{(N)}$.
- (ii) \bar{P}_{j1} and \bar{P}_{j2} strongly commute if and only if the magnetic flux is partially quantized with respect to $\Lambda_j \subset \Lambda^{(N)}$, where

$$\Lambda_j := \{(i, j), (j, k) \in \Lambda^{(N)} \mid i < j < k\}. \quad (6)$$

- (iii) $\{\bar{P}_{j\alpha} \mid j = 1, \dots, N, \alpha = 1, 2\}$ is a family of strongly commuting self-adjoint operators if and only if the magnetic flux is locally quantized.

Proof. These are simple applications of Theorem 2.4. \square

2.2. Fibre direct integral decomposition of the momentum operators with the winding gauge potential

For later use, we represent the momentum operators with the winding gauge potential as a fibre direct integral of Arai's momentum operators [1].

For each $i = 1, \dots, N$, let \mathbb{R}_i^2 is a copy of \mathbb{R}^2 . Then, clearly, we have

$$\mathbb{R}^{2N} = \mathbb{R}_1^2 \times \dots \times \mathbb{R}_N^2.$$

For each $j = 1, \dots, N$, we define

$$\Omega_j := \mathbb{R}_1^2 \times \dots \times \widehat{\mathbb{R}_j^2} \times \dots \times \mathbb{R}_N^2,$$

where $\widehat{\mathbb{R}}_j^2$ indicates the omission of \mathbb{R}_j^2 .

Let $\omega_j := (\mathbf{a}_1, \dots, \mathbf{a}_{j-1}, \mathbf{a}_{j+1}, \dots, \mathbf{a}_N) \in \Omega_j$. Then we define a multiplication operator $\tilde{A}_{j\alpha}(\omega_j)$ on $L^2(\widehat{\mathbb{R}}_j^2)$ by

$$\tilde{A}_{j\alpha}(\omega_j)(\mathbf{r}_j) := A_{j\alpha}(\mathbf{a}_1, \dots, \mathbf{a}_{j-1}, \mathbf{r}_j, \mathbf{a}_{j+1}, \dots, \mathbf{a}_N).$$

Then relative to the direct integral decomposition

$$L^2(\mathbb{R}^{2N}) = \int_{\Omega_j}^{\oplus} L^2(\widehat{\mathbb{R}}_j^2) d\omega_j, \quad (7)$$

we can represent the multiplication operator $A_{j\alpha}$ as

$$A_{j\alpha} = \int_{\Omega_j}^{\oplus} \tilde{A}_{j\alpha}(\omega_j) d\omega_j.$$

On the other hand, it is clear that

$$p_{j\alpha} = \int_{\Omega_j}^{\oplus} \tilde{p}_{j\alpha} d\omega_j,$$

where we denote the free momentum operator $-iD_{x_j}$, $-iD_{y_j}$ acting in $L^2(\widehat{\mathbb{R}}_j^2)$ by \tilde{p}_{j1} and \tilde{p}_{j2} , respectively. For each $\omega_j \in \Omega_j$, we define the linear operator acting in $L^2(\widehat{\mathbb{R}}_j^2)$ by

$$\begin{aligned} P_{j\alpha}(\omega_j) &:= \tilde{p}_{j\alpha} - q\tilde{A}_{j\alpha}(\omega_j), \\ \text{dom}(P_{j\alpha}(\omega_j)) &:= \text{dom}(\tilde{p}_{j\alpha}) \cap \text{dom}(\tilde{A}_{j\alpha}(\omega_j)), \end{aligned}$$

where, for a linear operator A , $\text{dom}(A)$ denotes the domain of A .

Definition 2.7 The operator $P_{j\alpha}(\omega_j)$ ($j = 1, \dots, N$, $\alpha = 1, 2$) is called *Arai's momentum operator* [1].

Lemma 2.8 Let $P_{j\alpha}(\omega_j)$ be Arai's momentum operator.

- (i) For all $\omega_j \in \Omega_j$ and $\alpha = 1, 2$, $P_{j\alpha}(\omega_j)$ is essentially self-adjoint.
- (ii) The mapping $\omega_j \in \Omega_j \rightarrow \overline{P}_{j\alpha}(\omega_j)$ is measurable, where $\overline{P}_{j\alpha}(\omega_j)$ is the closure of $P_{j\alpha}(\omega_j)$.

Proof. See [13, Lemma 2.8, Proposition 2.9]. □

By the above lemma, we can define a direct integral operator by

$$\int_{\Omega_j}^{\oplus} \overline{P}_{j\alpha}(\omega_j) d\omega_j.$$

Moreover we have the following theorem.

Theorem 2.9 *For each $j = 1, \dots, N$, $\alpha = 1, 2$, we have*

$$\bar{P}_{j\alpha} = \int_{\Omega_j}^{\oplus} \bar{P}_{j\alpha}(\omega_j) d\omega_j.$$

Proof. See [13, Proposition 2.8]. □

2.3. A Hamiltonian with no ground states

As a Hamiltonian of the quantum system under consideration, we can take the Schrödinger Hamiltonian $H_N(\mathbb{A})$ defined as the self-adjoint operator associated with the non-negative, closed, quadratic form

$$s(f, g) := \sum_{j=1}^N \{ \langle \bar{P}_{j1}f, \bar{P}_{j1}g \rangle + \langle \bar{P}_{j2}f, \bar{P}_{j2}g \rangle \},$$

$$f, g \in \bigcap_{j=1}^N (\text{dom}(\bar{P}_{j1}) \cap \text{dom}(\bar{P}_{j2}))$$

so that

$$\text{dom}(H_N(\mathbb{A})^{1/2}) = \bigcap_{j=1}^N \bigcap_{\alpha=1,2} \text{dom}(\bar{P}_{j\alpha})$$

and

$$\langle H_N(\mathbb{A})^{1/2}f, H_N(\mathbb{A})^{1/2}g \rangle = s(f, g), \quad f, g \in \text{dom}(H_N(\mathbb{A})^{1/2}).$$

Theorem 2.10 *The Hamiltonian $H_N(\mathbb{A})$ has no zero-energy states, i.e.,*

$$\ker(H_N(\mathbb{A})) = \{0\}.$$

Proof. By (4), it is clear that

$$\ker(\bar{P}_{j\alpha}) = \{0\} \tag{8}$$

for each $j = 1, \dots, N$ and $\alpha = 1, 2$. On the other hand, we can show that

$$\ker(H_N(\mathbb{A})) = \bigcap_{j=1}^N \bigcap_{\alpha=1,2} \ker(\bar{P}_{j\alpha}).$$

Hence we have the desired result. □

3. Dirac-Weyl operators with the winding gauge potential

Throughout this paper, the domain $\text{dom}(S + T)$ of the sum $S + T$ of two linear operators S and T from a Hilbert space to another is always taken to be $\text{dom}(S) \cap \text{dom}(T)$ unless otherwise stated.

Let σ_j ($j = 1, 2, 3$) be the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

For each $j = 1, \dots, N$, we introduce

$$\sigma_\alpha^{(j)} := \overbrace{\sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \underbrace{\sigma_\alpha}_{j \text{th}} \otimes I_2 \otimes \cdots \otimes I_2}^N \quad (\alpha = 1, 2),$$

where $I_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. It is easy to see that

$$\{\sigma_\alpha^{(j)}, \sigma_\beta^{(j)}\} = 2\delta_{\alpha\beta}, \quad \{\sigma_\alpha^{(j)}, \sigma_\beta^{(k)}\} = 0, \quad j \neq k$$

for $\alpha, \beta = 1, 2$, $j, k = 1, \dots, N$, where $\{A, B\} = AB + BA$ and δ_{ab} is the Kronecker delta.

Definition 3.1 For each $j = 1, \dots, N$, we define a linear operator acting in $\mathcal{H}_N := \mathbb{C}^{2^N} \otimes L^2(\mathbb{R}^{2N})$ by

$$Q_j := \sigma_1^{(j)} \otimes \bar{P}_{j1} + \sigma_2^{(j)} \otimes \bar{P}_{j2}.$$

Q_j is called the *Dirac-Weyl operator with the winding gauge potential*.

Let \mathcal{X} be a Hilbert space. If \mathcal{X} is a direct sum of \mathcal{X}_0 and \mathcal{X}_1 (i.e., $\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{X}_1$), then \mathcal{X} is said to be \mathbb{Z}_2 -graded Hilbert space. Let P_0 and P_1 be the orthogonal projections onto \mathcal{X}_0 and \mathcal{X}_1 , respectively. Then we define an operator τ on \mathcal{X} by

$$\tau := P_0 - P_1.$$

It is not difficult to show that τ is self-adjoint and unitary.

Conversely, if there exists a self-adjoint unitary operator τ on a Hilbert space \mathcal{X} , then the operators P_0 and P_1 defined by

$$P_0 := \frac{1}{2}(I + \tau), \quad P_1 := \frac{1}{2}(I - \tau)$$

are orthogonal projections such that $P_0P_1 = P_1P_0 = 0$. Hence, \mathcal{X} is a \mathbb{Z}_2 -graded Hilbert space with the direct sum decomposition $\mathcal{X} = \text{ran}(P_0) \oplus \text{ran}(P_1)$, where, for a linear operator T , $\text{ran}(T)$ denotes the range of T .

The operator τ is called the *grading operator for \mathcal{X}* .

Let \mathcal{X} be a \mathbb{Z}_2 -graded Hilbert space and τ be the grading operator for \mathcal{X} . Let T be a linear operator acting in \mathcal{X} satisfying $\tau \text{dom}(T) \subset \text{dom}(T)$. We say that T is *even* if it satisfies

$$\tau T \tau = T.$$

On the other hand, we say that T is *odd*, if it satisfies

$$\tau T \tau = -T.$$

It is not hard to check that the linear operator τ_N on \mathcal{H}_N defined by

$$\tau_N := \overbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}^N \otimes I_{L^2(\mathbb{R}^{2N})}$$

is self-adjoint and unitary. Hence, τ_N is a grading operator for \mathcal{H}_N . Thus \mathcal{H}_N is a \mathbb{Z}_2 -graded Hilbert space and each Q_j can be regarded as a linear operator acting in \mathbb{Z}_2 -graded Hilbert space.

From the definition of τ_N , it follows that

$$\{\tau_N, \sigma_\alpha^{(j)} \otimes I_{L^2(\mathbb{R}^{2N})}\} = 0$$

for each $\alpha = 1, 2$, $j = 1, \dots, N$, i.e., $\sigma_\alpha^{(j)} \otimes I_{L^2(\mathbb{R}^{2N})}$ is an *odd* operator.

We introduce

$$\widehat{H}_N(\mathbb{A}) := I_{2N} \otimes H_N(\mathbb{A}),$$

where $I_{2N} = \overbrace{I_2 \otimes \cdots \otimes I_2}^N$.

Proposition 3.2 (i) For each $\Psi \in \mathbb{C}^{2N} \otimes_{\text{alg}} C_0^\infty(\mathcal{M}_N)$ (the symbol \otimes_{alg} means algebraic tensor product) and $j, k = 1, \dots, N$, $j \neq k$,

$$\{Q_j, Q_k\} \Psi = 0.$$

(ii) Each Q_j is an *odd* operator i.e.,

$$\{Q_j, \tau_N\} \Psi = 0$$

for all $\Psi \in \text{dom}(Q_j)$.

(iii) For each $\Psi \in \mathbb{C}^{2^N} \otimes_{\text{alg}} C_0^\infty(\mathcal{M}_N)$,

$$\begin{aligned} \widehat{H}_N(\mathbb{A})\Psi &= \sum_{j=1}^N Q_j^2 \Psi \\ &= \left(\sum_{j=1}^N Q_j \right)^2 \Psi. \end{aligned}$$

Proof. Direct calculations. \square

Let $\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{X}_1$ be a \mathbb{Z}_2 -graded Hilbert space. For a self-adjoint operator T in \mathcal{X} , we denote its spectral measure by $E_T(J)$ ($J \in \mathbb{B}^1$ the Borel field of \mathbb{R}). If T is *odd*, then we define

$$E_T(J)_{\bar{0}} := \frac{1}{2} \{E_T(J) + E_T(-J)\}, \quad E_T(J)_{\bar{1}} := \frac{1}{2} \{E_T(J) - E_T(-J)\}$$

for each $J \in \mathbb{B}^1$.

Let A and B be odd self-adjoint operators in \mathcal{X} . We say that A and B *strongly anticommute* if the following hold:

$$\begin{aligned} [E_A(J_1)_{\bar{0}}, E_B(J_2)_{\bar{0}}] &= [E_A(J_1)_{\bar{0}}, E_B(J_2)_{\bar{1}}] = [E_A(J_1)_{\bar{1}}, E_B(J_2)_{\bar{0}}] = 0, \\ [E_A(J_1)_{\bar{1}}, E_B(J_2)_{\bar{1}}] &= 0 \end{aligned}$$

for each J_1, J_2 in \mathbb{B}^1 , where $\{a, b\} := ab + ba$. For reader's convenience, we summarize the basic properties of strongly anticommuting self-adjoint operators in a \mathbb{Z}_2 -graded Hilbert space in Appendix A.

Theorem 3.3 *Let Λ_j ($j = 1, \dots, N$) be a subset of $\Lambda^{(N)}$ defined by (6).*

(i) *If the magnetic flux is partially quantized with respect to Λ_j , then Q_j is self-adjoint and*

$$Q_j^2 = I_{2^N} \otimes (\overline{P}_{j1}^2 + \overline{P}_{j2}^2).$$

(ii) *If the magnetic flux is partially quantized with respect to $\Lambda_j \cup \Lambda_k$, then Q_j and Q_k strongly anticommute.*

Proof. (i) Suppose that $\gamma_{lm}/\theta_0 \in \mathbb{Z}$ ($(l, m) \in \Lambda_j$). Then by Corollary 2.6 (ii), \overline{P}_{j1} and \overline{P}_{j2} strongly commute. Hence, applying Proposition A.5, we can conclude that $\sigma_1^{(j)} \otimes \overline{P}_{j1}$ and $\sigma_2^{(j)} \otimes \overline{P}_{j2}$ strongly anticommute. Therefore we have the desired result by Proposition A.4 (ii).

(ii) Suppose that $\gamma_{lm}/\theta_0 \in \mathbb{Z}$ ($(l, m) \in \Lambda_j \cup \Lambda_k$). Then by an argument similar to the proof of (i), Q_j and Q_k are self-adjoint. Moreover, applying Proposition A.4 (iv), we can conclude that Q_j and Q_k strongly anticommute. \square

Theorem 3.4 *Suppose that the magnetic flux is locally quantized. Then the following hold:*

- (i) $\{Q_j\}_{j=1}^N$ is a family of strongly anticommuting self-adjoint operators.
- (ii) For each Ψ in $\text{dom}(Q_j) \cap \text{dom}(Q_k)$ ($j \neq k$),

$$\{Q_j, Q_k\}\Psi = 0.$$

- (iii) For each $\Psi \in \text{dom}(Q_j)$,

$$\{\tau_N, Q_j\}\Psi = 0.$$

- (iv) The linear operator $Q := \sum_{j=1}^N Q_j$ is self-adjoint.

$$\begin{aligned} \text{(v)} \quad \widehat{H}_N(\mathbb{A}) &= \sum_{j=1}^N Q_j^2 \\ &= Q^2 \end{aligned}$$

Proof. These are simple applications of Theorem 3.3, Proposition A.4. \square

Definition 3.5 Let \mathcal{H} be a \mathbb{Z}^2 -graded Hilbert space and τ be the grading operator for \mathcal{H} . Let $\{Q_j\}_{j=1}^N$ be a family of self-adjoint operators on \mathcal{H} . We say that the quadruple $\{\mathcal{H}, \tau, H, \{Q_j\}_{j=1}^N\}$ is a *model of N -fold supersymmetric quantum mechanics*, if the following (i)–(iii) hold:

- (i) Q_j is an odd operator i.e.,

$$\{Q_j, \tau\}\Psi = 0$$

for each $\Psi \in \text{dom}(Q_j)$.

- (ii) For each $\Psi \in \text{dom}(Q_j) \cap \text{dom}(Q_k)$ ($j \neq k$),

$$\{Q_j, Q_k\}\Psi = 0.$$

- (iii) For all $\Psi \in \bigcap_{j=1}^N \text{dom}(Q_j^2)$,

$$H\Psi = \sum_{j=1}^N Q_j^2\Psi.$$

The self-adjoint operators Q_j and H are called the j th *supercharge* and the *supersymmetric Hamiltonian*, respectively.

If $\{Q_j\}_{j=1}^N$ is a family of strongly anticommuting self-adjoint operators, then we say that the model $\{\mathcal{H}, \tau, H, \{Q_j\}_{j=1}^N\}$ is *integrable*.

Corollary 3.6 *Suppose that the magnetic flux is locally quantized, i.e., $\gamma_{ij}/\theta_0 \in \mathbb{Z}$ for each $(i, j) \in \Lambda^{(N)}$. Then the quadruple $\{\mathcal{H}_N, \tau_N, \widehat{H}_N(\mathbb{A}), \{Q_j\}_{j=1}^N\}$ is an integrable model of N -fold supersymmetric quantum mechanics.*

Proposition 3.7 *Let $\Pi = \{\mathcal{H}, \tau, H, \{Q_j\}_{j=1}^N\}$ be a model of N -fold supersymmetric quantum mechanics. If Π is integrable, then the total supercharge*

$$Q := \sum_{j=1}^N Q_j$$

is self-adjoint and satisfies

$$\{Q, \tau\}\Psi = 0, \quad Q^2 = H.$$

Hence the quadruple $\{\mathcal{H}, \tau, H, Q\}$ is a model of supersymmetric quantum mechanics.

Proof. This immediately follows from Proposition A.4. □

Corollary 3.8 *Let Q_j ($j = 1, \dots, N$) be the Dirac-Weyl operator with the winding gauge potential. Suppose that the magnetic flux is locally quantized. Then the total supercharge*

$$Q = \sum_{j=1}^N Q_j$$

is odd, self-adjoint and satisfies

$$\widehat{H}_N(\mathbb{A}) = Q^2.$$

Moreover, the quadruple $\{\mathcal{H}_N, \tau_N, \widehat{H}_N(\mathbb{A}), Q\}$ is a model of supersymmetric quantum mechanics.

4. Zero-energy states of the Dirac-Weyl operators with the winding gauge potential (I) — the case where the magnetic flux is locally quantized

Throughout this section, we assume the following condition:

$$\frac{\gamma_{ij}}{\theta_0} \in \mathbb{Z} \quad \text{for each } (i, j) \in \Lambda^{(N)}, \quad (9)$$

i.e., the magnetic flux is locally quantized.

Relative to the fibre direct integral decomposition (7), we have

$$\mathcal{H}_N = \int_{\Omega_j}^{\oplus} \mathbb{C}^{2^N} \otimes L^2(\mathbb{R}_j^2) d\omega_j.$$

For each $\omega_j \in \Omega_j$, we define a linear operator acting in $L^2(\mathbb{R}_j^2)$ by

$$Q_j(\omega_j) := \sigma_1^{(j)} \otimes \bar{P}_{j1}(\omega_j) + \sigma_2^{(j)} \otimes \bar{P}_{j2}(\omega_j). \quad (10)$$

Lemma 4.1 *Assume that (9). Then, for each $j = 1, \dots, N$, the following hold:*

- (i) *For each $\omega_j \in \Omega_j$, $Q_j(\omega_j)$ is self-adjoint.*
- (ii) *The mapping $\omega_j \in \Omega_j \rightarrow Q_j(\omega_j)$ is measurable.*
- (iii) $\ker(Q_j(\omega_j)) = \{0\}$.

Proof. (i) By [1, Theorem 2.1], $\bar{P}_{j1}(\omega_j)$ and $\bar{P}_{j2}(\omega_j)$ strongly commute if and only if the magnetic flux is partially quantized with respect to Λ_j . Hence $\sigma_1^{(j)} \otimes \bar{P}_{j1}(\omega_j)$ and $\sigma_2^{(j)} \otimes \bar{P}_{j2}(\omega_j)$ strongly anticommute by Proposition A.5 (iii). Applying Proposition A.4 (ii), $Q_j(\omega_j)$ is self-adjoint.

(ii) It suffices to show that the mapping $\omega_j \in \Omega_j \rightarrow e^{isQ_j(\omega_j)}$ is measurable for each $s \in \mathbb{R}$ (see [13, Appendix A]). By the Trotter product formula, we have

$$e^{isQ_j(\omega_j)} = \text{s-lim}_{n \rightarrow \infty} \left(e^{is/n\sigma_1^{(j)} \otimes \bar{P}_{j1}(\omega_j)} e^{is/n\sigma_2^{(j)} \otimes \bar{P}_{j2}(\omega_j)} \right)^n. \quad (11)$$

On the other hand, by the measurability of $\bar{P}_{j\alpha}(\omega_j)$ ($\alpha = 1, 2$), we can conclude that the mapping $\omega_j \rightarrow e^{is/n\sigma_1^{(j)} \otimes \bar{P}_{j1}(\omega_j)} e^{is/n\sigma_2^{(j)} \otimes \bar{P}_{j2}(\omega_j)}$ is measurable. Combining this with (11), we have the desired result.

(iii) By the strong anticommutativity of $\sigma_1^{(j)} \otimes \bar{P}_{j1}(\omega_j)$ and $\sigma_2^{(j)} \otimes \bar{P}_{j2}(\omega_j)$, we obtain

$$Q_j(\omega_j)^2 = I_{2^N} \otimes (\bar{P}_{j1}(\omega_j)^2 + \bar{P}_{j2}(\omega_j)^2). \quad (12)$$

On the other hand, using Theorem 2.9 and (8), we can conclude that $\ker(\overline{P}_{j\alpha}(\omega_j)) = \{0\}$ by Proposition B.10 (iii) and Proposition B.5 (iv). Combining this with (12), we can show that $\{0\} = \ker(Q_j(\omega_j)^2) = \ker(Q_j(\omega_j))$. \square

By the above lemma, we can define a self-adjoint operator by

$$\int_{\Omega_j}^{\oplus} Q_j(\omega_j) d\omega_j \quad (j = 1, \dots, N).$$

Furthermore, we obtain a following useful theorem.

Theorem 4.2 *Assume that (9). Then Q_j is decomposable and*

$$Q_j = \int_{\Omega_j}^{\oplus} Q_j(\omega_j) d\omega_j.$$

Proof. Under the assumption (9), $\sigma_1^{(j)} \otimes \overline{P}_{j1}$ and $\sigma_2^{(j)} \otimes \overline{P}_{j2}$ are decomposable and odd self-adjoint operators. Moreover, $\sigma_1^{(j)} \otimes \overline{P}_{j1}$ and $\sigma_2^{(j)} \otimes \overline{P}_{j2}$ strongly anticommute. Hence we can apply Proposition B.13 (iv) and conclude the assertion in the theorem. \square

Corollary 4.3 *Assume that (9). For each $j = 1, \dots, N$, we have*

$$\ker(Q_j) = \{0\}.$$

Hence, the total supercharge $Q := \sum_{j=1}^N Q_j$ has no zero-energy states.

Proof. By Proposition B.10 (iii) and Theorem 4.2, we obtain

$$\ker(Q_j) = \int_{\Omega_j}^{\oplus} \ker(Q_j(\omega_j)) d\omega_j.$$

Hence, applying Lemma 4.1 (iii) and Proposition B.5 (iv), we have $\ker(Q_j) = \{0\}$ ($j = 1, \dots, N$). By the strong anticommutativity of $\{Q_j\}_{j=1}^N$, we have $\ker(Q) = \bigcap_{j=1}^N \ker(Q_j) = \{0\}$. \square

5. Zero-energy states of the Dirac-Weyl operators with the winding gauge potential (II) — the case where the magnetic flux is not locally quantized

In Section 4, the strong anticommutativity played an important role. We also saw that the strong anticommutativity was closely connected with

the local quantization. In this section, we do not assume the local quantization. Hence, we can not apply the strong anticommutativity. By this reason, to analyze the zero-energy states of the Dirac-Weyl operators is more difficult in this case.

Let \mathcal{X} be a Hilbert space. For a linear operator T on \mathcal{X} and a subspace of $\mathcal{D} \subset \text{dom}(T)$, $T|_{\mathcal{D}}$ denotes the restriction of T to \mathcal{D} .

We first consider the minimal version of the Dirac-Weyl operators Q_j

$$Q_{j,\min} := Q_j|_{\mathbb{C}^{2^N} \otimes_{\text{alg}} C_0^\infty(\mathcal{M}_N)}.$$

It is easy to check that each $Q_{j,\min}$ is symmetric and hence closable. We denote its closure by $\overline{Q}_{j,\min}$.

Lemma 5.1 For each $j = 1, \dots, N$,

$$\ker(\overline{Q}_{j,\min}) = \{0\}.$$

Proof. Let $\Psi \in \ker(\overline{Q}_{j,\min})$. Then there exists a sequence $\{\Psi_n\}_{n=1}^\infty \in \mathbb{C}^{2^N} \otimes_{\text{alg}} C_0^\infty(\mathcal{M}_N)$ such that $\Psi_n \rightarrow \Psi$ and $\overline{Q}_{j,\min} \Psi_n \rightarrow 0$ in $\mathbb{C}^{2^N} \otimes L^2(\mathbb{R}^{2N})$ as $n \rightarrow \infty$. By (5) and the fact $\{\sigma_1^{(j)}, \sigma_2^{(j)}\} = 0$ ($j = 1, \dots, N$), we have

$$\|Q_{j,\min} \Psi_n\|^2 = \|I_{2^N} \otimes \overline{P}_{j1} \Psi_n\|^2 + \|I_{2^N} \otimes \overline{P}_{j2} \Psi_n\|^2.$$

Hence $\overline{P}_{j\alpha} \Psi_n \rightarrow 0$ ($\alpha=1,2$) as $n \rightarrow \infty$. This implies that $\Psi \in \text{dom}(I_{2^N} \otimes \overline{P}_{j1}) \cap \text{dom}(I_{2^N} \otimes \overline{P}_{j2})$ and $\overline{P}_{j\alpha} \Psi = 0$. Thus, by (8), we have $\Psi = 0$. \square

Although the above lemma shows that $Q_{j,\min}$ has no zero-energy states, self-adjoint extensions of $Q_{j,\min}$ may have zero-energy states. In fact this is true, as is shown below.

Relative to the natural identification

$$\mathcal{H}_N = \mathbb{C}^{2^{(j-1)}} \otimes (\mathbb{C}^2 \otimes L^2(\mathbb{R}^{2N})) \otimes \mathbb{C}^{2^{(N-j)}}, \quad (13)$$

we can represent Q_j as

$$Q_j := \overbrace{\sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \underbrace{Q_j}_{j \text{ th}} \otimes I_2 \otimes \cdots \otimes I_2}_N,$$

where

$$Q_j := \sigma_1 \otimes \overline{P}_{j1} + \sigma_2 \otimes \overline{P}_{j2}$$

acting in $\mathbb{C}^2 \otimes L^2(\mathbb{R}^{2N})$. We also note that under the identification $\mathbb{C}^2 \otimes L^2(\mathbb{R}^{2N}) = L^2(\mathbb{R}^{2N}; \mathbb{C}^2)$, Q_j is written as

$$Q_j = \begin{pmatrix} 0 & Q_j^- \\ Q_j^+ & 0 \end{pmatrix}$$

with

$$Q_j^\pm := \bar{P}_{j1} \pm i\bar{P}_{j2}.$$

Hence, $Q_{j,\min}$ is written as

$$Q_{j,\min} = \overbrace{\sigma_3 \otimes \cdots \otimes \sigma_3 \otimes Q_{j,\min} \otimes I_2 \otimes \cdots \otimes I_2}^N, \\ \text{\scriptsize } j \text{ th}$$

on $\mathbb{C}^{2(j-1)} \otimes_{\text{alg}} (\mathbb{C}^2 \otimes_{\text{alg}} C_0^\infty(\mathcal{M}_N)) \otimes_{\text{alg}} \mathbb{C}^{2(N-j)}$, where we set

$$Q_{j,\min} := Q_j \upharpoonright \mathbb{C}^2 \otimes_{\text{alg}} C_0^\infty(\mathcal{M}_N).$$

Let

$$Q_{j,\min}^\pm := Q_j^\pm \upharpoonright C_0^\infty(\mathcal{M}_N).$$

Then we have

$$\bar{Q}_{j,\min} = \begin{pmatrix} 0 & \bar{Q}_{j,\min}^- \\ \bar{Q}_{j,\min}^+ & 0 \end{pmatrix},$$

where $\bar{Q}_{j,\min}$ and $\bar{Q}_{j,\min}^\pm$ are the closure of $Q_{j,\min}$ and $Q_{j,\min}^\pm$, respectively.

Now we introduce the operators defined by

$$Q_{j,\min}^{(1)} := \overbrace{\sigma_3 \otimes \cdots \otimes \sigma_3 \otimes Q_{j,\min}^{(1)} \otimes I_2 \otimes \cdots \otimes I_2}^N, \\ \text{\scriptsize } j \text{ th}$$

$$Q_{j,\min}^{(1)} := \begin{pmatrix} 0 & Q_{j,\min}^{+*} \\ \bar{Q}_{j,\min}^+ & 0 \end{pmatrix},$$

and

$$Q_{j,\min}^{(2)} := \overbrace{\sigma_3 \otimes \cdots \otimes \sigma_3 \otimes Q_{j,\min}^{(2)} \otimes I_2 \otimes \cdots \otimes I_2}^N,$$

$$Q_{j,\min}^{(2)} := \begin{pmatrix} 0 & \overline{Q_{j,\min}^-} \\ Q_{j,\min}^{-*} & 0 \end{pmatrix}.$$

It is not difficult to show that $Q_{j,\min}^{(1)}$ and $Q_{j,\min}^{(2)}$ are self-adjoint extensions of $Q_{j,\min}$.

To state the main result of this section, we introduce some notations. Let \mathbb{Z}_+ be the set of non-negative integers. We introduce

$$W_{j,\pm} := \left\{ (p, k_1, \dots, k_N) \in \mathbb{Z}_+ \times \mathbb{Z}^N \mid p + \sum_{\nu=1}^N k_\nu < \pm \frac{\Phi_j}{\theta_0}, \right. \\ \left. k_\nu > \pm \frac{\gamma_{\nu j}}{\theta_0}, \nu \neq j \right\},$$

$$\Phi_j := \sum_{(l,m) \in \Lambda_j} \gamma_{lm}$$

and set

$$N_{j,\pm}(N; q) := \#W_{j,\pm},$$

the cardinality of $W_{j,\pm}$. We have

$$N_{j,-}(N; q) = N_{j,+}(N; -q).$$

Theorem 5.2 *For each $j = 1, \dots, N$, the following hold:*

- (i) $\ker(Q_{j,\min}^{(1)}) \neq \{0\}$ if and only if $N_{j,-}(N; q) \neq 0$.
- (ii) $\ker(Q_{j,\min}^{(2)}) \neq \{0\}$ if and only if $N_{j,+}(N; q) \neq 0$.
- (iii) If the magnetic flux is partially quantized with respect to Λ_j , then $\ker(Q_{j,\min}^{(\alpha)}) = \{0\}$ ($\alpha = 1, 2$).

We will give a proof of this theorem later.

Definition 5.3 Let $\Pi_N := \{\mathcal{H}, \tau, H, \{Q\}_{j=1}^N\}$ be a model of N -fold supersymmetric quantum mechanics.

- (i) If each Q_j has no zero-energy states, then we say that N -fold supersymmetry of Π_N is *broken*.

- (ii) Let E be a subset of $\{1, \dots, N\}$. If each Q_j ($j \in E$) has no zero-energy states and each Q_k ($k \in E^c := \{1, \dots, N\} \setminus E$) has zero-energy states, then we say that N -fold supersymmetry of Π_N is *partially broken with respect to E* .

Corollary 5.4 *Let*

$$\Pi_N^{(\alpha)} := \{\mathbb{C}^{2^N} \otimes L^2(\mathbb{R}^{2N}), \tau_N, \widehat{H}_N(\mathbb{A}), \{Q_{j,\min}^{(\alpha)}\}_{j=1}^N\} \quad (\alpha = 1, 2).$$

Let E be a subset of $\{1, \dots, N\}$ such that $\bigcup_{j \in E} \Lambda_j \neq \Lambda^{(N)}$. If the magnetic flux is partially quantized with respect to $\bigcup_{j \in E} \Lambda_j$ and for each $j \in E^c$, $N_{j,-}(N; q) \neq 0$ (resp. $N_{j,+}(N; q) \neq 0$), then the N -fold supersymmetry of $\Pi_N^{(1)}$ (resp. $\Pi_N^{(2)}$) is partially broken with respect to E .

To prove Theorem 5.2, we need some preparations.

We want to investigate the zero-energy states of $Q_{j,\min}^{(\alpha)}$ ($\alpha = 1, 2$). Since

$$\ker(Q_{j,\min}^{(\alpha)}) = \mathbb{C}^{2^{(j-1)}} \otimes \ker(Q_{j,\min}^{(\alpha)}) \otimes \mathbb{C}^{2^{(N-j)}}, \quad (\alpha = 1, 2),$$

it suffices to investigate the zero-energy states of $Q_{j,\min}^{(\alpha)}$ ($\alpha = 1, 2$).

Lemma 5.5 *Relative to the fibre direct integral decomposition (7), $\overline{Q}_{j,\min}^{\pm}$ is decomposable.*

Proof. Let \mathcal{N}_j ($j = 1, \dots, N$) be the von Neumann algebra generated by the diagonalizable operators on $L^2(\mathbb{R}^{2N}) = \int_{\Omega_j}^{\oplus} L^2(\mathbb{R}_j^2) d\omega_j$. Then it is easy to prove that

$$\mathcal{N}_j \subseteq (\overline{Q}_{j,\min}^{\pm})'_s,$$

where, for a linear operator T , $(T)'_s$ denotes the strong commutant of T (see Proposition B.9). Hence, by Proposition B.9 (i), we have the desired result. \square

By the above lemma, $\overline{Q}_{j,\min}^{\pm}$ can be represented as

$$\overline{Q}_{j,\min}^{\pm} = \int_{\Omega_j}^{\oplus} \overline{Q}_{j,\min}^{\pm}(\omega_j) d\omega_j.$$

Therefore, our next task is to determine the fibre $\overline{Q}_{j,\min}^{\pm}(\omega_j)$ of $\overline{Q}_{j,\min}^{\pm}$. For this purpose, we introduce

$$D_j^{(\pm)}(\omega) := (\overline{P}_{j1}(\omega) \pm i\overline{P}_{j2}(\omega)) [C_0^{\infty}(\mathbb{R}_j^2(\omega))$$

for $\omega = (\mathbf{a}_1, \dots, \widehat{\mathbf{a}}_j, \dots, \mathbf{a}_N) \in \Omega_j$, where

$$\mathbb{R}_j^2(\omega) := \mathbb{R}^2 \setminus \{\mathbf{a}_i\}_{i \neq j}.$$

For each $\Psi \in C_0^\infty(\mathcal{M}_N)$ and $\omega = (\mathbf{a}_1, \dots, \widehat{\mathbf{a}}_j, \dots, \mathbf{a}_N) \in \Omega_j$ ($\mathbf{a}_i \neq \mathbf{a}_k$, if $i \neq k$), we define a vector $\tilde{\Psi}(\omega) \in C_0^\infty(\mathbb{R}^2(\omega))$ defined by

$$\tilde{\Psi}(\omega)(\mathbf{r}_j) := \Psi(\mathbf{a}_1, \dots, \mathbf{a}_{j-1}, \mathbf{r}_j, \mathbf{a}_{j+1}, \dots, \mathbf{a}_N).$$

Moreover, we introduce

$$C_0^\infty(\mathcal{M}_N)|_\omega := \{\tilde{\Psi}(\omega) \in C_0^\infty(\mathbb{R}_j^2(\omega)) \mid \Psi \in C_0^\infty(\mathcal{M}_N)\}.$$

Lemma 5.6 For each $\omega = (\mathbf{a}_1, \dots, \widehat{\mathbf{a}}_j, \dots, \mathbf{a}_N) \in \Omega_j$ ($\mathbf{a}_i \neq \mathbf{a}_k$, if $i \neq k$), we have

$$C_0^\infty(\mathcal{M}_N)|_\omega = C_0^\infty(\mathbb{R}_j^2(\omega)).$$

Proof. For each $f \in C_0^\infty(\mathbb{R}_j^2(\omega))$, we define a vector $T_f \in C^\infty(\mathbb{R}^{2N})$ by

$$T_f(\mathbf{r}_1, \dots, \mathbf{r}_N) = f(\mathbf{r}_j), \quad (\mathbf{r}_1, \dots, \mathbf{r}_N) \in \mathbb{R}^{2N}.$$

Taking a vector $\eta \in C_0^\infty(\mathcal{M}_N)$ such that $\text{supp } \tilde{\eta}(\omega) \supseteq \text{supp } f$ and $\tilde{\eta}(\omega)(\mathbf{r}_j) = 1$ ($\mathbf{r}_j \in \text{supp } f$), we can show that $T_f \eta \in C_0^\infty(\mathcal{M}_N)$ and

$$\widetilde{(T_f \eta)}(\omega) = f \tilde{\eta}(\omega) = f.$$

Therefore, we can conclude that $f \in C_0^\infty(\mathcal{M}_N)|_\omega$ which imply $C_0^\infty(\mathbb{R}_j^2(\omega)) \subseteq C_0^\infty(\mathcal{M}_N)|_\omega$.

Conversely, for each $\Psi \in C_0^\infty(\mathcal{M}_N)$, it is clear that $\tilde{\Psi}(\omega) \in C_0^\infty(\mathbb{R}_j^2(\omega))$ which implies $C_0^\infty(\mathbb{R}_j^2(\omega)) \supseteq C_0^\infty(\mathcal{M}_N)|_\omega$. \square

Proposition 5.7 For each $j = 1, \dots, N$, we have

$$\overline{\mathcal{Q}}_{j,\min}^\pm(\omega) = \overline{D}_j^\pm(\omega) \quad \text{a.e. } \omega$$

Proof. For a linear operator T , we denote its graph by $\text{gr}(T)$. Since $C_0^\infty(\mathcal{M}_N)$ is a core of $\overline{\mathcal{Q}}_{j,\min}^\pm$, we have

$$\text{gr}(\overline{\mathcal{Q}}_{j,\min}^\pm) = \{\Psi \oplus \overline{\mathcal{Q}}_{j,\min}^\pm \Psi \mid \Psi \in C_0^\infty(\mathcal{M}_N)\}^-,$$

where we consider the graph norm topology. Hence it follows from the definition of the decomposable operator (i.e., Definition B.7) that

$$\text{gr}(\overline{\mathcal{Q}}_{j,\min}^\pm(\omega)) = \{\tilde{\Psi}(\omega) \oplus \overline{\mathcal{Q}}_{j,\min}^\pm(\omega) \tilde{\Psi}(\omega) \mid \Psi \in C_0^\infty(\mathcal{M}_N)\}^-$$

for a.e. ω . Combining this with Lemma 5.6, we have

$$\begin{aligned} \text{gr}(\overline{\mathbf{Q}}_{j,\min}^{\pm}(\omega)) &= \{\Psi \oplus \overline{\mathbf{Q}}_{j,\min}^{\pm}(\omega)\Psi \mid \Psi \in C_0^\infty(\mathcal{M}_N)|_\omega\}^- \\ &= \{\Psi \oplus \overline{D}_j^{(\pm)}(\omega)\Psi \mid \Psi \in C_0^\infty(\mathbb{R}_j^2(\omega))\}^- \\ &= \text{gr}(\overline{D}_j^{(\pm)}(\omega)) \end{aligned}$$

for a.e. ω , where we use the fact

$$\overline{\mathbf{Q}}_{j,\min}^{\pm}(\omega)\tilde{\Psi}(\omega) = \overline{D}_j^{(\pm)}(\omega)\tilde{\Psi}(\omega) \quad \text{a.e. } \omega$$

for $\Psi \in C_0^\infty(\mathcal{M}_N)$. Hence, we have the desired result. \square

Let A be a closed linear operator on a Hilbert space \mathcal{X} . We introduce a following notation:

$$L[A] := \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}.$$

Note that $L[A]$ is a self-adjoint operator acting in $\mathcal{X} \oplus \mathcal{X}$.

Theorem 5.8 *For each $j = 1, \dots, N$, we have*

$$\mathbf{Q}_{j,\min}^{(1)} = \int_{\Omega_j}^{\oplus} L[\overline{D}_j^{(+)}(\omega)] d\omega, \quad \mathbf{Q}_{j,\min}^{(2)} = \int_{\Omega_j}^{\oplus} L[D_j^{(-)}(\omega)^*] d\omega.$$

Therefore, under the identification (13), we have

$$\begin{aligned} \mathbf{Q}_{j,\min}^{(1)} &= \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{j-1} \otimes \int_{\Omega_j}^{\oplus} L[\overline{D}_j^{(+)}(\omega)] d\omega \otimes \underbrace{I_2 \otimes \cdots \otimes I_2}_{N-j}, \\ \mathbf{Q}_{j,\min}^{(2)} &= \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{j-1} \otimes \int_{\Omega_j}^{\oplus} L[D_j^{(-)}(\omega)^*] d\omega \otimes \underbrace{I_2 \otimes \cdots \otimes I_2}_{N-j}. \end{aligned}$$

Proof. This is a simple application of Proposition 5.7. \square

Remark 5.9 The self-adjoint operators $L[\overline{D}_j^{(+)}(\omega)]$ and $L[D_j^{(-)}(\omega)^*]$ are called *Arai's supercharge* [2].

Corollary 5.10 *For each $j = 1, \dots, N$, we have*

$$\ker(\mathbf{Q}_{j,\min}^{(1)}) = \int_{\Omega_j}^{\oplus} \ker(L[\overline{D}_j^{(+)}(\omega)]) d\omega,$$

$$\ker(Q_{j,\min}^{(2)}) = \int_{\Omega_j}^{\oplus} \ker(L[D_j^{(-)}(\omega)^*]) \, d\omega.$$

Therefore, under the identification (13), we have

$$\begin{aligned} \ker(Q_{j,\min}^{(1)}) &= \mathbb{C}^{2^{(j-1)}} \otimes \int_{\Omega_j}^{\oplus} \ker(L[\bar{D}_j^{(+)}(\omega)]) \, d\omega \otimes \mathbb{C}^{2^{(N-j)}}, \\ \ker(Q_{j,\min}^{(2)}) &= \mathbb{C}^{2^{(j-1)}} \otimes \int_{\Omega_j}^{\oplus} \ker(L[D_j^{(-)}(\omega)^*]) \, d\omega \otimes \mathbb{C}^{2^{(N-j)}}. \end{aligned}$$

The following lemma is proven by A. Arai in [2].

Lemma 5.11 *For each $j = 1, \dots, N$, the following hold:*

- (i) $\dim \ker(L[\bar{D}_j^{(+)}(\omega)]) = N_-(N; q)$,
 $\dim \ker(L[D_j^{(-)}(\omega)^*]) = N_+(N; q)$.
- (ii) *If the magnetic flux is partially quantized with respect to Λ_j , then*

$$\ker(L[\bar{D}_j^{(+)}(\omega)]) = \{0\}, \quad \ker(L[D_j^{(-)}(\omega)^*]) = \{0\}$$

Proof. See [2, Theorem 4.7]. □

Now, we are ready to prove Theorem 5.2.

Proof of Theorem 5.2.

- (i) By Corollary 5.10, Lemma 5.11 and Proposition B.10 (iii), we have

$$\begin{aligned} \ker(Q_{j,\min}^{(1)}) \neq \{0\} &\iff \ker(L[\bar{D}_j^{(+)}(\omega)]) \neq \{0\} \quad \text{a.e. } \omega \\ &\iff N_{j,-}(N; q) \neq 0. \end{aligned}$$

Similarly, we can prove (ii).

- (iii) If the magnetic flux is partially quantized with respect to Λ_j , then, it follows from Lemma 5.11 that

$$\ker(L[\bar{D}_j^{(+)}(\omega)]) = \{0\}, \quad \ker(L[D_j^{(-)}(\omega)^*]) = \{0\} \quad \text{a.e. } \omega.$$

Hence by Proposition B.10 (iii), we have the desired assertion. □

A. Strongly anticommuting self-adjoint operators on a \mathbb{Z}_2 -graded Hilbert space

Let \mathbb{Z}_2 be the residue class ring mod 2, with the elements $\bar{0}$ and $\bar{1}$. When applied to elements of \mathbb{Z}_2 , the symbol “+” always denotes addition

modulo 2.

Suppose that \mathcal{H} is a Hilbert space. If \mathcal{H} is a direct sum of $\mathcal{H}_{\bar{0}}$ and $\mathcal{H}_{\bar{1}}$ (i.e., $\mathcal{H} = \mathcal{H}_{\bar{0}} \oplus \mathcal{H}_{\bar{1}}$), where $\mathcal{H}_{\bar{0}}$ and $\mathcal{H}_{\bar{1}}$ are closed subspaces of \mathcal{H} , then \mathcal{H} is said to be \mathbb{Z}_2 -graded Hilbert space. Throughout this section, we fix the \mathbb{Z}_2 -grading $\mathcal{H} = \mathcal{H}_{\bar{0}} \oplus \mathcal{H}_{\bar{1}}$. Let $P_{\bar{0}}$ and $P_{\bar{1}}$ be the orthogonal projections onto $\mathcal{H}_{\bar{0}}$ and $\mathcal{H}_{\bar{1}}$, respectively. We define an operator τ on \mathcal{H} by

$$\tau := P_{\bar{0}} - P_{\bar{1}}.$$

It is not difficult to see that τ is self-adjoint and unitary. We refer to the operator τ as the *grading operator* for \mathcal{H} .

We introduce

$$\mathcal{L}(\mathcal{H}) := \{B : \text{a linear operator on } \mathcal{H} \text{ s.t. } P_{\alpha} \text{dom}(B) \subset \text{dom}(B) \text{ for each } \alpha \in \mathbb{Z}_2\}.$$

Note that $\mathcal{B}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ and hence $\mathcal{L}(\mathcal{H}) \neq \emptyset$. If $B \in \mathcal{L}(\mathcal{H})$ satisfies

$$B = \tau B \tau,$$

then B is said to be *even*. On the other hand, if

$$B = -\tau B \tau,$$

then B is said to be *odd*. An element in $\mathcal{L}(\mathcal{H})$ is said to be *homogeneous*, if it is even or odd. For a homogeneous operator B in $\mathcal{L}(\mathcal{H})$, we define

$$\rho(B) := \begin{cases} 0 & \text{if } B \text{ is even} \\ 1 & \text{if } B \text{ is odd} \end{cases}.$$

The value $\rho(B)$ is called the *parity* of B .

For an element B in $\mathcal{L}(\mathcal{H})$, we introduce

$$\begin{aligned} B_{\bar{0}} &:= P_{\bar{0}} B P_{\bar{0}} + P_{\bar{1}} B P_{\bar{1}}, & \text{dom}(B_{\bar{0}}) &= \text{dom}(B), \\ B_{\bar{1}} &:= P_{\bar{0}} B P_{\bar{1}} + P_{\bar{1}} B P_{\bar{0}}, & \text{dom}(B_{\bar{1}}) &= \text{dom}(B). \end{aligned}$$

Then it is clear that $B_{\bar{0}}$ (resp. $B_{\bar{1}}$) is even (resp. odd) and

$$B = B_{\bar{0}} + B_{\bar{1}}.$$

We say that $B_{\bar{0}}$ (resp. $B_{\bar{1}}$) is *even* (resp. *odd*) *part* of B . Note that if B is self-adjoint, then $B_{\bar{0}}$ and $B_{\bar{1}}$ are also self-adjoint.

For each A, B in $\mathcal{L}(\mathcal{H})$, we define

$$[A, B]_S \Psi = [A_0, B_0] \Psi + [A_0, B_1] \Psi + [A_1, B_0] \Psi + \{A_1, B_1\} \Psi,$$

for $\Psi \in \text{dom}(AB) \cap \text{dom}(BA)$, where $[a, b] := ab - ba$ (commutator) and $\{a, b\} := ab + ba$ (anticommutator). The operation $[\cdot, \cdot]_S$ is said to be *super-commutator*.

Let $M(\mathbb{R})$ be the set of all complex valued Borel measurable functions on \mathbb{R} . If $f \in M(\mathbb{R})$ satisfies

$$f(-x) = f(x)$$

for all $x \in \mathbb{R}$, then f is said to be *even*. If f satisfies

$$f(-x) = -f(x)$$

for all $x \in \mathbb{R}$, then f is said to be *odd*. We can also define the notions of homogenous element and parity of Borel measurable functions by the same way as in the case of linear operators.

Proposition A.1 *Suppose that A is a homogeneous self-adjoint operator on \mathcal{H} and f is a homogeneous element in $M(\mathbb{R})$. Then $f(A)$ is also homogeneous with*

$$\rho(f(A)) = \rho(f)\rho(A).$$

Here $f(A)$ is given by the operational calculus.

Proof. Since τ is unitary and self-adjoint, we have

$$\tau f(A) \tau = f(\tau A \tau) = f((-1)^{\rho(A)} A) = (-1)^{\rho(A)\rho(f)} f(A)$$

by the operational calculus. □

Given a self-adjoint operator S on a Hilbert space, we denote its spectral measure by $E_S(J)$ for an arbitrary J in \mathbb{B}^1 (the Borel field of \mathbb{R}).

Let A be an odd self-adjoint operator on \mathcal{H} . Then, by Proposition A.1, we have

$$E_A(J)_{\bar{0}} = \frac{1}{2}\{E_A(J) + E_A(-J)\}, \quad E_A(J)_{\bar{1}} = \frac{1}{2}\{E_A(J) - E_A(-J)\}$$

for each $J \in \mathbb{B}^1$, where $-J := \{-x \in \mathbb{R} \mid x \in J\}$.

Definition A.2 Suppose that A and B are *odd* self-adjoint operators on \mathcal{H} . We say that A and B *strongly anticommute*, when

$$[E_A(J_1), E_B(J_2)]_S = 0$$

for each J_1 and J_2 in \mathbb{B}^1 .

It is not hard to see that A and B strongly anticommute if and only if

$$\begin{aligned} [E_A(J_1)_{\bar{0}}, E_B(J_2)_{\bar{0}}] &= [E_A(J_1)_{\bar{0}}, E_B(J_2)_{\bar{1}}] = [E_A(J_1)_{\bar{1}}, E_B(J_2)_{\bar{0}}] = 0, \\ \{E_A(J_1)_{\bar{1}}, E_B(J_2)_{\bar{1}}\} &= 0 \end{aligned}$$

for each J_1, J_2 in \mathbb{B}^1 .

Following theorem is a fundamental characterization of the strong anti-commutativity.

Theorem A.3 Let A and B be *odd* self-adjoint operators on \mathcal{H} . The following conditions are equivalent to each other.

- (i) A and B strongly anticommute.
- (ii) $[e^{isA}, e^{itB}]_S = 0$ for each s, t in \mathbb{R} .
- (iii) $[R_z(A), R_w(B)]_S = 0$ for each z, w in $\mathbb{C} \setminus \mathbb{R}$, where $R_z(T) := (T - z)^{-1}$.

Proof. See [12]. □

Proposition A.4 Suppose that A, B, A_1, \dots, A_n are *odd* self-adjoint operators on \mathcal{H} .

- (i) If A and B are bounded, then A and B strongly anticommute if and only if

$$\{A, B\} = 0.$$

- (ii) If $\{A_j\}_{j=1}^n$ is a family of strongly anticommuting self-adjoint operators, then $\sum_{j=1}^n A_j$ is self-adjoint and

$$\left(\sum_{j=1}^n A_j \right)^2 = \sum_{j=1}^n A_j^2.$$

- (iii) If A and B strongly anticommute, then for each $\Psi \in \text{dom}(A) \cap \text{dom}(B)$,

$$\{A, B\}\Psi = 0.$$

- (iv) Let A_1, A_2, A_3, A_4 be *odd* operators in \mathcal{H} . If $\{A_1, A_2, A_3, A_4\}$ is a family of strongly anticommuting self-adjoint operators, then two

self-adjoint operators $B := A_1 + A_2$ and $C := A_3 + A_4$ strongly anti-commute.

Proof. (i) This follows from Theorem A.3.

(ii), (iii) See [14].

(iv) By Theorem A.3, we have

$$[e^{isA_j}, e^{itA_k}]_{\mathcal{S}} = 0 \quad (14)$$

for all $s, t \in \mathbb{R}$ and $j, k = 1, \dots, 4$ ($j \neq k$). On the other hand, we have

$$e^{isB} = \text{s-lim}_{n \rightarrow \infty} (e^{isA_1/n} e^{isA_2/n})^n, \quad e^{itC} = \text{s-lim}_{n \rightarrow \infty} (e^{itA_3/n} e^{itA_4/n})^n \quad (15)$$

by the Trotter product formula. Using (14), it is not hard to show that

$$\left[(e^{isA_1/n} e^{isA_2/n})^n, (e^{itA_3/n} e^{itA_4/n})^n \right]_{\mathcal{S}} = 0$$

for each $n \in \mathbb{N}$. Combining this with (15), we can prove that

$$[e^{isB}, e^{itC}]_{\mathcal{S}} = 0$$

for all $s, t \in \mathbb{R}$. Hence we have the desired result by Theorem A.3. \square

Proposition A.5 *Let \mathcal{K} be a Hilbert space. Let $\{A_j\}_{j=1}^n$ be a family of odd self-adjoint operators on \mathcal{H} and $\{B_j\}_{j=1}^n$ be a family of self-adjoint operators on \mathcal{K} .*

(i) *The Hilbert space $\mathcal{H} \otimes \mathcal{K}$ has a following natural \mathbb{Z}_2 -grading structure:*

$$\mathcal{H} \otimes \mathcal{K} = (\mathcal{H}_0 \otimes \mathcal{K}) \oplus (\mathcal{H}_1 \otimes \mathcal{K}). \quad (16)$$

Relative to this \mathbb{Z}_2 -grading, the operator

$$\tau_{\otimes} := \tau \otimes I_{\mathcal{K}}$$

is the grading operator for $\mathcal{H} \otimes \mathcal{K}$, where τ is the grading operator for \mathcal{H} .

- (ii) *The self-adjoint operator $A_j \otimes B_j$ ($j = 1, \dots, n$) is odd (relative to the \mathbb{Z}_2 -grading (16)).*
- (iii) *If $\{A_j\}_{j=1}^n$ is strongly anticommuting and $\{B_j\}_{j=1}^n$ is strongly commuting, then $\{A_j \otimes B_j\}_{j=1}^n$ is strongly anticommuting.*

Proof. (i) and (ii) are very easy.

(iii) By the spectral theorem for strongly commuting self-adjoint operators (the unbounded version of [9, problem VII, 4]), there is a σ -finite

measure space (M, ν) , unitary operator $U: \mathcal{K} \rightarrow L^2(M, d\nu)$ and real-valued Borel functions F_1, \dots, F_n on M with

$$[(UB_jU^{-1})f](m) = F_j(m)f(m).$$

Hence, we can identify $\mathcal{H} \otimes \mathcal{K}$ with $\int_M^\oplus \mathcal{H} d\nu$. Under this identification, we have

$$A_j \otimes B_j = \int_M^\oplus F_j(m)A_j d\nu(m).$$

Since $\{F_j(m)A_j\}_{j=1}^n$ is a family of strongly anticommuting self-adjoint operators, we have the desired assertion by Proposition B.13 (iii). \square

B. Decomposable operators

Let (Λ, μ) be a σ -finite measure space. Let

$$\mathcal{K} := \int_\Lambda^\oplus \mathcal{H} d\mu(\lambda)$$

be the direct integral of \mathcal{H} which will be fixed in what follows.

First, we begin with the following lemma.

Lemma B.1 *Let $\{\phi_n \mid n \in \mathbb{N}\} \subset \mathcal{K}$ be a family of vectors in \mathcal{K} .*

- (i) *If a family of vectors $\{\phi_n(\lambda) \mid n \in \mathbb{N}\} \subset \mathcal{H}$ is total in \mathcal{H} for μ -a.e. λ (i.e., linear span of $\{\phi_n(\lambda) \mid n \in \mathbb{N}\}$ is dense in \mathcal{H} for μ -a.e. λ), then $\{f\phi_n \mid f \in L^\infty(\Lambda, d\mu), n \in \mathbb{N}\}$ is total in \mathcal{K} .*
- (ii) *If $\{\phi_n \mid n \in \mathbb{N}\}$ is total in \mathcal{K} , then $\{\phi_n(\lambda) \mid n \in \mathbb{N}\}$ is total in \mathcal{H} for μ -a.e. λ .*

Proof. (i) We set

$$\mathcal{D} := \text{Lin}\{f\phi_n \mid f \in L^\infty(\Lambda, d\mu)\}^-,$$

where, for each subset \mathcal{V} , $\text{Lin}(\mathcal{V})$ means the subspace algebraically generated by \mathcal{V} . For each $\Psi \in \mathcal{D}^\perp$, $f \in L^\infty(\Lambda, d\mu)$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} 0 &= \langle \Psi, f\phi_n \rangle_{\mathcal{K}} \\ &= \int_\Lambda f(\lambda) \langle \Psi(\lambda), \phi_n(\lambda) \rangle_{\mathcal{H}} d\mu(\lambda) \\ &= \int_\Lambda g_n(\lambda)^* f(\lambda) d\mu(\lambda), \end{aligned}$$

where $g_n(\lambda) := \langle \phi_n(\lambda), \Psi(\lambda) \rangle_{\mathcal{H}}$. Note that $g_n \in L^1(\Lambda, d\mu)$. Since f is arbitrary, we can take f to be a real valued function. Then we have $\langle \operatorname{Im}(g_n), f \rangle_{L^2(\Lambda, d\mu)} = 0$. Hence, taking $f = \operatorname{sgn}(\operatorname{Im}(g_n))$, we get

$$\int_{\Lambda} |\operatorname{Im}(g_n(\lambda))| d\mu(\lambda) = 0,$$

which implies

$$\operatorname{Im}(g_n) = 0. \quad (17)$$

On the other hand, if we take $f = ih$, where h is a real valued function, then we have $\langle \operatorname{Re}(g_n), h \rangle_{L^2(\Lambda, d\mu)} = 0$. Therefore, taking $h = \operatorname{sgn}(\operatorname{Re}(g_n))$, we have

$$\int_{\Lambda} |\operatorname{Re}(g_n(\lambda))| d\mu(\lambda) = 0.$$

Hence we conclude that $\operatorname{Re}(g_n) = 0$. Combining this with (17), we obtain that $g_n = 0$. Since $\{\phi_n(\lambda) \mid n \in \mathbb{N}\}$ is total in \mathcal{H} for μ -a.e. λ , we have

$$\begin{aligned} g_n = 0 &\iff g_n(\lambda) = 0 \quad \mu\text{-a.e. } \lambda \\ &\iff \Psi(\lambda) = 0 \quad \mu\text{-a.e. } \lambda. \end{aligned}$$

Hence $\mathcal{D}^{\perp} = \{0\}$.

(ii) Since (Λ, μ) is σ -finite, there is a family of measurable sets $\{B_n \mid n \in \mathbb{N}\}$ such that

$$\Lambda = \bigcup_{n=1}^{\infty} B_n, \quad \mu(B_n) < \infty. \quad (18)$$

For each $\phi \in \mathcal{H}$, we set $\Phi := \chi_{B_n} \phi$. Then it is clear that $\Phi \in \mathcal{K}$. Therefore, from the assumption, it follows that there is a sequence $\{\Phi_n\}_{n=1}^{\infty} \subset \operatorname{Lin}\{\phi_n \mid n \in \mathbb{N}\}$ such that $\|\Phi - \Phi_n\|_{\mathcal{K}} \rightarrow 0$ ($n \rightarrow \infty$). Hence, there exists a subsequence $\{\Phi_{n_k}\}_k$ such that

$$\|\Phi(\lambda) - \Phi_{n_k}(\lambda)\|_{\mathcal{H}} \rightarrow 0 \quad \mu\text{-a.e. } \lambda,$$

which implies that

$$\|\phi - \Phi_{n_k}(\lambda)\| \rightarrow 0 \quad \mu\text{-a.e. } \lambda \text{ in } B_n.$$

Since $\{\Phi_{n_k}(\lambda)\}_k \subset \operatorname{Lin}\{\phi_n(\lambda) \mid n \in \mathbb{N}\}$ μ -a.e. λ , we conclude that $\{\phi_n \mid n \in \mathbb{N}\}$ is total in \mathcal{H} for μ -a.e. λ in B_n . Combining this with (18), we have the

desired result. \square

Let $F: \Lambda \rightarrow \mathcal{H}$ be a \mathcal{H} -valued mapping on Λ . We say that F is *measurable* if \mathbb{C} -valued function $\lambda \rightarrow \langle \Psi, F(\lambda) \rangle_{\mathcal{H}}$ is measurable for each $\Psi \in \mathcal{H}$.

For a $\mathcal{B}(\mathcal{H})$ -valued mapping $B: \Lambda \ni \lambda \rightarrow B(\lambda) \in \mathcal{B}(\mathcal{H})$, we say that the mapping $\lambda \rightarrow B(\lambda)$ is *measurable* if the mapping $\lambda \rightarrow \langle \phi, B(\lambda)\phi \rangle_{\mathcal{H}}$ is measurable for each $\phi \in \mathcal{H}$.

Definition B.2 For each $\lambda \in \Lambda$, let $\mathcal{D}(\lambda)$ be the closed subspace of \mathcal{H} and $P(\lambda)$ be an orthogonal projection onto $\mathcal{D}(\lambda)$. If the mapping $\lambda \rightarrow P(\lambda)$ is measurable, then we say that the field $\lambda \rightarrow \mathcal{D}(\lambda)$ is *measurable*.

Proposition B.3 *The field $\lambda \rightarrow \mathcal{D}(\lambda)$ is measurable if and only if there is a sequence $\{\phi_n\}_{n=1}^{\infty} \subset \mathcal{K}$ such that $\{\phi_n(\lambda)\}_{n=1}^{\infty}$ is total in $\mathcal{D}(\lambda)$ for μ -a.e. λ .*

Proof. If the field $\lambda \rightarrow \mathcal{D}(\lambda)$ is measurable, then a linear operator $P := \int_{\Lambda}^{\oplus} P(\lambda) d\mu(\lambda)$ is an orthogonal projection. Since \mathcal{K} is separable, so is $\text{ran}(P)$. Hence we can choose a sequence $\{\phi_n\}_{n=1}^{\infty} \subset \text{ran}(P)$ which is total in $\text{ran}(P)$. Applying Lemma B.1, we can conclude that $\{\phi_n(\lambda)\}_{n=1}^{\infty}$ is total in $\mathcal{D}(\lambda)$ for μ -a.e. λ .

Conversely, assume that there is a sequence $\{\phi_n\}_{n=1}^{\infty} \subset \mathcal{K}$ such that $\{\phi_n(\lambda)\}_{n=1}^{\infty}$ is total in $\mathcal{D}(\lambda)$ for μ -a.e. λ . By Lemma B.1 (i), $\{f\phi_n \mid f \in L^{\infty}(\Lambda, d\mu), n \in \mathbb{N}\}$ is total in \mathcal{K} . On the other hand, for each $\Psi \in \text{Lin}\{f\phi_n \mid f \in L^{\infty}(\Lambda, d\mu), n \in \mathbb{N}\}$, we can easily check that

$$P(\lambda)\Psi(\lambda) = \Psi(\lambda).$$

Hence, the mapping $\lambda \rightarrow P(\lambda)\Psi(\lambda)$ is measurable. Therefore we can conclude that the mapping $\lambda \rightarrow P(\lambda)\Psi(\lambda)$ is measurable for each Ψ in \mathcal{K} . By this, we can easily prove that $\lambda \rightarrow P(\lambda)$ is measurable. \square

Definition B.4 For each $\lambda \in \Lambda$, let $\mathcal{D}(\lambda)$ be a closed subspace of \mathcal{H} and $P(\lambda)$ be the orthogonal projection onto $\mathcal{D}(\lambda)$. If the field $\lambda \rightarrow \mathcal{D}(\lambda)$ is measurable, we can define the orthogonal projection $P := \int_{\Lambda}^{\oplus} P(\lambda) d\mu(\lambda)$. The closed subspace $\text{ran}(P)$ is called the *direct integral of $\mathcal{D}(\lambda)$* and denoted by $\int_{\Lambda}^{\oplus} \mathcal{D}(\lambda) d\mu(\lambda)$.

Proposition B.5 *Suppose that the fields $\lambda \rightarrow \mathcal{G}(\lambda)$ and $\lambda \rightarrow \mathcal{D}(\lambda)$ are measurable. Let*

$$\mathcal{G} := \int_{\Lambda}^{\oplus} \mathcal{G}(\lambda) d\mu(\lambda), \quad \mathcal{D} = \int_{\Lambda}^{\oplus} \mathcal{D}(\lambda) d\mu(\lambda).$$

Then the following hold.

- (i) $\mathcal{G}^\perp = \int_\Lambda^\oplus \mathcal{G}(\lambda)^\perp d\mu(\lambda)$.
- (ii) $\mathcal{G} \subseteq \mathcal{D} \iff \mathcal{G}(\lambda) \subseteq \mathcal{D}(\lambda) \text{ } \mu\text{-a.e. } \lambda$.
- (iii) $\mathcal{G} \cap \mathcal{D} = \int_\Lambda^\oplus \mathcal{G}(\lambda) \cap \mathcal{D}(\lambda) d\mu(\lambda)$.
- (iv) $\mathcal{G} = \{0\} \iff \mathcal{G}(\lambda) = \{0\} \text{ } \mu\text{-a.e. } \lambda$.

Proof. Easy (or see [11]). □

Lemma B.6 Let $\lambda \rightarrow \mathcal{G}(\lambda)$ be a measurable field of closed subspaces of

$$\mathcal{K} \oplus \mathcal{K} = \int_\Lambda^\oplus \mathcal{H} \oplus \mathcal{H} d\mu(\lambda)$$

and let

$$\mathcal{G} := \int_\Lambda^\oplus \mathcal{G}(\lambda) d\mu(\lambda).$$

Then \mathcal{G} is the graph of a closed linear operator in \mathcal{K} if and only if $\mathcal{G}(\lambda)$ is the graph of a closed linear operator in \mathcal{H} μ -a.e. λ .

Proof. By Proposition B.5 (iii), we have

$$\mathcal{G} \cap (\{0\} \oplus \mathcal{K}) = \int_\Lambda^\oplus \mathcal{G}(\lambda) \cap (\{0\} \oplus \mathcal{H}) d\mu(\lambda).$$

Hence

$$\mathcal{G} \cap (\{0\} \oplus \mathcal{K}) = \{0 \oplus 0\} \iff \mathcal{G}(\lambda) \cap (\{0\} \oplus \mathcal{H}) = \{0 \oplus 0\} \text{ } \mu\text{-a.e. } \lambda,$$

which means the desired assertion in the proposition. □

For a linear operator T , we denote its graph by $\text{gr}(T)$.

Definition B.7 (i) For each $\lambda \in \Lambda$, let $A(\lambda)$ be a closed operator on \mathcal{H} .

We say that the mapping $\lambda \rightarrow A(\lambda)$ is said to be *measurable* if the field $\lambda \rightarrow \text{gr}(A(\lambda))$ is measurable.

(ii) Let $\lambda \rightarrow A(\lambda)$ be a measurable field of closed operators. Then, by Lemma B.6, there exists a closed operator A on \mathcal{K} such that

$$\text{gr}(A) = \int_\Lambda^\oplus \text{gr}(A(\lambda)) d\mu(\lambda).$$

We say that the closed operator A is *decomposable* and denote it by

$$A = \int_{\Lambda}^{\oplus} A(\lambda) \, d\mu(\lambda).$$

(iii) Let $\lambda \rightarrow A(\lambda)$ be a measurable field of closed operators. If

$$A(\lambda) = f(\lambda)I_{\mathcal{H}}$$

with $f(\lambda) \in \mathbb{C}$, for each $\lambda \in \Lambda$, then we say that the closed operator $\int_{\Lambda}^{\oplus} A(\lambda) \, d\mu(\lambda)$ is *diagonalizable*.

Remark B.8 Let $A = \int_{\Lambda}^{\oplus} A(\lambda) \, d\mu(\lambda)$ be a decomposable operator. Then we can easily check that

$$\text{dom}(A) = \left\{ \phi \in \mathcal{K} \mid \begin{array}{l} \phi(\lambda) \in \text{dom}(A(\lambda)) \quad \mu\text{-a.e. } \lambda, \\ \int_{\Lambda} \|A(\lambda)\phi(\lambda)\|_{\mathcal{H}}^2 \, d\mu(\lambda) < \infty \end{array} \right\}$$

and

$$(A\phi)(\lambda) = A(\lambda)\phi(\lambda) \quad \mu\text{-a.e. } \lambda,$$

for each $\phi \in \text{dom}(A)$.

Let A be a closed operator on \mathcal{K} . We set

$$(A)'_s := \left\{ B \in \mathcal{B}(\mathcal{K}) \mid \begin{array}{l} B \text{ dom}(A) \subseteq \text{dom}(A), \\ BA\phi = AB\phi \quad (\phi \in \text{dom}(A)) \end{array} \right\},$$

where $\mathcal{B}(\mathcal{K})$ denotes the set of all bounded operators on \mathcal{K} . We say $(A)'_s$ the *strong commutant* of A .

The following proposition is often useful.

Proposition B.9 *Let $\mathcal{N}(\mathcal{K})$ be the abelian von Neumann algebra of bounded diagonalizable operators on \mathcal{K} and $\mathcal{R}(\mathcal{K})$ be the von Neumann algebra of bounded decomposable operators on \mathcal{K} . Let A be a closed operator on \mathcal{K} .*

- (i) *A is decomposable if and only if $\mathcal{N}(\mathcal{K}) \subseteq (A)'_s$.*
- (ii) *A is diagonalizable if and only if $\mathcal{R}(\mathcal{K}) \subseteq (A)'_s$.*

Proof. See [11]. □

Proposition B.10 Let $A = \int_{\Lambda}^{\oplus} A(\lambda) d\mu(\lambda)$ and $B = \int_{\Lambda}^{\oplus} B(\lambda) d\mu(\lambda)$ be decomposable operators on \mathcal{K} . Then:

- (i) $A \subseteq B$ if and only if $A(\lambda) \subseteq B(\lambda)$ for μ -a.e. λ .
- (ii) $A = B$ if and only if $A(\lambda) = B(\lambda)$ for μ -a.e. λ .
- (iii) $\ker(A) = \int_{\Lambda}^{\oplus} \ker(A(\lambda)) d\mu(\lambda)$
- (iv) $A^* = \int_{\Lambda}^{\oplus} A(\lambda)^* d\mu(\lambda)$.

Proof. Easy (or see [11]). □

Proposition B.11 Let $A = \int_{\Lambda}^{\oplus} A(\lambda) d\mu(\lambda)$ be a decomposable operator on \mathcal{K} . Then the following hold.

- (i) A is symmetric if and only if $A(\lambda)$ is symmetric for μ -a.e. λ .
- (ii) A is self-adjoint if and only if $A(\lambda)$ is self-adjoint for μ -a.e. λ .
- (iii) A is unitary if and only if $A(\lambda)$ is unitary for μ -a.e. λ .

Proof. These are simple applications of Proposition B.10. □

Proposition B.12 Let $A = \int_{\Lambda}^{\oplus} A(\lambda) d\mu(\lambda)$ and $B = \int_{\Lambda}^{\oplus} B(\lambda) d\mu(\lambda)$ be self-adjoint decomposable operators on \mathcal{K} . Then A and B strongly commute if and only if $A(\lambda)$ and $B(\lambda)$ strongly commute for μ -a.e. λ .

Proof. By Proposition B.10 (ii), we have

A and B strongly commute

$$\iff [e^{isA}, e^{itB}] = 0 \quad \text{for each } s, t \in \mathbb{R}.$$

$$\iff [e^{isA(\lambda)}, e^{itB(\lambda)}] = 0 \quad \mu\text{-a.e. } \lambda \quad \text{for each } s, t \in \mathbb{R}.$$

$$\iff A(\lambda) \text{ and } B(\lambda) \text{ strongly commute for } \mu\text{-a.e. } \lambda. \quad \square$$

Proposition B.13 Let $\mathcal{H} = \mathcal{H}_{\bar{0}} \oplus \mathcal{H}_{\bar{1}}$ be a \mathbb{Z}_2 -graded Hilbert space and let $A = \int_{\Lambda}^{\oplus} A(\lambda) d\mu(\lambda)$ and $B = \int_{\Lambda}^{\oplus} B(\lambda) d\mu(\lambda)$ be decomposable operators on $\mathcal{K} = \int_{\Lambda}^{\oplus} \mathcal{H} d\mu(\lambda)$. Then the following hold:

- (i) The Hilbert space \mathcal{K} has the following natural \mathbb{Z}_2 -grading structure:

$$\mathcal{K} = \mathcal{K}_{\bar{0}} \oplus \mathcal{K}_{\bar{1}},$$

$$\mathcal{K}_{\bar{0}} := \int_{\Lambda}^{\oplus} \mathcal{H}_{\bar{0}} d\mu(\lambda), \quad \mathcal{K}_{\bar{1}} := \int_{\Lambda}^{\oplus} \mathcal{H}_{\bar{1}} d\mu(\lambda).$$

The grading operator for $\mathcal{K} = \mathcal{K}_{\bar{0}} \oplus \mathcal{K}_{\bar{1}}$ is given by $\tau_{\mathcal{K}} = \int_{\Lambda}^{\oplus} \tau_{\mathcal{H}} d\mu(\lambda)$, where $\tau_{\mathcal{H}}$ is the grading operator for \mathcal{H} .

- (ii) A is odd if and only if $A(\lambda)$ is odd for μ -a.e. λ .

- (iii) Suppose that A and B are odd self-adjoint operators. Then A and B strongly anticommute if and only if $A(\lambda)$ and $B(\lambda)$ strongly anticommute for μ -a.e. λ .
- (iv) Suppose that A and B are odd self-adjoint operators. If A and B strongly anticommute, then a self-adjoint operator $C := A + B$ is decomposable with

$$C = \int_{\Lambda}^{\oplus} (A(\lambda) + B(\lambda)) d\mu(\lambda).$$

Proof. (i) Easy.

(ii) By Proposition B.10 (ii), we have

A is odd.

$$\iff \tau_{\mathcal{K}} A \tau_{\mathcal{K}} = -A.$$

$$\iff \tau_{\mathcal{H}} A(\lambda) \tau_{\mathcal{H}} = -A(\lambda) \quad \text{for } \mu\text{-a.e. } \lambda.$$

$$\iff A(\lambda) \text{ is odd for } \mu\text{-a.e. } \lambda.$$

(iii) In the same way as the proof of Proposition B.12, we can prove the assertion.

(iv) By Proposition B.9 (i), we obtain

$$(A)'_s \supseteq \mathcal{N}(\mathcal{K}), \quad (B)'_s \supseteq \mathcal{N}(\mathcal{K}).$$

Hence, we can easily check that

$$(C)'_s \supseteq (A)'_s \cap (B)'_s \supseteq \mathcal{N}(\mathcal{K}).$$

Applying Proposition B.9 (i) again, we can conclude that C is decomposable. For each $\Psi \in \text{dom}(C) = \text{dom}(A) \cap \text{dom}(B)$, we have

$$\begin{aligned} (C\Psi)(\lambda) &= [(A + B)\Psi](\lambda) \\ &= A(\lambda)\Psi(\lambda) + B(\lambda)\Psi(\lambda) \quad \mu\text{-a.e. } \lambda. \end{aligned}$$

Hence we have the desired result. \square

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